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Research article

A remark on the velocity averaging lemma of the transport equation with general case

Ming-Jiea Lyu[1] **and Baoyan Sun**[2,*]

[1] Department of Applied Mathematics, Chung Yuan Christian University, Taoyuan, Taiwan
[2] School of Mathematics and Information Sciences, Yantai University, Yantai, China

* **Correspondence:** Email: bysun@ytu.edu.cn.

Abstract: In this paper, we proved a new result for the celebrated velocity averaging lemma of the free transport equation with general case

$$\partial_t f + a(v) \cdot \nabla_x f = 0 \,.$$

After averaging with some weight functions $\varphi(v)$, we proved that the average quantity $\rho_\varphi(t, x) = \int_{\mathbb{R}^3_v} f(t, x, v)\, \varphi(v)\, \mathrm{d}v$ is in $W^{1,p}_x$, $p \in [1, +\infty]$. This result revealed the regularizing effect for the mean value with respect to the velocity of the solution. Our strategy was taking advantage of a modified vector field method to build up a bridge between the x-derivative and v-derivative. One significant point was that we first observed that the operator $t\,\nabla_x + \left([\nabla_v a(v)]^T\right)^{-1} \nabla_v$ commuted with $\partial_t + a(v) \cdot \nabla_x$.

Keywords: velocity averaging lemma; regularity; vector field method; nonrelativistic transport equation; relativistic transport equation

1. Introduction

1.1. The models and main results

Our goal of this paper is to consider the velocity averaging lemma of the following transport equation with general case $a(v)$:

$$\partial_t f + a(v) \cdot \nabla_x f = 0 \,, \qquad (t, x, v) \in \mathbb{R}_+ \times \Omega_x \times \mathbb{R}^3_v \,, \tag{1.1}$$

$$f(0, x, v) = f_{in}(x, v) \geq 0 \,, \qquad (x, v) \in \Omega_x \times \mathbb{R}^3_v \,, \tag{1.2}$$

where the function $f = f(t, x, v) \geq 0$ describes the gas density distribution of particles at time $t \in \mathbb{R}_+$, position $x = (x_1, x_2, x_3) \in \Omega_x$ and the microscopic velocity $v = (v_1, v_2, v_3) \in \mathbb{R}^3_v$. Here the spatial domain Ω_x is either the whole space $\mathbb{R}_x^3$ or the three-dimensional torus $\mathbb{T}_x^3$. The vector-valued function $a(v) = (a_1(v), a_2(v), a_3(v)) : \mathbb{R}_v^3 \to \mathbb{R}_v^3$ with $a_i(v) \in C^2(\mathbb{R}_v^3)$ $(i = 1, 2, 3)$ and $\det(\nabla_v a(v)) \neq 0$, where $\det(\nabla_v a(v))$ is the determinant of the matrix $\nabla_v a(v)$.

For the later presentation, we need to give the following notations. Let $m(v) > 0$ be some given positive weight function on $\mathbb{R}_v^3$. For the velocity variable v, we denote

$$\|f\|_{L_v^q(m(v))} = \left(\int_{\mathbb{R}_v^3} |f|^q\, m(v)^q\, \mathrm{d}v\right)^{\frac{1}{q}}, \ q \in [1, +\infty),$$

and

$$\|f\|_{L_v^\infty(m(v))} = \sup_{v \in \mathbb{R}_v^3} |f(v)|\ m(v).$$

Next, let's consider the notations involving the space variable x. The higher-order Sobolev space $W_x^{\sigma,p}$ for $\sigma \in \mathbb{N}$ ($\mathbb{N}$ means the set of natural numbers) is defined by

$$\|f\|_{W_x^{\sigma,p}} := \sum_{|\alpha| \leq \sigma} \|\partial_x^\alpha f\|_{L_x^p}.$$

Let $C_0^\infty(\mathbb{R}_v^3)$ be the space of infinitely differentiable functions with compact support.

Finally, we denote $L_v^q L_x^p(m(v))$ with exponent $p, q \in [1, +\infty]$, through the norms

$$\|f\|_{L_v^q L_x^p(m(v))} := \Big\| \|f\|_{L_x^p}\, m(v) \Big\|_{L_v^q}.$$

Now, we state the main result on the velocity averaging lemma of the free transport problem (1.1) and (1.2).

Theorem 1.1. *Consider $f \in L^1([0, T]; L_v^1 L_x^p(m(v)))$ and $\nabla_v f_{in} \in L_v^1 L_x^p(m(v))$ with $p \in [1, +\infty]$ such that the solution f satisfies the problem* (1.1) *and* (1.2) *in the weak sense. For any fixed weight fuction $\varphi(v) \in C_0^\infty(\mathbb{R}_v^3)$, let us define the average quantity with $\varphi(v)$ as*

$$\rho_\varphi(t, x) := \int_{\mathbb{R}_v^3} f(t, x, v)\, \varphi(v)\, \mathrm{d}v,$$

then ρ_φ satisfies:

$$\|\rho_\varphi\|_{W_x^{1,p}} \leq \left(1 + \frac{3\sqrt{3}}{t}\right) \|\varphi\|_{W_v^{1,\infty}} \left(\|f_{in}\|_{L_v^1 L_x^p(m(v))} + \|\nabla_v f_{in}\|_{L_v^1 L_x^p(m(v))}\right),$$

where the function $m(v) > 0$ is defined in Eq (2.8) *below.*

Remark 1.1.

(1) *This theorem extends the result of the velocity averaging lemma for the free transport equation in [21] (see Lemma 4.17 for the non-relativistic case, that is, $a(v) = v$) to a more general case. Furthermore, $\rho_\varphi \in W_x^{1,p}$ holds for whole space (see Theorem 2 in [25] for locally space with general $a(v)$ and a singular source term $(-\Delta_v)^{\frac{\alpha}{2}} g$), but in this paper, we need assume some extra regularity in v of the initial data f_{in}.*

(2) *If* $a(v) = \hat{v} = \frac{v}{\sqrt{1+|v|^2}}$*, this is the relativistic free transport equation. For this case, we can choose* $m(v) = \left(1 + |v|^2\right)^{\frac{3}{2}}$ *in Theorem 1.1; see also [35], which includes a detailed argument.*

(3) *If* $a(v) = \frac{v}{|v|}$*, this is the massless relativistic free transport equation [17]. Because* $\det\left(\nabla_v \frac{v}{|v|}\right) = 0$*, our approach cannot be adapted to this case.*

(4) *Indeed, to avoid a lengthy discussion, we can extend our results to higher space dimension without essential difficulty.*

1.2. Review of previously known works

The celebrated velocity averaging lemmas concern the regularity results of solutions to the kinetic transport equation developed in [16, 18, 20, 33, 38], which reveal that the combination of transport operator and averaging in velocity variable v of the solution yields regularity with respect to the space variable x. Such results are an interesting and powerful mathematical tool in kinetic theory that have been extensively used to obtain regularity, global weak solutions, spectral analysis, and hydrodynamic limits of the kinetic equations. It is worth mentioning that another major application consists in showing the regularizing effect of solutions wherever the kinetic formulations exist, such as the isentropic gas dynamics, the Ginzburg-Landau model, and scalar conservation laws [15, 19, 24, 26, 31, 36]. There are several types of proof provided, such as the Fourier transform, the Hörmander's commutators, the commutator method, the Harmonic analysis, the energy method, the real space method, and so on. For more on this topic, the reader may consult [1–4, 13, 14, 25, 28, 29] for detailed discussion. There have been several variants, extensions and generalizations of velocity averaging lemmas made available, such as the kinetic transport equation with a force term, time discrete kinetic equations and stochastic case, and the phenomena of dispersion and hypoellipticity. We refer to [5–10, 12, 22, 27, 37] and the references therein.

Now, we mention some literature that is relatively closer to our discussion. The velocity average lemma of the nonrelativistic transport equation has been investigated in the L^2 framework. If one considers the solutions $f \in L^2$ of the initial-value problem:

$$\partial_t f + v \cdot \nabla_x f = 0\,,$$

with suitable initial data f_{in}, then $\rho_\varphi \in H_x^{\frac{1}{2}}$ for any $\varphi(v) \in C_0^\infty(\mathbb{R}_v^3)$. Here, $H_x^{\frac{1}{2}} = W_x^{\frac{1}{2},2}$ denotes the usual fractional order Sobolev space defined by the Fourier transform. We note that there is no regularity assumption on f_{in}. See also DiPerna, Lions and Meyer [16] for general L^p, $1 < p < +\infty$ by applying the interpolation method. In [21], Gualdani, Mischler and Mouhot proved that $\rho_\varphi \in W_x^{1,p}$, $p \in [1, +\infty]$. They obtained that a full derivative in the x variable is stronger than the previous half-derivative, but they assumed some additional regularity in v of the initial data f_{in}.

Compared with the non-relativistic case, the version of the relativistic transport equation has a relatively short history. In 2004, Rein [34] proved the global weak solution of the relativistic Vlasov-Maxwell system by the velocity averaging. Huang and Jiang [23] investigated the average regularity of the solution to the relativistic transport equation by adopting the same method in [18]. The analogous result as in [21] also holds for the relativistic free transport equation in [35]. Moreover, the authors also showed the quantitative effects of the particle mass and the speed of light.

For the general case $a(v)$, Eq (1.1) is typical of kinetic formulations for the multidimensional scalar conservation laws, and also in kinetic models under relativistic and quantum setting. The hyperbolic system can be reformulated as a kinetic equation by using an additional kinetic variable. Those formulations were derived in [31] and in [11] for a more complicated situation. In [32], Lions, Perthame, and Tadmor provide kinetic formulations for the isentropic gas dynamics and the p-systems. The derivation of these regularity results employs a kinetic formulation like Eq (1.1). The typical example of an application of velocity averaging lemmas to scalar conservation laws is [19, 31]. The averaging lemmas for the Eq (1.1) were studied in [24, 33]. In [27, 28], Jabin and Vega introduced a new method that is performed in the real space to prove the velocity averaging lemma. Zhu [39] showed that in the setting of general transport operators, velocity averaging lemmas yield local boundedness and Hölder regularity of solutions under a suitable nondegeneracy condition on the rough vector field. Recently, Jabin, Lin, and Tadmor [25] found a commutator method with multipliers to prove averaging lemmas, the regularizing effect for the velocity average of solutions.

Motivated by the above works, the main contribution of the current paper is to render a simple method with the vector field inspired from the methodological approach developed in [21]. We prove a new result for the celebrated velocity averaging lemma of Eq (1.1). These results are completely new and some of our calculations appear to be new too. The key point is employing the differential operator D_t in Lemma 2.2.

1.3. Strategy of the proof of Theorem 1.1

One good strategy for the proof of the velocity averaging lemma, is to build up a bridge between the x-derivative and v-derivative. Gualdani, Mischler and Mouhot [21] proved that the operator $t\nabla_x + \nabla_v$ can commute with the nonrelativistic transport operator $\partial_t + v \cdot \nabla_x$. Thanks to this crucial differential operator $t\nabla_x + \nabla_v$, one can realize the exchange of regularity between the velocity variable v and the space variable x. However, for the relativistic model, it's difficult to find the associated operator commutes with the relativistic transport operator $\partial_t + \hat{v} \cdot \nabla_x$, even for general case. Recently, Lin, Lyu and Wu [30] significantly observed that the operator $t\nabla_x + [\nabla_v(\hat{v})]^{-1}\nabla_v$ can commute with the operator $\partial_t + \hat{v} \cdot \nabla_x$. Motivated by [21, 30], we try to find out the corresponding operator that can commute with $\partial_t + a(v) \cdot \nabla_x$. Indeed, the novelty of this paper is that we can also find an anticipant operator $t\nabla_x + \left([\nabla_v a(v)]^T\right)^{-1} \nabla_v$. This observation will play an important role in the proof of Theorem 1.1.

2. Preliminaries

In this section, we give the following lemma which will be used in the later proof. First, we will present some basic properties. For the sake of brevity and readability, the proof of Lemma 2.1 is shown in the Appendix.

Lemma 2.1. *For any σ, τ is equal to x or v, we have*

$$(a(v) \cdot \nabla_\sigma) \nabla_\tau f = \left[\nabla_\sigma (\nabla_\tau f)\right] a(v), \tag{2.1}$$

$$(a(v) \cdot \nabla_x) B \nabla_v f = B (a(v) \cdot \nabla_x) \nabla_v f = B \left[\nabla_x (\nabla_v f)\right] a(v), \tag{2.2}$$

$$\nabla_x (a(v) \cdot \nabla_x f) = \left[\nabla_x (\nabla_x f)\right] a(v), \tag{2.3}$$

and

$$\nabla_v \left(a(v) \cdot \nabla_x f\right) = \left[\nabla_v a(v)\right]^T \nabla_x f + \left[\nabla_x \left(\nabla_v f\right)\right] a(v)\,, \tag{2.4}$$

where B is a 3×3 matrix and $[\nabla_v a(v)]^T$ is the transpose of the matrix $\nabla_v a(v)$.

By means of Lemma 2.1, we prove the crucial idea, which is the cornerstone of the proof of Theorem 1.1.

Lemma 2.2. *Let us define the differential operator $D_t := t\nabla_x + \left([\nabla_v a(v)]^T\right)^{-1} \nabla_v$, then we can prove that D_t commutes with the operator $\partial_t + a(v) \cdot \nabla_x$, namely,*

$$[\partial_t + a(v) \cdot \nabla_x,\ D_t] = 0\,,$$

where $[\mathcal{A}, \mathcal{B}] = \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}$ is the commutator.

Proof. Set $B = \left([\nabla_v a(v)]^T\right)^{-1}$. Applying Eqs (2.1)–(2.4), we can get

$$\begin{aligned}
[\partial_t + a(v) \cdot \nabla_x,\ D_t]\, f =& (\partial_t + a(v) \cdot \nabla_x)\, D_t f - D_t\, (\partial_t + a(v) \cdot \nabla_x)\, f \\
=& \partial_t(t\nabla_x f + B\nabla_v f) + (a(v) \cdot \nabla_x)\,(t\nabla_x f + B\nabla_v f) \\
& - t\nabla_x\,(\partial_t f + a(v) \cdot \nabla_x f) - B\nabla_v\,(\partial_t f + a(v) \cdot \nabla_x f) \\
=& \nabla_x f + t\nabla_x\,(\partial_t f) + B\nabla_v\,(\partial_t f) + t\,(a(v) \cdot \nabla_x)\,\nabla_x f \\
& + (a(v) \cdot \nabla_x)\, B\nabla_v f - t\nabla_x\,(\partial_t f) - t\nabla_x\,(a(v) \cdot \nabla_x f) \\
& - B\nabla_v\,(\partial_t f) - B\nabla_v\,(a(v) \cdot \nabla_x f) \\
=& \nabla_x f + t\nabla_x\,(\partial_t f) + B\nabla_v\,(\partial_t f) + t\left[\nabla_x\,(\nabla_x f)\right] a(v) \\
& + B\left[\nabla_x\,(\nabla_v f)\right] a(v) - t\nabla_x\,(\partial_t f) - t\left[\nabla_x\,(\nabla_x f)\right] a(v) \\
& - B\nabla_v\,(\partial_t f) - B\left[\nabla_x\,(\nabla_v f)\right] a(v) - B\left[\nabla_v a(v)\right]^T \nabla_x f \\
=& \nabla_x f - B\left[\nabla_v a(v)\right]^T \nabla_x f\,.
\end{aligned}$$

Since $\det(\nabla_v a(v)) \neq 0$, then $B\,[\nabla_v a(v)]^T = I$. Therefore,

$$[\partial_t + a(v) \cdot \nabla_x,\ D_t]\, f = 0\,.$$

□

Next, we obtain a priori estimate.

Lemma 2.3. *For any positive weight function $\omega(v) > 0$, and for any $t \geq 0$, $1 \leq p, q \leq +\infty$, then the solution f of the problem* (1.1) *and* (1.2) *verifies*

$$\|f(t, x, v)\|_{L^q_v L^p_x(\omega(v))} = \|f_{in}(x, v)\|_{L^q_v L^p_x(\omega(v))}\ , \tag{2.5}$$

and

$$\|D_t f(t, x, v)\|_{L^q_v L^p_x(\omega(v))} = \|B\nabla_v f_{in}(x, v)\|_{L^q_v L^p_x(\omega(v))}\ . \tag{2.6}$$

Proof. We consider first that $1 \leq p, q < +\infty$,

$$\frac{d}{dt}\, \|f\|_{L^q_v L^p_x(\omega(v))} = \frac{d}{dt}\left(\int_{\mathbb{R}^3_v} \left(\int_{\Omega_x} |f|^p\, dx\right)^{\frac{q}{p}}\ \omega(v)^q\, \mathrm{d}v\right)^{\frac{1}{q}}$$

$$
\begin{aligned}
&= \|f\|^{1-q}_{L^q_v L^p_x(\omega(v))} \int_{\mathbb{R}^3_v} \|f\|^{q-p}_{L^p_x} \Big(\int_{\Omega_x} |f|^{p-1} \operatorname{sign}(f)\, \partial_t f \, dx \Big) \omega(v)^q \, \mathrm{d}v \\
&= - \|f\|^{1-q}_{L^q_v L^p_x(\omega(v))} \int_{\mathbb{R}^3_v} \|f\|^{q-p}_{L^p_x} \Big(\int_{\Omega_x} \frac{1}{p} a(v) \cdot \nabla_x (|f|^p) \, dx \Big) \omega(v)^q \, \mathrm{d}v \\
&= 0 \, .
\end{aligned} \tag{2.7}
$$

Following the integration by parts, the term involving $a(v) \cdot \nabla_x$ vanishes.

Similarly, by taking the limits $p \to +\infty$ and $q \to +\infty$ in Eq (2.7), the cases $p = +\infty$ and $q = +\infty$ are still holds.

Thus,

$$
\|f\|_{L^q_v L^p_x(\omega(v))} = \|f_{in}\|_{L^q_v L^p_x(\omega(v))} \, .
$$

According to Lemma 2.2, we know that the differential operator $D_t := t\nabla_x + B\nabla_v$ can commute with $\partial_t + a(v) \cdot \nabla_x$. Thus,

$$
\partial_t (D_t f) + a(v) \cdot \nabla_x (D_t f) = 0 \, .
$$

By taking the similar arguments as employed in Eq (2.7), it turns out that

$$
\begin{aligned}
\frac{d}{dt} \|D_t f\|_{L^q_v L^p_x(\omega(v))} &= - \|D_t f\|^{1-q}_{L^q_v L^p_x(\omega(v))} \int_{\mathbb{R}^3_v} \|D_t f\|^{q-p}_{L^p_x} \Big(\int_{\Omega_x} \frac{1}{p} a(v) \cdot \nabla_x (|D_t f|^p) \, dx \Big) \omega(v)^q \, \mathrm{d}v \\
&= 0 \, .
\end{aligned}
$$

Consequently, we have

$$
\|D_t f\|_{L^q_v L^p_x(\omega(v))} = \|D_{t=0} f_{in}\|_{L^q_v L^p_x(\omega(v))} = \|B \nabla_v f_{in}\|_{L^q_v L^p_x(\omega(v))} \, .
$$

Thus, we conclude the proof of Lemma 2.3. □

Finally, let's introduce the weight function $m(v) > 0$ in Theorem 1.1. For clarity, we write

$$
\begin{aligned}
[\nabla_v a(v)]^T &= (a_{ij}(v))_{1 \le i,j \le 3} \\
&= \begin{bmatrix} \partial_{v_1} a_1(v) & \partial_{v_1} a_2(v) & \partial_{v_1} a_3(v) \\ \partial_{v_2} a_1(v) & \partial_{v_2} a_2(v) & \partial_{v_2} a_3(v) \\ \partial_{v_3} a_1(v) & \partial_{v_3} a_2(v) & \partial_{v_3} a_3(v) \end{bmatrix},
\end{aligned}
$$

where the entry $a_{ij}(v) = \partial_{v_i} a_j(v)$, $i, j = 1, 2, 3$.

A straightforward computation gives

$$
\begin{aligned}
B &= \Big([\nabla_v a(v)]^T \Big)^{-1} = (b_{ij}(v))_{1 \le i,j \le 3} \\
&= \frac{1}{\det \Big([\nabla_v a(v)]^T \Big)} \begin{bmatrix} a_{22} a_{33} - a_{23} a_{32} & a_{13} a_{32} - a_{12} a_{33} & a_{12} a_{23} - a_{13} a_{22} \\ a_{23} a_{31} - a_{21} a_{33} & a_{11} a_{33} - a_{13} a_{31} & a_{13} a_{21} - a_{11} a_{23} \\ a_{21} a_{32} - a_{22} a_{31} & a_{12} a_{31} - a_{11} a_{32} & a_{11} a_{22} - a_{12} a_{21} \end{bmatrix}.
\end{aligned}
$$

Next, we set

$$
m(v) = \max \Big\{ \max_{1 \le i,j \le 3} \{ |b_{ij}(v)| \}, \max_{1 \le i,j \le 3} \{ |\partial_{v_j} b_{ij}(v)| \} \Big\} > 0. \tag{2.8}
$$

3. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. In order to present it clear, we shall divide it into two steps.

Proof. Step 1: We estimate the L_x^p-norm of ρ_φ. By using the Minkowski's integral inequality and estimate Eq (2.5), we obtain that, for any $1 \le p < +\infty$,

$$\begin{aligned}
\|\rho_\varphi\|_{L_x^p} &= \left(\int_{\Omega_x} \left| \int_{\mathbb{R}_v^3} f(t,x,v)\,\varphi(v)\,\mathrm{d}v \right|^p \mathrm{d}x \right)^{\frac{1}{p}} \\
&\le \int_{\mathbb{R}_v^3} \left(\int_{\Omega_x} |f(t,x,v)\,\varphi(v)|^p \,\mathrm{d}x \right)^{\frac{1}{p}} \mathrm{d}v \\
&\le \|\varphi\|_{L_v^\infty} \|f\|_{L_v^1 L_x^p} \\
&= \|\varphi\|_{L_v^\infty} \|f_{in}\|_{L_v^1 L_x^p}\,.
\end{aligned} \tag{3.1}$$

The case $p = +\infty$ can be proved by the straightforward calculations as the reasoning above; hence, we omit the details.

Step 2: We estimate the derivatives in x. We first compute the x-derivatives of ρ_φ as follows, for any $i \in \{1,2,3\}$,

$$\begin{aligned}
\partial_{x_i}\rho_\varphi &= \int_{\mathbb{R}_v^3} \partial_{x_i} f\,\varphi(v)\,\mathrm{d}v \\
&= \int_{\mathbb{R}_v^3} \frac{1}{t} \Big[D_{t_i} - (b_{i1}\partial_{v_1} + b_{i2}\partial_{v_2} + b_{i3}\partial_{v_3}) \Big] f\,\varphi(v)\,\mathrm{d}v \\
&= \frac{1}{t} \int_{\mathbb{R}_v^3} D_{t_i} f\,\varphi(v)\,\mathrm{d}v + \frac{1}{t} \int_{\mathbb{R}_v^3} f \Big[\partial_{v_1}(b_{i1}\varphi(v)) + \partial_{v_2}(b_{i2}\varphi(v)) + \partial_{v_3}(b_{i3}\varphi(v)) \Big] \mathrm{d}v \\
&\le \frac{1}{t} \int_{\mathbb{R}_v^3} \left| D_{t_i} f \right| |\varphi(v)|\,\mathrm{d}v \\
&\quad + \frac{1}{t} \int_{\mathbb{R}_v^3} |f|\,m(v) \Big[(|\partial_{v_1}\varphi(v)| + |\partial_{v_2}\varphi(v)| + |\partial_{v_3}\varphi(v)|) + |\varphi(v)| \Big] \mathrm{d}v \\
&\le \frac{1}{t} \|\varphi(v)\|_{L_v^\infty} \|D_{t_i} f\|_{L_v^1} + \frac{1}{t} \|\varphi(v)\|_{W_v^{1,\infty}} \|f\|_{L_v^1(m(v))}\,.
\end{aligned}$$

Next, we continue by the Minkowski's integral inequality once again and the equality (2.6). It holds that

$$\begin{aligned}
\|\partial_{x_i}\rho_\varphi\|_{L_x^p} &\le \frac{1}{t} \|\varphi(v)\|_{L_v^\infty} \left(\int_{\Omega_x} \|D_{t_i} f\|_{L_v^1}^p \,\mathrm{d}x \right)^{\frac{1}{p}} + \frac{1}{t} \|\varphi(v)\|_{W_v^{1,\infty}} \left(\int_{\Omega_x} \|f\|_{L_v^1(m(v))}^p \,\mathrm{d}x \right)^{\frac{1}{p}} \\
&\le \frac{1}{t} \|\varphi(v)\|_{L_v^\infty} \|D_{t_i} f\|_{L_v^1 L_x^p} + \frac{1}{t} \|\varphi(v)\|_{W_v^{1,\infty}} \|f\|_{L_v^1 L_x^p(m(v))} \\
&= \frac{1}{t} \|\varphi(v)\|_{L_v^\infty} \|(b_{i1}\partial_{v_1} + b_{i2}\partial_{v_2} + b_{i3}\partial_{v_3}) f_{in}\|_{L_v^1 L_x^p} \\
&\quad + \frac{1}{t} \|\varphi(v)\|_{W_v^{1,\infty}} \|f_{in}\|_{L_v^1 L_x^p(m(v))}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\sqrt{3}}{t} \|\varphi(v)\|_{L_v^\infty} \|\nabla_v f_{in}\|_{L_v^1 L_x^p(m(v))} + \frac{1}{t} \|\varphi(v)\|_{W_v^{1,\infty}} \|f_{in}\|_{L_v^1 L_x^p(m(v))} \\
&\leq \frac{\sqrt{3}}{t} \|\varphi(v)\|_{W_v^{1,\infty}} \left(\|f_{in}\|_{L_v^1 L_x^p(m(v))} + \|\nabla_v f_{in}\|_{L_v^1 L_x^p(m(v))} \right) . \qquad (3.2)
\end{aligned}
$$

We now gather the above estimates Eqs (3.1) and (3.2) together to deduce that, for any $p \in [1, +\infty]$,

$$
\begin{aligned}
\|\rho_\varphi\|_{W_x^{1,p}} =& \|\rho_\varphi\|_{L_x^p} + \sum_{i=1}^{3} \|\partial_{x_i} \rho_\varphi\|_{L_x^p} \\
\leq& \left(1 + \frac{3\sqrt{3}}{t} \right) \|\varphi(v)\|_{W_v^{1,\infty}} \left(\|f_{in}\|_{L_v^1 L_x^p(m(v))} + \|\nabla_v f_{in}\|_{L_v^1 L_x^p(m(v))} \right) .
\end{aligned}
$$

Hence the proof of Theorem 1.1 is finished. □

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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Appendix A

In this appendix, we shall give a complete proof of Lemma 2.1.

Proof of the equality (2.1). We only show for the case $\sigma = x$ and $\tau = v$. Since

$$
\begin{aligned}
(a(v)\cdot\nabla_x)\nabla_v f &= (a_1(v)\partial_{x_1} + a_2(v)\partial_{x_2} + a_3(v)\partial_{x_3})\begin{bmatrix}\partial_{v_1} f\\ \partial_{v_2} f\\ \partial_{v_3} f\end{bmatrix}\\
&= \begin{bmatrix} a_1(v)\partial_{x_1}\partial_{v_1} f + a_2(v)\partial_{x_2}\partial_{v_1} f + a_3(v)\partial_{x_3}\partial_{v_1} f\\ a_1(v)\partial_{x_1}\partial_{v_2} f + a_2(v)\partial_{x_2}\partial_{v_2} f + a_3(v)\partial_{x_3}\partial_{v_2} f\\ a_1(v)\partial_{x_1}\partial_{v_3} f + a_2(v)\partial_{x_2}\partial_{v_3} f + a_3(v)\partial_{x_3}\partial_{v_3} f\end{bmatrix}\\
&= \begin{bmatrix}\partial_{x_1}\partial_{v_1} f & \partial_{x_2}\partial_{v_1} f & \partial_{x_3}\partial_{v_1} f\\ \partial_{x_1}\partial_{v_2} f & \partial_{x_2}\partial_{v_2} f & \partial_{x_3}\partial_{v_2} f\\ \partial_{x_1}\partial_{v_3} f & \partial_{x_2}\partial_{v_3} f & \partial_{x_3}\partial_{v_3} f\end{bmatrix}\begin{bmatrix}a_1(v)\\ a_2(v)\\ a_3(v)\end{bmatrix},
\end{aligned}
$$

then we have

$$
(a(v)\cdot\nabla_x)\nabla_v f = \left[\nabla_x(\nabla_v f)\right] a(v).
$$

Proof of the equality (2.2).

$$
\begin{aligned}
(a(v)\cdot\nabla_x) B\nabla_v f &= (a_1(v)\partial_{x_1} + a_2(v)\partial_{x_2} + a_3(v)\partial_{x_3})\begin{bmatrix} b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{bmatrix}\begin{bmatrix}\partial_{v_1} f\\ \partial_{v_2} f\\ \partial_{v_3} f\end{bmatrix}\\
&= (a_1(v)\partial_{x_1} + a_2(v)\partial_{x_2} + a_3(v)\partial_{x_3})\begin{bmatrix} b_{11}\partial_{v_1} f + b_{12}\partial_{v_2} f + b_{13}\partial_{v_3} f\\ b_{21}\partial_{v_1} f + b_{22}\partial_{v_2} f + b_{23}\partial_{v_3} f\\ b_{31}\partial_{v_1} f + b_{32}\partial_{v_2} f + b_{33}\partial_{v_3} f\end{bmatrix}\\
&= \begin{bmatrix} a_1(v)b_{11}\partial_{x_1}\partial_{v_1} f + a_1(v)b_{12}\partial_{x_1}\partial_{v_2} f + \cdots + a_3(v)b_{12}\partial_{x_3}\partial_{v_2} f + a_3(v)b_{13}\partial_{x_3}\partial_{v_3} f\\ a_1(v)b_{21}\partial_{x_1}\partial_{v_1} f + a_1(v)b_{22}\partial_{x_1}\partial_{v_2} f + \cdots + a_3(v)b_{22}\partial_{x_3}\partial_{v_2} f + a_3(v)b_{23}\partial_{x_3}\partial_{v_3} f\\ a_1(v)b_{31}\partial_{x_1}\partial_{v_1} f + a_1(v)b_{32}\partial_{x_1}\partial_{v_2} f + \cdots + a_3(v)b_{32}\partial_{x_3}\partial_{v_2} f + a_3(v)b_{33}\partial_{x_3}\partial_{v_3} f\end{bmatrix}\\
&= \begin{bmatrix} b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{bmatrix}\begin{bmatrix}\partial_{x_1}\partial_{v_1} f & \partial_{x_2}\partial_{v_1} f & \partial_{x_3}\partial_{v_1} f\\ \partial_{x_1}\partial_{v_2} f & \partial_{x_2}\partial_{v_2} f & \partial_{x_3}\partial_{v_2} f\\ \partial_{x_1}\partial_{v_3} f & \partial_{x_2}\partial_{v_3} f & \partial_{x_3}\partial_{v_3} f\end{bmatrix}\begin{bmatrix}a_1(v)\\ a_2(v)\\ a_3(v)\end{bmatrix}\\
&= B\left[\nabla_x(\nabla_v f)\right] a(v).
\end{aligned}
$$

Proof of the equality (2.3).

$$
\nabla_x(a(v)\cdot\nabla_x f) = \begin{bmatrix}\partial_{x_1}(a_1(v)\partial_{x_1} f + a_2(v)\partial_{x_2} f + a_3(v)\partial_{x_3} f)\\ \partial_{x_2}(a_1(v)\partial_{x_1} f + a_2(v)\partial_{x_2} f + a_3(v)\partial_{x_3} f)\\ \partial_{x_3}(a_1(v)\partial_{x_1} f + a_2(v)\partial_{x_2} f + a_3(v)\partial_{x_3} f)\end{bmatrix}
$$

$$
\begin{aligned}
&= \begin{bmatrix} a_1(v)\partial_{x_1}\partial_{x_1} f + a_2(v)\partial_{x_1}\partial_{x_2} f + a_3(v)\partial_{x_1}\partial_{x_3} f \\ a_1(v)\partial_{x_2}\partial_{x_1} f + a_2(v)\partial_{x_2}\partial_{x_2} f + a_3(v)\partial_{x_2}\partial_{x_3} f \\ a_1(v)\partial_{x_3}\partial_{x_1} f + a_2(v)\partial_{x_3}\partial_{x_2} f + a_3(v)\partial_{x_3}\partial_{x_3} f \end{bmatrix} \\
&= \begin{bmatrix} \partial_{x_1}\partial_{x_1} f & \partial_{x_1}\partial_{x_2} f & \partial_{x_1}\partial_{x_3} f \\ \partial_{x_2}\partial_{x_1} f & \partial_{x_2}\partial_{x_2} f & \partial_{x_2}\partial_{x_3} f \\ \partial_{x_3}\partial_{x_1} f & \partial_{x_3}\partial_{x_2} f & \partial_{x_3}\partial_{x_3} f \end{bmatrix} \begin{bmatrix} a_1(v) \\ a_2(v) \\ a_3(v) \end{bmatrix} \\
&= \nabla_x \left(\nabla_x f\right) a(v) .
\end{aligned}
$$

Proof of the equality (2.4).

$$
\begin{aligned}
\nabla_v \left(a(v) \cdot \nabla_x f\right) &= \begin{bmatrix} \partial_{v_1}\left(a_1(v)\partial_{x_1} f + a_2(v)\partial_{x_2} f + a_3(v)\partial_{x_3} f\right) \\ \partial_{v_2}\left(a_1(v)\partial_{x_1} f + a_2(v)\partial_{x_2} f + a_3(v)\partial_{x_3} f\right) \\ \partial_{v_3}\left(a_1(v)\partial_{x_1} f + a_2(v)\partial_{x_2} f + a_3(v)\partial_{x_3} f\right) \end{bmatrix} \\
&= \begin{bmatrix} \partial_{v_1} a_1(v)\partial_{x_1} f + \partial_{v_1} a_2(v)\partial_{x_2} f + \partial_{v_1} a_3(v)\partial_{x_3} f \\ \partial_{v_2} a_1(v)\partial_{x_1} f + \partial_{v_2} a_2(v)\partial_{x_2} f + \partial_{v_2} a_3(v)\partial_{x_3} f \\ \partial_{v_3} a_1(v)\partial_{x_1} f + \partial_{v_3} a_2(v)\partial_{x_2} f + \partial_{v_3} a_3(v)\partial_{x_3} f \end{bmatrix} \\
&\quad + \begin{bmatrix} a_1(v)\partial_{v_1}\partial_{x_1} f + a_2(v)\partial_{v_1}\partial_{x_2} f + a_3(v)\partial_{v_1}\partial_{x_3} f \\ a_1(v)\partial_{v_2}\partial_{x_1} f + a_2(v)\partial_{v_2}\partial_{x_2} f + a_3(v)\partial_{v_2}\partial_{x_3} f \\ a_1(v)\partial_{v_3}\partial_{x_1} f + a_2(v)\partial_{v_3}\partial_{x_2} f + a_3(v)\partial_{v_3}\partial_{x_3} f \end{bmatrix} \\
&= \begin{bmatrix} \partial_{v_1} a_1(v) & \partial_{v_1} a_2(v) & \partial_{v_1} a_3(v) \\ \partial_{v_2} a_1(v) & \partial_{v_2} a_2(v) & \partial_{v_2} a_3(v) \\ \partial_{v_3} a_1(v) & \partial_{v_3} a_2(v) & \partial_{v_3} a_3(v) \end{bmatrix} \begin{bmatrix} \partial_{x_1} f \\ \partial_{x_2} f \\ \partial_{x_3} f \end{bmatrix} \\
&\quad + \begin{bmatrix} \partial_{x_1}\partial_{v_1} f & \partial_{x_2}\partial_{v_1} f & \partial_{x_3}\partial_{v_1} f \\ \partial_{x_1}\partial_{v_2} f & \partial_{x_2}\partial_{v_2} f & \partial_{x_3}\partial_{v_2} f \\ \partial_{x_1}\partial_{v_3} f & \partial_{x_2}\partial_{v_3} f & \partial_{x_3}\partial_{v_3} f \end{bmatrix} \begin{bmatrix} a_1(v) \\ a_2(v) \\ a_3(v) \end{bmatrix} \\
&= \left[\nabla_v a(v)\right]^T \nabla_x f + \left[\nabla_x \left(\nabla_v f\right)\right] a(v) .
\end{aligned}
$$