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Research article

Approximation of solutions to integro-differential time fractional wave equations in L^p-space

Yongqiang Zhao[1] **and Yanbin Tang**[1,2,*]

[1] School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan, Hubei, 430074, China

[2] Hubei Key Laboratory of Engineering Modeling and Scientific Computing, Huazhong University of Science and Technology, Wuhan, Hubei, 430074, China

* **Correspondence:** Email: tangyb@hust.edu.cn.

Abstract: In this paper, we investigate the abstract integro-differential time-fractional wave equation with a small positive parameter ε. The $L^p - L^q$ estimates for the resolvent operator family are obtained using the Laplace transform, the Mittag-Leffler operator family, and the C_0-semigroup. These estimates serve as the foundation for some fixed point theorems that demonstrate the local-in-time existence of the solution in weighted function space. We first demonstrate that, for acceptable indices $p \in [1, +\infty)$ and $s \in (1, +\infty)$, the mild solution of the approximation problem converges to the solution of the associated limit problem in $L^p((0, T), L^s(\mathbf{R}^n))$ as $\varepsilon \to 0^+$. The resolvent operator family and a set of kernel $k(t)$ assumptions form the foundation of the proof's primary methodology for evaluating norms. Moreover, we consider the asymptotic behavior of solutions as $\alpha \to 2^-$.

Keywords: time fractional derivative; integro-differential equation; time fractional wave equation; heterogeneous parameter; homogeneous limit problem; analytic semigroup; C_0-semigroup

1. Introduction

In this paper we will consider an initial value problem (IVP) to the following integro-differential time-fractional hyperbolic equations with a tiny positive parameter ε in a suitable L^p-space

$$\begin{cases} D_t^\alpha u_\varepsilon(t) - \chi(\varepsilon)\Delta u_\varepsilon(t) - \Delta(k^\varepsilon * u_\varepsilon)(t) = f_\varepsilon(t), \ t > 0, x \in \mathbf{R}^n, \\ u_\varepsilon(0) = u_{0,\varepsilon}, u'_\varepsilon(0) = u_{1,\varepsilon}, \ x \in \mathbf{R}^n, \end{cases} \tag{1.1}$$

where Δ is Laplacian in $\mathbf{R}^n$, the kernel $k(t)$ is a continuous function from $\mathbf{R}$ to $\mathbf{R}$ and

$$k^\varepsilon(t) = \frac{1}{\varepsilon}k(\frac{t}{\varepsilon}), \ (k^\varepsilon * u_\varepsilon)(t) = \int_0^t k^\varepsilon(t-s)u_\varepsilon(s)ds, \tag{1.2}$$

$\alpha \in (1,2)$, $D_t^\alpha u(t)$ represents the Caputo fractional derivative of order α of u, is defined by

$$D_t^\alpha u(t) = \int_0^t g_{2-\alpha}(t-s)u''(s)ds, \ g_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}, t > 0, \beta > 0, \tag{1.3}$$

and $\chi(\varepsilon)$ is a positive scalar function defined on $(0, \varepsilon_0]$ for a given real number $\varepsilon_0 > 0$ such that

$$\chi(\varepsilon) \to \chi_0 \text{ as } \varepsilon \to 0^+. \tag{1.4}$$

In mathematical physics and mechanical engineering, fractional partial differential equations are widely used. For instance, we mention the books [1,2] and [3] for the abstract evolution equations. We mention the sources [4–9] for the fractional integro-differential equations.

The theory of cosine operator functions is developed by applying the Caputo fractional derivative described in Eq (1.3) to the limit situation $\alpha = 2$, Eq (1.1) is a second order abstract Cauchy problem. We consult [10, 11] and their sources for further information on the theory of the cosine operator function. Lorenzi-Messina [12] considered the approximation of solutions to linear integro-differential parabolic Eq (1.1) in L^p-spaces and extended the findings to the corresponding non-linear equations [13], in contrast to the first order abstract Cauchy problem with $\alpha = 1$. Using almost sectorial operators, Wang et al. [14] investigated abstract fractional Cauchy problems. When $k(t)$ exhibits sub-exponential growth, Ahmed and Mohamed [15] examined the maximal regularity and continuity of the solution for the Eq (1.1) using the solution operator. The multiplier theorem has been used by Rodrigo Ponce [16] to describe the existence and uniqueness of solutions to an abstract fractional differential equation where $k(t)$ is an infinite delay in Hölder spaces. According to Conti-Pata-Squassina [17], the differential systems with memory terms that might explain the previous history of u up to time t. They primarily focused on the convergence of the reaction-diffusion equation solution over a finite time period utilizing semigroup and energy approaches. Agarwal-Santos-Uevas [18] utilized the solution operator to investigate the existence and qualitative characteristics of an analytical $\alpha-$resolvent operator for an abstract. When $B(t) = k(t)A$ is a closed linear operator for $\alpha \in (0,1)$, Santos-Henríquez-Henáandez [19] utilized perturbation theory of sectorial operators to analyze the abstract fractional integro-differential Cauchy problem. Nasser-eddine Tatar [20, 21] used the solution operator and energy approach, respectively, to examine the stability of a fractional Euler-Bernoulli problem and fractional viscoelastic telegraph issue.

Recently, a coupled system of hybrid FDEs with Caputo-Hadamard fractional derivatives was taken into consideration by P. Bedi et al. [22]. Also, they used the Dhage fixed point approach to demonstrate the existence of mild solutions. Atangana-Baleanu-Caputo fractional Volterra integro-differential equations were considered firstly by H. Khan et al. [23] by using a Mittag-Leffler kernel, they investigated the existence, stability, and numerical simulations of these equations. O. Martnez-Fuentes et al. [24] have shown asymptotic stability in the sense of operators with general analytic kernels by investigating the recently suggested fractional-order operators with general analytic kernels, leading to Lyapunov-like conclusions and a Lyapunov direct approach. A novel $\psi-$Hilfer differential equation with integral-type subsidiary conditions was studied by Asma et al. [25] using the Picard operator approach, the Banach contraction principle, and Gronwall's inequality to analyze the stability of the solution. R. Dhayal et al. [26] discussed the existence and uniqueness of the mild solution and stability criteria for a new class of Atangana-Baleanu fractional stochastic differential systems driven by fractional Brownian motion with non-instantaneous impulsive effects using resolvent family, fixed point technique, and fractional calculus. The fractional Gompertz equation was examined using the hyperbolic-numerical inverse Laplace transform approach by Gonzlez-Calderna et al. [27]. They discovered a separate fractional order formula, which is utilized to optimize the hyperbolic-NILT method's parameter together with the starting condition.

In this research, we take into account time-fractional order super-diffusion equations in Banach spaces along with α-order time-fractional wave equations. We know that the idea of the $\alpha-$resolvent family produced by the Laplace operator $\triangle$, which will be represented by $\{S_\alpha(t)\}_{t\geq 0}$, is connected to the existences of solutions of the α order fractional abstract Cauchy problem. Many studies have been conducted on the existence and some features of $\{S_\alpha(t)\}_{t\geq 0}$, see for example, [28–37]. Kim [38] also took into account the time fractional evolution equations with variable order derivatives.

The linear integro-differential wave Eq (1.1) with $\alpha \in (1,2)$ is the first thing we take into account. Our main goal is to demonstrate that the mild solution $u_\varepsilon(t)$ of problem (1.1) converges to the mild solution $u(t)$ of the following limit problem as $\varepsilon \to 0^+$ under appropriate assumptions on the function f_ε and the initial data $u_{0,\varepsilon}$, $u_{1,\varepsilon}$,

$$\begin{cases} D_t^\alpha u(t) - (1+\chi_0)\Delta u(t) = f(t), \ t \in (0,T], x \in \mathbf{R}^n, \\ u(0) = u_0, u'(0) = u_1, \ x \in \mathbf{R}^n. \end{cases} \tag{1.5}$$

Any $\alpha-$order Caputo fractional derivative is defined by

$$D_t^\alpha u(t) = \int_0^t g_{n-\alpha}(t-s)\frac{d^n}{ds^n}u(s)ds,$$

where n is the smallest integer greater than or equal to $\alpha > 0$ and Gelfand-Shilov function $g_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$ satisfies the semigroup property $g_\alpha * g_\beta = g_{\alpha+\beta}$. Similarly, the Riemann-Liouville fractional integral of order $\alpha \geq$ is defined by

$$J_t^\alpha f(t) = (g_\alpha * f)(t) = \int_0^t g_\alpha(t-s)f(s)ds,$$

thus

$$D_t^\alpha J_t^\alpha f(t) = f(t), \ J_t^\alpha D_t^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0)\frac{t^k}{k!}. \tag{1.6}$$

Applying the properties of Laplace transform and taking into account that $\widehat{g_\alpha}(\lambda) = \lambda^{-\alpha}$, and $\widehat{(J_t^\alpha f)}(\lambda) = \widehat{(g_\alpha * f)}(\lambda) = \widehat{g_\alpha}(\lambda)\hat{f}(\lambda) = \lambda^{-\alpha}\hat{f}(\lambda)$, then Eq (1.6) implies that

$$\widehat{D_t^\alpha f}(\lambda) = \lambda^\alpha \widehat{f}(\lambda) - \sum_{k=0}^{n-1} f^{(k)}(0)\lambda^{\alpha-1-k}. \tag{1.7}$$

Following the results on the linear time-fractional hyperbolic Eq (1.1) with a small positive parameter ε, we consider a semi-linear approximating problem

$$\begin{cases} D_t^\alpha u_\varepsilon(t) - \chi(\varepsilon)\Delta u_\varepsilon(t) - \Delta(k^\varepsilon * u_\varepsilon)(t) = f_\varepsilon(t) + N[u_\varepsilon](t), \ t > 0, x \in \mathbf{R}^n, \\ u_\varepsilon(0) = u_{0,\varepsilon}(x), \ u_\varepsilon'(0) = u_{1,\varepsilon}(x), \ x \in \mathbf{R}^n, \end{cases} \tag{1.8}$$

and the corresponding semi-linear limit problem as $\varepsilon \to 0^+$

$$\begin{cases} D_t^\alpha u(t) + (1+\chi_0)(-\triangle)u(t) = f(t) + N[u](t), \ t > 0, x \in \mathbf{R}^n, \\ u(0) = u_0(x), \ u'(0) = u_1(x), \ x \in \mathbf{R}^n, \end{cases} \tag{1.9}$$

where N is a nonlinear operator admitting the following representation

$$N(u)(t,x) = \psi(t, u(t,x)), \tag{1.10}$$

and ψ is given in later.

The Eq (1.9) is the semilinear heat equation for $\alpha = 1$ and $u_1 = 0$, the crucial exponent is $p = 1 + \frac{2}{n}$ when the nonlinear component is $|u|^{p-1}u$. Every nontrivial solution of the equation blows up in a finite state if $1 < p \le 1 + \frac{2}{n}$ and $u_0 \ge 0$, but if $p > 1 + \frac{2}{n}$ and the starting value u_0 is small enough in $L^{q_c}(\mathbb{R}^n)$ where $q_c = \frac{n(p-1)}{2}$, then the solution of Eq (1.9) exists globally. We mention [39] for more information on these findings.

Equation (1.9) interpolates the heat equation and the wave equation. The Fujita critical exponent of the issue is $1 + \frac{2\alpha}{\alpha n+2-2\alpha} \to 1 + \frac{2}{n}$ which is for the case $u_1 \equiv 0$ and nonlinear term $N(u)(t,x) = |u|^{p-1}$. The critical exponents $1 + \frac{2\alpha}{\alpha n+2-2\alpha}$ and $1 + \frac{2\alpha}{\alpha n-2}$ tend to $\frac{n+1}{n-1}$ as $\alpha \to 1$, which is an exponent that may be found in the work by Kato [40]. The critical exponent in the situation $u_1 \ne 0$ and nonlinear term $N(u)(t,x) = |u|^p$ is $\overline{p}$, where $\overline{p} := 1 + \frac{2}{n-2(1+\alpha)^{-1}}$. We shall demonstrate the global existence of the solution to Eq (1.9) when $p \ge \overline{p}$, and no global solution exists in the subcritical range $p \in (1, \overline{p})$. The details are available in [41].

When $\alpha = 2$, the problems (1.8) and (1.9) become the second-order abstract semi-linear approximating problem

$$\begin{cases} u_\varepsilon''(t) - \chi(\varepsilon)\Delta u_\varepsilon(t) - \Delta(k^\varepsilon * u_\varepsilon)(t) = f_\varepsilon(t) + N[u_\varepsilon](t), \ t > 0, x \in \mathbf{R}^n, \\ u_\varepsilon(0) = u_{0,\varepsilon}(x), \ u_\varepsilon'(0) = u_{1,\varepsilon}(x), \ x \in \mathbf{R}^n, \end{cases} \tag{1.11}$$

and the corresponding semi-linear limit problem as $\varepsilon \to 0^+$

$$\begin{cases} u''(t) - (1+\chi_0)\Delta u(t) = f(t) + N[u](t), \ t > 0, x \in \mathbf{R}^n, \\ u(0) = u_0(x), u'(0) = u_1(x), \ x \in \mathbf{R}^n. \end{cases} \tag{1.12}$$

The Eq (1.12) is the semilinear wave equation for $\alpha = 2$ and the nonlinear term $N(u)(t,x) = |u|^p$, the crucial exponent is $p_c(n)$, which is the positive root of $(n-1)p^2 - (n+1)p - 2 = 0$. Given that u_0, u_1 have compact support and meet a certain positivity condition, global solutions of the equation do not exist if $1 < p \le p_c(n)$, however if $p > p_c(n)$, solutions with tiny starting values exist for all time (see Yordanov [42] and the references therein). Using somewhat lower assumptions, Kato [40] obtained a little less precise conclusion. If $1 < p \le \frac{n+1}{n-1}$, then Kato demonstrated that the issue does not permit a global solution.

For Banach spaces X and Y, their norms are $\|\cdot\|_X, \|\cdot\|_Y$. For a closed linear operator $A : D(A) \subset X \to Y$, the notation $[D(A)]$ represents the domain of A endowed with graph norm $\|u\|_1 = \|u\|_X + \|Au\|_Y, u \in D(A)$. Recall that the Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} = \frac{1}{2\pi i} \int_C \frac{\mu^{\alpha-\beta} e^\mu}{\mu^\alpha - z} d\mu, \ \alpha, \beta > 0, z \in \mathbb{C}, \tag{1.13}$$

and $E_\alpha(z) = E_{\alpha,1}(z)$, where the path C is a loop which starts and ends at $-\infty$, and encircles the disc $|\mu| \leq |z|^{\frac{1}{\alpha}}$ in the positive sense, for $\mu \in \mathbb{C}$, μ^α denotes the principal branch of μ^α. Gamma function $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1}e^{-t}dt$. In the paper we will use another special Beta function $\mathrm{B} : (0,\infty) \times (0,\infty) \to (0,\infty)$ is defined by $\mathrm{B}(a,b) = \int_0^1 (1-s)^{a-1}s^{b-1}ds$. Using a subordination principle we can write the Mittag-Leffler family associated to the operator Δ in the form

$$E_\alpha(t\Delta) = \int_0^\infty \Psi_\alpha(s)e^{st^\alpha\Delta}ds, \tag{1.14}$$

where $\{e^{t\Delta}\}_{t\geq 0}$ is the analytic semigroup associated with the operator Δ and Ψ_α is the Wright function.

The novel results of this paper are described in detail as follows.

The first novel result concerns the convergence of approximating problem to $\alpha-$order time-fractional evolution equation $D_t^\alpha u_\varepsilon - \chi(\varepsilon)\Delta u_\varepsilon - \Delta(k^\varepsilon * u_\varepsilon) = N[u_\varepsilon]$ to the corresponding limit problem of $D_t^\alpha u - (1+\chi_0)\Delta u = N[u]$ when $\alpha \in (1,2)$ as scale parameter $\varepsilon \to 0^+$. The key tool is based on the resolvent operator family properties as well as the assumptions **(H5)**, **(H6)** about the initial data $u_{0,\varepsilon}, u_0$ and $u_{1,\varepsilon}, u_1$. Unlike the first-order abstract Cauchy problem with $\alpha = 1$, Lorenzi-Messina [12] studied the approximation of solutions to linear integro-differential parabolic Eq (1.1) in L^p-spaces and extended their results to the corresponding non-linear equation [13]. This method can be applied to a fractional viscoelastic telegraph equation [21] as well as a fractional Euler-Bernoulli problem [20].

The second novel result concerns the existence of a local solution to the approximating problems (1.1) and (1.8) and the corresponding limiting problems (1.5) and (1.9) in weighted function spaces respectively. We used a lemma from [14] on $L^p - L^q$ estimates for the semigroup of Laplace operator to deal with the $L^p - L^q$ estimates to the the resolvent operator family. This technique comes partly form the interesting recent work of Andrade-Siracusa-Viana [43].

The final novel result is related to the asymptotic behavior of the mild solutions to the problems (1.8) and (1.9) when $\alpha \to 2^-$. The convergence of solution of fractional sub-diffusion equation when α approaches 1 was first posed by Neto-Planas [44], and then it was considered by Andrade-Siracusa-Viana [43]. However, there are currently no similar studies on the fractional super-diffusion equation. This research is the first step in that direction. Our method described is applicable to other applied models [45, 46] by replacing the classical derivative with the non-integer order derivative. On the convergence of approximating IVP to $\alpha-$order time-fractional evolution equation

$$D_t^\alpha u_\varepsilon - \chi(\varepsilon)\Delta u_\varepsilon - \Delta(k^\varepsilon * u_\varepsilon) = N[u_\varepsilon]$$

to the corresponding limit IVP of

$$D_t^\alpha u - (1+\chi_0)\Delta u = N[u]$$

as scale parameter $\varepsilon \to 0^+$, the novelty of this paper is to extend the results in [12, 13] from $\alpha = 1$ to $\alpha \in (1,2)$, as well as the more general kernel $k(t)$ is considered in the problem (1.1) instead of the kernel including the series representation in [12]. We also extend the results that the existence of the unique maximal solution, a blow-up alternative and the asymptotic behavior in [43] from $\alpha \in (0,1)$ to $\alpha \in (1,2)$.

For simplicity, throughout the paper C denotes a positive constant which may vary from one line to another line, but it is not essential in analysis of the corresponding problems.

This paper is mainly divided into five sections. Section 2 is devoted to showing some basic properties of the resolvent operators F^{ε}_{α}, G^{ε}_{α} defined in Eqs (2.2)–(2.5) and the non-linear operator N. Our main results are stated in Section 3, and their proofs are given in Section 4. We give a conclusion and a discussion in Section 5.

2. Properties of resolvent operators

We first state some assumptions on the operator A and the initial data $u_{0,\varepsilon}(x), u_{1,\varepsilon}(x)$ to our abstract Cauchy problems of the linear Eq (1.1) and the semi-linear Eqs (1.8) and (1.11) respectively. Let X be a Banach space with norm $\|\cdot\|$, $\alpha \in (1,2)$.
(H1) The operator $A : D(A) \subset X \to X$ is a densely defined and closed linear operator. For some $\phi \in (\frac{\pi}{2}, \pi)$ there is a positive constant $C = C(\phi)$ such that

$$\Sigma_{0,\alpha\phi} = \{\lambda \in \mathbb{C} : |\arg(\lambda)| < \alpha\phi\} \subset \rho(A),$$

and the resolvent operator $R(\lambda, A) = (\lambda I - A)^{-1}$ satisfies that $\|R(\lambda, A)\| \le \frac{C}{|\lambda|}$, $\forall \lambda \in \Sigma_{0,\alpha\phi}$.
(H2) The kernel $k(t) \in L^1_{loc}(\mathbb{R}^+)$ such that $\widehat{k}(\lambda)$ exists for $\mathrm{Re}(\lambda) > 0$ and $\widehat{k}(\lambda)$ can be extended to $\sum_{0,\phi}$, and satisfies that $\|\widehat{k}(\lambda)\| = \mathrm{O}(\frac{1}{|\lambda|})$ as $|\lambda| \to +\infty$. For $p \in [1, +\infty)$, there exist constants $\theta_0 > \frac{1}{p} + \alpha, k_0, r > 0$ and $r_0 > r, C > 0$ such that $|\widehat{k}(\lambda) - k_0| \le C|\lambda|^{\theta_0}$, $\forall \lambda \in \Sigma_{0,\phi} \cap B(0, r_0)$.
(H3) There exists a constant $M > 0$ such that

$$\|(\lambda I - (1 + \chi(\varepsilon))A)^{-1}\|_{\mathcal{L}(X)} \le \frac{M}{|\lambda|},\ \forall \lambda \in \Sigma_{0,\alpha\phi},\ \forall \varepsilon \in (0, \varepsilon_0).$$

(H4) The sequence $f_\varepsilon \in C([0,T], X)$ converges to f in $C([0,T], X)$ as $\varepsilon \to 0^+$.
(H5) $u_{0,\varepsilon}, u_0 \in X$ and $u_{0,\varepsilon}$ converges to u_0 in X as $\varepsilon \to 0^+$.
(H6) $u_{1,\varepsilon}, u_1 \in X$ and $u_{1,\varepsilon}$ converges to u_1 in X as $\varepsilon \to 0^+$.

Remark 1. The assumptions **(H1)** and **(H2)** have been considered in [7–9, 18, 19, 47], where the operator $B(t) = k(t)\Delta$ for a given function $k(t)$ and $\alpha \in (0, 2)$.

In the sequel, for $r > 0$ and $\theta \in (\frac{\pi}{2}, \pi)$, we denote a sector by

$$\Sigma_{r,\theta} = \{\lambda \in \mathbb{C} : |\lambda| \ge r, |\arg(\lambda)| < \theta\}. \tag{2.1}$$

In this paper we consider the Banach space $X = L^q(\mathbf{R}^n)(q > 1)$, the operator $A = \triangle$ with domain $D(A) = \{u \in X | \triangle u \in X\}$ and $\sigma(A) = [0, +\infty)$. For $\alpha \in (1,2)$, $\rho(F^{\varepsilon}_{\alpha}), \rho(G^{\varepsilon}_{\alpha}), \rho(F_\alpha), \rho(G_\alpha)$ are the sets

$$\rho(F^{\varepsilon}_{\alpha}) = \{\lambda \in \mathbb{C} : F^{\varepsilon}_{\alpha}(\lambda) = (\lambda^{\alpha} I - \chi(\varepsilon)\triangle - \widehat{k}(\varepsilon\lambda)\triangle)^{-1} \in \mathcal{L}(X)\}, \tag{2.2}$$
$$\rho(G^{\varepsilon}_{\alpha}) = \{\lambda \in \mathbb{C} : G^{\varepsilon}_{\alpha}(\lambda) = \lambda^{\alpha-1}(\lambda^{\alpha} I - \chi(\varepsilon)\Delta - \widehat{k}(\varepsilon\lambda)\Delta)^{-1} \in \mathcal{L}(X)\}, \tag{2.3}$$
$$\rho(F_\alpha) = \{\lambda \in \mathbb{C} : F_\alpha(\lambda) = (\lambda^{\alpha} I - \chi_0\Delta - k_0\Delta)^{-1} \in \mathcal{L}(X)\}, \tag{2.4}$$
$$\rho(G_\alpha) = \{\lambda \in \mathbb{C} : G_\alpha(\lambda) = \lambda^{\alpha-1}(\lambda^{\alpha} I - \chi_0\Delta - k_0\Delta)^{-1} \in \mathcal{L}(X)\}. \tag{2.5}$$

We next collect some properties established in [18, 48].

Lemma 2.1. *[18,48] Under the assumptions* **(H1)** *and* **(H2)***, there exists* $r > 0$ *such that* $\sum_{r,\phi} \subset \rho(F_\alpha^\varepsilon)$, $\sum_{r,\phi} \subset \rho(G_\alpha^\varepsilon)$, $\sum_{r,\phi} \subset \rho(F_2^\varepsilon)$, $\sum_{r,\phi} \subset \rho(G_2^\varepsilon)$, *and the operator-value functions* $F_\alpha^\varepsilon, G_\alpha^\varepsilon, F_2^\varepsilon, G_2^\varepsilon : \sum_{r,\phi} \to \mathcal{L}(X)$ *are analytic. Moreover there exists a constant M Such that*

$$\|F_\alpha^\varepsilon(\lambda)\| \le \frac{M}{|\lambda|^\alpha},\ \|G_\alpha^\varepsilon(\lambda)\| \le \frac{M}{|\lambda|},\ \|F_2^\varepsilon(\lambda)\| \le \frac{M}{|\lambda|^2},\ \|G_2^\varepsilon(\lambda)\| \le \frac{M}{|\lambda|},\ \lambda \in \Sigma_{r,\phi}.$$

Definition 2.2. [1,18,48] For $\alpha \in (1,2)$, the operator families $\{S_\alpha^\varepsilon(t)\}_{t\ge0}, \{S_\alpha(t)\}_{t\ge0}, \{T_\alpha^\varepsilon(t)\}_{t\ge0}, \{T_\alpha(t)\}_{t\ge0}$ are defined by

$$\begin{aligned} S_\alpha^\varepsilon(t) &= \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t}\lambda^{\alpha-1}\Big(\lambda^\alpha I - \chi(\varepsilon)\Delta - \widehat{k}(\varepsilon\lambda)\Delta\Big)^{-1} d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t} G_\alpha^\varepsilon(\lambda) d\lambda, \end{aligned} \tag{2.6}$$

$$\begin{aligned} S_\alpha(t) &= \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t}\lambda^{\alpha-1}\Big(\lambda^\alpha I - \chi_0\Delta - k_0\Delta\Big)^{-1} d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t} G_\alpha(\lambda) d\lambda, \end{aligned} \tag{2.7}$$

$$\begin{aligned} T_\alpha^\varepsilon(t) &= \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t}\Big(\lambda^\alpha I - \chi(\varepsilon)\Delta - \widehat{k}(\varepsilon\lambda)\Delta\Big)^{-1} d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t} F_\alpha^\varepsilon(\lambda) d\lambda, \end{aligned} \tag{2.8}$$

$$\begin{aligned} T_\alpha(t) &= \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t}\Big(\lambda^\alpha I - \chi_0\Delta - k_0\Delta\Big)^{-1} d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t} F_\alpha(\lambda) d\lambda, \end{aligned} \tag{2.9}$$

where $\Gamma_1(r,\theta) = \{te^{i\theta} : t \ge r\}$, $\Gamma_2(r,\theta) = \{re^{i\xi} : |\xi| \le \theta\}$, $\Gamma_3(r,\theta) = \{te^{-i\theta} : t \ge r\}$, $\theta \in (\frac{\pi}{2}, \phi)$, and $\Gamma_{r,\theta} = \bigcup_{i=1}^3 \Gamma_i(r,\theta)$ oriented counterclockwise.

By the Laplace transform, we give the definitions of L^q-mild solutions to the problems (1.1) and (1.5), (1.8) and (1.9), (1.11) and (1.12) respectively.

Definition 2.3. Let $T > 0, \varepsilon \in (0, \varepsilon_0]$, $u_{0,\varepsilon}, u_{1,\varepsilon}, u_0, u_1 \in L^q(\mathbf{R}^n), q \in (1, +\infty)$. Functions $u_\varepsilon \in C([0,T], L^q(\mathbf{R}^n))$ and $u \in C([0,T], L^q(\mathbf{R}^n))$ are called L^q-mild solutions of Eqs (1.1) and (1.5) in $[0,T]$ if u_ε and u satisfy the following equations respectively

$$u_\varepsilon(t) = \mathrm{S}_\alpha^\varepsilon(t)u_{0,\varepsilon} + \int_0^t \mathrm{S}_\alpha^\varepsilon(s)u_{1,\varepsilon}ds + \int_0^t \mathrm{T}_\alpha^\varepsilon(t-s)f_\varepsilon(s)ds, \tag{2.10}$$

$$u(t) = \mathrm{S}_\alpha(t)u_0 + \int_0^t \mathrm{S}_\alpha(s)u_1 ds + \int_0^t \mathrm{T}_\alpha(t-s)f(s)ds. \tag{2.11}$$

Definition 2.4. Let $T > 0, \varepsilon \in (0, \varepsilon_0]$, $u_{0,\varepsilon}, u_{1,\varepsilon}, u_0, u_1 \in L^q(\mathbf{R}^n)$, $q \in (1, +\infty)$. Functions $u_\varepsilon, u \in \mathrm{C}([0,T], L^q(\mathbf{R}^n))$ are called L^q-mild solutions of Eqs (1.8) and (1.9) in $[0,T]$ if u_ε and u satisfy the following equations respectively

$$\begin{aligned} u_\varepsilon(t) &= \mathrm{S}^\varepsilon_\alpha(t)u_{0,\varepsilon} + \int_0^t \mathrm{S}^\varepsilon_\alpha(s)u_{1,\varepsilon}ds \\ &+ \int_0^t [\mathrm{T}^\varepsilon_\alpha(t-s)f_\varepsilon(s) + \mathrm{T}^\varepsilon_\alpha(t-s)Nu_\varepsilon(s)]ds, \qquad (2.12) \\ u(t) &= \mathrm{S}_\alpha(t)u_0 + \int_0^t \mathrm{S}_\alpha(s)u_1 ds \\ &+ \int_0^t [\mathrm{T}_\alpha(t-s)f(s) + \mathrm{T}_\alpha(t-s)Nu(s)]ds. \qquad (2.13) \end{aligned}$$

Definition 2.5. Let $T > 0, \varepsilon \in (0, \varepsilon_0]$, $u_{0,\varepsilon}, u_{1,\varepsilon}, u_0, u_1 \in L^q(\mathbf{R}^n)$, $q \in (1, +\infty)$. Functions $u_\varepsilon \in \mathrm{C}([0,T], L^q(\mathbf{R}^n))$ and $u \in \mathrm{C}([0,T], L^q(\mathbf{R}^n))$ are called L^q-mild solutions of Eqs (1.11) and (1.12) in $[0,T]$ if u_ε and u satisfy the following equations respectively

$$\begin{aligned} u_\varepsilon(t) &= \mathrm{S}^\varepsilon_2(t)u_{0,\varepsilon} + \int_0^t \mathrm{S}^\varepsilon_2(s)u_{1,\varepsilon}ds \\ &+ \int_0^t [\mathrm{T}^\varepsilon_2(t-s)f_\varepsilon(s) + \mathrm{T}^\varepsilon_2(t-s)Nu_\varepsilon(s)]ds, \qquad (2.14) \\ u(t) &= \mathrm{S}_2(t)u_0 + \int_0^t \mathrm{S}_2(s)u_1 ds \\ &+ \int_0^t [\mathrm{T}_2(t-s)f(s) + \mathrm{T}_2(t-s)Nu(s)]ds. \qquad (2.15) \end{aligned}$$

Lemma 2.6. *[18] Under the assumptions* **(H1)**, **(H2)**, *the operator-valued function $S^\varepsilon_\alpha(t)$ defined in Definition 2.2 is (i) exponentially bounded in $\mathcal{L}(X)$; (ii) exponentially bounded in $\mathcal{L}([D(A)])$; (iii) strongly continuous on $[0,\infty)$ and uniformly continuous on $(0,\infty)$ in $\mathcal{L}(X)$; (iv) strongly continuous on $[0,\infty)$ in $\mathcal{L}([D(A)])$; (v) The operator function $T^\varepsilon_\alpha(t)$ defined in Definition 2.2 is exponentially bounded in $\mathcal{L}(X)$; (vi) exponentially bounded in $\mathcal{L}([D(A)])$.*

The cosine and sine operator families $\{S_\alpha(t)\}_{t\geq 0}$ and $\{T_\alpha(t)\}_{t\geq 0}$ are defined in Definition 2.2, their properties can be found in [48–50]. We give the following $L^p - L^q$ estimates of the operators.

Lemma 2.7. *[50, Lemma 4.3] For $1 < q \leq r \leq +\infty$ and $q < +\infty$, there exists a constant $C > 0$ such that for every $\varphi \in L^q(\mathbf{R}^n)$ and $t > 0$ we have*

$$\|S_\alpha(t)\varphi\|_{L^r(\mathbf{R}^n)} \leq Ct^{-\frac{\alpha n}{2}(\frac{1}{q}-\frac{1}{r})}\|\varphi\|_{L^q(\mathbf{R}^n)}, \ \frac{n}{2}(\frac{1}{q}-\frac{1}{r}) < 1, \qquad (2.16)$$

$$\|T_\alpha(t)\varphi\|_{L^r(\mathbf{R}^n)} \leq Ct^{-\frac{\alpha n}{2}(\frac{1}{q}-\frac{1}{r})+\alpha-1}\|\varphi\|_{L^q(\mathbf{R}^n)}, \ \frac{n}{4}(\frac{1}{q}-\frac{1}{r}) < 1. \qquad (2.17)$$

Moreover, the operator family $\{S_\alpha(t)\}_{t\geq 0}$ is strongly continuously in $L^r(\mathbf{R}^n)$.

From Lemma 2.6, we can easily obtain the following lemma.

Lemma 2.8. *[51] For $\alpha \in (1,2]$, there exists a positive constant C depending only on α, T, M, θ, r such that*

$$\|S^{\varepsilon}_{\alpha}(t)\|_{\mathcal{L}(X)} \leq C, \ \|T^{\varepsilon}_{\alpha}(t)\|_{\mathcal{L}(X)} \leq C, \ \forall t \in [0,T], \tag{2.18}$$

$$\|S_{\alpha}(t)\|_{\mathcal{L}(X)} \leq C, \ \|T_{\alpha}(t)\|_{\mathcal{L}(X)} \leq C, \ \forall t \in [0,T]. \tag{2.19}$$

Lemma 2.9. *Under the assumptions* **(H1)–(H3)***, for $p \in [1,+\infty)$, $\alpha \in (1,2)$, $\theta_0 \in (\frac{1}{p}+\alpha,+\infty)$ and $\varepsilon_1 \in (0, \min\{\varepsilon_0, \frac{r_0}{r}\})$, there exists a constant $C > 0$ depending only on $\alpha, T, M, \theta, r, \varepsilon_0, \theta_0, p, r_0$ such that*

$$\|S^{\varepsilon}_{\alpha} - S_{\alpha}\|_{L^p((0,T),\mathcal{L}(X))} \leq C(\varepsilon^{\frac{1}{p}} + \varepsilon^{\theta_0} + |\chi(\varepsilon) - \chi_0|), \ \forall \varepsilon \in (0,\varepsilon_1], \tag{2.20}$$

$$\|T^{\varepsilon}_{\alpha} - T_{\alpha}\|_{L^p((0,T),\mathcal{L}(X))} \leq C(\varepsilon^{\frac{1}{p}+\alpha-1} + \varepsilon^{\theta_0} + |\chi(\varepsilon) - \chi_0|), \ \forall \varepsilon \in (0,\varepsilon_1]. \tag{2.21}$$

Proof. From Lemma 2.1, we have

$$\begin{aligned}
&\|(\lambda^{\alpha}I - (\chi(\varepsilon) + \widehat{k}(\varepsilon\lambda))\Delta)^{-1} - (\lambda^{\alpha}I - (k_0+\chi_0)\Delta)^{-1}\|_{\mathcal{L}(X)} \\
\leq\ & \Big\|\Big\{\lambda^{\alpha}I - (\chi(\varepsilon) + \widehat{k}(\varepsilon\lambda))\Delta\Big\}^{-1}\Big\{(\chi(\varepsilon)-\chi_0) + (\widehat{k}(\varepsilon\lambda) - k_0)\Big\}\Big\|_{\mathcal{L}(X)} \\
&\cdot\Big\|\Delta(\lambda^{\alpha}I - (k_0+\chi_0)\Delta)^{-1}\Big\|_{\mathcal{L}(X)} \\
\leq\ & \frac{M}{|\lambda|^{\alpha}}\Big(|\chi(\varepsilon)-\chi_0| + |\widehat{k}(\varepsilon\lambda) - k_0|\Big)\Big\|\Delta(\lambda^{\alpha} - (k_0+\chi_0)\Delta)^{-1}\Big\|_{\mathcal{L}(X)} \\
\leq\ & \frac{M(|\chi(\varepsilon)-\chi_0| + |\widehat{k}(\varepsilon\lambda) - k_0|)}{|\lambda|^{\alpha}(k_0+\chi_0)}\Big\|I - \lambda^{\alpha}(\lambda^{\alpha} - (k_0+\chi_0)\Delta)^{-1}\Big\|_{\mathcal{L}(X)} \\
\leq\ & \frac{M(1+C)}{|\lambda|^{\alpha}(k_0+\chi_0)}\Big(|\chi(\varepsilon)-\chi_0| + |\widehat{k}(\varepsilon\lambda) - k_0|\Big).
\end{aligned} \tag{2.22}$$

According to Definition 2.2, inequality (2.22) implies that

$$\begin{aligned}
&\|S^{\varepsilon}_{\alpha}(t) - S_{\alpha}(t)\|_{\mathcal{L}(X)} \\
\leq\ & \frac{1}{2\pi}\Big\|\int_{\Gamma_{r,\theta}} e^{\lambda t}\lambda^{\alpha-1}\Big\{(\lambda^{\alpha}I - (\chi(\varepsilon) + \widehat{k}(\varepsilon\lambda))\Delta)^{-1} - (\lambda^{\alpha} - (k_0+\chi_0)\Delta)^{-1}\Big\}d\lambda\Big\|_{\mathcal{L}(X)} \\
\leq\ & \frac{M(1+C)}{2\pi(k_0+\chi_0)}\int_{\Gamma_{r,\theta}} \frac{e^{tRe\lambda}}{|\lambda|}\Big(|\chi(\varepsilon)-\chi_0| + |\widehat{k}(\varepsilon\lambda) - k_0|\Big)d\lambda,
\end{aligned}$$

$$\begin{aligned}
&\|T^{\varepsilon}_{\alpha}(t) - T_{\alpha}(t)\|_{\mathcal{L}(X)} \\
\leq\ & \frac{1}{2\pi}\Big\|\int_{\Gamma_{r,\theta}} e^{\lambda t}\Big\{[\lambda^{\alpha}I - (\chi(\varepsilon) + \widehat{k}(\varepsilon\lambda))\Delta]^{-1} - [\lambda^{\alpha} - (k_0+\chi_0)\Delta]^{-1}\Big\}d\lambda\Big\|_{\mathcal{L}(X)} \\
\leq\ & \frac{M(1+C)}{2\pi(k_0+\chi_0)}\int_{\Gamma_{r,\theta}} \frac{e^{tRe\lambda}}{|\lambda|^{\alpha}}\Big(|\chi(\varepsilon)-\chi_0| + |\widehat{k}(\varepsilon\lambda) - k_0|\Big)d\lambda.
\end{aligned}$$

Hence, for $p \geq 1$, we have

$$\begin{aligned}
&\|S^{\varepsilon}_{\alpha} - S_{\alpha}\|_{L^p((0,T),\mathcal{L}(X))} \\
\leq\ & \frac{M(1+C)}{2\pi(k_0+\chi_0)}\int_{\Gamma_{r,\theta}} \frac{|\chi(\varepsilon)-\chi_0| + |\widehat{k}(\varepsilon\lambda) - k_0|}{|\lambda|}\Big(\int_0^{+\infty} e^{tpRe(\lambda)}dt\Big)^{\frac{1}{p}}d\lambda
\end{aligned}$$

$$= p^{-\frac{1}{p}} \frac{M(1+C)}{2\pi(k_0+\chi_0)} \sum_{j=1}^{3} \int_{\Gamma_j(r,\theta)} \frac{|\chi(\varepsilon)-\chi_0| + |\widehat{k}(\varepsilon\lambda)-k_0|}{|Re(\lambda)|^{\frac{1}{p}}|\lambda|} d\lambda, \tag{2.23}$$

$$\begin{aligned} & \|T_\alpha^\varepsilon - T_\alpha\|_{L^p((0,T),\mathcal{L}(X))} \\ \leq\ & \frac{M(1+C)}{2\pi(k_0+\chi_0)} \int_{\Gamma_{r,\theta}} \frac{|\chi(\varepsilon)-\chi_0| + |\widehat{k}(\varepsilon\lambda)-k_0|}{|\lambda|^\alpha} \Big(\int_0^{+\infty} e^{tpRe(\lambda)} dt\Big)^{\frac{1}{p}} d\lambda \\ =\ & p^{-\frac{1}{p}} \frac{M(1+C)}{2\pi(k_0+\chi_0)} \sum_{j=1}^{3} \int_{\Gamma_j(r,\theta)} \frac{|\chi(\varepsilon)-\chi_0| + |\widehat{k}(\varepsilon\lambda)-k_0|}{|Re(\lambda)|^{\frac{1}{p}}|\lambda|^\alpha} d\lambda. \end{aligned} \tag{2.24}$$

We first estimate the integrals over $\Gamma_j(r,\theta)$, $j = 1, 3$. From the assumption **(H2)**, there exists a positive constant C such that for $\Gamma_1(r,\theta) = \{te^{i\theta} : t \geq r\}$ and $\Gamma_3(r,\theta) = \{te^{-i\theta} : t \geq r\}$, we have

$$\begin{aligned} & \int_{\Gamma_j(r,\theta)} \frac{|\widehat{k}(\varepsilon\lambda)-k_0|}{|Re\lambda|^{\frac{1}{p}}|\lambda|} d\lambda \leq \cos^{-\frac{1}{p}}(\theta)\varepsilon^{\frac{1}{p}} \int_{\varepsilon r}^{+\infty} \frac{|\widehat{k}(se^{i\theta})-k_0|}{s^{1+\frac{1}{p}}} ds \\ & \leq C\cos^{-\frac{1}{p}}(\theta)\varepsilon^{\frac{1}{p}}\Big(\int_0^{r_0} s^{\theta_0-1-\frac{1}{p}} ds + \int_{r_0}^{+\infty} s^{-1-\frac{1}{p}} ds\Big) \\ & \leq C\varepsilon^{\frac{1}{p}},\ j = 1, 3, \end{aligned} \tag{2.25}$$

where for $\theta_0 > \frac{1}{p}$, $I_1 = \int_0^{r_0} s^{\theta_0-1-\frac{1}{p}} ds$ converges and $\forall p \geq 1$, $I_2 = \int_{r_0}^{+\infty} s^{-1-\frac{1}{p}} ds$ converges.

Similarly, there exists a positive constant C such that

$$\begin{aligned} & \int_{\Gamma_j(r,\theta)} \frac{|\widehat{k}(\varepsilon\lambda)-k_0|}{|Re\lambda|^{\frac{1}{p}}|\lambda|^\alpha} d\lambda \leq \cos^{-\frac{1}{p}}(\theta)\varepsilon^{\frac{1}{p}+\alpha-1} \int_{\varepsilon r}^{+\infty} \frac{|\widehat{k}(se^{i\theta})-k_0|}{s^{\alpha+\frac{1}{p}}} ds \\ & \leq C\cos^{-\frac{1}{p}}(\theta)\varepsilon^{\frac{1}{p}+\alpha-1}\Big(\int_0^{r_0} s^{\theta_0-1-\frac{1}{p}-\alpha} ds + \int_{r_0}^{+\infty} s^{-1-\frac{1}{p}-\alpha} ds\Big) \\ & \leq C\varepsilon^{\frac{1}{p}+\alpha-1},\ j = 1, 3, \end{aligned} \tag{2.26}$$

where for $\theta_0 > \frac{1}{p} + \alpha$, $I_3 = \int_0^{r_0} s^{\theta_0-1-\frac{1}{p}-\alpha} ds$ converges and $\forall p > 1, \alpha \in (1, 2)$, $I_4 = \int_{r_0}^{+\infty} s^{-1-\frac{1}{p}-\alpha} ds$ converges.

For the function $\chi(\varepsilon)$ and $j = 1, 3$ we can also get

$$\int_{\Gamma_j(r,\theta)} \frac{|\chi(\varepsilon)-\chi_0|}{|Re\lambda|^{\frac{1}{p}}|\lambda|} d\lambda = \int_r^{+\infty} \frac{|\chi(\varepsilon)-\chi|}{s^{1+\frac{1}{p}}\cos^{\frac{1}{p}}(\theta)} ds \leq C|\chi(\varepsilon)-\chi_0|r^{-\frac{1}{p}}, \tag{2.27}$$

$$\int_{\Gamma_j(r,\theta)} \frac{|\chi(\varepsilon)-\chi_0|}{|Re\lambda|^{\frac{1}{p}}|\lambda|^\alpha} d\lambda = \int_r^{+\infty} \frac{|\chi(\varepsilon)-\chi|}{s^{\alpha+\frac{1}{p}}\cos^{\frac{1}{p}}(\theta)} ds \leq C|\chi(\varepsilon)-\chi_0|r^{1-\alpha-\frac{1}{p}}. \tag{2.28}$$

From inequalities (2.25)–(2.28), we get the estimates of the integrals over $\Gamma_j(r,\theta)$ for $j = 1, 3$,

$$\int_{\Gamma_j(r,\theta)} \frac{|\chi(\varepsilon)-\chi_0| + |\widehat{k}(\varepsilon\lambda)-k_0|}{|Re\lambda|^{\frac{1}{p}}|\lambda|} d\lambda \leq C\varepsilon^{\frac{1}{p}} + C|\chi(\varepsilon)-\chi_0|r^{-\frac{1}{p}}, \tag{2.29}$$

$$\int_{\Gamma_j(r,\theta)} \frac{|\chi(\varepsilon)-\chi_0|+|\widehat{k}(\varepsilon\lambda)-k_0|}{|Re\lambda|^{\frac{1}{p}}|\lambda|^{\alpha}}d\lambda \leq C\varepsilon^{\frac{1}{p}+\alpha-1}+C|\chi(\varepsilon)-\chi_0|r^{1-\alpha-\frac{1}{p}}. \tag{2.30}$$

Now we estimate the integral over $\Gamma_2(r,\theta)$ in inequalities (2.23) and (2.24). Choosing $\varepsilon \in (0,\varepsilon_1]$ with $\varepsilon_1 \leq \min\{\varepsilon_0, \frac{r_0}{r}\}$ and making use of assumptions (iv) and (v) in Lemma 2.6, we have

$$\begin{aligned}
&\int_{\Gamma_2(r,\theta)} \frac{|\chi(\varepsilon)-\chi_0|+|\widehat{k}(\varepsilon\lambda)-k_0|}{|Re\lambda|^{\frac{1}{p}}|\lambda|}d\lambda \\
= &\int_{\Gamma_2(r,\theta)} \frac{|\widehat{k}(\varepsilon\lambda)-k_0|}{|Re\lambda|^{\frac{1}{p}}|\lambda|}d\lambda + \int_{\Gamma_2(r,\theta)} \frac{|\chi(\varepsilon)-\chi_0|}{|Re\lambda|^{\frac{1}{p}}|\lambda|}d\lambda \\
\leq\ & r^{-\frac{1}{p}}\Big[\int_{-\theta}^{\theta} \frac{|\widehat{k}(\varepsilon re^{i\xi})-k_0|}{\cos^{\frac{1}{p}}(\xi)}d\xi + \int_{-\theta}^{\theta} \frac{|\chi(\varepsilon)-\chi_0|}{\cos^{\frac{1}{p}}(\xi)}d\xi\Big] \\
\leq\ & Cr^{-\frac{1}{p}}\Big[(\varepsilon r)^{\theta_0}\int_{-\theta}^{\theta}\cos^{-\frac{1}{p}}(\xi)d\xi + |\chi(\varepsilon)-\chi_0|\int_{-\theta}^{\theta}\cos^{-\frac{1}{p}}(\xi)d\xi\Big] \\
\leq\ & C(\varepsilon^{\theta_0}+|\chi(\varepsilon)-\chi_0|),
\end{aligned} \tag{2.31}$$

$$\begin{aligned}
&\int_{\Gamma_2(r,\theta)} \frac{|\chi(\varepsilon)-\chi_0|+|\widehat{k}(\varepsilon\lambda)-k_0|}{|Re\lambda|^{\frac{1}{p}}|\lambda|^{\alpha}}d\lambda \\
= &\int_{\Gamma_2(r,\theta)} \frac{|\widehat{k}(\varepsilon\lambda)-k_0|}{|Re\lambda|^{\frac{1}{p}}|\lambda|^{\alpha}}d\lambda + \int_{\Gamma_2(r,\theta)} \frac{|\chi(\varepsilon)-\chi_0|}{|Re\lambda|^{\frac{1}{p}}|\lambda|^{\alpha}}d\lambda \\
\leq\ & r^{1-\alpha+\frac{1}{p}}\Big[\int_{-\theta}^{\theta} \frac{|\widehat{k}(\varepsilon re^{i\xi})-k_0|}{\cos^{\frac{1}{p}}(\xi)}d\xi + \int_{-\theta}^{\theta} \frac{|\chi(\varepsilon)-\chi_0|}{\cos^{\frac{1}{p}}(\xi)}d\xi\Big] \\
\leq\ & Cr^{1-\alpha+\frac{1}{p}}(\varepsilon r)^{\theta_0}\int_{-\theta}^{\theta}\cos^{-\frac{1}{p}}(\xi)d\xi + |\chi(\varepsilon)-\chi_0|\int_{-\theta}^{\theta}\cos^{-\frac{1}{p}}(\xi)d\xi \\
\leq\ & C(\varepsilon^{\theta_0}+|\chi(\varepsilon)-\chi_0|).
\end{aligned} \tag{2.32}$$

From inequalities (2.23), (2.29) and (2.31) we get the estimate inequality (2.20), and from inequalities (2.24), (2.30) and (2.32) we get the estimate inequality (2.21). This completes the proof of Lemma 2.9.

Lemma 2.10. *Under the assumptions* **(H1)–(H4)***, for $p \geq 1, \alpha \in (1,2)$, we have*

$$\begin{aligned}
&\|T_\alpha^\varepsilon * f_\varepsilon - T_\alpha * f\|_{L^r((0,T),X)} \\
&\leq C\|f_\varepsilon - f\|_{C([0,T],X)} + C(\varepsilon^{\frac{1}{p}+\alpha-1}+\varepsilon^{\theta_0}+|\chi(\varepsilon)-\chi_0|)\|f\|_{C([0,T],X)},
\end{aligned} \tag{2.33}$$

where C is a positive constant depending only on $\alpha, T, M, \theta, r, \varepsilon_0, \theta_0, p, r_0$, and

$$\frac{1}{r} = \begin{cases} \frac{1}{p}+\frac{1}{q}-1, & if\ \ \frac{1}{p}+\frac{1}{q}\geq 1, p,q\geq 1, \\ \frac{1}{p}, & if\ \ \frac{1}{p}+\frac{1}{q}<1, p,q>1, \end{cases}$$

the convolution $(T*f)(t) = \int_0^t T(t-s)f(s)ds$ and we have

$$T_\alpha^\varepsilon * f_\varepsilon \to T_\alpha * f \ \ in\ L^r((0,T);X)\ as\ \varepsilon\to 0^+. \tag{2.34}$$

Proof. From the identity $T^{\varepsilon}_{\alpha} * f_{\varepsilon} - T_{\alpha} * f = T^{\varepsilon}_{\alpha} * (f_{\varepsilon} - f) + (T^{\varepsilon}_{\alpha} - T_{\alpha}) * f$, we have

$$\begin{aligned}
&||T^{\varepsilon}_{\alpha} * f_{\varepsilon} - T_{\alpha} * f||_{L^r((0,T),X)} \\
&= ||T^{\varepsilon}_{\alpha} * (f_{\varepsilon} - f) + (T^{\varepsilon}_{\alpha} - T_{\alpha}) * f||_{L^r((0,T),X)} \\
&\le ||T^{\varepsilon}_{\alpha} * (f_{\varepsilon} - f)||_{L^r((0,T),X)} + ||(T^{\varepsilon}_{\alpha} - T_{\alpha}) * f||_{L^r((0,T),X)}.
\end{aligned} \tag{2.35}$$

For $\frac{1}{p} + \frac{1}{q} \ge 1 (p, q \ge 1)$, we can use Young's inequality for convolution, together with Minkowski's inequality, Lemmas 2.8 and 2.9, we have

$$\begin{aligned}
&||T^{\varepsilon}_{\alpha} * f_{\varepsilon} - T_{\alpha} * f||_{L^r((0,T),X)} \\
&\le ||T^{\varepsilon}_{\alpha}||_{L^p((0,T),\mathcal{L}(X))}||f_{\varepsilon} - f||_{L^q((0,T),X)} + ||T^{\varepsilon}_{\alpha} - T_{\alpha}||_{L^p((0,T),\mathcal{L}(X))}||f||_{L^q((0,T),X)} \\
&\le C||f_{\varepsilon} - f||_{C([0,T],X)} + C(\varepsilon^{\frac{1}{p}+\alpha-1} + \varepsilon^{\theta_0} + |\chi(\varepsilon) - \chi_0|)||f||_{C([0,T],X)}.
\end{aligned}$$

For $\frac{1}{p} + \frac{1}{q} < 1 (p, q > 1)$, we first choose $r = p$ and then use Young's inequality for convolution, together with Lemmas 2.8 and 2.9, we have

$$\begin{aligned}
&||T^{\varepsilon}_{\alpha} * f_{\varepsilon} - T_{\alpha} * f||_{L^p((0,T),X)} \\
&\le ||T^{\varepsilon}_{\alpha}||_{L^p((0,T),\mathcal{L}(X))}||f_{\varepsilon} - f||_{L^1((0,T),X)} + ||T^{\varepsilon}_{\alpha} - T_{\alpha}||_{L^p((0,T),\mathcal{L}(X))}||f||_{L^1((0,T),X)} \\
&\le C||f_{\varepsilon} - f||_{C([0,T],X)} + C(\varepsilon^{\frac{1}{p}+\alpha-1} + \varepsilon^{\theta_0} + |\chi(\varepsilon) - \chi_0|)||f||_{C([0,T],X)}.
\end{aligned}$$

Thus the inequality (2.33) holds true for $\frac{1}{p} + \frac{1}{q} < 1 (p, q > 1)$. Since $L^q((0,T), X) \hookrightarrow L^1((0,T), X)$, we easily get estimate inequality (2.34). This completes the proof of Lemma 2.10.

Lemma 2.11. *Under the assumptions* **(H1)–(H3)**, **(H5)**, *for any* $p \ge 1, \varepsilon \in (0, \varepsilon_1]$, *we have*

$$||S^{\varepsilon}_{\alpha}u_{0,\varepsilon} - S_{\alpha}u_0||_{L^p((0,T),X)} \le C||u_{0,\varepsilon} - u_0|| + C(\varepsilon^{\frac{1}{p}} + \varepsilon^{\theta_0} + |\chi(\varepsilon) - \chi_0|)||u_0||, \tag{2.36}$$

for some positive constant C depending only on $\alpha, T, M, \theta, r, \varepsilon_0, \theta_0, p, r_0$. *Moreover,*

$$S^{\varepsilon}_{\alpha}u_{0,\varepsilon} \to S_{\alpha}u_0 \text{ in } L^p((0,T); X) \text{ as } \varepsilon \to 0^+. \tag{2.37}$$

Proof. Due to the identity $S^{\varepsilon}_{\alpha}u_{0,\varepsilon} - S_{\alpha}u_0 = S^{\varepsilon}_{\alpha}(u_{0,\varepsilon} - u_0) + (S^{\varepsilon}_{\alpha} - S_{\alpha})u_0$, Minkowski's inequality, Lemmas 2.8 and 2.9, we obtain

$$\begin{aligned}
&||S^{\varepsilon}_{\alpha}u_{0,\varepsilon} - S_{\alpha}u_0||_{L^p((0,T),X)} \\
&= ||S^{\varepsilon}_{\alpha}(u_{0,\varepsilon} - u_0) + (S^{\varepsilon}_{\alpha} - S_{\alpha})u_0||_{L^p((0,T),X)} \\
&\le ||S^{\varepsilon}_{\alpha}(u_{0,\varepsilon} - u_0)||_{L^p((0,T),X)} + ||(S^{\varepsilon}_{\alpha} - S_{\alpha})u_0||_{L^p((0,T),X)} \\
&\le ||S^{\varepsilon}_{\alpha}||_{L^p((0,T),\mathcal{L}(X))}||u_{0,\varepsilon} - u_0|| + ||S^{\varepsilon}_{\alpha} - S_{\alpha}||_{L^p((0,T),\mathcal{L}(X))}||u_0|| \\
&\le C||u_{0,\varepsilon} - u_0|| + (\varepsilon^{\frac{1}{p}} + \varepsilon^{\theta_0} + |\chi_0 - \chi(\varepsilon)|)||u_0||.
\end{aligned} \tag{2.38}$$

This completes the proof of Lemma 2.11.

Lemma 2.12. *Under the assumptions* **(H1)–(H3)** *and* **(H6)**, *for any* $p \ge 1, \varepsilon \in (0, \varepsilon_1]$, *we have*

$$\Big\| \int_0^t (S^{\varepsilon}_{\alpha}u_{1,\varepsilon} - S_{\alpha}u_1)ds \Big\|_{L^p((0,T),X)}$$

$$\leq C\Big\{\|u_{1,\varepsilon}-u_1\|+\Big(\varepsilon^{\frac{1}{p}}+\varepsilon^{\theta_0}+|\chi(\varepsilon)-\chi_0|\Big)\|u_1\|\Big\}, \tag{2.39}$$

for some positive constant C depending only on $\alpha, T, M, \theta, r, \varepsilon_0, \theta_0, p, r_0$. Moreover,

$$\int_0^t S_\alpha^\varepsilon u_{1,\varepsilon}ds \to \int_0^t S_\alpha u_1 ds \text{ as } \varepsilon \to 0^+ \text{ in } L^p((0,T),X). \tag{2.40}$$

Proof. Due to the identity

$$\int_0^t [S_\alpha^\varepsilon u_{1,\varepsilon}-S_\alpha u_1]ds=\int_0^t S_\alpha^\varepsilon(u_{1,\varepsilon}-u_1)ds+\int_0^t(S_\alpha^\varepsilon-S_\alpha)u_1ds,$$

Minkowski's inequality, Hölder inequality, Lemmas 2.8 and 2.9, we obtain

$$\begin{aligned}
&\Big\|\int_0^t(S_\alpha^\varepsilon u_{1,\varepsilon}-S_\alpha u_1)ds\Big\|_{L^p((0,T),X)}\\
&\leq \Big\|\int_0^t S_\alpha^\varepsilon(u_{1,\varepsilon}-u_1)ds\|_{L^p((0,T),X)}+\int_0^t(S_\alpha^\varepsilon-S_\alpha)u_1ds\Big\|_{L^p((0,T),X)}\\
&\leq C\|S_\alpha^\varepsilon\|_{L^p((0,T),\mathcal{L}(X))}\|u_{1,\varepsilon}-u_1\|+\|S_\alpha^\varepsilon-S_\alpha\|_{L^p((0,T),\mathcal{L}(X))}\|u_1\|\\
&\leq C\|u_{1,\varepsilon}-u_1\|+C(\varepsilon^{\frac{1}{p}}+\varepsilon^{\theta_0}+|\chi_0-\chi(\varepsilon)|)\|u_1\|.
\end{aligned} \tag{2.41}$$

This completes the proof of Lemma 2.12.

Lemma 2.13. *Under the assumptions* **(H1)**–**(H6)** *with $X=L^q(\mathbf{R}^n)(q>1)$, $\alpha\in(1,2)$, then for $t\geq 0$ we have as $\alpha\to 2^-$:*

$$\begin{aligned}
&\Big\|\mathrm{S}_\alpha^\varepsilon(t)u_{0,\varepsilon}+\int_0^t \mathrm{S}_\alpha^\varepsilon(s)u_{1,\varepsilon}ds+\int_0^t \mathrm{T}_\alpha^\varepsilon(t-s)f_\varepsilon(s)ds\\
&-\Big(\mathrm{S}_2^\varepsilon(t)u_{0,\varepsilon}+\int_0^t \mathrm{S}_2^\varepsilon(s)u_{1,\varepsilon}ds+\int_0^t \mathrm{T}_2^\varepsilon(t-s)f_\varepsilon(s)ds\Big)\Big\|_{L^q(\mathbf{R}^n)}\to 0,
\end{aligned} \tag{2.42}$$

$$\begin{aligned}
&\Big\|\mathrm{S}_\alpha(t)u_0+\int_0^t \mathrm{S}_\alpha(s)u_1ds+\int_0^t \mathrm{T}_\alpha(t-s)f(s)ds\\
&-\Big(\mathrm{S}_2(t)u_0+\int_0^t \mathrm{S}_2(s)u_1ds+\int_0^t \mathrm{T}_2(t-s)f(s)ds\Big)\Big\|_{L^q(\mathbf{R}^n)}\to 0.
\end{aligned} \tag{2.43}$$

Furthermore, this convergence is unform for t in bounded subintervals and u_0, u_1, $u_{0,\varepsilon}$, $u_{1,\varepsilon}$ in bounded subsets of $L^q(\mathbf{R}^n)$ and $f(t)$, $f_\varepsilon(t)$ in bounded subsets of $C([0,T],L^q(\mathbf{R}^n))$.

Proof. We only prove Eq (2.43) since the proof of Eq (2.42) is similar.

$$\begin{aligned}
&2\pi i\Big(\mathrm{S}_\alpha^\varepsilon(t)u_{0,\varepsilon}-\mathrm{S}_2^\varepsilon(t)u_{0,\varepsilon}+\int_0^t \mathrm{S}_\alpha^\varepsilon(s)u_{1,\varepsilon}ds-\int_0^t \mathrm{S}_2^\varepsilon(s)u_{1,\varepsilon}ds\\
&+\int_0^t \mathrm{T}_\alpha^\varepsilon(t-s)f_\varepsilon(s)ds-\int_0^t \mathrm{T}_2^\varepsilon(t-s)f_\varepsilon(s)ds\Big)\\
=&\int_{\Gamma_{r,\theta}} e^{\lambda t}\Big\{\lambda^{\alpha-1}\Big(\lambda^\alpha I-\chi(\varepsilon)\Delta-\widehat{k}(\varepsilon\lambda)\Delta\Big)^{-1}-\lambda\Big(\lambda^2 I-\chi(\varepsilon)\Delta-\widehat{k}(\varepsilon\lambda)\Delta\Big)^{-1}\Big\}u_{0,\varepsilon}d\lambda
\end{aligned}$$

$$+ \int_0^t \int_{\Gamma_{r,\theta}} e^{\lambda s}\Big\{\lambda^{\alpha-1}\Big(\lambda^\alpha I - \chi(\varepsilon)\Delta - \widehat{k}(\varepsilon\lambda)\Delta\Big)^{-1} - \lambda\Big(\lambda^2 I - \chi(\varepsilon)\Delta - \widehat{k}(\varepsilon\lambda)\Delta\Big)^{-1}\Big\}u_{1,\varepsilon}d\lambda ds$$
$$+ \int_0^t \int_{\Gamma_{r,\theta}} e^{\lambda(t-s)}\Big\{\Big(\lambda^\alpha I - \chi(\varepsilon)\Delta - \widehat{k}(\varepsilon\lambda)\Delta\Big)^{-1} - \Big(\lambda^2 I - \chi(\varepsilon)\Delta - \widehat{k}(\varepsilon\lambda)\Delta\Big)^{-1}\Big\}f_\varepsilon(s)d\lambda ds.$$

Due to Lemma 2.1, assumptions **(H1)** and **(H3)**, we obtain

$$\begin{aligned}
&\Big\|\lambda^{\alpha-1}\Big(\lambda^\alpha I - \chi(\varepsilon)\Delta - \widehat{k}(\varepsilon\lambda)\Delta\Big)^{-1}u_{0,\varepsilon} \\
&\quad -\lambda\Big(\lambda^2 I - \chi(\varepsilon)\Delta - \widehat{k}(\varepsilon\lambda)\Delta\Big)^{-1}u_{0,\varepsilon}\Big\|_{L^q(\mathbf{R}^n)} \le \frac{2C}{|\lambda|}\, \| u_{0,\varepsilon}\|_{L^q(\mathbf{R}^n)}, \\
&\Big\|\lambda^{\alpha-1}\Big(\lambda^\alpha I - \chi(\varepsilon)\Delta - \widehat{k}(\varepsilon\lambda)\Delta\Big)^{-1}u_{1,\varepsilon} \\
&\quad -\lambda\Big(\lambda^2 I - \chi(\varepsilon)\Delta - \widehat{k}(\varepsilon\lambda)\Delta\Big)^{-1}u_{1,\varepsilon}\Big\|_{L^q(\mathbf{R}^n)} \le \frac{2C}{|\lambda|}\, \| u_{1,\varepsilon}\|_{L^q(\mathbf{R}^n)},
\end{aligned}$$

and

$$\begin{aligned}
&\Big\|\Big(\lambda^\alpha I - \chi(\varepsilon)\Delta - \widehat{k}(\varepsilon\lambda)\Delta\Big)^{-1}f_\varepsilon - \Big(\lambda^2 I - \chi(\varepsilon)\Delta - \widehat{k}(\varepsilon\lambda)\Delta\Big)^{-1}f_\varepsilon\Big\|_{L^q(\mathbf{R}^n)} \\
&\le (\frac{C}{|\lambda|^\alpha} + \frac{C}{|\lambda|^2})\|f_\varepsilon\|_{C([0,T],L^q(\mathbf{R}^n))} \le \frac{2C}{|\lambda|^\alpha}\|f_\varepsilon\|_{C([0,T],L^q(\mathbf{R}^n))}.
\end{aligned}$$

Furthermore, for each $\lambda \in \rho(\Delta)$ we have

$$\begin{aligned}
&\Big\|\lambda^{\alpha-1}\Big(\lambda^\alpha I - \chi(\varepsilon)\triangle - \widehat{k}(\varepsilon\lambda)\triangle\Big)^{-1}u_{0,\varepsilon} - \lambda\Big(\lambda^2 I - \chi(\varepsilon)\triangle - \widehat{k}(\varepsilon\lambda)\triangle\Big)^{-1}u_{0,\varepsilon}\Big\|_{L^q(\mathbf{R}^n)} \to 0, \\
&\Big\|\lambda^{\alpha-1}\Big(\lambda^\alpha I - \chi(\varepsilon)\triangle - \widehat{k}(\varepsilon\lambda)\triangle\Big)^{-1}u_{1,\varepsilon} - \lambda\Big(\lambda^2 I - \chi(\varepsilon)\triangle - \widehat{k}(\varepsilon\lambda)\triangle\Big)^{-1}u_{1,\varepsilon}\Big\|_{L^q(\mathbf{R}^n)} \to 0, \\
&\Big\|\Big(\lambda^\alpha I - \chi(\varepsilon)\triangle - \widehat{k}(\varepsilon\lambda)\triangle\Big)^{-1}f_\varepsilon - \Big(\lambda^2 I - \chi(\varepsilon)\triangle - \widehat{k}(\varepsilon\lambda)\triangle\Big)^{-1}f_\varepsilon\Big\|_{L^q(\mathbf{R}^n)} \to 0,
\end{aligned}$$

as $\alpha \to 2^-$ uniform for t in bounded subintervals of $[0, +\infty]$ and $u_0, u_1, u_{0,\varepsilon}, u_{1,\varepsilon}$ in bounded subsets of $L^q(\mathbf{R}^n)$ and $f(t), f_\varepsilon(t)$ in bounded subsets of $C([0, T], L^q(\mathbf{R}^n))$. Thus, we conclude the proof of Eq (2.42) by using the Lebesgue dominated convergence theorem.

This completes the proof of Lemma 2.13.

3. Statement of the problems and main results

Now we consider the existence of the solutions u_ε and u to problem (1.1) with a small parameter ε and the corresponding limit problem (1.5) as $\varepsilon \to 0^+$ respectively, and then we focus on the convergence of u_ε and u in the space X_ρ for $\rho > 1$, where the space

$$X_\rho = \{u \in C((0, T], L^{\rho q}(\mathbf{R}^n)) : \sup_{t\in(0,T]} t^{\frac{\alpha n}{2\rho q}(\rho-1)}\|u(t)\|_{L^{\rho q}(\mathbf{R}^n)} < \infty\} \tag{3.1}$$

endowed with the norm

$$\|u\|_{X_\rho} = \sup_{t\in(0,T]} t^{\frac{\alpha n}{2\rho q}(\rho-1)}\|u(t)\|_{L^{\rho q}(\mathbf{R}^n)}. \tag{3.2}$$

Now let $\beta = \frac{\alpha n}{2\rho q}(\rho - 1)$, we state our main results.

Theorem 3.1. *Under the assumptions* **(H1)**, **(H3)** *with* $A = \Delta, X = L^q(\mathbf{R}^n)$, *for* $q > 1$, $\alpha \in (1, 2)$, $\rho > 1$, $q > \frac{\alpha n}{2}(\rho - 1)$, *there exist constants* $T > 0$ *and* $R > 0$ *such that the problem (1.5) admits an* L^q*−mild solution* $u : [0, T] \to X$ *which is unique in* $C([0, T], X) \cap X_\rho$, *for any* $(f, u_0, u_1) \in C([0, T], X) \times X \times X$.

Theorem 3.2. *Under the assumptions* **(H1)**, **(H3)** *with* $A = \Delta, X = L^q(\mathbf{R}^n)$, *the nonlinear operator* $N : [0, T] \times X \to X$ *defined by Eq (1.10) is continuous with respect to t and there exists a constant* $M > 0$ *such that*

$$\begin{aligned} \|N(t, x)\| &\leq M(1 + \|x\|^{\rho-1}),\ \forall x, y \in X, && (3.3)\\ \|N(t, x) - N(t, y)\| &\leq M(1 + \|x\|^{\rho-1} + \|y\|^{\rho-1})\|x - y\|,\ \forall x, y \in X. && (3.4) \end{aligned}$$

For $q > 1, \alpha \in (1, 2), \rho > 1, q > \frac{\alpha n}{2}(\rho - 1)$, *there exist constants* $T > 0$ *and* $R > 0$ *such that the problem (1.9) admits an* L^q*−mild solution* $u : [0, T] \to X$ *which are unique in* $C([0, T], X) \cap X_\rho$ *for any* $(f, u_0, u_1) \in C([0, T], X) \times X \times X$.

Theorem 3.3. *Under the assumptions* **(H1)–(H3)** *with* $A = \Delta, X = L^q(\mathbf{R}^n)$, *the coefficient* χ *satisfies Eq (1.4), for* $q > 1, \alpha \in (1, 2)$, *there exist constants* $T, R > 0$ *such that the problem (1.1) admits an* L^q*−mild solution* $u_\varepsilon : [0, T] \to X$ *which is unique in* $C([0, T], X)$ *for any* $(f_\varepsilon, u_{0,\varepsilon}, u_{1,\varepsilon}) \in C([0, T], X) \times X \times X$, $\varepsilon \in [0, \varepsilon_0]$. *For T and R, the problem (1.5) admits an* L^q*−mild solution* $u : [0, T] \to X$ *which is unique in* $C([0, T], X)$ *for any* $(f, u_0, u_1) \in C([0, T], X) \times X \times X$.

Theorem 3.4. *Under the assumptions* **(H1)–(H3)** *with* $A = \Delta, X = L^q(\mathbf{R}^n)$, *the nonlinear operator* $N : [0, T] \times X \to X$ *defined by Eq (1.10) is continuous with respect to t and satisfies Eqs (3.3) and (3.4), the coefficient* χ *satisfies Eq (1.4), for* $\rho > 1, q > 1, \alpha \in (1, 2)$, *there exist constants* $T, R > 0$ *such that the problem (1.8) admits an* L^q*−mild solution* $u_\varepsilon : [0, T] \to X$ *which is unique in* $C([0, T], X)$ *for any* $(f_\varepsilon, u_{0,\varepsilon}, u_{1,\varepsilon}) \in C([0, T], X) \times X \times X$, $\varepsilon \in [0, \varepsilon_0]$, *while the problem (1.9) admits an* L^q*−mild solution* $u : [0, T] \to X$ *which is unique in* $C([0, T], X)$ *for any* $(f, u_0, u_1) \in C([0, T], X) \times X \times X$.

Theorem 3.5. *Under the assumptions* **(H1)–(H6)** *with* $A = \Delta, X = L^q(\mathbf{R}^n)$, *and the coefficient* χ *satisfies Eq (1.4). For* $\alpha \in (1, 2), p \geq 1, q > 1$, *there exists* $T > 0$ *such that the mild solutions* u_ε *of the approximating problem (1.1) converges in* $C([0, T], X)$ *to the mild solution u of the limit problem (1.5) as* $\varepsilon \to 0^+$. *More exactly, there exists a positive constant C such that*

$$\begin{aligned} \|u_\varepsilon - u\|_{C([0,T]),X)} &\leq C(\|u_{0,\varepsilon} - u_0\|_X + \|u_{1,\varepsilon} - u_1\|_X + \|f_\varepsilon - f\|_{C([0,T],X)})\\ &+ C(\varepsilon^{\frac{1}{p}+\alpha-1} + \varepsilon^{\theta_0} + |\chi(\varepsilon) - \chi_0|)\|f\|_{C([0,T],X)}\\ &+ C(\varepsilon^{\frac{1}{p}} + \varepsilon^{\theta_0} + |\chi(\varepsilon) - \chi_0|)(\|u_0\|_X + \|u_1\|_X). && (3.5) \end{aligned}$$

Theorem 3.6. *Under the assumptions* **(H1)–(H6)** *with* $A = \Delta, X = L^q(\mathbf{R}^n)$, *the nonlinear operator* $N : [0, T] \times X \to X$ *defined by Eq (1.10) is continuous with respect to t and satisfies inequalities (3.3) and (3.4), the coefficient* χ *satisfies (1.4). For* $\rho > 1, \alpha \in (1, 2), p \geq 1, q > 1$, *there exists* $T > 0$ *such that the mild solutions* u_ε *of the approximating problem (1.8) converges in* $C([0, T], L^q(\mathbf{R}^n))$ *to the mild solution u of the limit problem (1.9) as* $\varepsilon \to 0^+$. *More exactly, there exists a constant* $C > 0$ *such that*

$$\begin{aligned} &\|u_\varepsilon - u\|_{C([0,T]),X)}\\ \leq\ &C(\|u_{0,\varepsilon} - u_0\|_X + \|u_{1,\varepsilon} - u_1\|_X + \|f_\varepsilon - f\|_{C([0,T],X)}) \end{aligned}$$

$$
\begin{aligned}
&+ \quad C(\varepsilon^{\frac{1}{p}+\alpha-1} + \varepsilon^{\theta_0} + |\chi(\varepsilon) - \chi_0|)\|f\|_{C([0,T],X)} \\
&+ \quad C(\varepsilon^{\frac{1}{p}} + \varepsilon^{\theta_0} + |\chi(\varepsilon) - \chi_0|)(\|u_0\|_X + \|u_1\|_X) \\
&+ \quad C(\varepsilon^{\frac{1}{p}+\alpha-1} + \varepsilon^{\theta_0} + |\chi(\varepsilon) - \chi_0|)(\|u_0\|_X^\rho + \|u_1\|_X^\rho + \|f\|^\rho_{C([0,T],X)}).
\end{aligned} \tag{3.6}
$$

Theorem 3.7. *Let u be the solution of IVP (1.9) given in Theorem 3.2. Then u can be uniquely continued to a maximal time $T_{\max} > T$.*

Theorem 3.8. *Let u be the solution of IVP (1.9) given in Theorem 3.2. If $T_{\max} < \infty$, then*

$$
\limsup_{t\to T_{\max}^-} t^\beta \|u(t)\|_{L^{\rho q}(\mathbf{R}^n)} = \infty, \ \limsup_{t\to T_{\max}^-} \|u(t)\|_{L^q(\mathbf{R}^n)} = \infty.
$$

Theorem 3.9. *If $u_\alpha^\varepsilon, u_\alpha, u_2^\varepsilon, u_2$ are mild solutions of the problems (1.8), (1.9), (1.11) and (1.12) respectively, then*

$$
\|u_\alpha^\varepsilon(t) - u_2^\varepsilon(t)\|_{L^q(\mathbf{R}^n)} \to 0, \ \|u_\alpha(t) - u_2(t)\|_{L^q(\mathbf{R}^n)} \to 0,
$$

as $\alpha \to 2^-$ uniformly for t in bounded subintervals of $[0,T]$, where $T > 0$ is any common time of existence for $u_\alpha^\varepsilon, u_\alpha, u_2^\varepsilon, u_2$.

4. Proof of Theorems 3.1–3.9

For the approximating problem (1.1) with a small positive parameter ε and the limit problem (1.5) as $\varepsilon \to 0^+$, due to Definition 2.3 we can transform them into the integral Eq (2.10) by Laplace transform and the operator families $\{S_\alpha(t)\}_{t\geq 0}$ and $\{T_\alpha(t)\}_{t\geq 0}$ defined in Eqs (2.7) and (2.9).

Proof of Theorem 3.1. For the linear limit problem (1.5), we consider the integral Eq (2.11). Use the space

$$
X_\rho = \{u \in C((0,T], L^{\rho q}(\mathbf{R}^n)) : \sup_{t\in(0,T]} t^{\frac{\alpha n}{2\rho q}(\rho-1)} \|u(t)\|_{L^{\rho q}(\mathbf{R}^n)} < \infty\}
$$

defined in Eq (3.1), endowed with the norm given in Eq (3.2):

$$
\|u\|_{X_\rho} = \sup_{t\in(0,T]} t^{\frac{\alpha n}{2\rho q}(\rho-1)} \|u(t)\|_{L^{\rho q}(\mathbf{R}^n)},
$$

let $\mathbf{B}$ be the closed ball in X_ρ centered in the origin with radius R. Since

$$
\rho\beta = \rho\frac{\alpha n}{2\rho q}(\rho-1) = \frac{\alpha n}{2q}(\rho-1) < 1, \ \frac{n}{2}(\frac{1}{q} - \frac{1}{\rho q}) = \frac{n(\rho-1)}{2\rho q} = \frac{\beta}{\alpha} < 1,
$$

using Lemma 2.7 and the integral Eq (2.11), we have

$$
\begin{aligned}
\|u\|_{X_\rho} &\leq \|S_\alpha(t)u_0\|_{X_\rho} + \|\int_0^t S_\alpha(s)u_1 ds\|_{X_\rho} + \|\int_0^t \mathrm{T}_\alpha(t-s)f(s)ds\|_{X_\rho} \\
&= \sup_{t\in(0,T]} t^\beta \|S_\alpha(t)u_0\|_{L^{\rho q}(\mathbf{R}^n)} + \sup_{t\in(0,T]} t^\beta \|\int_0^t S_\alpha(s)u_1 ds\|_{L^{\rho q}(\mathbf{R}^n)} \\
&+ \sup_{t\in(0,T]} t^\beta \|\int_0^t \mathrm{T}_\alpha(t-s)f(s)ds\|_{L^{\rho q}(\mathbf{R}^n)}
\end{aligned}
$$

$$
\begin{aligned}
&\le \sup_{t\in(0,T]} t^{\beta}t^{-\beta}\|u_0\|_{L^q(\mathbf{R}^n)} + \sup_{t\in(0,T]} t^{\beta}\frac{t^{1-\beta}}{1-\beta}\|u_1\|_{L^q(\mathbf{R}^n)} \\
&+ \sup_{t\in(0,T]} t^{\beta}\int_0^t \|\mathrm{T}_\alpha(t-s)f(s)\|_{L^{\rho q}(\mathbf{R}^n)}ds \\
&\le C\|u_0\|_{L^q(\mathbf{R}^n)} + \frac{CT}{1-\beta}\|u_1\|_{L^q(\mathbf{R}^n)} \\
&+ C\sup_{t\in(0,T]} t^{\beta}\int_0^t (t-s)^{-\beta+\alpha-1}\|f(s)\|_{L^q(\mathbf{R}^n)}ds \\
&\le C\|u_0\|_{L^q(\mathbf{R}^n)} + \frac{CT}{1-\beta}\|u_1\|_{L^q(\mathbf{R}^n)} \\
&+ C\|f\|_{C([0,T],L^q(\mathbf{R}^n))}\sup_{t\in(0,T]} t^{\beta}\int_0^t (t-s)^{\alpha-1-\beta}ds \\
&\le C\|u_0\|_{L^q(\mathbf{R}^n)} + \frac{CT}{1-\beta}\|u_1\|_{L^q(\mathbf{R}^n)} + CT^{\alpha}\|f\|_{C([0,T],L^q(\mathbf{R}^n))} \\
&\le R.
\end{aligned}
$$

The contraction principle implies existence of the mild solution u in the ball $\mathbf{B}$. We will prove that u is an L^q-mild solution and is unique in X_ρ. Indeed,

$$
\begin{aligned}
&\|u(t)-u_0\|_{L^q(\mathbf{R}^n)} \\
\le\;& \|S_\alpha(t)u_0-u_0\|_{L^q(\mathbf{R}^n)} + \int_0^t \|S_\alpha(s)u_1\|_{L^q(\mathbf{R}^n)}ds + \int_0^t \|\mathrm{T}_\alpha(t-s)f(s)\|_{L^q(\mathbf{R}^n)}ds \\
\le\;& \|S_\alpha(t)u_0-u_0\|_{L^q(\mathbf{R}^n)} + C\int_0^t \|u_1\|_{L^q(\mathbf{R}^n)}ds + C\int_0^t (t-s)^{\alpha-1}\|f(s)\|_{L^q(\mathbf{R}^n)}ds \\
\le\;& \|S_\alpha(t)u_0-u_0\|_{L^q(\mathbf{R}^n)} + Ct\|u_1\|_{L^q(\mathbf{R}^n)} + C\|f\|_{C([0,T],L^q(\mathbf{R}^n))}\int_0^t (t-s)^{\alpha-1}ds \\
=\;& \|S_\alpha(t)u_0-u_0\|_{L^q(\mathbf{R}^n)} + Ct\|u_1\|_{L^q(\mathbf{R}^n)} + \frac{C}{\alpha}t^{\alpha}\|f\|_{C([0,T],L^q(\mathbf{R}^n))} \\
\to\;& 0 \text{ as } t\to 0^+.
\end{aligned}
$$

Therefore, $u \in \mathrm{C}([0,T],L^q(\mathbf{R}^n))\bigcap \mathrm{C}((0,T],L^{\rho q}(\mathbf{R}^n))$ is an L^q-mild solution the linear limit problem (1.5). We finish the proof of Theorem 3.1.

Proof of Theorem 3.2. For $\rho\beta<1$, $\frac{n}{2}(\frac{1}{q}-\frac{1}{\rho q})<1$, we can choose T such that

$$
\begin{gathered}
C\|u_0\|_{L^q(\mathbf{R}^n)} + C\frac{T}{1-\beta}\|u_1\|_{L^q(\mathbf{R}^n)} + \frac{CT^{\alpha}}{\alpha-\beta}\|f\|_{C([0,T],L^q(\mathbf{R}^n))} \\
+CMT^{\alpha+\beta}\Big(\frac{1}{\alpha}+R^{\rho}T^{-\rho\beta}\mathrm{B}(\alpha,1-\rho\beta)\Big)\le R, \\
MCT^{\alpha}\mathrm{B}(\alpha,1-\beta)+2R^{\rho-1}T^{\alpha+\beta-\rho\beta}\mathrm{B}(\alpha,1-\rho\beta)\le\frac{1}{2},
\end{gathered}
$$

where $\Gamma(\alpha)$ and $\mathrm{B}(\alpha,\beta)$ are Gamma and Beta functions respectively. Let $\mathbf{B}$ be the closed ball in X_ρ centered in the origin with radius R. For the nonlinear limit problem (1.9), define a map $\Gamma:\mathbf{B}\to\mathbf{B}$ by

$$
\Gamma u(t)=\mathrm{S}_\alpha(t)u_0+\int_0^t \mathrm{S}_\alpha(s)u_1ds+\int_0^t \mathrm{T}_\alpha(t-s)f(s)ds+\int_0^t \mathrm{T}_\alpha(t-s)Nu(s)ds. \tag{4.1}
$$

By the Lemmas 2.7 and 2.8, for $u \in X_\rho$ we have

$$\begin{aligned}
\|\Gamma u\|_{X_\rho} &\leq \|\mathrm{S}_\alpha(t)u_0\|_{X_\rho} + \|\int_0^t \mathrm{S}_\alpha(s)u_1 ds\|_{X_\rho} + \|\int_0^t \mathrm{T}_\alpha(t-s)f(s)ds\|_{X_\rho} \\
&+ \|\int_0^t \mathrm{T}_\alpha(t-s)Nu(s)ds\|_{X_\rho} \\
&\leq C\sup_{t\in(0,T]} t^\beta t^{-\beta}\|u_0\|_{L^q(\mathbf{R}^n)} + \sup_{t\in(0,T]} t^\beta \int_0^t \|\mathrm{S}_\alpha(s)u_1 ds\|_{L^{\rho q}(\mathbf{R}^n)} \\
&+ \sup_{t\in(0,T]} t^\beta \int_0^t \|\mathrm{T}_\alpha(t-s)f(s)\|_{L^{\rho q}(\mathbf{R}^n)} ds \\
&+ \sup_{t\in(0,T]} t^\beta \int_0^t \|\mathrm{T}_\alpha(t-s)Nu(s)\|_{L^{\rho q}(\mathbf{R}^n)} ds,
\end{aligned}$$

thus we have

$$\begin{aligned}
&\|\Gamma u\|_{X_\rho} \leq C\|u_0\|_{L^q(\mathbf{R}^n)} + C\sup_{t\in(0,T]} t^\beta \int_0^t (t-s)^{\alpha-1-\beta}\|f(s)\|_{L^q(\mathbf{R}^n)} ds \\
&+CM\sup_{t\in(0,T]} t^\beta \int_0^t (t-s)^{\alpha-1}(1+\|u(s)\|^\rho_{L^{\rho q}(\mathbf{R}^n)})ds \\
&+C\sup_{t\in(0,T]} t^\beta \int_0^t s^{-\beta}\|u_1\|_{L^q(\mathbf{R}^n)} ds \\
&\leq C\|u_0\|_{L^q(\mathbf{R}^n)} + \frac{CT}{1-\beta}\|u_1\|_{L^q(\mathbf{R}^n)} + \frac{CT^\alpha}{\alpha-\beta}\|f(s)\|_{C([0,T],L^q(\mathbf{R}^n))} \\
&+\frac{CM}{\alpha}T^{\alpha+\beta} + CM\sup_{t\in(0,T]} t^\beta \int_0^t (t-s)^{\alpha-1}\|u(s)\|^\rho_{L^{\rho q}(\mathbf{R}^n)} ds \\
&\leq C\|u_0\|_{L^q(\mathbf{R}^n)} + \frac{CT}{1-\beta}\|u_1\|_{L^q(\mathbf{R}^n)} + \frac{CT^\alpha}{\alpha-\beta}\|f(s)\|_{C([0,T],L^q(\mathbf{R}^n))} \\
&+\frac{CM}{\alpha}T^{\alpha+\beta} + CMR^\rho\sup_{t\in(0,T]} t^\beta \int_0^t (t-s)^{\alpha-1}s^{-\rho\beta} ds \\
&\leq C\|u_0\|_{L^q(\mathbf{R}^n)} + \frac{CT}{1-\beta}\|u_1\|_{L^q(\mathbf{R}^n)} + \frac{CT^\alpha}{\alpha-\beta}\|f(s)\|_{C([0,T],L^q(\mathbf{R}^n))} \\
&+\frac{CM}{\alpha}T^{\alpha+\beta} + CMR^\rho T^{\alpha-\rho\beta+\beta}\mathrm{B}(\alpha, 1-\rho\beta) \leq R.
\end{aligned}$$

For $u, v \in \mathbf{B}$, we also get

$$\begin{aligned}
&\|\Gamma u - \Gamma v\|_{X_\rho} \leq \sup_{t\in(0,T]} t^\beta \int_0^t \|\mathrm{T}_\alpha(t-s)(Nu(s)-Nv(s))\|_{L^{\rho q}(\mathbf{R}^n)} ds \\
&\leq MC\sup_{t\in(0,T]} t^\beta \int_0^t (t-s)^{\alpha-1}(1+\|u(s)\|^{\rho-1}_{L^{\rho q}(\mathbf{R}^n)} + \|v(s)\|^{\rho-1}_{L^{\rho q}(\mathbf{R}^n)}) \\
&\qquad \cdot\|u(s)-v(s)\|_{L^{\rho q}(\mathbf{R}^n)} ds \\
&\leq MC\|u-v\|_{X_\rho}\sup_{t\in(0,T]} t^\beta \int_0^t (t-s)^{\alpha-1}s^{-\beta} ds
\end{aligned}$$

$$\leq +MC\|u-v\|_{X_\rho}\sup_{t\in[0,T]} t^{\beta}\int_0^t (t-s)^{\alpha-1}s^{-\beta}s^{-(\rho-1)\beta}(\|u\|_{X_\rho}^{\rho-1}+\|v)\|_{X_\rho}^{\rho-1})ds$$
$$\leq MC\|u-v\|_{X_\rho}T^{\alpha}\mathrm{B}(\alpha,1-\beta)+2MC\|u-v\|_{X_\rho}R^{\rho-1}T^{\alpha+\beta-\rho\beta}\mathrm{B}(\alpha,1-\rho\beta)$$
$$\leq MC\|u-v\|_{X_\rho}\Big(T^{\alpha}\mathrm{B}(\alpha,1-\beta)+2R^{\rho-1}T^{\alpha+\beta-\rho\beta}\mathrm{B}(\alpha,1-\rho\beta)\Big)$$
$$\leq \frac{1}{2}\|u-v\|_{X_\rho}.$$

The contraction principle implies the existence of mild solution u in $\mathbf{B}$. We will prove that u is an L^q-mild solution and is unique in X_ρ. Indeed,

$$\|u(t)-u_0\|_{L^q(\mathbf{R}^n)} \leq \|S_\alpha(t)u_0-u_0\|_{L^q(\mathbf{R}^n)}+\|\int_0^t S_\alpha(s)u_1\|_{L^q(\mathbf{R}^n)}$$
$$+\|\int_0^t \mathrm{T}_\alpha(t-s)f(s)ds\|_{L^q(\mathbf{R}^n)}+\|\int_0^t \mathrm{T}_\alpha(t-s)Nu(s)ds\|_{L^q(\mathbf{R}^n)},$$

thus we have

$$\|u(t)-u_0\|_{L^q(\mathbf{R}^n)} \leq \|S_\alpha(t)u_0-u_0\|_{L^q(\mathbf{R}^n)}+C\int_0^t \|u_1\|_{L^q(\mathbf{R}^n)}ds$$
$$+\int_0^t \|\mathrm{T}_\alpha(t-s)f(s)\|_{L^q(\mathbf{R}^n)}ds+\int_0^t \|\mathrm{T}_\alpha(t-s)Nu(s)\|_{L^q(\mathbf{R}^n)}ds$$
$$\leq \|S_\alpha(t)u_0-u_0\|_{L^q(\mathbf{R}^n)}+Ct\|u_1\|_{L^q(\mathbf{R}^n)}$$
$$+C\int_0^t (t-s)^{\alpha-1}\|f(s)\|_{L^q(\mathbf{R}^n)}ds$$
$$+CM\int_0^t (t-s)^{\alpha-1}(1+\|u(s)\|^{\rho}_{L^q(\mathbf{R}^n)})ds$$
$$\leq \|S_\alpha(t)u_0-u_0\|_{L^q(\mathbf{R}^n)}+Ct\|u_1\|_{L^q(\mathbf{R}^n)}$$
$$+C\|f\|_{C([0,T],L^q(\mathbf{R}^n))}\int_0^t (t-s)^{\alpha-1}ds$$
$$+CM\int_0^t (t-s)^{\alpha-1}ds+CM\int_0^t (t-s)^{\alpha-1}\|u(s)\|^{\rho}_{L^q(\mathbf{R}^n)}ds$$
$$\leq \|S_\alpha(t)u_0-u_0\|_{L^q(\mathbf{R}^n)}+Ct\|u_1\|_{L^q(\mathbf{R}^n)}+C\alpha^{-1}t^{\alpha}\|f\|_{C([0,T],L^q(\mathbf{R}^n))}$$
$$+CM\alpha^{-1}t^{\alpha}+CMt^{\alpha-\rho\beta}\mathrm{B}(\alpha,1-\rho\beta)\|u\|^{\rho}_{X_\rho}\to 0 \text{ as } t\to 0^+.$$

Therefore, $u\in \mathrm{C}([0,T],L^q(\mathbf{R}^n))\bigcap \mathrm{C}((0,T],L^{\rho q}(\mathbf{R}^n))$ is an L^q-mild solution for the limit problem (1.9). If $v\in X_\rho$ is a solution of problem (1.9), we may take $0<T'\leq T$ such that $\|v\|_{X_\rho}^{T'}\leq R$. Then, by uniqueness in $\mathbf{B}$, we have $u(t,x)=v(t,x)$ for all $t\in[0,T']$. Now, we set

$$R'=\max\Big\{\sup_{t\in(0,T)} t^{\beta}\|u(t)\|_{L^{\rho q}(\mathbf{R}^n)},\ \sup_{t\in(0,T)} t^{\beta}\|v(t)\|_{L^{\rho q}(\mathbf{R}^n)}\Big\}.$$

For $t\in[T',T]$, we have

$$t^{\beta}\|u(t,x)-v(t,x)\|_{L^{\rho q}(\mathbf{R}^n)}$$

$$
\begin{aligned}
&\le MCt^{\beta}\int_0^t (t-s)^{\alpha-1}\Big(1+\|u(s)\|_{L^{\rho q}(\mathbf{R}^n)}^{\rho-1}+\|v(s)\|_{L^{\rho q}(\mathbf{R}^n)}^{\rho-1}\Big)\\
&\qquad\qquad \cdot\|u(s)-v(s)\|_{L^{\rho q}(\mathbf{R}^n)}ds\\
&\le MCt^{\beta}\int_0^t (t-s)^{\alpha-1}\Big(1+s^{-(\rho-1)\beta}\|u\|_{X_\rho}^{\rho-1}+s^{-(\rho-1)\beta}\|v\|_{X_\rho}^{\rho-1}\Big)\\
&\qquad\qquad \cdot\|u(s)-v(s)\|_{L^{\rho q}(\mathbf{R}^n)}ds\\
&\le MC(1+2(T')^{-(\rho-1)\beta}R'^{\rho-1})(T')^{-\beta}t^{\beta}\\
&\qquad\qquad \cdot\int_0^t (t-s)^{\alpha-1}s^{\beta}\|u(s)-v(s)\|_{L^{\rho q}(\mathbf{R}^n)}ds.
\end{aligned}
$$

Now, let $\xi:[0,T]\to[0,+\infty)$ be defined by $\xi(t)=t^{\beta}\|u(t)-v(t)\|_{L^{\rho q}(\mathbf{R}^n)}$, thus we have

$$
\xi(t)\le MC(1+2(T')^{-(\rho-1)\beta}R'^{\rho-1})(T')^{-\beta}t^{\beta}\int_0^t (t-s)^{\alpha-1}\xi(s)ds,
$$

apply the singular Gronwall's Lemma [52, Theorem 4] we can obtain the uniqueness of the solution. We finish the proof of Theorem 3.2.

Proof of Theorem 3.3. For the approximating problem (1.1) with a small positive parameter ε and the limit problem (1.5) as $\varepsilon\to 0^+$, due to Definition 2.2 we can transform them into the integral Eqs (2.10) and (2.11) by the operator families $\{S_\alpha(t)\}_{t\ge0}$, $\{T_\alpha(t)\}_{t\ge0}$, $\{S_\alpha^{\varepsilon}(t)\}_{t\ge0}$, $\{T_\alpha^{\varepsilon}(t)\}_{t\ge0}$ and Laplace transform. From the assumptions **(H5)**, **(H6)** on $u_{0,\varepsilon}, u_{1,\varepsilon}$ and (i)(v) in Lemma 2.6 we obtain

$$
\begin{aligned}
&\|u_\varepsilon\|_{C([0,T],L^q(\mathbf{R}^n))}\le\|S_\alpha^{\varepsilon}(t)u_{0,\varepsilon}\|_{C([0,T],L^q(\mathbf{R}^n))}\\
&\quad+\Big\|\int_0^t \mathrm{S}_\alpha^{\varepsilon}(s)u_{1,\varepsilon}ds\Big\|_{C([0,T],L^q(\mathbf{R}^n))}+\Big\|\int_0^t \mathrm{T}_\alpha^{\varepsilon}(t-s)f_\varepsilon(s)ds\Big\|_{C([0,T],L^q(\mathbf{R}^n))}\\
&=\sup_{t\in[0,T]}\|S_\alpha^{\varepsilon}(t)u_{0,\varepsilon}\|_{L^q(\mathbf{R}^n)}+\sup_{t\in[0,T]}\Big\|\int_0^t \mathrm{S}_\alpha^{\varepsilon}(s)u_{1,\varepsilon}ds\Big\|_{L^q(\mathbf{R}^n)}\\
&\quad+\sup_{t\in[0,T]}\Big\|\int_0^t \mathrm{T}_\alpha^{\varepsilon}(t-s)f_\varepsilon(s)ds\Big\|_{L^q(\mathbf{R}^n)}\\
&\le\sup_{t\in[0,T]}Ce^{rt}\|u_{0,\varepsilon}\|_{L^q(\mathbf{R}^n)}+\sup_{t\in[0,T]}C\|u_{1,\varepsilon}\|_{L^q(\mathbf{R}^n)}\int_0^t e^{rs}ds\\
&\quad+\sup_{t\in[0,T]}\int_0^t\|\mathrm{T}_\alpha^{\varepsilon}(t-s)f_\varepsilon(s)\|_{L^q(\mathbf{R}^n)}ds\\
&\le Ce^{rT}\|u_{0,\varepsilon}\|_{L^q(\mathbf{R}^n)}+\frac{C}{r}(e^{rT}-1)\|u_{1,\varepsilon}\|_{L^q(\mathbf{R}^n)}\\
&\quad+C\sup_{t\in[0,T]}\int_0^t e^{r(t-s)}\|f_\varepsilon(s)\|_{L^q(\mathbf{R}^n)}ds,
\end{aligned}
$$

thus,

$$
\begin{aligned}
\|u_\varepsilon\|_{C([0,T],L^q(\mathbf{R}^n))}\;\le\;& Ce^{rT}\|u_{0,\varepsilon}\|_{L^q(\mathbf{R}^n)}+\frac{C}{r}(e^{rT}-1)\|u_{1,\varepsilon}\|_{L^q(\mathbf{R}^n)}\\
&+C\|f_\varepsilon\|_{C([0,T],L^q(\mathbf{R}^n))}\sup_{t\in[0,T]}e^{rt}\int_0^t e^{-rs}ds
\end{aligned}
$$

$$\leq \quad Ce^{rT}\|u_{0,\varepsilon}\|_{L^q(\mathbf{R}^n)} + \frac{C}{r}(e^{rT}-1)\|u_{1,\varepsilon}\|_{L^q(\mathbf{R}^n)}$$
$$+\frac{C}{r}(e^{rT}-1)\|f_\varepsilon\|_{C([0,T],L^q(\mathbf{R}^n))} \leq R.$$

Next, we will prove that u_ε is an L^q-mild solution. Indeed,

$$\|u_\varepsilon(t)-u_{0,\varepsilon}\|_{L^q(\mathbf{R}^n)} \leq \|S^\varepsilon_\alpha(t)u_{0,\varepsilon}-u_{0,\varepsilon}\|_{L^q(\mathbf{R}^n)}$$
$$+\int_0^t \|\mathrm{S}^\varepsilon_\alpha(s)u_{1,\varepsilon}\|_{L^q(\mathbf{R}^n)}ds + \int_0^t \|\mathrm{T}^\varepsilon_\alpha(t-s)f_\varepsilon(s)\|_{L^q(\mathbf{R}^n)}ds$$
$$\leq \|S^\varepsilon_\alpha(t)u_{0,\varepsilon}-u_{0,\varepsilon}\|_{L^q(\mathbf{R}^n)} + C\|u_{1,\varepsilon}\|_{L^q(\mathbf{R}^n)}\int_0^t e^{rs}ds$$
$$+C\int_0^t e^{r(t-s)}\|f_\varepsilon(s)\|_{L^q(\mathbf{R}^n)}ds$$
$$\leq \|S^\varepsilon_\alpha(t)u_{0,\varepsilon}-u_{0,\varepsilon}\|_{L^q(\mathbf{R}^n)} + \frac{C(e^{rt}-1)}{r}\|u_{1,\varepsilon}\|_{L^q(\mathbf{R}^n)}$$
$$+C\|f_\varepsilon\|_{C([0,T],L^q(\mathbf{R}^n)}e^{rt}\int_0^t e^{-rs}ds$$
$$= \|S^\varepsilon_\alpha(t)u_{0,\varepsilon}-u_{0,\varepsilon}\|_{L^q(\mathbf{R}^n)} + \frac{C(e^{rt}-1)}{r}\|u_{1,\varepsilon}\|_{L^q(\mathbf{R}^n)}$$
$$+\frac{C(e^{rt}-1)}{r}\|f_\varepsilon\|_{C([0,T],L^q(\mathbf{R}^n))} \to 0 \text{ as } t\to 0^+.$$

Therefore, $u_\varepsilon \in C([0,T],L^q(\Omega))$ is an L^q-mild solution of the approximating problem (1.1).

Next, the proof of the existence of solution to the linear limit problem (1.5) is similar to the proof of Theorem 3.1, so we omit it. We finish the proof of Theorem 3.3.

Proof of Theorem 3.4. We can choose T such that

$$Ce^{rT}\|u_{0,\varepsilon}\|_{L^q(\mathbf{R}^n)}$$
$$+\frac{C(e^{rT}-1)}{r}\Big(\|u_{1,\varepsilon}\|_{L^q(\mathbf{R}^n)} + \|f_\varepsilon\|_{C([0,T],L^q(\mathbf{R}^n))} + MR^\rho + M\Big) \leq R,$$
$$C\Big(\|u_0\|_{L^q(\mathbf{R}^n)} + T\|u_1\|_{L^q(\mathbf{R}^n)}\Big)$$
$$+\frac{CT^\alpha}{\alpha}\Big(\|f(t)\|_{C([0,T],L^q(\mathbf{R}^n))} + M + MR^\rho\Big) \leq R,$$
$$CM\Big(\frac{e^{rT}-1}{r} + 2R^{\rho-1}\frac{e^{rT}-1}{r}\Big) \leq R,$$
$$CM\Big(\frac{T^\alpha}{\alpha} + 2R^{\rho-1}\frac{T^\alpha}{\alpha}\Big) \leq R.$$

For the nonlinear approximating problem (1.8), define an operator $\Lambda : \mathbf{B}\to\mathbf{B}$ by

$$\Lambda u_\varepsilon(t) = \mathrm{S}^\varepsilon_\alpha(t)u_{0,\varepsilon} + \int_0^t \Big(\mathrm{S}^\varepsilon_\alpha(s)u_{1,\varepsilon}ds + \mathrm{T}^\varepsilon_\alpha(t-s)f_\varepsilon(s) + \mathrm{T}^\varepsilon_\alpha(t-s)Nu_\varepsilon(s)\Big)ds, \tag{4.2}$$

and $\mathbf{B}$ is a closed ball in $C([0,T],L^q(\mathbf{R}^n))$ centered in the origin with radius R. By the assumption on nonlinear operator N, due to Lemma 2.6 and $u_\varepsilon \in C([0,T],L^q(\mathbf{R}^n))$ we have we have

$$\|\Lambda u_\varepsilon\|_{C([0,T],L^q(\mathbf{R}^n))} \leq \|\mathrm{S}^\varepsilon_\alpha(t)u_{0,\varepsilon}\|_{C([0,T],L^q(\mathbf{R}^n))}$$

$$
\begin{aligned}
&+\|\int_0^t \mathrm{S}_\alpha^\varepsilon(s)u_{1,\varepsilon}ds\|_{C([0,T],L^q(\mathbf{R}^n))} + \|\int_0^t \mathrm{T}_\alpha^\varepsilon(t-s)f_\varepsilon(s)ds\|_{C([0,T],L^q(\mathbf{R}^n))}\\
&+\|\int_0^t \mathrm{T}_\alpha^\varepsilon(t-s)Nu_\varepsilon(s)ds\|_{C([0,T],L^q(\mathbf{R}^n))}\\
&\le \sup_{t\in[0,T]}\|\mathrm{S}_\alpha^\varepsilon(t)u_{0,\varepsilon}\|_{L^q(\mathbf{R}^n)} + \sup_{t\in[0,T]}\int_0^t \|\mathrm{S}_\alpha^\varepsilon(s)u_{1,\varepsilon}\|_{L^q(\mathbf{R}^n)}ds\\
&+\sup_{t\in[0,T]}\int_0^t \|\mathrm{T}_\alpha^\varepsilon(t-s)f_\varepsilon(s)\|_{L^q(\mathbf{R}^n)}ds + \sup_{t\in[0,T]}\int_0^t \|\mathrm{T}_\alpha^\varepsilon(t-s)Nu_\varepsilon(s)\|_{L^q(\mathbf{R}^n)}ds\\
&\le \sup_{t\in[0,T]} Ce^{rt}\|u_{0,\varepsilon}\|_{L^q(\mathbf{R}^n)} + \sup_{t\in[0,T]} C\frac{e^{rt}-1}{r}\|u_{1,\varepsilon}\|_{L^q(\mathbf{R}^n)}\\
&+\sup_{t\in[0,T]}\int_0^t \|\mathrm{T}_\alpha^\varepsilon(t-s)f_\varepsilon(s)\|_{L^q(\mathbf{R}^n)}ds + CM\sup_{t\in[0,T]}\int_0^t e^{r(t-s)}(1+\|u_\varepsilon(s)\|^\rho_{L^q(\mathbf{R}^n)})ds\\
&\le C\Big(e^{rT}\|u_{0,\varepsilon}\|_{L^q(\mathbf{R}^n)} + (e^{rT}-1)\frac{\|u_{1,\varepsilon}\|_{L^q(\mathbf{R}^n)}}{r}\Big) + CM\sup_{t\in[0,T]} e^{rt}\int_0^t e^{-rs}ds\\
&+CT^\beta\|f_\varepsilon\|_{C([0,T],L^q(\mathbf{R}^n)}\sup_{t\in[0,T]} e^{rt}\int_0^t e^{-rs}ds + CM\sup_{t\in[0,T]} e^{rt}\int_0^t e^{-rs}\|u_\varepsilon(s)\|^\rho_{L^q(\mathbf{R}^n)}ds\\
&\le C\Big(e^{rT}\|u_{0,\varepsilon}\|_{L^q(\mathbf{R}^n)} + (e^{rT}-1)\frac{\|u_{1,\varepsilon}\|_{L^q(\mathbf{R}^n)}}{r}\Big) + \frac{C}{r}(e^{rT}-1)\|f_\varepsilon\|_{C([0,T],L^q(\mathbf{R}^n))}\\
&+\frac{e^{rT}-1}{r}R^\rho CM + \frac{C(e^{rT}-1)}{r}M \le R.
\end{aligned}
$$

If $u_\varepsilon, v_\varepsilon \in C([0,T], L^q(\mathbf{R}^n))$, we have

$$
\begin{aligned}
&\|\Lambda u_\varepsilon - \Lambda v_\varepsilon\|_{C([0,T],L^q(\mathbf{R}^n))}\\
&\le \sup_{t\in[0,T]}\int_0^t \|\mathrm{T}_\alpha^\varepsilon(t-s)(Nu_\varepsilon(s)-Nv_\varepsilon(s))\|_{L^q(\mathbf{R}^n)}ds\\
&\le MC\sup_{t\in[0,T]}\int_0^t e^{r(t-s)}\Big(1+\|u_\varepsilon(s)\|^{\rho-1}_{L^q(\mathbf{R}^n)} + \|v_\varepsilon(s)\|^{\rho-1}_{L^q(\mathbf{R}^n)}\Big)\\
&\quad\cdot\|u_\varepsilon(s)-v_\varepsilon(s)\|_{L^q(\mathbf{R}^n)}ds\\
&\le MC\|u_\varepsilon - v_\varepsilon\|_{C([0,T],L^q(\mathbf{R}^n))}\sup_{t\in[0,T]} e^{rt}\int_0^t e^{-rs}ds\\
&+MC\|u_\varepsilon - v_\varepsilon\|_{C([0,T],L^q(\mathbf{R}^n))}\sup_{t\in[0,T]} e^{rt}\int_0^t e^{-rs}\|u_\varepsilon\|^{\rho-1}_{C([0,T],L^q(\mathbf{R}^n))}ds\\
&+MC\|u_\varepsilon - v_\varepsilon\|_{C([0,T],L^q(\mathbf{R}^n))}\sup_{t\in(0,T]} e^{rt}\int_0^t e^{-rs}\|v_\varepsilon\|^{\rho-1}_{C([0,T],L^q(\mathbf{R}^n))}ds\\
&\le CM(e^{rT}-1)\|u_\varepsilon - v_\varepsilon\|_{C([0,T],L^q(\mathbf{R}^n))}\Big(\frac{1}{r}+\frac{2R^{\rho-1}}{r}\Big)\\
&\le \frac{1}{2}\|u_\varepsilon - v_\varepsilon\|_{C([0,T],L^q(\mathbf{R}^n))}.
\end{aligned}
$$

This yields that Λ is a contraction operator on $C([0,T], L^q(\mathbf{R}^n))$, we shall prove that u_ε is an L^q-mild solution and is unique in $C([0,T],L^q(\mathbf{R}^n))$. Indeed,

$$\begin{aligned}
&\|u_\varepsilon(t)-u_{0,\varepsilon}\|_{L^q(\mathbf{R}^n)} \le \|S_\alpha^\varepsilon(t)u_{0,\varepsilon}-u_{0,\varepsilon}\|_{L^q(\mathbf{R}^n)} + \|\int_0^t S_\alpha^\varepsilon(s)u_{1,\varepsilon}ds\|_{L^q(\mathbf{R}^n)}\\
&+\|\int_0^t \mathrm{T}_\alpha^\varepsilon(t-s)f_\varepsilon(s)ds\|_{L^q(\mathbf{R}^n)} + \|\int_0^t \mathrm{T}_\alpha^\varepsilon(t-s)Nu_\varepsilon(s)ds\|_{L^q(\mathbf{R}^n)}\\
&\le \|S_\alpha^\varepsilon(t)u_{0,\varepsilon}-u_{0,\varepsilon}\|_{L^q(\mathbf{R}^n)} + C\int_0^t e^{rs}\|u_{1,\varepsilon}\|_{L^q(\mathbf{R}^n)}ds\\
&+C\int_0^t e^{r(t-s)}\|f_\varepsilon(s)\|_{L^q(\mathbf{R}^n)}ds + MC\int_0^t e^{r(t-s)}(1+\|u_\varepsilon(s)\|^\rho_{L^q(\mathbf{R}^n)})ds\\
&\le \|S_\alpha^\varepsilon(t)u_{0,\varepsilon}-u_{0,\varepsilon}\|_{L^q(\mathbf{R}^n)} + \frac{C}{r}(e^{rt}-1)[\|f_\varepsilon\|_{C([0,T],L^q(\mathbf{R}^n))} + \|u_{1,\varepsilon}\|_{L^q(\mathbf{R}^n)}]\\
&+\frac{MC}{r}(e^{rt}-1) + \frac{MC}{r}(e^{rt}-1)\|u_\varepsilon\|_{C([0,T],L^q(\mathbf{R}^n))} \to 0 \text{ as } t\to 0^+.
\end{aligned}$$

Therefore, $u_\varepsilon(t,x) \in C([0,T],L^q(\mathbf{R}^n))$ is an L^q-mild solution for the nonlinear approximating problem (1.8). If $v_\varepsilon \in C([0,T],L^q(\mathbf{R}^n))$ is a solution of problem (1.8), taking $0<T'\le T$ such that $\|v_\varepsilon\|^{T'}_{C([0,T],L^q(\mathbf{R}^n))} \le R$, the uniqueness in $\mathbf{B}$ implies that $u_\varepsilon(t,x)=v_\varepsilon(t,x)$ for all $t\in[0,T']$. Now, we set

$$R' = \max\{\sup_{t\in(0,T)}\|u_\varepsilon(t)\|_{L^q(\mathbf{R}^n)}, \sup_{t\in(0,T)}\|v_\varepsilon(t)\|_{L^q(\mathbf{R}^n)}\}.$$

For $t\in[T',T]$, we have

$$\begin{aligned}
&\|u_\varepsilon(t)-v_\varepsilon(t)\|_{L^q(\mathbf{R}^n)}\\
\le\ & MC\int_0^t e^{r(t-s)}\Big(1+\|u_\varepsilon(s)\|^{\rho-1}_{L^q(\mathbf{R}^n)} + \|v_\varepsilon(s)\|^{\rho-1}_{L^q(\mathbf{R}^n)}\Big)\|u_\varepsilon(s)-v_\varepsilon(s)\|_{L^q(\mathbf{R}^n)}ds\\
\le\ & MCe^{rt}\int_0^t e^{-rs}\Big(1+\|u_\varepsilon\|^{\rho-1}_{C([0,T],L^q(\mathbf{R}^n))} + \|v_\varepsilon\|^{\rho-1}_{C([0,T],L^q(\mathbf{R}^n))}\Big)\\
&\cdot\|u_\varepsilon(s)-v_\varepsilon(s)\|_{L^q(\mathbf{R}^n)}ds\\
\le\ & MC(1+2R'^{\rho-1})e^{rt}\int_{T'}^t e^{-rs}\|u_\varepsilon(s)-v_\varepsilon(s)\|_{L^q(\mathbf{R}^n)}ds.
\end{aligned}$$

Define a function $\eta:[0,T]\to[0,+\infty)$ by $\eta_\varepsilon(t)=\|u_\varepsilon(t)-v_\varepsilon(t)\|_{L^q(\mathbf{R}^n)}$, thus we have

$$\eta_\varepsilon(t)\le MC\Big(1+2R'^{\rho-1}\Big)e^{rt}\int_0^t e^{-rs}\eta_\varepsilon(s)ds,$$

due to the Gronwall's inequality we obtain the uniqueness.

Next for the nonlinear approximating problem (1.9), the proof is similar to the proof of Theorem 3.2, so we only show primary difference. Define an operator $\Gamma:\mathbf{B}\to\mathbf{B}$ by Eq (4.1). We use Lemma 2.7, so we have

$$\begin{aligned}
\|\Gamma u\|_{C([0,T],L^q(\mathbf{R}^n))} &\le \sup_{t\in[0,T]}\|\mathrm{S}_\alpha(t)u_0\|_{L^q(\mathbf{R}^n)} + \sup_{t\in[0,T]}\int_0^t\|\mathrm{S}_\alpha(s)u_1ds\|_{L^q(\mathbf{R}^n)}\\
&+\sup_{t\in[0,T]}\int_0^t\|\mathrm{T}_\alpha(t-s)f(s)\|_{L^q(\mathbf{R}^n)}ds + \sup_{t\in[0,T]}\int_0^t\|\mathrm{T}_\alpha(t-s)Nu(s)\|_{L^q(\mathbf{R}^n)}ds
\end{aligned}$$

$$\leq C\|u_0\|_{L^q(\mathbf{R}^n)} + CT\|u_1\|_{L^q(\mathbf{R}^n)} + \frac{CT^\alpha}{\alpha}\|f(t)\|_{C([0,T],L^q(\mathbf{R}^n))}$$
$$+\frac{CMT^\alpha}{\alpha} + CMR^\rho\frac{T^\alpha}{\alpha} \leq R.$$

For $u, v \in \mathbf{B}$, we also get

$$\begin{aligned}
&\|\Gamma u - \Gamma v\|_{C([0,T],L^q(\mathbf{R}^n))}\\
\leq\ & \sup_{t\in[0,T]}\int_0^t \|\mathrm{T}_\alpha(t-s)(Nu(s) - Nv(s))\|_{L^q(\mathbf{R}^n)}ds\\
\leq\ & MC\|u - v\|_{C([0,T],L^q(\mathbf{R}^n))}\Big(\frac{T^\alpha}{\alpha} + 2R^{\rho-1}\frac{T^\alpha}{\alpha}\Big)\\
\leq\ & \frac{1}{2}\|u - v\|_{C([0,T],L^q(\mathbf{R}^n))}.
\end{aligned}$$

We finish the proof of Theorem 3.4.

To give the proof of Theorem 3.5 and Theorem 3.6, we need the result that the solutions $u, u_\varepsilon \in L^\infty((0,T), L^{\rho q}(\mathbf{R}^n))$, where the space

$$L^\infty((0,T), L^q(\mathbf{R}^n)) = \Big\{u \in L^\infty((0,T), L^q(\mathbf{R}^n)) : \sup_{t\in(0,T)}\|u(t)\|_{L^q(\mathbf{R}^n)} < \infty\Big\} \tag{4.3}$$

endowed with the norm

$$\|u\|_{L^\infty((0,T),L^q(\mathbf{R}^n))} = \sup_{t\in(0,T)}\|u(t)\|_{L^q(\mathbf{R}^n)}. \tag{4.4}$$

Lemma 4.1. *If u_ε, u are defined by Eqs (2.10) and (2.11) respectively, then we have*

$$\begin{aligned}
&\|u_\varepsilon\|_{L^\infty((0,T),L^{\rho q}(\mathbf{R}^n))}\\
&\leq C(\|u_{0,\varepsilon}\|_{L^{\rho q}(\mathbf{R}^n)} + \|u_{1,\varepsilon}\|_{L^{\rho q}(\mathbf{R}^n)} + \|f_\varepsilon\|_{L^1((0,T),L^{\rho q}(\mathbf{R}^n))}),
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
&\|u\|_{L^\infty((0,T),L^{\rho q}(\mathbf{R}^n))}\\
&\leq C\|u_0\|_{L^q(\mathbf{R}^n)} + CT\|u_1\|_{L^q(\mathbf{R}^n)} + \frac{C}{\alpha}T^\alpha\|f\|_{C((0,T),L^q(\mathbf{R}^n))}.
\end{aligned} \tag{4.6}$$

Proof. From Eqs (2.10) and (2.11), inequalities (2.16)–(2.18), we get

$$\begin{aligned}
&\|u_\varepsilon(t)\|_{L^\infty((0,T),L^q(\mathbf{R}^n))} \leq \sup_{t\in(0,T)}\|\mathrm{S}_\alpha^\varepsilon(t)\|_{\mathcal{L}(L^q(\mathbf{R}^n))}\|u_{0,\varepsilon}\|_{L^q(\mathbf{R}^n)}\\
&+\sup_{t\in(0,T)}\Big\|\int_0^t \mathrm{S}_\alpha^\varepsilon(s)ds\Big\|_{\mathcal{L}(L^q(\mathbf{R}^n))}\|u_{1,\varepsilon}\|_{L^q(\mathbf{R}^n)}\\
&+\sup_{t\in(0,T)}\|\mathrm{T}_\alpha^\varepsilon(t)\|_{\mathcal{L}(L^q(\mathbf{R}^n))} * \|f_\varepsilon(t)\|_{L^q(\mathbf{R}^n)}\\
&\leq C(\|u_{0,\varepsilon}\|_{L^q(\mathbf{R}^n)} + \|u_{1,\varepsilon}\|_{L^q(\mathbf{R}^n)} + \|f_\varepsilon\|_{L^1((0,T),L^q(\mathbf{R}^n))}),
\end{aligned}$$

and

$$\|u(t)\|_{L^\infty((0,T),L^{\rho q}(\mathbf{R}^n))}$$

$$\leq \sup_{t\in(0,T)} \|\mathrm{S}_\alpha(t)u_0\|_{L^q(\mathbf{R}^n)} + \sup_{t\in(0,T)} \|\int_0^t \mathrm{S}_\alpha(s)u_1 ds\|_{L^q(\mathbf{R}^n)}$$
$$+ \sup_{t\in(0,T)} \|\mathrm{T}_\alpha(t) * f(t)\|_{L^q(\mathbf{R}^n)}$$
$$\leq C\|u_0\|_{L^q(\mathbf{R}^n)} + CT\|u_1\|_{L^q(\mathbf{R}^n)} + \frac{C}{\alpha}T^\alpha\|f(t)\|_{C((0,T),L^q(\mathbf{R}^n))},$$

This ends the proof of Lemma 4.1.

Lemma 4.2. *If u_ε and u are defined by Eqs (2.12) and (2.13) respectively, then*

$$\|u_\varepsilon\|_{L^\infty((0,T),L^q(\mathbf{R}^n))} \leq C\Big(\|u_{0,\varepsilon}\|_{L^q(\mathbf{R}^n)} + \|u_{1,\varepsilon}\|_{L^q(\mathbf{R}^n)} + \|f_\varepsilon\|_{L^1((0,T),L^q(\mathbf{R}^n))}\Big), \tag{4.7}$$
$$\|u\|_{L^\infty((0,T),L^q(\mathbf{R}^n))} \leq C\Big(\|u_0\|_{L^q(\mathbf{R}^n)} + \|u_1\|_{L^q(\mathbf{R}^n)} + \|f\|_{C((0,T),L^q(\mathbf{R}^n))}\Big), \tag{4.8}$$

where $C = C(\alpha, r, \theta, T, \rho) > 0$ is a constant.

Proof. From Eqs (2.12) and (2.13) and Lemmas 2.6–2.8, we get

$$\|u_\varepsilon(t)\|_{L^q(\mathbf{R}^n)} \leq \|\mathrm{S}^\varepsilon_\alpha(t)\|_{\mathcal{L}(L^q(\mathbf{R}^n))}\|u_{0,\varepsilon}\|_{L^q(\mathbf{R}^n)} + \|\int_0^t \mathrm{S}^\varepsilon_\alpha(s)ds\|_{\mathcal{L}(L^q(\mathbf{R}^n))}\|u_{1,\varepsilon}\|_{L^q(\mathbf{R}^n)}$$
$$+\|\mathrm{T}^\varepsilon_\alpha(t)\|_{\mathcal{L}(L^q(\mathbf{R}^n))} * \|f_\varepsilon(t)\|_{L^q(\mathbf{R}^n)} + \|\mathrm{T}^\varepsilon_\alpha(t)\|_{\mathcal{L}(L^q(\mathbf{R}^n))} * \|Nu_\varepsilon(t)\|_{L^q(\mathbf{R}^n)}$$
$$\leq C\Big(\|u_{0,\varepsilon}\|_{L^q(\mathbf{R}^n)} + T\|u_{1,\varepsilon}\|_{L^q(\mathbf{R}^n)} + \|f_\varepsilon\|_{L^1((0,T),L^q(\mathbf{R}^n))} + MT\Big)$$
$$+MC\int_0^t \|u_\varepsilon(s)\|^\rho_{L^q(\mathbf{R}^n)}ds.$$

Applying generalisation type of Gronwall's inequality in [53], we get

$$\begin{aligned}\|u_\varepsilon(t)\|_{L^q(\mathbf{R}^n)} &\leq \Big[C\Big(\|u_{0,\varepsilon}\|_{L^q(\mathbf{R}^n)} + T\|u_{1,\varepsilon}\|_{L^q(\mathbf{R}^n)} + \|f_\varepsilon\|_{L^1((0,T),L^q(\mathbf{R}^n))} + MT\Big)^{1-\rho} + (\rho-1)tMC\Big]^{\rho-1}\\ &\leq C\Big(\|u_{0,\varepsilon}\|_{L^q(\mathbf{R}^n)} + T\|u_{1,\varepsilon}\|_{L^q(\mathbf{R}^n)} + \|f_\varepsilon\|_{L^1((0,T),L^q(\mathbf{R}^n))} + MT\Big) + C\Big[(\rho-1)TMC\Big]^{\rho-1},\end{aligned}$$

thus

$$\|u_\varepsilon\|_{L^\infty((0,T),L^q)} \leq C\Big(\|u_{0,\varepsilon}\|_{L^q(\mathbf{R}^n)} + \|u_{1,\varepsilon}\|_{L^q(\mathbf{R}^n)} + \|f_\varepsilon\|_{L^1((0,T),L^q(\mathbf{R}^n))}\Big),$$

and

$$\begin{aligned}\|u(t)\|_{L^q(\mathbf{R}^n)} &\leq C\Big(\|u_0\|_{L^q(\mathbf{R}^n)} + CT\|u_1\|_{L^q(\mathbf{R}^n)} + \frac{T^\alpha}{\alpha}\|f\|_{C((0,T),L^q(\mathbf{R}^n))} + \frac{T^\alpha}{\alpha}\Big)\\ &\quad +C\int_0^t (t-s)^{\alpha-1}\|u(s)\|^\rho_{L^q(\mathbf{R}^n)}ds.\end{aligned}$$

Applying the Gronwall's inequality in [54], we get

$$\begin{aligned}\|u(t)\|_{L^q(\mathbf{R}^n)} &\leq C\Big(\|u_0\|_{L^q(\mathbf{R}^n)} + CT\|u_1\|_{L^q(\mathbf{R}^n)}\\ &\quad +\frac{T^\alpha}{\alpha}\|f\|_{C((0,T),L^q(\mathbf{R}^n))} + \frac{T^\alpha}{\alpha}\Big) + \Big((\rho-1)C\frac{t^\alpha}{\alpha}\Big)^{\rho-1},\end{aligned}$$

therefore,

$$\|u\|_{L^\infty((0,T),L^q(\mathbf{R}^n))} \leq C\Big(\|u_0\|_{L^q(\mathbf{R}^n)} + \|u_1\|_{L^q(\mathbf{R}^n)} + \|f\|_{C((0,T),L^q(\mathbf{R}^n))}\Big).$$

This ends the proof of Lemma 4.2.

Proof of Theorem 3.5. Using the formulas (2.10) and (2.11), from Lemmas 2.8–2.12, we have

$$\begin{aligned}
&\|u_\varepsilon - u\|_{L^p(0,T),L^q(\mathbf{R}^n))} = \|\mathrm{S}^\varepsilon_\alpha(t)u_{0,\varepsilon} + \int_0^t \mathrm{S}^\varepsilon_\alpha(s)u_{1,\varepsilon}ds + \int_0^t \mathrm{T}^\varepsilon_\alpha(t-s)f_\varepsilon(s)ds \\
&-\mathrm{S}_\alpha(t)u_0 - \int_0^t \mathrm{S}_\alpha(s)u_1 ds - \int_0^t \mathrm{T}_\alpha(t-s)f(s)ds\|_{L^p(0,T),L^q(\mathbf{R}^n))} \\
&\leq \|\mathrm{S}^\varepsilon_\alpha(t)u_{0,\varepsilon} - \mathrm{S}_\alpha(t))u_0\|_{L^p(0,T),L^q(\mathbf{R}^n))} \\
&+\|\int_0^t \Big(\mathrm{S}^\varepsilon_\alpha(s)u_{1,\varepsilon} - \mathrm{S}_\alpha(t))u_1 ds\Big)\|_{L^p(0,T),L^q(\mathbf{R}^n))} \\
&+\|\int_0^t (\mathrm{T}^\varepsilon_\alpha(t-s)f_\varepsilon(s) - \mathrm{T}_\alpha(t-s)f(s))ds\|_{L^p(0,T),L^q(\mathbf{R}^n))} \\
&\leq C\Big\{\|u_{0,\varepsilon} - u_0\|_{L^q(\mathbf{R}^n)} + \Big(\varepsilon^{\frac{1}{p}} + \varepsilon^{\theta_0} + |\chi(\varepsilon) - \chi_0|\Big)\|u_0\|_{L^q(\mathbf{R}^n)}\Big\} \\
&+C\Big\{\|u_{1,\varepsilon} - u_1\|_{L^q(\mathbf{R}^n)} + \Big(\varepsilon^{\frac{1}{p}} + \varepsilon^{\theta_0} + |\chi(\varepsilon) - \chi_0|\Big)\|u_1\|_{L^q(\mathbf{R}^n)}\Big\} \\
&+C\Big\{\|f_\varepsilon - f\|_{L^p((0,T),L^1(\mathbf{R}^n))} + [\varepsilon^{\frac{1}{p}+\alpha-1} + \varepsilon^{\theta_0} + |\chi_0 - \chi(\varepsilon)|]\|f\|_{L^p((0,T),L^1(\mathbf{R}^n))}\Big\},
\end{aligned}$$

thus we have

$$\begin{aligned}
\|u_\varepsilon - u\|_{L^p(0,T),L^q(\mathbf{R}^n))} \leq\ & C\Big\{\|u_{0,\varepsilon} - u_0\|_{L^q(\mathbf{R}^n)} + \|u_{1,\varepsilon} - u_1\|_{L^q(\mathbf{R}^n)} + \|f_\varepsilon - f\|_{C((0,T),L^q((\mathbf{R}^n))} \\
&+\Big(\varepsilon^{\frac{1}{p}} + \varepsilon^{\theta_0} + |\chi(\varepsilon) - \chi_0|\Big)\Big(\|u_0\|_{L^q(\mathbf{R}^n)} + \|u_1\|_{L^q(\mathbf{R}^n)}\Big) \\
&+\Big(\varepsilon^{\frac{1}{p}+\alpha-1} + \varepsilon^{\theta_0} + |\chi_0 - \chi(\varepsilon)|\Big)\|f\|_{C((0,T),L^q(\mathbf{R}^n))}\Big\}.
\end{aligned}$$

This ends the proof of Theorem 3.5.

Proof of Theorem 3.6. From the Lemma 4.2, we have

$$\begin{aligned}
\|u_\varepsilon\|_{L^\infty((0,T),L^q(\mathbf{R}^n))} &\leq C(\|u_{0,\varepsilon}\|_{L^q(\mathbf{R}^n)} + \|u_{1,\varepsilon}\|_{L^q(\mathbf{R}^n)} + \|f_\varepsilon\|_{L^1((0,T),L^q(\mathbf{R}^n))}) \leq C, \\
\|u\|_{L^\infty((0,T),L^q(\mathbf{R}^n))} &\leq C\Big(\|u_0\|_{L^q(\mathbf{R}^n)} + \|u_1\|_{L^q(\mathbf{R}^n)} + \|f\|_{C((0,T),L^q(\mathbf{R}^n))}\Big) \leq C.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\|u_\varepsilon(t) - u(t)\|_{L^q(\mathbf{R}^n)} \leq \|\mathrm{S}^\varepsilon_\alpha(t)(u_{0,\varepsilon} - u_0)\|_{L^q(\mathbf{R}^n)} + \|(\mathrm{S}^\varepsilon_\alpha(t) - \mathrm{S}_\alpha(t))u_0\|_{L^q(\mathbf{R}^n)} \\
&+ \int_0^t \|\mathrm{S}^\varepsilon_\alpha(t)(u_{1,\varepsilon} - u_1)\|_{L^q(\mathbf{R}^n)}ds + \int_0^t \|(\mathrm{S}^\varepsilon_\alpha(t) - \mathrm{S}_\alpha(t))u_1\|_{L^q(\mathbf{R}^n)}ds \\
&+ \int_0^t \|\mathrm{T}^\varepsilon_\alpha(t-s)(f_\varepsilon(s) - f(s))\|_{L^q(\mathbf{R}^n)}ds \\
&+ \int_0^t \|(\mathrm{T}^\varepsilon_\alpha(t-s) - \mathrm{T}_\alpha(t-s))f(s)\|_{L^q(\mathbf{R}^n)}ds \\
&+ \int_0^t \|\mathrm{T}^\varepsilon_\alpha(t-s)(Nu_\varepsilon(s) - Nu(s))\|_{L^q(\mathbf{R}^n)}ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \|(\mathrm{T}_\alpha^\varepsilon(t-s)-\mathrm{T}_\alpha(t-s))Nu(s)\|_{L^q(\mathbf{R}^n)}ds\\
&\le D_\varepsilon^1(t)+M\|\mathrm{T}_\alpha^\varepsilon(t)-\mathrm{T}_\alpha(t)\|_{\mathcal{L}(L^q(\mathbf{R}^n))}*(1+\|u(t)\|_{L^q(\mathbf{R}^n)}^\rho)\\
&+Me^{rt}\int_0^t e^{-rs}(1+\|u_\varepsilon(s)\|_{L^q(\mathbf{R}^n)}^{\rho-1}+\|u(s)\|_{L^q(\mathbf{R}^n)}^{\rho-1})\|u_\varepsilon(s)-u(s)\|_{L^q(\mathbf{R}^n)}ds\\
&\le D_\varepsilon^1(t)+D_\varepsilon^2(t)+Me^{rT}\int_0^t e^{-rs}(1+2C)\|u_\varepsilon(s)-u(s)\|_{L^q(\mathbf{R}^n)}ds,
\end{aligned}
$$

where

$$
\begin{aligned}
D_\varepsilon^1(t) &= \|\mathrm{S}_\alpha^\varepsilon(t)(u_{0,\varepsilon}-u_0)\|_{L^q(\mathbf{R}^n)}+\|(\mathrm{S}_\alpha^\varepsilon(t)-\mathrm{S}_\alpha(t))u_0\|_{L^q(\mathbf{R}^n)}\\
&+ \int_0^t \|\mathrm{S}_\alpha^\varepsilon(t)(u_{1,\varepsilon}-u_1)\|_{L^q(\mathbf{R}^n)}ds+\int_0^t \|(\mathrm{S}_\alpha^\varepsilon(t)-\mathrm{S}_\alpha(t))u_1\|_{L^q(\mathbf{R}^n)}ds\\
&+ \int_0^t \|\mathrm{T}_\alpha^\varepsilon(t-s)(f_\varepsilon(s)-f(s))\|_{L^q(\mathbf{R}^n)}ds\\
&+ \int_0^t \|(\mathrm{T}_\alpha^\varepsilon(t-s)-\mathrm{T}_\alpha(t-s))f(s)\|_{L^q(\mathbf{R}^n)}ds,\\
D_\varepsilon^2(t) &= M\|\mathrm{T}_\alpha^\varepsilon(t)-\mathrm{T}_\alpha(t)\|_{\mathcal{L}(L^q(\mathbf{R}^n))}*(1+\|u(t)\|_{L^q(\mathbf{R}^n)}^\rho).
\end{aligned}
$$

Applying Gronwall's inequality, we obtain

$$
\begin{aligned}
&\|u_\varepsilon(t)-u(t)\|_{L^q(\mathbf{R}^n)}\le D_\varepsilon^1(t)+D_\varepsilon^2(t)\\
&+Me^{rT}(1+2C)\int_0^t \exp\Big(\frac{1-e^{-rs}}{r}\Big)[D_\varepsilon^1(s)+D_\varepsilon^2(s)](e^{-rs})ds.
\end{aligned}
$$

Taking the $L^p((0,T),L^q(\mathbf{R}^n))$-norm and using Young inequality, we get

$$
\begin{aligned}
&\|u_\varepsilon-u\|_{L^p((0,T),L^q(\mathbf{R}^n))}\le\Big(\|D_\varepsilon^1(t)\|_{L^p(0,T)}+\|D_\varepsilon^2(t)\|_{L^p(0,T)}\Big)\\
&+Me^{rT}(1+2C)\exp(\frac{1-e^{-rT}}{r})\Big(\|D_\varepsilon^1(t)\|_{L^p(0,T)}+\|D_\varepsilon^2(t)\|_{L^p(0,T)}\Big)\frac{1-e^{-rT}}{r}\\
&\le C\Big(\|D_\varepsilon^1(t)\|_{L^p(0,T)}+\|D_\varepsilon^2(t)\|_{L^p(0,T)}\Big).
\end{aligned}
$$

Applying Young inequality again, from Lemmas 2.9–2.12, we can deduce that

$$
\begin{aligned}
&\|D_\varepsilon^1(t)\|_{L^p(0,T)}\le C\Big\{\|u_{0,\varepsilon}-u_0\|_{L^q(\mathbf{R}^n)}+\|u_{1,\varepsilon}-u_1\|_{L^q(\mathbf{R}^n)}\\
&+\|f_\varepsilon-f\|_{C((0,T),L^q(\mathbf{R}^n))}+\Big(\varepsilon^{\frac{1}{p}}+\varepsilon^{\theta_0}+|\chi(\varepsilon)-\chi_0|\Big)\|u_1\|_{L^q(\mathbf{R}^n)}\\
&+\Big(\varepsilon^{\frac{1}{p}}+\varepsilon^{\theta_0}+|\chi(\varepsilon)-\chi_0|\Big)\|u_0\|_{L^q(\mathbf{R}^n)}\\
&+\Big(\varepsilon^{\frac{1}{p}+\alpha-1}+\varepsilon^{\theta_0}+|\chi_0-\chi(\varepsilon)|\Big)\|f\|_{C((0,T),L^q(\mathbf{R}^n))}\Big\},
\end{aligned}
$$

$$
\begin{aligned}
\|D_\varepsilon^2(t)\|_{L^p(0,T)} &\le C\|(\mathrm{T}_\alpha^\varepsilon(t)-\mathrm{T}_\alpha(t))\|_{L^p((0,T),\mathcal{L}(L^q(\mathbf{R}^n)))}(T+\|u(t)\|_{L^1((0,T),L^q(\mathbf{R}^n))}^\rho)\\
&\le C\|(\mathrm{T}_\alpha^\varepsilon(t)-\mathrm{T}_\alpha(t))\|_{L^p((0,T),\mathcal{L}(L^q(\mathbf{R}^n)))}
\end{aligned}
$$

$$\cdot\Big\{T + C\Big(\|u_0\|^{\rho}_{L^q(\mathbf{R}^n)} + \|u_1\|^{\rho}_{L^q(\mathbf{R}^n)} + \|f\|^{\rho}_{C((0,T),L^q(\mathbf{R}^n))}\Big)\Big\},$$

therefore,

$$\begin{aligned}
&\|u_\varepsilon - u\|_{L^p((0,T),L^q(\mathbf{R}^n))}\\
&\le C\Big\{\|u_{0,\varepsilon} - u_0\|_{L^q(\mathbf{R}^n)} + \|u_{1,\varepsilon} - u_1\|_{L^q(\mathbf{R}^n)} + \|f_\varepsilon - f\|_{C((0,T),L^q(\mathbf{R}^n))}\\
&+(\varepsilon^{\frac{1}{p}} + \varepsilon^{\theta_0} + |\chi(\varepsilon) - \chi_0|)(\|u_0\|_{L^q(\mathbf{R}^n)} + \|u_1\|_{L^q(\mathbf{R}^n)})\\
&+(\varepsilon^{\frac{1}{p}+\alpha-1} + \varepsilon^{\theta_0} + |\chi_0 - \chi(\varepsilon)|)\|f\|_{C((0,T),L^q(\mathbf{R}^n))}\}\\
&+(\varepsilon^{\frac{1}{p}+\alpha-1} + \varepsilon^{\theta_0} + |\chi(\varepsilon) - \chi_0|)(\|u_0\|^{\rho}_{L^q(\mathbf{R}^n)} + \|u_1\|^{\rho}_{L^q(\mathbf{R}^n)} + \|f\|^{\rho}_{C((0,T),L^q(\mathbf{R}^n))}).
\end{aligned}$$

We complete the proof of Theorem 3.6.

Proof of Theorem 3.7. Let $u \in X_\rho$ be the solution given by Theorem 3.2. We proceed similarly to the proof of Theorem 3.2 so that we point out the differences. Indeed, we define $\Gamma : \mathbf{C} \to \mathbf{C}$ by 4.1, where the complete metric space $\mathbf{C}$ is defined by

$$\begin{aligned}
\mathbf{C} = \Big\{v \in C([0,\bar{T}], L^{\rho q}(\mathbf{R}^n)) \cap X_\beta : \sup_{t\in[T,\bar{T}]} \|v(t) - u(T)\|_{L^q(\mathbf{R}^n)} < \bar{R},\\
\sup_{t\in[T,\bar{T}]} t^{\beta}\|v(t) - u(T)\|_{L^{\rho q}(\mathbf{R}^n)} < \bar{R},\ u \equiv v \ on\ t \in [0,T]\Big\},
\end{aligned}$$

Given $v \in \mathbf{C}$, the continuity of $\Gamma v : (0,\bar{T}] \to L^q(\mathbf{R}^n)$ follows as in Theorem 3.2. Moreover, it is easy to see that $\Gamma v(\cdot,t) = u(\cdot,t)$, for every $t \in [0,T]$. Next, for $T < t < \bar{T}$, we have

$$\begin{aligned}
&\Gamma v(t,x) - u(T,x)\\
&= \Big(S_\alpha(t) - S_\alpha(T)\Big)u_0(x) + \int_0^T \Big(T_\alpha(t-s) - T_\alpha(T-s)\Big)f(s,x)ds\\
&+ \int_0^T \Big(T_\alpha(t-s) - T_\alpha(T-s)\Big)Nu(s,x)ds + \int_T^t S_\alpha(s)u_1(x)ds\\
&+ \int_T^t T_\alpha(t-s)f(s,x)ds + \int_T^t T_\alpha(t-s)Nv(s,x)ds. \qquad (4.9)
\end{aligned}$$

Now, we note that the first, the second and the third terms on right hand side of Eq (4.9) are in $L^{\rho q}(\mathbf{R}^n)\cap L^q(\mathbf{R}^n)$ because of Lemma 2.7 so that Lebesgue's Dominated Convergence Theorem can be applied to prove that these three terms go to zero in the norm $\|\cdot\|_{L^{\rho q}(\mathbf{R}^n)\cap L^q(\mathbf{R}^n)}$ as $t \to T^+$. Furthermore

$$\begin{aligned}
&t^{\beta}\Big\|\int_T^t S_\alpha(s)u_1ds + \int_T^t T_\alpha(t-s)f(s)ds + \int_T^t T_\alpha(t-s)Nv(s)ds\Big\|_{L^{\rho q}(\mathbf{R}^n)}\\
&\le Ct^{\beta}\int_T^t s^{-\beta}\|u_1\|_{L^q(\mathbf{R}^n)}ds + Ct^{\beta}\int_T^t (t-s)^{-\beta+\alpha-1}\|f(s)\|_{L^q(\mathbf{R}^n)}ds\\
&+t^{\beta}CM\int_T^t (t-s)^{\alpha-1}(1 + \|v(s)\|^{\rho}_{L^{\rho q}(\mathbf{R}^n)})ds\\
&\le Ct^{\beta}\|u_1\|_{L^q(\mathbf{R}^n)}(\frac{t^{1-\beta} - T^{1-\beta}}{1-\beta}) + Ct^{\beta}(M + \|f(T)\|_{L^q(\mathbf{R}^n)})\frac{(t-T)^{\alpha-\beta}}{\alpha-\beta}
\end{aligned}$$

$$+Ct^{\beta}M\frac{(t-T)^{\alpha}}{\alpha}+Ct^{\beta}M(\bar{R}+T^{\rho\beta}\|u(T)\|^{\rho}_{L^{\rho q}(\mathbf{R}^n)})t^{\alpha-\rho\beta}\int_{T/t}^{1}(1-s)^{\alpha-1}s^{-\rho\beta}ds$$
$$\to 0 \text{ as } t\to T^{+}.$$

$$\begin{aligned}
&\Big\|\int_T^t S_{\alpha}(s)u_1ds+\int_T^t T_{\alpha}(t-s)f(s)ds+\int_T^t T_{\alpha}(t-s)Nv(s)ds\Big\|_{L^q(\mathbf{R}^n)}\\
&\le C\int_T^t\|u_1\|_{L^q(\mathbf{R}^n)}ds+C\int_T^t(t-s)^{\alpha-1}\|f(s)\|_{L^q(\mathbf{R}^n)}ds\\
&\quad +CM\int_T^t(t-s)^{\alpha-1}(1+\|v(s)\|^{\rho}_{L^q(\mathbf{R}^n)})ds\\
&\le C\|u_1\|_{L^q(\mathbf{R}^n)}(t-T)+C(M+\|f(T)\|_{L^q(\mathbf{R}^n)})\frac{(t-T)^{\alpha}}{\alpha}\\
&\quad +CM\frac{(t-T)^{\alpha}}{\alpha}+CM(\bar{R}+\|u(T)\|^{\rho}_{L^q(\mathbf{R}^n)})t^{\alpha}\int_{T/t}^{1}(1-s)^{\alpha-1}ds\\
&\to 0 \text{ as } t\to T^{+}.
\end{aligned}$$

Therefore, we can take $\bar{T}$ so close to T such that the norm $\|\cdot\|_{L^{\rho q}(\mathbf{R}^n)\cap L^q(\mathbf{R}^n)}$ of each term on right hand side of Eq (4.9) is less than $\bar{R}/5$ and we have

$$\sup_{t\in[T,\bar{T}]} t^{\beta}\|v(t)-u(T)\|_{L^{\rho q}(\mathbf{R}^n)}<\bar{R},\quad \sup_{t\in[T,\bar{T}]}\|v(t)-u(T)\|_{L^q(\mathbf{R}^n)}<\bar{R}.$$

It is similar to prove that Γ is a contraction and the uniqueness is proved as in Theorem 3.2. We complete the proof of Theorem 3.7.

Proof of Theorem 3.8. Suppose that $T_{max}<\infty$ and there exists a constant $M>0$ such that $\|t^{\beta}u(t)\|_{L^{\rho q}(\mathbf{R}^n)}\le M$, $\|u(t)\|_{L^q(\mathbf{R}^n)}\le M$, for all $t\in[a,T_{max})$, with $a>0$. Thus, given a sequence of positive real number $t_n\to T^{-}_{max}$, we will show that $\{u(t_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^{\rho q}(\mathbf{R}^n)\cap L^q(\mathbf{R}^n)$. Indeed, for $0<t_m<t_m<T_{max}$, we have

$$\begin{aligned}
&u(t_n,x)-u(t_m,x)\\
=\;&\Big(S_{\alpha}(t_n)-S_{\alpha}(t_m)\Big)u_0(x)+\int_0^{t_m}\Big(T_{\alpha}(t_n-s)-T_{\alpha}(t_m-s)\Big)f(s,x)ds\\
+\;&\int_0^{t_m}\Big(T_{\alpha}(t_n-s)-T_{\alpha}(t_m-s)\Big)Nu(s,x)ds+\int_{t_m}^{t_n}S_{\alpha}(s)u_1(x)ds\\
+\;&\int_{t_m}^{t_n}T_{\alpha}(t_n-s)f(s,x)ds+\int_{t_m}^{t_n}T_{\alpha}(t_n-s)Nv(s,x)ds.
\end{aligned}$$

Similar to Eq (4.9), we have $\|u(t_n,x)-u(t_m,x)\|_{L^{\rho q}(\mathbf{R}^n)\cap L^q(\mathbf{R}^n)}\to 0$ as $m,n\to\infty$. Hence, there exists the limit $\lim_{n\to\infty}u(t_n)=u(T_{max})$ in $L^{\rho q}(\mathbf{R}^n)\cap L^q(\mathbf{R}^n)$. Thus, $u(T_{max})$ exists in $L^{\rho q}(\mathbf{R}^n)\cap L^q(\mathbf{R}^n)$ and Theorem 3.7 yields a contradiction. We complete the proof of Theorem 3.8.

Proof of Theorem 3.9. For $X=L^q(\mathbf{R}^n)$, denote

$$I(t)\;=\;\|\mathrm{S}^{\varepsilon}_{\alpha}(t)u_{0,\varepsilon}+\int_0^t\mathrm{S}^{\varepsilon}_{\alpha}(s)u_{1,\varepsilon}ds+\int_0^t\mathrm{T}^{\varepsilon}_{\alpha}(t-s)f_{\varepsilon}(s)ds$$

$$-(\mathrm{S}_2^\varepsilon(t)u_{0,\varepsilon} + \int_0^t \mathrm{S}_2^\varepsilon(s)u_{1,\varepsilon}ds + \int_0^t \mathrm{T}_2^\varepsilon(t-s)f_\varepsilon(s)ds)\|_X,$$

$$J(t) = \int_0^t \|T_\alpha^\varepsilon(t-s) - T_2^\varepsilon(t-s)\|_{\mathcal{L}(X)}(1 + \|u_\alpha^\varepsilon(s)\|_X^\rho)ds.$$

From Theorem 3.4, it is easy to obtain that there exists a $T > 0$ such that for $t \in [0, T]$, we have

$$\|u_\alpha^\varepsilon(t) - u_2^\varepsilon(t)\|_X \le I(t) + \int_0^t \|(T_\alpha^\varepsilon(t-s) - T_2^\varepsilon(t-s))Nu_\alpha^\varepsilon(s)\|_X ds$$
$$+ \int_0^t \|T_2^\varepsilon(t-s)(Nu_\alpha^\varepsilon(s) - Nu_2^\varepsilon(s))\|_X ds \le I(t) + J(t)$$
$$+ \int_0^t \|T_2^\varepsilon(t-s)\|_{\mathcal{L}(X)}(1 + \|u_\alpha^\varepsilon(s)\|_X^{\rho-1} + \|u_2^\varepsilon(s)\|_X^{\rho-1})\|u_\alpha^\varepsilon(s) - u_2^\varepsilon(s)\|_X ds.$$

Due to Lemma 2.13, we have $I(t) \to 0$ as $\alpha \to 2^-$ uniformly for $t \in [0, T]$. Furthermore, since $\alpha \mapsto 1 + \|u_\alpha^\varepsilon(s)\|_{L^q(\mathbf{R}^n)}^\rho$ remains bounded as $\alpha \to 2^-$, and $t - s \in [0, t] \subset [0, T]$, the uniform convergence in Lemma 2.9 gives that $J(t) \to 0$ as $\alpha \to 2^-$ uniformly for $t \in [0, T]$. There exists $R > 0$ such that $\|u_\alpha^\varepsilon, u_2^\varepsilon\|_{C([0,T],L^q(\mathbf{R}^n))} \le R$, then

$$\int_0^t \|T_2^\varepsilon(t-s)\|_{\mathcal{L}(X)}\Big(1 + \|u_\alpha^\varepsilon(s)\|_X^{\rho-1} + \|u_2^\varepsilon(s)\|_X^{\rho-1}\Big)\|u_\alpha^\varepsilon(s) - u_2^\varepsilon(s)\|_X ds$$
$$\le C(1 + 2R^{\rho-1}) \int_0^t e^{r(t-s)}\|u_\alpha^\varepsilon(s) - u_2^\varepsilon(s)\|_X ds,$$

thus we have

$$\|u_\alpha^\varepsilon(t) - u_2^\varepsilon(t)\|_{L^q(\mathbf{R}^n)} \le I(t) + J(t) + C\int_0^t e^{r(t-s)}\|u_\alpha^\varepsilon(s) - u_2^\varepsilon(s)\|_{L^q(\mathbf{R}^n)}ds,$$

Gronwall's inequality implies that

$$\|u_\alpha^\varepsilon(t) - u_2^\varepsilon(t)\|_{L^q(\mathbf{R}^n)} \le (I(t) + J(t))\exp(C\int_0^t e^{r(t-s)}ds).$$

Since $I(t), J(t) \to 0$ as $\alpha \to 2^-$, thus we have $\|u_\alpha^\varepsilon(t) - u_2^\varepsilon(t)\|_{L^q(\mathbf{R}^n)} \to 0$. Similarly we can get $\|u_\alpha(t) - u_2(t)\|_{L^q(\mathbf{R}^n)} \to 0$ as $\alpha \to 2^-$ uniformly for t in bounded subintervals of $[0, T]$. We complete the proof of Theorem 3.9.

5. Conclusion and discussion

This paper mainly discusses three problems. The first question is the convergence of mild solution $u_\varepsilon(t, x)$ to the initial value problems (1.1) and (1.8) which contain the small scale positive parameter ε, and the asymptotic behaviour between the approximating mild solution $u_\varepsilon(t, x)$ and the limit mild solution $u(t, x)$ as $\varepsilon \to 0^+$, where $u(t, x)$ is mild solution of the limit problems (1.5) and (1.9) respectively. When $\alpha \in (1, 2)$, the Eqs (1.5) and (1.9) are fractional wave equations govern the propagation of mechanical diffusion in viscoelastic media, revealing a power-law creep and thus providers us a physical interpretation in the frame work of dynamical viscoelasticity [55–57]. However, in many concert situations, only the most recent past history of u has an effective impact on the future dynamics. In mathematical terms, this translates into having a rapidly fading memory kernel $k(t)$. The equation becomes (1.5) in the limiting situation where $k(t)$ is the dirac mass at zero. It is then reasonable to view Eq (1.5) as a good approximation of an evolution system that keeps a very short memory of the past. The second question contains two aspects. On the one hand, by using the properties of the resolvent operators and the representations of the mild solutions, we have obtained the existence and uniqueness of the mild solutions. On the other hand, we demonstrate the existence of a unique maximal solution and a blow-up alternative for the semi-linear approximating problem with ε and the limit problem using the $L^p - L^q$ estimates of the resolvent operator family. The last problem is the convergence of the fractional super-diffusion Eqs (1.5) and (1.9) solutions as α approaches 2^-. Our interest in studying problems (1.5) and (1.9) that come from their applications as a model for physical systems exhibiting anomalous diffusion. In many complex processes, the behavior usually no longer follows Gaussian statics, and thus, the Fick's second law fails to describe the related transport behavior. In classical diffusion, the linear time dependence of the mean squared displacement can be observed, which indicates how fast particles diffuse, whereas, in anomalous diffusion the mean squared displacement of a diffusive particle usually behaves like const$\cdot t^\alpha$ as $t \to \infty$. So by studying the asymptotic behavior of the mild of the problems (1.5) and (1.9), we can gain insight into the rate of convergence of the solutions to the problems (1.5) and (1.9).

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Conflict of interest

The authors have no conflicts to disclose.

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