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A SINGULAR LIMIT PROBLEM FOR CONSERVATION LAWS RELATED TO THE KAWAHARA-KORTEWEG-DE VRIES EQUATION

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Abstract. We consider the Kawahara-Korteweg-de Vries equation, which contains nonlinear dispersive effects. We prove that as the dispersion parameter tends to zero, the solutions of the dispersive equation converge to discontinuous weak solutions of the Burgers equation. The proof relies on deriving suitable a priori estimates together with an application of the compensated compactness method in the L^p setting.

1. **Introduction.** Nonlinear evolution equations have been used to model many physical phenomena in various fields such as fluid mechanics, solid state physics, plasma physics, chemical physics, optical fiber and geochemistry. An example is given by the Kawahara-Korteweg-de Vries equation:

$$\partial_t u + au\partial_x u + b\partial^3_{xxx} u + c\partial^5_{xxxxx} u = 0, \tag{1}$$

where $u = u(t,x)$ is a real function, and a, b, $c \in \mathbb{R}$ are constants.

It is a model for water waves in the long wave regime for moderate values of surface tension (see [16]), or for the propagation of the magnet-acoustic waves in a cold collision free plasma (see [17]).

To obtain the exact solutions for (1), a number of methods has been proposed in the literature, some of them include solitary wave ansatz method, inverse scattering, Hirotas bilinear method, homogeneous balance method, Lie group analysis, etc. Among the above mentioned, the Lie group analysis method, which is also called the symmetry method, is one of the most effective to determine solutions of nonlinear partial differential equations [18].

In [1], the authors use Lie group analysis to obtain some exact solutions for (1), the Kawahara equation

$$\partial_t u + au\partial_x u + c\partial^5_{xxxxx} u = 0, \tag{2}$$

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the modified Kawahara equation

$$\partial_t u + a u^2 \partial_x u + c \partial^5_{xxxxx} u = 0,$$

the modified Kawahara-Korteweg-de Vries equation

$$\partial_t u + a u^2 \partial_x u + b \partial^3_{xxx} u + c \partial^5_{xxxxx} u = 0,$$

and, the Rosenau-Kawahara equation

$$\partial_t u + a u \partial_x u + b \partial^3_{xxx} u + c \partial^5_{xxxxx} u + d \partial^5_{txxxx} u = 0. \tag{3}$$

The Kawahara equation (2) describes small–amplitude gravity capillary waves in water of finite depth when the Weber number is close to $\frac{1}{3}$ (see [23]). In [16], the author deduced (2) and (1) describing one–dimensional propagation of small–amplitude long waves in various problems of fluid dynamics and plasma physics. (1) is also known as the fifth-order Korteweg-de Vries equation, or a special version of the Benney-Lin equation (see [2]). In [20], the author used the exp-function method to find some exact solution for (3). In [21], the authors proved that the solution of (2) converges to the solution of the Korteweg-de Vries equation

$$\partial_t u + \partial_x u^2 + \beta \partial^3_{xxx} u = 0. \tag{4}$$

We consider (4), and observe that, if we send $\beta \to 0$ in (4), we pass from (4) to the Burgers equation

$$\partial_t u + \partial_x u^2 = 0. \tag{5}$$

In [19, 25], the convergence of the solution of

$$\partial_t u + \partial_x u^2 + \beta \partial^3_{xxx} u = \varepsilon \partial^2_{xx} u$$

to the unique entropy solution of (5) is proven, under the assumption

$$u_0 \in L^2(\mathbb{R}) \cap L^4(\mathbb{R}), \quad \beta = o\left(\varepsilon^2\right). \tag{6}$$

[7, Appendices A and B] show that it is possible to obtain the same result of convergence, under the following assumptions

$$\begin{aligned} &u_0 \in L^2(\mathbb{R}), \quad -\infty < \int_{\mathbb{R}} u_0(x) dx < \infty, \quad \beta = o\left(\varepsilon^3\right), \\ &u_0 \in L^2(\mathbb{R}), \quad \beta = o\left(\varepsilon^4\right). \end{aligned}$$

Several of the ideas used in this paper were inspired by the analysis of the following generalization of (4)

$$\partial_x (\partial_t u + \partial_x u^2 - \beta \partial^3_{xxx} u) = \gamma u, \quad \beta, \gamma \in \mathbb{R}, \tag{7}$$

that is the Ostrovsky equation (see [24]). Equation (7) describes small-amplitude long waves in a rotating fluid of a finite depth by the additional term induced by the Coriolis force. If we send $\beta \to 0$ in (7), we pass from (7) to the Ostrovsky-Hunter equation (see [3])

$$\partial_x (\partial_t u + \partial_x u^2) = \gamma u, \qquad t > 0, \quad x \in \mathbb{R}. \tag{8}$$

In [9, 12, 15], the wellposedness of the entropy solutions of (8) is proven, in the sense of the following definition:

Definition 1.1. We say that $u \in L^\infty((0,T) \times \mathbb{R})$, $T > 0$, is an entropy solution of (8) if

i) u is a distributional solution of (8);

ii) for every convex function $\eta \in C^2(\mathbb{R})$ the entropy inequality

$$\partial_t \eta(u) + \partial_x q(u) - \gamma \eta'(u) P \leq 0, \qquad q(u) = 2\int^u \xi \eta'(\xi)\, d\xi,$$

holds in the sense of distributions in $(0,\infty)\times\mathbb{R}$, where $\partial_x P = u$.

Under the assumption (6), in [10], the convergence of the solutions of (7) to the unique entropy solution of (8) is proven.

Consider (3). Choosing $a = 2$, $b = d = 1$, $c = 0$, we have the Rosenau-Korteweg-de Vries equation

$$\partial_t u + \partial_x u^2 + \partial^3_{xxx} u + \partial^5_{txxxx} u = 0. \tag{9}$$

Arguing as in [11], we re–scale the equation as follows

$$\partial_t u + \partial_x u^2 + \beta \partial^3_{xxx} u + \beta^2 \partial^5_{txxxx} u = 0, \tag{10}$$

where β is the dispersion parameter. In [5], the authors proved that the solution of

$$\partial_t u + \partial_x u^2 + \beta \partial^3_{xxx} u + \beta^2 \partial^5_{txxxx} u = \varepsilon \partial^2_{xx} u, \tag{11}$$

converge to the unique entropy solution of (5), choosing the initial datum in two different ways. The first one is:

$$u_0 \in L^2(\mathbb{R}), \quad \beta = o\left(\varepsilon^4\right). \tag{12}$$

The second choice is:

$$u_0 \in L^2(\mathbb{R}) \cap L^4(\mathbb{R}), \quad \beta \leq const\left(\|u_0\|_{L^2(\mathbb{R})}, \|u_0\|_{L^4(\mathbb{R})}\right)\left(\varepsilon^4\right). \tag{13}$$

Consider (2) with $a = 2$, $c = 1$. Arguing as in [11], we re–scale the equation as follows

$$\partial_t u + \partial_x u^2 + \beta^2 \partial^5_{xxxxx} u = 0, \tag{14}$$

where β is the dispersion parameter. Assuming (12), or (13), in [4], the authors proved that the solution of

$$\partial_t u + \partial_x u^2 + \beta^2 \partial^5_{xxxxx} u = \varepsilon \partial^2_{xx} u, \tag{15}$$

converge to the unique entropy solution of (5).

[4, Appendices A and B] show that, using the approximation introduced in [6], we have the same result of convergence, under the following assumptions

$$u_0 \in L^2(\mathbb{R}), \quad \beta = o\left(\varepsilon^8\right), \tag{16}$$

$$u_0 \in L^2(\mathbb{R}) \cap L^4(\mathbb{R}), \quad \beta \leq const\left(\|u_0\|_{L^2(\mathbb{R})}, \|u_0\|_{L^4(\mathbb{R})}\right)\left(\varepsilon^8\right). \tag{17}$$

In this paper, we consider (1) with $a = 2$, $b = c = 1$. Arguing as in [11], we re–scale the equations as follows

$$\partial_t u + \partial_x u^2 + \beta \partial^3_{xxx} u + \beta^2 \partial^5_{xxxxx} u = 0, \tag{18}$$

where β is the dispersion parameter.

We are interested in the dispersion-diffusion limit, we send $\beta \to 0$ in (18). In this way, we pass from (18) to (5). We prove that, as $\beta, \varepsilon \to 0$, the solution of (18) converge to the unique entropy solution of (5). In other to do this, using the following approximation (see (21) below)

$$\partial_t u + \partial_x u^2 + \beta \partial^3_{xxx} u + \beta^2 \partial^5_{xxxxx} u = \varepsilon \partial^2_{xx} u - \beta\varepsilon \partial^4_{xxxx} u, \tag{19}$$

where ε, β are two small numbers. The form of the right hand side of (19) has been chosen for mathematical reason and there is no deep physical meaning behind it. The two terms are designed to preserve the $\|\cdot\|_{L^4}$.

We can choose the initial datum and β in two different ways Following [14, Theorem 7.1], the first choice is given by (12) (see Theorem 2.2). Since $\|\cdot\|_{L^4}$ is a conserved quantity for (19), the second choice is given by (13) (see Theorem 3.1).

It is interesting to observe that, while the summability on the initial datum in (13) is greater than the one of (12), the assumption on β in (13) is weaker than the one in (12). From the mathematical point of view, the two assumptions require two different arguments for the $L^\infty-$estimate (see Lemmas 2.3 and 3.2). Indeed, the proof of Lemma 2.3, under the assumption (12), is more technical than the one of Lemma 3.2. Moreover, due to the presence of the third order term, Lemma 3.3 is finer than [4, Lemmas 3.2]. Indeed, in Lemma 3.3 we need to prove the existence of two positive constants, that is not the case in [4, Lemma 3.2].

Alternatively we can consider the approximation introduced in [6]

$$\partial_t u+\partial_x u^2+\beta\partial^3_{xxx}u+\beta^2\partial^5_{xxxxx}u=\varepsilon\partial^2_{xx}u-\beta^{\frac{3}{2}}\varepsilon\partial^4_{xxxx}u. \tag{20}$$

We consider (19), in lieu of (20), because it gives us shaper estimates. Indeed if we work with (20), we have to replace (12) and (13) with (16) and (17), respectively. To better show that (19) works better than (20) we give the details on (19) in the main part of the paper and the ones on (20) are briefly discussed in the final appendices.

The paper is organized in five sections. In Section 2, we prove the convergence of (18) to (5) in the L^p setting, with $1\le p<2$. In Section 3, we prove the convergence of the solutions of (18) to the ones of (5) in the L^p setting, with $1\le p<4$. Sections A and B are the appendices where, using the approximation (20), we prove the convergence of the solutions of (18) to the ones of (5) in the L^p setting, with $1\le p<2$, and in the L^p setting with $1\le p<4$, respectively.

2. The Kawahara-KDV-equation: $u_0\in L^2(\mathbb{R})$, $\beta=o\left(\varepsilon^4\right)$.

In this section, we consider (18), and assume (12) on the initial datum.

We study the dispersion-diffusion limit for (18). Therefore, we fix two small numbers ε, β and consider the following fifth order approximation

$$\begin{cases}\partial_t u_{\varepsilon,\beta}+\partial_x u^2_{\varepsilon,\beta}+\beta\partial^3_{xxx}u_{\varepsilon,\beta}+\beta^2\partial^5_{xxxxx}u_{\varepsilon,\beta}\\ \qquad\qquad=\varepsilon\partial^2_{xx}u_{\varepsilon,\beta}-\beta\varepsilon\partial^4_{xxxx}u_{\varepsilon,\beta}, & t>0,\ x\in\mathbb{R},\\ u_{\varepsilon,\beta}(0,x)=u_{\varepsilon,\beta,0}(x), & x\in\mathbb{R},\end{cases} \tag{21}$$

where $u_{\varepsilon,\beta,0}$ is a C^∞ approximation of u_0 such that

$$\begin{aligned}&u_{\varepsilon,\,\beta,\,0}\to u_0\quad\text{ in } L^p_{loc}(\mathbb{R}),\ 1\le p<2,\text{ as }\varepsilon,\,\beta\to0,\\ &\|u_{\varepsilon,\beta,0}\|^2_{L^2(\mathbb{R})}+\beta^{\frac12}\left\|\partial_x u_{\varepsilon,\beta,0}\right\|^2_{L^2(\mathbb{R})}+\beta\varepsilon^2\left\|\partial^2_{xx}u_{\varepsilon,\beta,0}\right\|^2_{L^2(\mathbb{R})}\le C_0,\quad \varepsilon,\beta>0,\end{aligned} \tag{22}$$

and C_0 is a constant independent on ε and β. Such sequence $\{u_{\varepsilon,\beta,0}\}$ can be constructed using standard mollifiers and (12). The well-posedness of the smooth solutions $u_{\varepsilon,\beta}\in C^\infty$ can be proven following the same argument of [25].

We consider the following definition.

Definition 2.1. A pair of functions (η,q) is called an entropy–entropy flux pair if $\eta:\mathbb{R}\to\mathbb{R}$ is a C^2 function and $q:\mathbb{R}\to\mathbb{R}$ is defined by

$$q(u)=2\int_0^u\xi\eta'(\xi)d\xi.$$

An entropy–entropy flux pair $(\eta,\,q)$ is called convex/compactly supported if, in addition, η is convex/compactly supported.

The main result of this section is the following theorem.

Theorem 2.2. *Assume that (12) and (22) hold. Fix $T > 0$, if*

$$\beta = o\left(\varepsilon^4\right), \tag{23}$$

then, there exist two sequences $\{\varepsilon_n\}_{n\in\mathbb{N}}$, $\{\beta_n\}_{n\in\mathbb{N}}$, with $\varepsilon_n, \beta_n \to 0$, and a limit function

$$u \in L^\infty((0,T); L^2(\mathbb{R})), \tag{24}$$

such that

$$u_{\varepsilon_n,\beta_n} \to u \quad \text{strongly in } L^p_{loc}(\mathbb{R}^+ \times \mathbb{R}), \text{ for each } 1 \le p < 2, \tag{25}$$

$$u \quad \text{is the unique entropy solution of (5)}, \tag{26}$$

where $u_{\varepsilon,\beta}$ solves (21).

Let us prove some a priori estimates on $u_{\varepsilon,\beta}$, denoting with C_0 the constants which depend only on the initial data.

Lemma 2.3. *For each $t > 0$,*

$$\begin{aligned} \|u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} &+ 2\varepsilon \int_0^t \|\partial_x u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}\, ds \\ &+ 2\beta\varepsilon \int_0^t \left\|\partial^2_{xx} u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C_0. \end{aligned} \tag{27}$$

For every $T > 0$, we have

$$\|u_{\varepsilon,\beta}\|_{L^2((0,T)\times\mathbb{R})} \le C_0\sqrt{T}, \tag{28}$$

$$\|u_{\varepsilon,\beta}\|_{L^\infty((0,T)\times\mathbb{R})} \le C_0\beta^{-\frac{1}{4}}. \tag{29}$$

Moreover

$$\begin{aligned} \beta \|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} &+ \beta^{\frac{3}{2}}\varepsilon^2 \left\|\partial^2_{xx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\ &+ \frac{3\beta\varepsilon}{2} \int_0^t \left\|\partial^2_{xx} u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \\ &+ 2\beta^{\frac{3}{2}}\varepsilon^3 \int_0^t \left\|\partial^3_{xxx} u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \\ &+ 2\beta^2\varepsilon \int_0^t \left\|\partial^3_{xxx} u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \\ &+ \frac{3\beta^{\frac{5}{2}}\varepsilon^3}{2} \int_0^t \left\|\partial^4_{xxxx} u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C_0. \end{aligned} \tag{30}$$

Proof. Let $0 < t < T$. We begin by proving that (27), and (28) hold. Multiplying (21) by $2u_{\varepsilon,\beta}$, since

$$2\beta \int_{\mathbb{R}} u_{\varepsilon,\beta}\partial^3_{xxx} u_{\varepsilon,\beta} dx = 0,$$

arguing as in [4, Lemmas 2.1], we have (27), and (28).

Finally, we prove (29), and (30). Multiplying (21) by $-2\beta^{\frac{1}{2}}\partial^2_{xx}u_{\varepsilon,\beta}+2\beta\varepsilon^2\partial^4_{xxxx}u_{\varepsilon,\beta}$, we have

$$\begin{aligned}
&\left(-2\beta^{\frac{1}{2}}\partial^2_{xx}u_{\varepsilon,\beta}+2\beta\varepsilon^2\partial^4_{xxxx}u_{\varepsilon,\beta}\right)\partial_t u_{\varepsilon,\beta}\\
&\quad+2\left(-2\beta^{\frac{1}{2}}\partial^2_{xx}u_{\varepsilon,\beta}+2\beta\varepsilon^2\partial^4_{xxxx}u_{\varepsilon,\beta}\right)u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\\
&\quad+\beta\left(-2\beta^{\frac{1}{2}}\partial^2_{xx}u_{\varepsilon,\beta}+2\beta\varepsilon^2\partial^4_{xxxx}u_{\varepsilon,\beta}\right)\partial^3_{xxx}u_{\varepsilon,\beta},\\
&\quad+\beta^2\left(-2\beta^{\frac{1}{2}}\partial^2_{xx}u_{\varepsilon,\beta}+2\beta\varepsilon^2\partial^4_{xxxx}u_{\varepsilon,\beta}\right)\partial^5_{xxxxx}u_{\varepsilon,\beta}\\
&=\varepsilon\left(-2\beta^{\frac{1}{2}}\partial^2_{xx}u_{\varepsilon,\beta}+2\beta\varepsilon^2\partial^4_{xxxx}u_{\varepsilon,\beta}\right)\partial^2_{xx}u_{\varepsilon,\beta}\\
&\quad-\beta\varepsilon\left(-2\beta^{\frac{1}{2}}\partial^2_{xx}u_{\varepsilon,\beta}+2\beta\varepsilon^2\partial^4_{xxxx}u_{\varepsilon,\beta}\right)\partial^4_{xxxx}u_{\varepsilon,\beta}.
\end{aligned}$$

Since

$$\begin{aligned}
-2\beta^{\frac{3}{2}}\int_{\mathbb{R}}\partial^2_{xx}u_{\varepsilon,\beta}\partial^3_{xxx}u_{\varepsilon,\beta}dx=0,\\
2\beta^2\varepsilon^2\int_{\mathbb{R}}\partial^4_{xxxx}u_{\varepsilon,\beta}\partial^3_{xxx}u_{\varepsilon,\beta}dx=0,
\end{aligned}$$

arguing as in [4, Lemmas 2.2], we have (27), (28), (29) and (30). □

To prove Theorem 2.2, the Murat Lemma is needed [22].

Following [19], we prove Theorem 2.2.

Proof Theorem 2.2. Let us consider a compactly supported entropy-entropy flux pair $(\eta,\,q)$. Multiplying (21) by $\eta'(u_{\varepsilon,\beta})$, we have

$$\begin{aligned}
\partial_t\eta(u_{\varepsilon,\beta})+\partial_x q(u_{\varepsilon,\beta})=&\varepsilon\eta'(u_{\varepsilon,\beta})\partial^2_{xx}u_{\varepsilon,\beta}-\beta\varepsilon\eta'(u_{\varepsilon,\beta})\partial^4_{xxxx}u_{\varepsilon,\beta}\\
&-\beta^2\eta'(u_{\varepsilon,\beta})\partial^5_{xxxxx}u_{\varepsilon,\beta}-\beta\eta'(u_{\varepsilon,\beta})\partial^3_{xxx}u_{\varepsilon,\beta}\\
=&I_{1,\,\varepsilon,\,\beta}+I_{2,\,\varepsilon,\,\beta}+I_{3,\,\varepsilon,\,\beta}+I_{4,\,\varepsilon,\,\beta}+I_{5,\,\varepsilon,\,\beta}+I_{6,\,\varepsilon,\,\beta}\\
&+I_{7,\,\varepsilon,\,\beta}+I_{8,\,\varepsilon,\,\beta},
\end{aligned}$$

where

$$\begin{aligned}
I_{1,\,\varepsilon,\,\beta}&=\partial_x(\varepsilon\eta'(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}),\\
I_{2,\,\varepsilon,\,\beta}&=-\varepsilon\eta''(u_{\varepsilon,\beta})(\partial_x u_{\varepsilon,\beta})^2,\\
I_{3,\,\varepsilon,\,\beta}&=-\partial_x\left(\beta\varepsilon\eta'(u_{\varepsilon,\beta})\partial^3_{xxx}u_{\varepsilon,\beta}\right),\\
I_{4,\,\varepsilon,\,\beta}&=\beta\varepsilon\eta''(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}\partial^3_{xxx}u_{\varepsilon,\beta},\\
I_{5,\,\varepsilon,\,\beta}&=-\partial_x(\beta^2\eta'(u_{\varepsilon,\beta})\partial^4_{xxxx}u_{\varepsilon,\beta}),\\
I_{6,\,\varepsilon,\,\beta}&=\beta^2\eta''(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}\partial^4_{xxxx}u_{\varepsilon,\beta},\\
I_{7,\,\varepsilon,\,\beta}&=-\partial_x\left(\beta\eta'(u_{\varepsilon,\beta})\partial^2_{xx}u_{\varepsilon,\beta}\right),\\
I_{8,\,\varepsilon,\,\beta}&=\beta\eta''(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}\partial^2_{xx}u_{\varepsilon,\beta}.
\end{aligned}\tag{31}$$

Fix $T>0$. Arguing as in [4, Theorem 2.1], we have that $I_{1,\,\varepsilon,\,\beta}\to 0$ in $H^{-1}((0,T)\times\mathbb{R})$, $\{I_{2,\,\varepsilon,\,\beta}\}_{\varepsilon,\beta>0}$ is bounded in $L^1((0,T)\times\mathbb{R})$, $I_{3,\,\varepsilon,\,\beta}\to 0$ in $H^{-1}((0,T)\times\mathbb{R})$, $I_{4,\,\varepsilon,\,\beta}\to 0$ in $L^1((0,T)\times\mathbb{R})$, $I_{5,\,\varepsilon,\,\beta}\to 0$ in $H^{-1}((0,T)\times\mathbb{R})$, and $I_{6,\,\varepsilon,\,\beta}\to 0$ in $L^1((0,T)\times\mathbb{R})$. Due to (23), Lemma 2.3, and the Hölder inequality, we have that $I_{7,\,\varepsilon,\,\beta}\to 0$ in $H^{-1}((0,T)\times\mathbb{R})$, and $I_{8,\,\varepsilon,\,\beta}\to 0$ in $L^1((0,T)\times\mathbb{R})$.

Therefore, (25) follows from the Murat Lemma and the L^p compensated compactness of [25].

Arguing as in [4, Theorem 2.1] and using (22), we can prove that for every convex entropy-entropy flux (η, q) we have

$$\partial_t \eta(u) + \partial_x q(u) \leq 0$$

in the sense of distributions. Therefore (26) is proved.

Finally, arguing as in [14, Theorem 7.1], we get (24). □

3. **The Kawahara -KDV equation:** $u_0 \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$, $\beta = \mathcal{O}\left(\varepsilon^4\right)$**.** In this section, we consider (18), and assume (13) on the initial datum.

We study the dispersion-diffusion limit for (18). Therefore, we fix two small numbers ε, β, and consider the approximation (21), where $u_{\varepsilon,\beta,0}$ is a C^∞ approximation of u_0 such that

$$\begin{aligned} &u_{\varepsilon,\beta,0} \to u_0 \quad \text{in } L^p_{loc}(\mathbb{R}),\ 1 \leq p < 4, \text{ as } \varepsilon, \beta \to 0, \\ &\|u_{\varepsilon,\beta,0}\|^4_{L^4(\mathbb{R})} + \|u_{\varepsilon,\beta,0}\|^2_{L^2(\mathbb{R})} \leq C_0, \quad \varepsilon, \beta > 0, \\ &\left(\beta^{\frac{1}{2}} + \varepsilon^2\right) \|\partial_x u_{\varepsilon,\beta,0}\|^2_{L^2(\mathbb{R})} + \beta\varepsilon^2 \left\|\partial^2_{xx} u_{\varepsilon,\beta,0}\right\|^2_{L^2(\mathbb{R})} \leq C_0, \quad \varepsilon, \beta > 0, \end{aligned} \tag{32}$$

and C_0 is a constant independent on ε and β.

The main result of this section is the following theorem.

Theorem 3.1. *Assume that (13) and (32) hold. Fix $T > 0$, if*

$$\beta \leq \frac{25}{32 C_0} \varepsilon^4, \tag{33}$$

then, there exist two sequences $\{\varepsilon_n\}_{n\in\mathbb{N}}$, $\{\beta_n\}_{n\in\mathbb{N}}$, with $\varepsilon_n, \beta_n \to 0$, and a limit function

$$u \in L^\infty((0,T); L^2(\mathbb{R}) \cap L^4(\mathbb{R})),$$

such that

$$u_{\varepsilon_n,\beta_n} \to u, \quad \text{strongly in } L^p_{loc}(\mathbb{R}^+ \times \mathbb{R}), \text{ for each } 1 \leq p < 4, \tag{34}$$

$$u \quad \text{the unique entropy solution of (5)}, \tag{35}$$

where $u_{\varepsilon,\beta}$ solves (21).

Let us prove some a priori estimates on $u_{\varepsilon,\beta}$, denoting with C_0 the constants which depend only on the initial data.

Lemma 3.2. *Fix $T > 0$. Assume (33) holds. There exists $C_0 > 0$ such that (29) holds. Moreover,*

$$\begin{aligned} \beta^{\frac{1}{2}} \|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{3\beta^{\frac{1}{2}}\varepsilon}{2} \int_0^t \left\|\partial^2_{xx} u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \\ + 2\beta^{\frac{3}{2}}\varepsilon \int_0^t \left\|\partial^3_{xxx} u_{\varepsilon,\beta}(s,\cdot)\right\|_{L^2(\mathbb{R})} ds \leq C_0 \beta^{-\frac{1}{2}}. \end{aligned} \tag{36}$$

Proof of Lemma 3.2. Let $0 < t < T$. Multiplying (18) by $-2\beta^{\frac{1}{2}}\partial^2_{xx} u_{\varepsilon,\beta}$, we have

$$\begin{aligned} &-2\beta^{\frac{1}{2}}\partial^2_{xx} u_{\varepsilon,\beta}\partial_t u_{\varepsilon,\beta} - 4\beta^{\frac{1}{2}} u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial^2_{xx} u_{\varepsilon,\beta} - 2\beta^{\frac{3}{2}}\partial^2_{xx} u_{\varepsilon,\beta}\partial^3_{xxx} u_{\varepsilon,\beta} \\ &\quad - 2\beta^{\frac{5}{2}}\partial^2_{xx} u_{\varepsilon,\beta}\partial^5_{xxxxx} u_{\varepsilon,\beta} = -2\beta^{\frac{1}{2}}\varepsilon(\partial^2_{xx} u_{\varepsilon,\beta})^2 + 2\beta^{\frac{3}{2}}\varepsilon\partial^2_{xx} u_{\varepsilon,\beta}\partial^4_{xxxx} u_{\varepsilon,\beta}. \end{aligned}$$

Since,

$$-2\beta^{\frac{3}{2}}\int_{\mathbb{R}}\partial_{xx}^{2}u_{\varepsilon,\beta}\partial_{xxx}^{3}u_{\varepsilon,\beta}dx=0,$$

arguing as in [4, Lemma 3.1], we have (29) and (36). □

Following [13, Lemma 4.2], or [8, Lemma 2.2], we prove the following result.

Lemma 3.3. *Fix $T>0$. Assume (32) and (33) hold. Then:*

i) the family $\{u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$ is bounded in $L^{\infty}((0,T);L^{4}(\mathbb{R}))$;

ii) the families $\{\varepsilon\partial_{x}u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$, $\{\beta^{\frac{1}{2}}\varepsilon\partial_{xx}^{2}u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$ are bounded in $L^{\infty}((0,T);L^{2}(\mathbb{R}))$;

iii) the families $\{\beta^{\frac{1}{2}}\varepsilon\partial_{xxx}^{3}u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$, $\{\beta^{\frac{1}{2}}\varepsilon^{\frac{3}{2}}\partial_{xxx}^{3}u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$, $\{\beta\varepsilon^{\frac{3}{2}}\partial_{xxxx}^{4}u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$, $\{\varepsilon^{\frac{1}{2}}u_{\varepsilon,\beta}\partial_{x}u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$, $\{\varepsilon^{\frac{3}{2}}\partial_{xx}^{2}u_{\varepsilon,\beta}\}_{\varepsilon,\beta}$ are bounded in $L^{2}((0,T)\times\mathbb{R})$.

Proof. Let $0<t<T$. Let A, B be some positive constants which be specified later. Multiplying (18) by

$$u_{\varepsilon,\beta}^{3}-A\varepsilon^{2}\partial_{xx}^{2}u_{\varepsilon,\beta}+B\beta\varepsilon^{2}\partial_{xxxx}^{4}u_{\varepsilon,\beta},$$

we have

$$\begin{aligned}
&\left(u_{\varepsilon,\beta}^{3}-A\varepsilon^{2}\partial_{xx}^{2}u_{\varepsilon,\beta}+B\beta\varepsilon^{2}\partial_{xxxx}^{4}u_{\varepsilon,\beta}\right)\partial_{t}u_{\varepsilon,\beta}\\
&\quad+2\left(u_{\varepsilon,\beta}^{3}-A\varepsilon^{2}\partial_{xx}^{2}u_{\varepsilon,\beta}+B\beta\varepsilon^{2}\partial_{xxxx}^{4}u_{\varepsilon,\beta}\right)u_{\varepsilon,\beta}\partial_{x}u_{\varepsilon,\beta}\\
&\quad+\beta\left(u_{\varepsilon,\beta}^{3}-A\varepsilon^{2}\partial_{xx}^{2}u_{\varepsilon,\beta}+B\beta\varepsilon^{2}\partial_{xxxx}^{4}u_{\varepsilon,\beta}\right)\partial_{xxx}^{3}u_{\varepsilon,\beta}\\
&\quad+\beta^{2}\left(u_{\varepsilon,\beta}^{3}-A\varepsilon^{2}\partial_{xx}^{2}u_{\varepsilon,\beta}+B\beta\varepsilon^{2}\partial_{xxxx}^{4}u_{\varepsilon,\beta}\right)\partial_{xxxxx}^{5}u_{\varepsilon,\beta}\\
&=\varepsilon\left(u_{\varepsilon,\beta}^{3}-A\varepsilon^{2}\partial_{xx}^{2}u_{\varepsilon,\beta}+B\beta\varepsilon^{2}\partial_{xxxx}^{4}u_{\varepsilon,\beta}\right)\partial_{xx}^{2}u_{\varepsilon,\beta}\\
&\quad-\beta\varepsilon\left(u_{\varepsilon,\beta}^{3}-A\varepsilon^{2}\partial_{xx}^{2}u_{\varepsilon,\beta}+B\beta\varepsilon^{2}\partial_{xxxx}^{4}u_{\varepsilon,\beta}\right)\partial_{xxxx}^{4}u_{\varepsilon,\beta}.
\end{aligned}\tag{37}$$

Since

$$\begin{aligned}
&\beta\int_{\mathbb{R}}\left(u_{\varepsilon,\beta}^{3}-A\varepsilon^{2}\partial_{xx}^{2}u_{\varepsilon,\beta}+B\beta\varepsilon^{2}\partial_{xxxx}^{4}u_{\varepsilon,\beta}\right)\partial_{xxx}^{3}u_{\varepsilon,\beta}dx\\
&\qquad=-3\beta\int_{\mathbb{R}}u_{\varepsilon,\beta}^{2}\partial_{x}u_{\varepsilon,\beta}\partial_{xx}^{2}u_{\varepsilon,\beta}dx,
\end{aligned}$$

argument as in [4, Lemma 3.2], an integration on $\mathbb{R}$ of (37) gives

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{4}\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^{4}(\mathbb{R})}^{4}+\frac{A\varepsilon^{2}}{2}\left\|\partial_{x}u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^{2}(\mathbb{R})}^{2}+\frac{B\beta\varepsilon^{2}}{2}\left\|\partial_{xx}^{2}u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^{2}(\mathbb{R})}^{2}\right)\\
&\quad+3\varepsilon\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_{x}u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^{2}(\mathbb{R})}^{2}+A\varepsilon^{3}\left\|\partial_{xx}^{2}u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^{2}(\mathbb{R})}^{2}\\
&\quad+B\beta\varepsilon^{3}\left\|\partial_{xxx}^{3}u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^{2}(\mathbb{R})}^{2}+A\beta\varepsilon^{2}\left\|\partial_{xxx}^{3}u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^{2}(\mathbb{R})}^{2}\\
&\quad+B\beta^{2}\varepsilon^{3}\left\|\partial_{xxxx}^{4}u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^{2}(\mathbb{R})}^{2}\\
&=2A\varepsilon^{2}\int_{\mathbb{R}}u_{\varepsilon,\beta}\partial_{x}u_{\varepsilon,\beta}\partial_{xx}^{2}u_{\varepsilon,\beta}dx-2B\beta\varepsilon^{2}\int_{\mathbb{R}}u_{\varepsilon,\beta}\partial_{x}u_{\varepsilon,\beta}\partial_{xxxx}^{4}u_{\varepsilon,\beta}dx\\
&\quad+3\beta\int_{\mathbb{R}}u_{\varepsilon,\beta}^{2}\partial_{x}u_{\varepsilon,\beta}\partial_{xx}^{2}u_{\varepsilon,\beta}dx+3\beta^{2}\int_{\mathbb{R}}u_{\varepsilon,\beta}^{2}\partial_{x}u_{\varepsilon,\beta}\partial_{xxxx}^{4}u_{\varepsilon,\beta}dx\\
&\quad+3\beta\varepsilon\int_{\mathbb{R}}u_{\varepsilon,\beta}^{2}\partial_{x}u_{\varepsilon,\beta}\partial_{xxx}^{3}u_{\varepsilon,\beta}dx.
\end{aligned}$$

Due the Young inequality,

$$2A\varepsilon^{2}\int_{\mathbb{R}}|u_{\varepsilon,\beta}\partial_{x}u_{\varepsilon,\beta}|\partial_{xx}^{2}u_{\varepsilon,\beta}|dx=2\int_{\mathbb{R}}\left|\varepsilon^{\frac{1}{2}}u_{\varepsilon,\beta}\partial_{x}u_{\varepsilon,\beta}\right|\left|A\varepsilon^{\frac{3}{2}}\partial_{xx}^{2}u_{\varepsilon,\beta}\right|dx$$

$$
\begin{aligned}
&\leq \varepsilon \left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + A^2\varepsilon^3 \left\|\partial^2_{xx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
2B\beta\varepsilon^2 \int_{\mathbb{R}} |u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}||\partial^4_{xxxx} u_{\varepsilon,\beta}|dx &= B\int_{\mathbb{R}} \left|2\varepsilon^{\frac{1}{2}} u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\right| \left|\beta\varepsilon^{\frac{3}{2}}\partial^4_{xxxx} u_{\varepsilon,\beta}\right| dx\\
&\leq 2B\varepsilon \left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{B\beta^2\varepsilon^3}{2}\left\|\partial^4_{xxxx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{4}\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|^4_{L^4(\mathbb{R})} + \frac{A\varepsilon^2}{2}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right.\\
&\quad\left. + \frac{B\beta\varepsilon^2}{2}\left\|\partial^2_{xx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right) + (2-2B)\,\varepsilon\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad + \left(A - A^2\right)\varepsilon^3\left\|\partial^2_{xx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + B\beta\varepsilon^3\left\|\partial^3_{xxx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad + A\beta\varepsilon^2\left\|\partial^3_{xxx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{B\beta^2\varepsilon^3}{2}\left\|\partial^4_{xxxx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq 3\beta\int_{\mathbb{R}} u^2_{\varepsilon,\beta}|\partial_x u_{\varepsilon,\beta}||\partial^2_{xx} u_{\varepsilon,\beta}|dx + 3\beta^2\int_{\mathbb{R}} u^2_{\varepsilon,\beta}|\partial_x u_{\varepsilon,\beta}||\partial^4_{xxxx} u_{\varepsilon,\beta}|dx\\
&\quad + 3\beta\varepsilon\int_{\mathbb{R}} u^2_{\varepsilon,\beta}|\partial_x u_{\varepsilon,\beta}||\partial^3_{xxx} u_{\varepsilon,\beta}|dx.
\end{aligned}
\tag{38}
$$

By (33),

$$
\beta \leq D^2\varepsilon^4, \tag{39}
$$

where D is a positive constant which will be specified later. Thanks to (29), (39), and the Young inequality,

$$
\begin{aligned}
3\beta\int_{\mathbb{R}} u^2_{\varepsilon,\beta}|\partial_x u_{\varepsilon,\beta}||\partial^2_{xx} u_{\varepsilon,\beta}|dx &\leq 3\beta\left\|u_{\varepsilon,\beta}\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x u_{\varepsilon,\beta}||\partial^2_{xx} u_{\varepsilon,\beta}|dx\\
&\leq C_0\beta^{\frac{1}{2}}\int_{\mathbb{R}}|\partial_x u_{\varepsilon,\beta}||\partial^2_{xx} u_{\varepsilon,\beta}|dx \leq C_0 D\varepsilon^2\int_{\mathbb{R}}|\partial_x u_{\varepsilon,\beta}||\partial^2_{xx} u_{\varepsilon,\beta}|dx\\
&= \int_{\mathbb{R}}\left|\frac{C_0 D\varepsilon^{\frac{1}{2}}\partial_x u_{\varepsilon,\beta}}{\sqrt{A}}\right|\left|\sqrt{A}\varepsilon^{\frac{3}{2}}\partial^2_{xx} u_{\varepsilon,\beta}\right| dx\\
&\leq \frac{C_0 D^2\varepsilon}{A}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{A^2\varepsilon^3}{2}\left\|\partial^2_{xx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
3\beta^2\int_{\mathbb{R}} u^2_{\varepsilon,\beta}|\partial_x u_{\varepsilon,\beta}||\partial^4_{xxxx} u_{\varepsilon,\beta}|dx &= \beta^2\int_{\mathbb{R}}\left|\frac{3\sqrt{3}u^2_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}}{\sqrt{2B}\varepsilon^{\frac{3}{2}}}\right|\left|\frac{\sqrt{2B}\varepsilon^{\frac{3}{2}}\partial^4_{xxxx} u_{\varepsilon,\beta}}{\sqrt{3}}\right| dx\\
&\leq \frac{27\beta^2}{4B\varepsilon^3}\int_{\mathbb{R}} u^4_{\varepsilon,\beta}(\partial_x u_{\varepsilon,\beta})^2 dx + \frac{B\beta^2\varepsilon^3}{3}\left\|\partial^4_{xxxx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq \frac{27\beta^2}{4B\varepsilon^3}\left\|u_{\varepsilon,\beta}\right\|^4_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad + \frac{B\beta^2\varepsilon^3}{3}\left\|\partial^4_{xxxx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq \frac{C_0\beta}{\varepsilon^3}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{B\beta^2\varepsilon^3}{3}\left\|\partial^4_{xxxx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq C_0\varepsilon\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{B\beta^2\varepsilon^3}{3}\left\|\partial^4_{xxxx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

$$
\begin{aligned}
&3\beta\varepsilon\int_{\mathbb{R}} u_{\varepsilon,\beta}^2|\partial_x u_{\varepsilon,\beta}||\partial_{xxx}^3 u_{\varepsilon,\beta}|dx\\
&\quad=\beta\varepsilon\int_{\mathbb{R}}\left|\frac{3u_{\varepsilon,\beta}^2\partial_x u_{\varepsilon,\beta}}{\sqrt{B}\varepsilon}\right|\left|\sqrt{B}\varepsilon\partial_{xxx}^3 u_{\varepsilon,\beta}\right|dx\\
&\quad\leq\frac{9\beta}{2B\varepsilon}\int_{\mathbb{R}} u_{\varepsilon,\beta}^4(\partial_x u_{\varepsilon,\beta})^2dx+\frac{B\beta\varepsilon^3}{2}\left\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\\
&\quad\leq\frac{9\beta}{2B\varepsilon}\left\|u_{\varepsilon,\beta}\right\|_{L^\infty((0,T)\times\mathbb{R})}^2\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\\
&\qquad+\frac{B\beta\varepsilon^3}{2}\left\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\\
&\quad\leq\frac{C_0\beta^{\frac{1}{2}}}{B\varepsilon}\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2+\frac{B\beta\varepsilon^3}{2}\left\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\\
&\quad\leq\frac{C_0D^2\varepsilon}{B}\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2+\frac{B\beta\varepsilon^3}{2}\left\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2.
\end{aligned}
$$

As a consequence (38) gives

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{4}\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^4(\mathbb{R})}^4+\frac{A\varepsilon^2}{2}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\right.\\
&\qquad\left.+\frac{B\beta\varepsilon^2}{2}\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\right)\\
&\qquad+\left(3-2B-\frac{C_0D^2}{B}\right)\varepsilon\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\\
&\qquad+\left(A-\frac{3A^2}{2}\right)\varepsilon^3\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2+\frac{B\beta\varepsilon^3}{2}\left\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\\
&\qquad+A\beta\varepsilon^2\left\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2+\frac{B\beta^2\varepsilon^3}{6}\left\|\partial_{xxxx}^4 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\\
&\quad\leq C_0\varepsilon\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2.
\end{aligned}
\tag{40}
$$

We search A, B such that

$$
\begin{cases}
A-\dfrac{3A^2}{2}>0,\\
3-2B-\dfrac{C_0D^2}{B}>0,
\end{cases}
$$

that is

$$
\begin{cases}
A<\dfrac{2}{3},\\
2B^2-3B+C_0D^2<0.
\end{cases}
\tag{41}
$$

We choose

$$
A=\frac{1}{3}. \tag{42}
$$

The second equation of (41) admits solution if

$$
D<\frac{6\sqrt{2}}{8\sqrt{C_0}}.
$$

Choosing

$$
D=\frac{5\sqrt{2}}{8\sqrt{C_0}}, \tag{43}
$$

it follows from (41) and (43) that there exist $0 < B_1 < B_2$, such that for every

$$B_1 < B < B_2, \tag{44}$$

(41) holds. Hence, from (40), (42) and (44), we get

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{4}\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|^4_{L^4(\mathbb{R})}+\frac{\varepsilon^2}{6}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{B\beta\varepsilon^2}{2}\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)\\
&\quad+K\varepsilon\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{\varepsilon^3}{6}\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad+\frac{B\beta\varepsilon^3}{2}\left\|\partial^3_{xxx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{\beta\varepsilon^2}{3}\left\|\partial^3_{xxx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad+\frac{\beta\varepsilon^3}{6}\left\|\partial^3_{xxx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{B\beta^2\varepsilon^3}{6}\left\|\partial^4_{xxxx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le C_0\varepsilon\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}$$

where K is a positive constant.

(32), (27), an integration on $(0,t)$ give

$$\begin{aligned}
&\frac{1}{4}\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|^4_{L^4(\mathbb{R})}+\frac{\varepsilon^2}{6}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{B\beta\varepsilon^2}{2}\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad++K\varepsilon\int_0^t\left\|u_{\varepsilon,\beta}(s,\cdot)\partial_x u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds+\frac{\varepsilon^3}{6}\int_0^t\left\|\partial^2_{xx}u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\
&\quad+\frac{B\beta\varepsilon^3}{2}\int_0^t\left\|\partial^3_{xxx}u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds+\frac{\beta\varepsilon^2}{3}\int_0^t\left\|\partial^3_{xxx}u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\
&\quad+\frac{\beta\varepsilon^3}{6}\int_0^t\left\|\partial^3_{xxx}u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds+\frac{B\beta^2\varepsilon^3}{6}\int_0^t\left\|\partial^4_{xxxx}u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\
&\le C_0+C_0\varepsilon\int_0^t\left\|\partial_x u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\le C_0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^4(\mathbb{R})}&\le C_0,\\
\varepsilon\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}&\le C_0,\\
\beta^{\frac{1}{2}}\varepsilon\left\|\partial^2_{xx}u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})}&\le C_0,\\
\varepsilon\int_0^t\left\|u_{\varepsilon,\beta}(s,\cdot)\partial_x u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds&\le C_0,\\
\varepsilon^3\int_0^t\left\|\partial^2_{xx}u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds&\le C_0,\\
\beta\varepsilon^2\int_0^t\left\|\partial^3_{xxx}u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds&\le C_0,\\
\beta\varepsilon^3\int_0^t\left\|\partial^3_{xxx}u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds&\le C_0,\\
\beta^2\varepsilon^3\int_0^t\left\|\partial^4_{xxxx}u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds&\le C_0,
\end{aligned}$$

for every $0 < t < T$. □

We are ready for the proof of Theorem 3.1.

Proof of Theorem 3.1. Let us consider a compactly supported entropy-entropy flux pair (η, q). Multiplying (21) by $\eta'(u_{\varepsilon,\beta})$, we have

$$\begin{aligned}\partial_t\eta(u_{\varepsilon,\beta})+\partial_x q(u_{\varepsilon,\beta}) =&\varepsilon\eta'(u_{\varepsilon,\beta})\partial^2_{xx}u_{\varepsilon,\beta}-\beta\varepsilon\eta'(u_{\varepsilon,\beta})\partial^4_{xxxx}u_{\varepsilon,\beta}\\ &-\beta\eta'(u_{\varepsilon,\beta})\partial^3_{xxx}u_{\varepsilon,\beta}-\beta^2\eta'(u_{\varepsilon,\beta})\partial^5_{xxxxx}u_{\varepsilon,\beta}\\ =&I_{1,\varepsilon,\beta}+I_{2,\varepsilon,\beta}+I_{3,\varepsilon,\beta}+I_{4,\varepsilon,\beta}+I_{5,\varepsilon,\beta}+I_{6,\varepsilon,\beta}\\ &+I_{7,\varepsilon,\beta}+I_{8,\varepsilon,\beta},\end{aligned}$$

where $I_{1,\varepsilon,\beta}$, $I_{2,\varepsilon,\beta}$, $I_{3,\varepsilon,\beta}$, $I_{4,\varepsilon,\beta}$, $I_{5,\varepsilon,\beta}$, $I_{6,\varepsilon,\beta}$, $I_{7,\varepsilon,\beta}$, $I_{8,\varepsilon,\beta}$ are defined in (31).

Fix $T>0$. Arguing as in [4, Theorem 3.1], we have that $I_{1,\varepsilon,\beta}\to 0$ in $H^{-1}((0,T)\times\mathbb{R})$, $\{I_{2,\varepsilon,\beta}\}_{\varepsilon,\beta>0}$ is bounded in $L^1((0,T)\times\mathbb{R})$, $I_{3,\varepsilon,\beta}\to 0$ in $H^{-1}((0,T)\times\mathbb{R})$, $I_{4,\varepsilon,\beta}\to 0$ in $L^1((0,T)\times\mathbb{R})$, $I_{7,\varepsilon,\beta}\to 0$ in $H^{-1}((0,T)\times\mathbb{R})$ and $I_{8,\varepsilon,\beta}\to 0$ in $L^1((0,T)\times\mathbb{R})$.

We claim that

$$I_{5,\varepsilon,\beta}\to 0 \quad \text{in } H^{-1}((0,T)\times\mathbb{R}),\ T>0,\ \text{as } \varepsilon\to 0.$$

Due to (33) and Lemma 3.3,

$$\begin{aligned}&\left\|\beta\eta'(u_{\varepsilon,\beta})\partial^2_{xx}u_{\varepsilon,\beta}\right\|^2_{L^2((0,T)\times\mathbb{R})}\\ &\quad\le\beta^2\left\|\eta'\right\|_{L^\infty(\mathbb{R})}\left\|\partial^2_{xx}u_{\varepsilon,\beta}\right\|^2_{L^2((0,T)\times\mathbb{R})}\\ &\quad=\frac{\beta^2\varepsilon^3}{\varepsilon^3}\left\|\eta'\right\|_{L^\infty(\mathbb{R})}\left\|\partial^2_{xx}u_{\varepsilon,\beta}\right\|^2_{L^2((0,T)\times\mathbb{R})}\\ &\quad\le C_0\left\|\eta'\right\|_{L^\infty(\mathbb{R})}\varepsilon^5\to 0.\end{aligned}$$

We have that

$$I_{6,\varepsilon,\beta}\to 0 \quad \text{in } L^1((0,T)\times\mathbb{R}),\ T>0,\ \text{as } \varepsilon\to 0.$$

Thanks to (33), Lemmas 2.3 and 3.3, and the Hölder inequality,

$$\begin{aligned}&\left\|\beta\eta''(u_{\varepsilon,\beta})\partial_x u_{\varepsilon,\beta}\partial^3_{xxx}u_{\varepsilon,\beta}\right\|_{L^1((0,T)\times\mathbb{R})}\\ &\quad\le\beta\left\|\eta''\right\|_{L^\infty(\mathbb{R})}\int_0^T\int_{\mathbb{R}}|\partial_x u_{\varepsilon,\beta}||\partial^2_{xx}u_{\varepsilon,\beta}|dtdx\\ &\quad=\frac{\varepsilon^2\beta}{\varepsilon^4}\left\|\eta''\right\|_{L^\infty(\mathbb{R})}\varepsilon^{\frac12}\left\|\partial_x u_{\varepsilon,\beta}\right\|_{L^2((0,T)\times\mathbb{R})}\varepsilon^{\frac32}\left\|\partial^2_{xx}u_{\varepsilon,\beta}\right\|_{L^2((0,T)\times\mathbb{R})}\\ &\quad\le C_0\left\|\eta''\right\|_{L^\infty(\mathbb{R})}\varepsilon^2\to 0.\end{aligned}$$

Therefore, (34) follows from Lemmas 2.3, 3.3, and the L^p compensated compactness of [25]. Arguing as in [4, Theorem 3.1], we have (35). □

Appendix A. The Kawahara-KdV equation: $u_0\in L^2(\mathbb{R})$, $\beta=o\left(\varepsilon^8\right)$. In this appendix, we consider (10), and assume

$$u_0\in L^2(\mathbb{R}), \tag{45}$$

on the initial datum. We study the dispersion-diffusion limit for (10). Therefore, we fix two small numbers $0<\varepsilon,\beta<1$, and, following [6], consider the following fifth order problem

$$\begin{cases}\partial_t u_{\varepsilon,\beta}+\partial_x u^2_{\varepsilon,\beta}+\beta\partial^3_{xxx}u_{\varepsilon,\beta}\\ \qquad+\beta^2\partial^5_{xxxxx}u_{\varepsilon,\beta}=\varepsilon\partial^2_{xx}u_{\varepsilon,\beta}-\beta^{\frac32}\varepsilon\partial^4_{xxxx}u_{\varepsilon,\beta}, & t>0,\ x\in\mathbb{R},\\ u_{\varepsilon,\beta}(0,x)=u_{\varepsilon,\beta,0}(x), & x\in\mathbb{R},\end{cases} \tag{46}$$

where $u_{\varepsilon,\beta,0}$ is a C^∞ approximation of u_0 such that

$$\begin{aligned} &u_{\varepsilon,\,\beta,\,0} \to u_0 \quad \text{in } L^p_{loc}(\mathbb{R}),\ 1 \le p < 4, \text{ as } \varepsilon,\ \beta \to 0, \\ &\|u_{\varepsilon,\beta,0}\|^2_{L^2(\mathbb{R})} + \beta^{\frac{1}{2}} \|\partial_x u_{\varepsilon,\beta,0}\|^2_{L^2(\mathbb{R})} + \beta^{\frac{3}{2}}\varepsilon^2 \left\|\partial^2_{xx} u_{\varepsilon,\beta,0}\right\|^2_{L^2(\mathbb{R})} \le C_0, \quad \varepsilon,\beta > 0, \end{aligned} \tag{47}$$

and C_0 is a constant independent on ε and β.

The main result of this section is the following theorem.

Theorem A.1. *Assume that (45), and (47) hold. Fix $T > 0$, if*

$$\beta = o\left(\varepsilon^8\right), \tag{48}$$

then, there exist two sequences $\{\varepsilon_n\}_{n\in\mathbb{N}}$, $\{\beta_n\}_{n\in\mathbb{N}}$, with $\varepsilon_n, \beta_n \to 0$, and a limit function

$$u \in L^\infty((0,T); L^2(\mathbb{R})),$$

such that

i) $u_{\varepsilon_n,\beta_n} \to u$ strongly in $L^p_{loc}(\mathbb{R}^+ \times \mathbb{R})$, for each $1 \le p < 2$,
ii) u is the unique entropy solution of (5),

where $u_{\varepsilon,\beta}$ solves (46).

Let us prove some a priori estimates on $u_{\varepsilon,\beta}$, denoting with C_0 the constants which depend only on the initial data.

Lemma A.2. *For each $t > 0$,*

$$\begin{aligned} \|u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} &+ 2\varepsilon \int_0^t \|\partial_x u_{\varepsilon,\beta}(s,\cdot)\|^2_{L^2(\mathbb{R})}\, ds \\ &+ 2\beta^{\frac{3}{2}}\varepsilon \int_0^t \left\|\partial^2_{xx} u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C_0. \end{aligned} \tag{49}$$

Proof. We begin by observing that

$$\beta \int_{\mathbb{R}} u_{\varepsilon,\beta}\partial^3_{xxx} u_{\varepsilon,\beta} dx = -\beta \int_{\mathbb{R}} \partial_x u_{\varepsilon,\beta}\partial^2_{xx} u_{\varepsilon,\beta} dx = 0.$$

Therefore, arguing as in [4, Lemma $A.1$], we have (49). □

Lemma A.3. *Fix $T > 0$. Assume (48) holds. There exists $C_0 > 0$, independent on ε, β such that (29) holds. Moreover*

$$\begin{aligned} \beta \|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} &+ \beta^2\varepsilon^2 \left\|\partial^2_{xx} u_{\varepsilon,\beta}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\ &+ \frac{3\beta\varepsilon}{2} \int_0^t \left\|\partial^2_{xx} u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \\ &+ 2\beta^2\varepsilon^3 \int_0^t \left\|\partial^3_{xxx} u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \\ &+ 2\beta^{\frac{5}{2}}\varepsilon \int_0^t \left\|\partial^3_{xxx} u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \\ &+ \frac{3\beta^{\frac{7}{2}}\varepsilon^3}{2} \int_0^t \left\|\partial^4_{xxxx} u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C_0. \end{aligned} \tag{50}$$

Proof. Let $0 < t < T$. Multiplying (46) by $-2\beta^{\frac{1}{2}}\partial^2_{xx} u_{\varepsilon,\beta} + 2\beta^{\frac{3}{2}}\varepsilon^2\partial^4_{xxxx} u_{\varepsilon,\beta}$, we have

$$\left(-2\beta^{\frac{1}{2}}\partial^2_{xx} u_{\varepsilon,\beta} + 2\beta^{\frac{3}{2}}\varepsilon^2\partial^4_{xxxx} u_{\varepsilon,\beta}\right)\partial_t u_{\varepsilon,\beta}$$

$$
\begin{aligned}
&+2\left(-2\beta^{\frac{1}{2}}\partial_{xx}^2u_{\varepsilon,\beta}+2\beta^{\frac{3}{2}}\varepsilon^2\partial_{xxxx}^4u_{\varepsilon,\beta}\right)u_{\varepsilon,\beta}\partial_xu_{\varepsilon,\beta}\\
&+\beta\left(-2\beta^{\frac{1}{2}}\partial_{xx}^2u_{\varepsilon,\beta}+2\beta^{\frac{3}{2}}\varepsilon^2\partial_{xxxx}^4u_{\varepsilon,\beta}\right)\partial_{xxx}^3u_{\varepsilon,\beta}\\
&+\beta^2\left(-2\beta^{\frac{1}{2}}\partial_{xx}^2u_{\varepsilon,\beta}+2\beta^{\frac{3}{2}}\varepsilon^2\partial_{xxxx}^4u_{\varepsilon,\beta}\right)\partial_{xxxxx}^5u_{\varepsilon,\beta}\\
=&\,\varepsilon\left(-2\beta^{\frac{1}{2}}\partial_{xx}^2u_{\varepsilon,\beta}+2\beta^{\frac{3}{2}}\varepsilon^2\partial_{xxxx}^4u_{\varepsilon,\beta}\right)\partial_{xx}^2u_{\varepsilon,\beta}\\
&-\beta^{\frac{3}{2}}\varepsilon\left(-2\beta^{\frac{1}{2}}\partial_{xx}^2u_{\varepsilon,\beta}+2\beta^{\frac{3}{2}}\varepsilon^2\partial_{xxxx}^4u_{\varepsilon,\beta}\right)\partial_{xxxx}^4u_{\varepsilon,\beta}.
\end{aligned}
$$

We observe that

$$
\beta\int_{\mathbb{R}}\left(-2\beta^{\frac{1}{2}}\partial_{xx}^2u_{\varepsilon,\beta}+2\beta^{\frac{3}{2}}\varepsilon^2\partial_{xxxx}^4u_{\varepsilon,\beta}\right)\partial_{xxx}^3u_{\varepsilon,\beta}dx=0.
$$

Arguing as in [4, Lemma A.1], we get (50). □

We are ready for the proof of Theorem A.1.

Proof of Theorem A.1. Let us consider a compactly supported entropy–entropy flux pair $(\eta,\,q)$. Multiplying (21) by $\eta'(u_{\varepsilon,\beta})$, we have

$$
\begin{aligned}
\partial_t\eta(u_{\varepsilon,\beta})+\partial_xq(u_{\varepsilon,\beta})=&\varepsilon\eta'(u_{\varepsilon,\beta})\partial_{xx}^2u_{\varepsilon,\beta}-\beta^{\frac{3}{2}}\varepsilon\eta'(u_{\varepsilon,\beta})\partial_{xxxx}^4u_{\varepsilon,\beta}\\
&-\beta^2\eta'(u_{\varepsilon,\beta})\partial_{xxxxx}^5u_{\varepsilon,\beta}-\beta\eta'(u_{\varepsilon,\beta})\partial_{xxx}^3u_{\varepsilon,\beta}\\
=&I_{1,\,\varepsilon,\,\beta}+I_{2,\,\varepsilon,\,\beta}+I_{3,\,\varepsilon,\,\beta}+I_{4,\,\varepsilon,\,\beta}+I_{5,\,\varepsilon,\,\beta}+I_{6,\,\varepsilon,\,\beta}\\
&+I_{7,\,\varepsilon,\,\beta}+I_{8,\,\varepsilon,\,\beta},
\end{aligned}
$$

where

$$
\begin{aligned}
I_{1,\,\varepsilon,\,\beta}&=\partial_x\left(\varepsilon\eta'(u_{\varepsilon,\beta})\partial_xu_{\varepsilon,\beta}\right),\\
I_{2,\,\varepsilon,\,\beta}&=-\varepsilon\eta''(u_{\varepsilon,\beta})(\partial_xu_{\varepsilon,\beta})^2,\\
I_{3,\,\varepsilon,\,\beta}&=-\partial_x\left(\beta^{\frac{3}{2}}\varepsilon\eta'(u_{\varepsilon,\beta})\partial_{xxx}^3u_{\varepsilon,\beta}\right),\\
I_{4,\,\varepsilon,\,\beta}&=\beta^{\frac{3}{2}}\varepsilon\eta''(u_{\varepsilon,\beta})\partial_xu_{\varepsilon,\beta}\partial_{xxx}^3u_{\varepsilon,\beta},\\
I_{5,\,\varepsilon,\,\beta}&=-\partial_x\left(\beta^2\eta'(u_{\varepsilon,\beta})\partial_{xxxx}^4u_{\varepsilon,\beta}\right),\\
I_{6,\,\varepsilon,\,\beta}&=\beta^2\eta''(u_{\varepsilon,\beta})\partial_xu_{\varepsilon,\beta}\partial_{xxxx}^4u_{\varepsilon,\beta},\\
I_{7,\,\varepsilon,\,\beta}&=-\partial_x\left(\beta\eta'(u_{\varepsilon,\beta})\partial_{xx}^2u_{\varepsilon,\beta}\right),\\
I_{8,\,\varepsilon,\,\beta}&=\beta\eta''(u_{\varepsilon,\beta})\partial_xu_{\varepsilon,\beta}\partial_{xx}^2u_{\varepsilon,\beta}.
\end{aligned}
$$

Fix $T>0$. Arguing as in [4, Theorem A.1], we have $I_{1,\,\varepsilon,\,\beta}\to0$ in $H^{-1}((0,T)\times\mathbb{R})$, $\{I_{2,\,\varepsilon,\,\beta}\}_{\varepsilon,\beta>0}$ is bounded in $L^1((0,T)\times\mathbb{R})$, $I_{3,\,\varepsilon,\,\beta}\to0$ in $H^{-1}((0,T)\times\mathbb{R})$, $I_{4,\,\varepsilon,\,\beta}\to0$ in $L^1((0,T)\times\mathbb{R})$, $I_{5,\,\varepsilon,\,\beta}\to0$ in $H^{-1}((0,T)\times\mathbb{R})$, and $I_{6,\,\varepsilon,\,\beta}\to0$ in $L^1((0,T)\times\mathbb{R})$. Using (48), Lemmas A.2, A.3, and the Hölder inequality, we have $I_{7,\,\varepsilon,\,\beta}\to0$ in $H^{-1}((0,T)\times\mathbb{R})$, and $I_{8,\,\varepsilon,\,\beta}\to0$ in $L^1((0,T)\times\mathbb{R})$. Arguing as in [4, Theorem A.1], the proof is concluded. □

Appendix B. **The Kawahara-KdV equation:** $u_0\in L^2(\mathbb{R})\cap L^4(\mathbb{R})$, $\beta=\mathcal{O}\left(\varepsilon^8\right)$**.** In this appendix, we consider (18), and assume

$$
u_0\in L^2(\mathbb{R})\cap L^4(\mathbb{R}) \tag{51}
$$

on the initial datum. We study the dispersion-diffusion limit for (18). Therefore, we fix two small numbers $0 < \varepsilon,\ \beta < 1$ and, consider the approximation (46), where $u_{\varepsilon,\beta,0}$ is a C^∞ approximation of u_0 such that

$$\begin{aligned} &u_{\varepsilon,\,\beta,\,0} \to u_0 \quad \text{in } L^p_{loc}(\mathbb{R}),\ 1 \le p < 4, \text{ as } \varepsilon,\ \beta \to 0,\\ &\|u_{\varepsilon,\beta,0}\|^4_{L^4(\mathbb{R})} + \|u_{\varepsilon,\beta,0}\|^2_{L^2(\mathbb{R})} \le C_0, \quad \varepsilon,\beta > 0,\\ &(\beta^{\frac12} + \varepsilon^2)\,\|\partial_x u_{\varepsilon,\beta,0}\|^2_{L^2(\mathbb{R})} + \beta^{\frac32}\varepsilon^2\,\left\|\partial^2_{xx} u_{\varepsilon,\beta,0}\right\|^2_{L^2(\mathbb{R})} \le C_0, \quad \varepsilon,\beta > 0, \end{aligned} \tag{52}$$

and C_0 is a constant independent on ε and β.

The main result of this section is the following theorem.

Theorem B.1. *Assume that (51), and (52) hold. Fix $T > 0$, if*

$$\beta \le \frac{\varepsilon^8}{64 C_0^2}, \tag{53}$$

then, there exist two sequences $\{\varepsilon_n\}_{n\in\mathbb{N}}$, $\{\beta_n\}_{n\in\mathbb{N}}$, with $\varepsilon_n, \beta_n \to 0$, and a limit function

$$u \in L^\infty((0,T); L^2(\mathbb{R}) \cap L^4(\mathbb{R})),$$

such that

i) $u_{\varepsilon_n,\beta_n} \to u$ strongly in $L^p_{loc}(\mathbb{R}^+ \times \mathbb{R})$, for each $1 \le p < 4$,
ii) u is the unique entropy solution of (5),

where $u_{\varepsilon,\beta}$ solves (46).

Let us prove some a priori estimates on $u_{\varepsilon,\beta}$, denoting with C_0 the constants which depend only on the initial data.

Lemma B.2. *Fix $T > 0$. Assume (53) holds. There exists $C_0 > 0$, independent on ε, β such that (29) holds.*

Moreover

$$\begin{aligned} \beta^{\frac12}\,\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{3\beta^{\frac12}\varepsilon}{2}\int_0^t \left\|\partial^2_{xx} u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \\ + 2\beta^2\varepsilon^3 \int_0^t \left\|\partial^3_{xxx} u_{\varepsilon,\beta}(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C_0 \beta^{-\frac12}. \end{aligned} \tag{54}$$

Remark 1. Observe that the proof of Lemma B.2 is simpler than the one of Lemma A.3. Indeed, here we only need to prove (29).

Proof of Lemma B.2. Let $0 < t < T$. Multiplying (46) by $-2\beta^{\frac12} u_{\varepsilon,\beta}$, we have

$$\begin{aligned} &-2\beta^{\frac12}\partial^2_{xx}u_{\varepsilon,\beta}\partial_t u_{\varepsilon,\beta} + 4\beta^{\frac12}u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial^2_{xx}u_{\varepsilon,\beta} - \beta^{\frac32}\partial^2_{xx}u_{\varepsilon,\beta}\partial^3_{xxx}u_{\varepsilon,\beta}\\ &\quad + 2\beta^{\frac52}\partial^2_{xx}u_{\varepsilon,\beta}\partial^5_{xxxxx}u_{\varepsilon,\beta} = 2\beta^{\frac12}\varepsilon(\partial^2_{xx}u_{\varepsilon,\beta})^2 + 2\beta^2\varepsilon\partial^2_{xx}u_{\varepsilon,\beta}\partial^4_{xxxx}u_{\varepsilon,\beta}. \end{aligned}$$

Since

$$-\beta^{\frac32}\int_{\mathbb{R}} \partial^2_{xx}u_{\varepsilon,\beta}\partial^3_{xxx}u_{\varepsilon,\beta}dx = 0,$$

arguing as in [4, Lemma B.1], we have (29) and (54). □

Lemma B.3. *Fix $T > 0$. Assume (52) and (53) hold. Then:*

i) the family $\{u_{\varepsilon,\beta}\}_{\varepsilon,\,\beta}$ is bounded in $L^\infty((0,T); L^4(\mathbb{R}))$;
ii) the families $\{\varepsilon\partial_x u_{\varepsilon,\beta}\}_{\varepsilon,\,\beta}$, $\{\beta^{\frac34}\varepsilon\partial^2_{xx}u_{\varepsilon,\beta}\}_{\varepsilon,\,\beta}$ are bounded in $L^\infty((0,T); L^2(\mathbb{R}))$;
iii) the families $\{\beta^{\frac34}\varepsilon^{\frac32}\partial^3_{xxx}u_{\varepsilon,\beta}\}_{\varepsilon,\,\beta}$, $\{\beta^{\frac32}\varepsilon^{\frac32}\partial^4_{xxxx}u_{\varepsilon,\beta}\}_{\varepsilon,\,\beta}$, $\{\varepsilon^{\frac12}u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\}_{\varepsilon,\,\beta}$, $\{\varepsilon^{\frac32}\partial^2_{xx}u_{\varepsilon,\beta}\}_{\varepsilon,\,\beta}$ are bounded in $L^2((0,T)\times\mathbb{R}))$.

Proof. Let $0 < t < T$. Let A, B be two positive constant which be specified later. Multiplying (46) by

$$u_{\varepsilon,\beta}^3 - A\varepsilon^2\partial_{xx}^2 u_{\varepsilon,\beta} - B\beta^{\frac{3}{2}}\varepsilon^2\partial_{xxxx}^4 u_{\varepsilon,\beta},$$

we have

$$\begin{aligned}
&\left(u_{\varepsilon,\beta}^3 - A\varepsilon^2\partial_{xx}^2 u_{\varepsilon,\beta} - B\beta^{\frac{3}{2}}\varepsilon^2\partial_{xxxx}^4 u_{\varepsilon,\beta}\right)\partial_t u_{\varepsilon,\beta}\\
&\quad + 2\left(u_{\varepsilon,\beta}^3 - A\varepsilon^2\partial_{xx}^2 u_{\varepsilon,\beta} - B\beta^{\frac{3}{2}}\varepsilon^2\partial_{xxxx}^4 u_{\varepsilon,\beta}\right)u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\\
&\quad + \beta\left(u_{\varepsilon,\beta}^3 - A\varepsilon^2\partial_{xx}^2 u_{\varepsilon,\beta} - B\beta^{\frac{3}{2}}\varepsilon^2\partial_{xxxx}^4 u_{\varepsilon,\beta}\right)\partial_{xxx}^3 u_{\varepsilon,\beta}\\
&\quad + \beta^2\left(u_{\varepsilon,\beta}^3 - A\varepsilon^2\partial_{xx}^2 u_{\varepsilon,\beta} - B\beta^{\frac{3}{2}}\varepsilon^2\partial_{xxxx}^4 u_{\varepsilon,\beta}\right)\partial_{xxxxx}^5 u_{\varepsilon,\beta}\\
&= \varepsilon\left(u_{\varepsilon,\beta}^3 - A\varepsilon^2\partial_{xx}^2 u_{\varepsilon,\beta} - B\beta^{\frac{3}{2}}\varepsilon^2\partial_{xxxx}^4 u_{\varepsilon,\beta}\right)\partial_{xx}^2 u_{\varepsilon,\beta}\\
&\quad - \beta^{\frac{3}{2}}\varepsilon\left(u_{\varepsilon,\beta}^3 - A\varepsilon^2\partial_{xx}^2 u_{\varepsilon,\beta} - B\beta^{\frac{3}{2}}\varepsilon^2\partial_{xxxx}^4 u_{\varepsilon,\beta}\right)\partial_{xxxx}^4 u_{\varepsilon,\beta}.
\end{aligned}$$

Since

$$\begin{aligned}
&\beta\int_{\mathbb{R}}\left(u_{\varepsilon,\beta}^3 - A\varepsilon^2\partial_{xx}^2 u_{\varepsilon,\beta} - B\beta^{\frac{3}{2}}\varepsilon^2\partial_{xxxx}^4 u_{\varepsilon,\beta}\right)\partial_{xxx}^3 u_{\varepsilon,\beta}dx\\
&= -3\beta\int_{\mathbb{R}} u_{\varepsilon,\beta}^2\partial_x u_{\varepsilon,\beta}\partial_{xx}^2 u_{\varepsilon,\beta}dx,
\end{aligned}$$

arguing as in [4, Lemma B.2], we get

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{4}\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^4(\mathbb{R})}^4 + \frac{A\varepsilon^2}{2}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\right)\\
&\quad + \frac{B\beta^{\frac{3}{2}}\varepsilon^2}{2}\frac{d}{dt}\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 + 3\varepsilon\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\\
&\quad + A\varepsilon^3\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 + (A+B)\,\beta^{\frac{3}{2}}\varepsilon^3\left\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\\
&\quad + B\beta^3\varepsilon^3\left\|\partial_{xxxx}^4 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 \qquad (55)\\
&= -2A\varepsilon^2\int_{\mathbb{R}} u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial_{xx}^2 u_{\varepsilon,\beta}dx - 2B\beta^{\frac{3}{2}}\varepsilon^2\int_{\mathbb{R}} u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\partial_{xxxx}^4 u_{\varepsilon,\beta}dx\\
&\quad + 3\beta\int_{\mathbb{R}} u_{\varepsilon,\beta}^2\partial_x u_{\varepsilon,\beta}\partial_{xx}^2 u_{\varepsilon,\beta}dx + 3\beta^2\int_{\mathbb{R}} u_{\varepsilon,\beta}^2\partial_x u_{\varepsilon,\beta}\partial_{xxxx}^4 u_{\varepsilon,\beta}dx\\
&\quad + 3\beta^{\frac{3}{2}}\varepsilon\int_{\mathbb{R}} u_{\varepsilon,\beta}^2\partial_x u_{\varepsilon,\beta}\partial_{xxx}^3 u_{\varepsilon,\beta}dx.
\end{aligned}$$

Due to the Young inequality,

$$\begin{aligned}
2A\varepsilon^2\int_{\mathbb{R}}|u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}||\partial_{xx}^2 u_{\varepsilon,\beta}|dx &= \int_{\mathbb{R}}\left|2\varepsilon^{\frac{1}{2}}u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\right|\left|A\varepsilon^{\frac{3}{2}}\partial_{xx}^2 u_{\varepsilon,\beta}\right|dx\\
&\le 2\varepsilon\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 + \frac{A^2\varepsilon^3}{2}\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2,\\
2B\beta^{\frac{3}{2}}\varepsilon^2\int_{\mathbb{R}}|u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}|\partial_{xxxx}^4 u_{\varepsilon,\beta}dx &= B\int_{\mathbb{R}}\left|2\varepsilon^{\frac{1}{2}}u_{\varepsilon,\beta}\partial_x u_{\varepsilon,\beta}\right|\left|\beta^{\frac{3}{2}}\varepsilon^{\frac{3}{2}}\partial_{xxxx}^4 u_{\varepsilon,\beta}\right|dx\\
&\le 2B\varepsilon\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 + \frac{B\beta^3\varepsilon^3}{2}\left\|\partial_{xxxx}^4 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2.
\end{aligned}$$

From (53),

$$\beta \le D^2\varepsilon^8, \tag{56}$$

where D is a positive constant which will be speficied later. Since $0<\varepsilon<1$, (29), (56), and the Young inequality,

$$\begin{aligned}
3\beta\int_{\mathbb{R}} u_{\varepsilon,\beta}^2|\partial_x u_{\varepsilon,\beta}||\partial_{xx}^2 u_{\varepsilon,\beta}|dx &\le 3\beta\left\|u_{\varepsilon,\beta}\right\|_{L^\infty((0,T)\times\mathbb{R})}^2\int_{\mathbb{R}}|\partial_x u_{\varepsilon,\beta}||\partial_{xx}^2 u_{\varepsilon,\beta}|dx\\
&\le C_0\beta^{\frac12}\int_{\mathbb{R}}|\partial_x u_{\varepsilon,\beta}||\partial_{xx}^2 u_{\varepsilon,\beta}|dx \le C_0D\varepsilon^4\int_{\mathbb{R}}|\partial_x u_{\varepsilon,\beta}||\partial_{xx}^2 u_{\varepsilon,\beta}|dx\\
&\le C_0D\varepsilon\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2 + C_0D\varepsilon^3\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2,\\
3\beta^2\int_{\mathbb{R}} u_{\varepsilon,\beta}^2|\partial_x u_{\varepsilon,\beta}||\partial_{xxxx}^4 u_{\varepsilon,\beta}|dx &= \int_{\mathbb{R}}\left|\frac{3\sqrt{2}\beta^{\frac12}u_{\varepsilon,\beta}^2\partial_x u_{\varepsilon,\beta}}{\varepsilon^{\frac32}\sqrt{B}}\right|\left|\frac{\sqrt{B}\beta^{\frac32}\varepsilon^{\frac32}\partial_{xxxx}^4 u_{\varepsilon,\beta}}{\sqrt{2}}\right|dx\\
&\le \frac{9\beta}{B\varepsilon^3}\left\|u_{\varepsilon,\beta}\right\|_{L^\infty((0,T)\times\mathbb{R})}^2\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\\
&\quad+\frac{B\beta^3\varepsilon^3}{4}\left\|\partial_{xxxx}^4 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\\
&\le \frac{C_0D\varepsilon}{B}\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2+\frac{B\beta^3\varepsilon^3}{4}\left\|\partial_{xxxx}^4 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2,\\
3\beta^{\frac32}\varepsilon\int_{\mathbb{R}} u_{\varepsilon,\beta}^2|\partial_x u_{\varepsilon,\beta}||\partial_{xxx}^3 u_{\varepsilon,\beta}|dx &= \beta^{\frac32}\varepsilon\int_{\mathbb{R}}\left|\frac{3u_{\varepsilon,\beta}^2\partial_x u_{\varepsilon,\beta}}{\sqrt{B}\varepsilon}\right|\left|\sqrt{B}\varepsilon^2\partial_{xxx}^3 u_{\varepsilon,\beta}\right|dx\\
&\le \frac{9\beta^{\frac32}}{2B\varepsilon}\left\|u_{\varepsilon,\beta}\right\|_{L^\infty((0,T)\times\mathbb{R})}^2\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\\
&\quad+\frac{B\beta^{\frac32}\varepsilon^3}{2}\left\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\\
&\le \frac{C_0\beta}{B\varepsilon}\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2+\frac{B\beta^{\frac32}\varepsilon^3}{2}\left\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\\
&\le \frac{C_0D\varepsilon}{B}\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2+\frac{B\beta^{\frac32}\varepsilon^3}{2}\left\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2.
\end{aligned}$$

Therefore, from (55), we gain

$$\begin{aligned}
&\frac{d}{dt}\left(\frac14\left\|u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^4(\mathbb{R})}^4+\frac{A\varepsilon^2}{2}\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\right)\\
&\quad+\frac{B\beta^{\frac32}\varepsilon^2}{2}\frac{d}{dt}\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2+\frac{B\beta^3\varepsilon^3}{4}\left\|\partial_{xxxx}^4 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\\
&\quad+\left(A+\frac{B}{2}\right)\beta^{\frac32}\varepsilon^3\left\|\partial_{xxx}^3 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\\
&\quad+\left(1-2B-\frac{C_0D}{B}\right)\varepsilon\left\|u_{\varepsilon,\beta}(t,\cdot)\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\\
&\quad+\left(A-\frac{A^2}{2}-C_0D\right)\varepsilon^3\left\|\partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2\\
&\le C_0\varepsilon\left\|\partial_x u_{\varepsilon,\beta}(t,\cdot)\right\|_{L^2(\mathbb{R})}^2.
\end{aligned}\tag{57}$$

We search A, B such that

$$\begin{cases} A^2 - 2A + 2C_0 D < 0, \\ 2B^2 - B + C_0 D < 0. \end{cases} \tag{58}$$

The first inequality of (58) admits solution if

$$D < \frac{1}{2C_0}. \tag{59}$$

The second inequality of (58) admits solution if

$$D < \frac{1}{8C_0}. \tag{60}$$

It follows from (59) and (60) that

$$D < \min\{\frac{1}{2C_0}, \frac{1}{8C_0}\} = \frac{1}{8C_0}. \tag{61}$$

Then, from (58) and (61), there exist $0 < A_1 < A_2$, $0 < B_1 < B_2$ such that for every

$$A_1 < A < A_2, \quad B_1 < B < B_2, \tag{62}$$

(58) holds. By (49), (52), (62), and an integration on $(0,t)$ of (57), we get

$$\begin{aligned} &\frac{1}{4} \left\| u_{\varepsilon,\beta}(t,\cdot) \right\|_{L^4(\mathbb{R})}^4 + \frac{A\varepsilon^2}{2} \left\| \partial_x u_{\varepsilon,\beta}(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{B\beta^{\frac{3}{2}}\varepsilon^2}{2} \left\| \partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 \\ &\quad + \frac{B\beta^3\varepsilon^3}{4} \int_0^t \left\| \partial_{xxxx}^4 u_{\varepsilon,\beta}(s,\cdot) \right\|_{L^2(\mathbb{R})}^2 ds + K_1 \beta^{\frac{3}{2}} \varepsilon^3 \int_0^t \left\| \partial_{xxx}^3 u_{\varepsilon,\beta}(s,\cdot) \right\|_{L^2(\mathbb{R})}^2 ds \\ &\quad + K_2 \varepsilon \int_0^t \left\| u_{\varepsilon,\beta}(s,\cdot) \partial_x u_{\varepsilon,\beta}(s,\cdot) \right\|_{L^2(\mathbb{R})}^2 ds + K_3 \varepsilon^3 \int_0^t \left\| \partial_{xx}^2 u_{\varepsilon,\beta}(s,\cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq C_0, \end{aligned}$$

for some K_1, K_2, $K_3 > 0$. Hence,

$$\begin{aligned} \left\| u_{\varepsilon,\beta}(t,\cdot) \right\|_{L^4(\mathbb{R})} \leq & C_0, \\ \varepsilon \left\| \partial_x u_{\varepsilon,\beta}(t,\cdot) \right\|_{L^2(\mathbb{R})} \leq & C_0, \\ \beta^{\frac{3}{4}} \varepsilon \left\| \partial_{xx}^2 u_{\varepsilon,\beta}(t,\cdot) \right\|_{L^2(\mathbb{R})} \leq & C_0, \\ \beta^3 \varepsilon^3 \int_0^t \left\| \partial_{xxxx}^4 u_{\varepsilon,\beta}(s,\cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq & C_0, \\ \beta^{\frac{3}{2}} \varepsilon^3 \int_0^t \left\| \partial_{xxx}^3 u_{\varepsilon,\beta}(s,\cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq & C_0, \\ \varepsilon \int_0^t \left\| u_{\varepsilon,\beta}(s,\cdot) \partial_x u_{\varepsilon,\beta}(s,\cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq & C_0, \\ \varepsilon^3 \int_0^t \left\| \partial_{xx}^2 u_{\varepsilon,\beta}(s,\cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq & C_0, \end{aligned}$$

for every $0 < t < T$. □

We are ready for the proof of Theorem B.1

Proof of Theorem B.1. Let us consider a compactly supported entropy–entropy flux pair (η, q). Multiplying (21) by $\eta'(u_{\varepsilon,\beta})$, we have

$$\begin{aligned} \partial_t \eta(u_{\varepsilon,\beta}) + \partial_x q(u_{\varepsilon,\beta}) = & \varepsilon \eta'(u_{\varepsilon,\beta}) \partial_{xx}^2 u_{\varepsilon,\beta} - \beta^{\frac{3}{2}} \varepsilon \eta'(u_{\varepsilon,\beta}) \partial_{xxxx}^4 u_{\varepsilon,\beta} \\ & - \beta^2 \eta'(u_{\varepsilon,\beta}) \partial_{xxxxx}^5 u_{\varepsilon,\beta} - \beta \eta'(u_{\varepsilon,\beta}) \partial_{xxx}^3 u_{\varepsilon,\beta} \end{aligned}$$

$$= I_{1,\varepsilon,\beta} + I_{2,\varepsilon,\beta} + I_{3,\varepsilon,\beta} + I_{4,\varepsilon,\beta} + I_{5,\varepsilon,\beta} + I_{6,\varepsilon,\beta} + I_{7,\varepsilon,\beta} + I_{8,\varepsilon,\beta},$$

where $I_{1,\varepsilon,\beta}$, $I_{2,\varepsilon,\beta}$, $I_{3,\varepsilon,\beta}$, $I_{4,\varepsilon,\beta}$, $I_{5,\varepsilon,\beta}$, $I_{6,\varepsilon,\beta}$, $I_{7,\varepsilon,\beta}$, $I_{8,\varepsilon,\beta}$ are defined in Theorem A.1.

Fix $T > 0$. Arguing as in [4, Theorem B.1], we have $I_{1,\varepsilon,\beta} \to 0$ in $H^{-1}((0,T) \times \mathbb{R})$, $\{I_{1,\varepsilon,\beta}\}_{\varepsilon,\beta>0}$ is bounded in $L^1((0,T) \times \mathbb{R})$, $I_{3,\varepsilon,\beta} \to 0$ in $H^{-1}((0,T) \times \mathbb{R})$, $I_{4,\varepsilon,\beta} \to 0$ in $L^1((0,T) \times \mathbb{R})$, $I_{5,\varepsilon,\beta} \to 0$ in $H^{-1}((0,T) \times \mathbb{R})$, $I_{6,\varepsilon,\beta} \to 0$ in $L^1((0,T) \times \mathbb{R})$. Due to (53), Lemmas 49, B.3, and the Hölder inequality, $I_{7,\varepsilon,\beta} \to 0$ in $H^{-1}((0,T) \times \mathbb{R})$, and $I_{8,\varepsilon,\beta} \to 0$ in $L^1((0,T) \times \mathbb{R})$. Arguing as in [4, Theorem B.1], the proof is concluded. □

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