MORREY SPACES NORMS AND CRITERIA FOR BLOWUP IN CHEMOTAXIS MODELS

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Abstract. Two-dimensional Keller–Segel models for the chemotaxis with fractional (anomalous) diffusion are considered. Criteria for blowup of solutions in terms of suitable Morrey spaces norms are derived. Similarly, a criterion for blowup of solutions in terms of the radial initial concentrations, related to suitable Morrey spaces norms, is shown for radially symmetric solutions of chemotaxis in several dimensions. Those conditions are, in a sense, complementary to the ones guaranteeing the global-in-time existence of solutions.

1. **Introduction.** We consider in this paper the following versions of the parabolic-elliptic Keller–Segel model of chemotaxis in $d \geq 2$ space dimensions

$$u_t + (-\Delta)^{\alpha/2} u + \nabla \cdot (u \nabla v) \;=\; 0, \quad x \in \mathbb{R}^d, \ t > 0, \tag{1}$$

$$\Delta v + u \;=\; 0, \tag{2}$$

supplemented with the initial condition

$$u(x, 0) = u_0(x). \tag{3}$$

Here the unknown variables $u = u(x, t)$ and $v = v(x, t)$ denote the density of the population and the density of the chemical secreted by the microorganisms, respectively. The diffusion operator is described either by the usual Laplacian with $\alpha = 2$ ($d \geq 3$) or by a fractional power of the Laplacian $(-\Delta)^{\alpha/2}$ with $\alpha \in (0, 2)$ (when $d = 2$). The system (1)–(2) with $\alpha \in (0, 2)$ can be interpreted as a nonlocal transport equation with nonlocal diffusion, cf. [7].

We choose those particular examples of the system (1)–(2) to show the role of the size of the initial data (3) (measured in terms of the quantities close to various Morrey space norms) on the blowup versus global-in-time existence of solutions to (1)–(3).

The initial data are nonnegative functions $u_0 \in L^1(\mathbb{R}^d)$ of the total mass

$$M = \int u_0(x) \, \mathrm{d}x, \tag{4}$$

radially symmetric when $d \geq 3$.

Recall that the homogeneous Morrey space $M^p(\mathbb{R}^d)$, $1 < p < \infty$, is defined as the space of all locally integrable functions that satisfy

$$\|f\|_{M^p} = \sup_{x_0 \in \mathbb{R}^d,\, R>0} R^{d(1/p-1)} \int\limits_{\{|y-x_0| \le R\}} |f(y)|\, \mathrm{d}y < \infty. \tag{5}$$

Many previous works have dealt with the existence of global-in-time solutions with small data in critical Morrey spaces, i.e. those which are scale-invariant under a natural scaling of the chemotaxis model, cf. e.g. [1] and [11]. Our criteria for a blowup of solutions with large concentration in [5] can be expressed using Morrey space norms $M^{2/\alpha}(\mathbb{R}^2)$ (see Remark 1 below for more details), and the size of such a norm is critical for the global-in-time existence versus finite time blowup. The analogous question for radially symmetric solutions of the d-dimensional Keller–Segel model with $d \ge 3$ in $M^{d/2}(\mathbb{R}^d)$ has been recently studied in [6].

We begin with that case improving the result [6, Th. 1.1]. Here we formulate some new sufficient conditions for blowup of solutions of (1)–(2).

Theorem 1.1 (Blowup of radial solutions with large concentration). *If $d \ge 2$, $\alpha = 2$, $u_0 \in L^1(\mathbb{R}^d)$ is a radially symmetric function and*

$$R^{2-d} \int\limits_{\{|x| \le R\}} \psi\left(\frac{x}{R}\right) u_0(x)\, \mathrm{d}x > \frac{C_d}{d}\sigma_d,$$

for $\psi(x) = \left(1 - |x|^2\right)_+^2$ and some $R > 0$, then the solution u of the problem (1)–(3) blows up in a finite time. Here, as usual, $\sigma_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ denotes the area of the unit sphere in $\mathbb{R}^d$, $C_d > 0$ is a constant which depends on the dimension only, and $\limsup_{d \to \infty} C_d \le 16$.

Thus, we show in the present work that the radial concentration of data is the critical quantity for the finite time blowup of nonnegative radial solutions of (1)–(3). Here, we define *the radial concentration* by

$$\|u_0\| \equiv \sup_{R>0} R^{2-d} \int\limits_{\{|x| \le R\}} \psi\left(\frac{x}{R}\right) u_0(x)\, \mathrm{d}x \tag{6}$$

with a fixed radial nonnegative function ψ piecewise $\mathcal{C}^2$, supported on the unit ball such that $\psi(0) = 1$. Clearly, those quantities for such weight functions ψ are comparable, so we fix in the following $\psi(x) = \left(1 - |x|^2\right)_+^2$, see (10). Of course, for $d \ge 3$ and $p = d/2$, the norm (5) in $M^{d/2}(\mathbb{R}^d)$ (relevant to the theory of existence of local-in-time solutions) dominates the radial concentration (6):

$$\|u_0\|_{M^{d/2}} \ge R^{2-d} \int\limits_{\{|x| \le R\}} u_0(x)\, \mathrm{d}x$$

but, in fact, for radially symmetric u_0 the quantities $\|u_0\|_{M^{d/2}}$ and $\|u_0\|$ are equivalent, see Remark 4.

This "criticality" implies that for the initial data with small $\|u_0\|$ solutions exist indefinitely in time (cf. [1], [11]), while the result of Theorem 1.1 shows that for sufficiently large $\|u_0\|$ regular solutions cease to exist in a finite time. This result seems be to new for $d \ge 3$ although related criteria appeared in, e.g., [2] and [4]. They have been, however, formulated in terms of "global quantities" like the second

moment $\int |x|^2 u_0(x)\,\mathrm{d}x$ while (6) is a local quantity, and its definition does not require supplementary properties of u_0 such as the assumption $\int |x|^2 u_0(x)\,\mathrm{d}x < \infty$.

In fact, this astonishingly simple proof, involving an analysis of a local moment of the solution, extends to the two-dimensional case ($x \in \mathbb{R}^2$) of arbitrary (not necessarily radially symmetric) nonnegative solutions, cf. [12], [10] for a similar argument and Theorem 1.2 below.

Theorem 1.2 (Blowup in two-dimensional models with fractional diffusion). *Consider a local-in-time nonnegative solution of problem (1)–(3) with a nonnegative function u_0 on $\mathbb{R}^2$.*

(i) If $\alpha = 2$ (the classical scaling invariant Keller–Segel model), then for each $M > 8\pi$ the solution u blows up in a finite time.

(ii) Let $\alpha \in (0,2)$ (the Keller–Segel model with fractional diffusion). If there exist $x_0 \in \mathbb{R}^2$ and $R > 0$ such that

$$R^{\alpha-2} \int\limits_{\{|y-x_0|\le R\}} u_0(y)\,\mathrm{d}y > C_\alpha \ \ \text{and} \ \ \int\limits_{\{|y-x_0|>R\}} u_0(y)\,\mathrm{d}y < \nu, \qquad (7)$$

for some explicit constants: small $\nu > 0$ and large $C_\alpha > 0$, then the solution u blows up in a finite time.

(iii) (radial blowup for the Keller–Segel model with fractional diffusion). If $\alpha \in [1,2)$ then there exists $\widetilde{C}_\alpha$ such that for each radially symmetric initial data u_0 satisfying

$$R^{\alpha-2} \int\limits_{|y|\le R} u_0(y)\,\mathrm{d}y > \widetilde{C}_\alpha \qquad (8)$$

for some $R > 0$, the solution u blows up in a finite time.

Remark 1. The cases (ii) and (iii): $1 \le \alpha < 2$. The first condition in (7) is equivalent to a sufficiently large Morrey norm of u_0 in the space $M^{2/\alpha}(\mathbb{R}^2)$. Indeed, we have the obvious relations

$$\|u_0\|_{M^{2/\alpha}} \ge R^{\alpha-2} \int\limits_{\{|y-x_0|\le R\}} u_0(y)\,\mathrm{d}y$$

for every x_0 and $R > 0$, but also there is $x_0 \in \mathbb{R}^2$ and $R > 0$ such that

$$\|u_0\|_{M^{2/\alpha}} \le 2R^{\alpha-2} \int\limits_{\{|y-x_0|\le R\}} u_0(y)\,\mathrm{d}y.$$

Thus, our blowup condition in terms of the Morrey norm $M^{2/\alpha}(\mathbb{R}^2)$ seems to be new, and in a sense complementary to that guaranteeing the global-in-time existence of solutions, where smallness of initial conditions in the $M^{2/\alpha}$-Morrey norm has to be imposed, cf. prototypes of such results in [1, Theorem 1], [4, Remark 2.7] and [11, Theorem 2]. Similarly, (8) implies that the Morrey norm $\|u_0\|_{M^{2/\alpha}}$ is large.

Remark 2. A natural scaling for system (1)–(2)

$$u_\lambda(x,t) = \lambda^\alpha u(\lambda x, \lambda^\alpha t),$$

leads to the equality $\int u_\lambda\,\mathrm{d}x = \lambda^{\alpha-2} \int u\,\mathrm{d}x$. In particular, when $\alpha \in (0,2)$, mass of rescaled solution u_λ can be chosen arbitrarily with a suitable $\lambda > 0$. Thus, the first part of the condition in Theorem 1.2(ii) is insensitive to the actual value of M, so w.l.o.g. we may suppose that $M = 1$. The second part of the condition (7) is

not scale invariant, however, we believe that this assumption is not necessary for the conclusion in Theorem 1.2(ii). In fact, one can prove it for α close to 2 by an inspection of methods from [1, 6].

Remark 3. In view of results in e.g. [8], assumptions on u_0 in Theorem 1.2 can be relaxed; it suffices that u_0 is a finite nonnegative measure satisfying some "small atoms" assumption.

Notation. In the sequel, $\|\cdot\|_q$ denotes the usual $L^q(\mathbb{R}^d)$ norm. The integrals with no integration limits are understood as over the whole space $\mathbb{R}^d$. The letter C denotes various constants which may vary from line to line but they are independent of solutions.

2. **Proof of blowup of radial solutions in d dimensions.** We begin with an elementary observation which will be used in the proof of Theorem 1.1.

Lemma 2.1. *Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ be a radially symmetric function, such that $v = E_d * u$ with $E_d(x) = \frac{1}{(d-2)\sigma_d}|x|^{2-d}$ for $d \geq 3$, solves the Poisson equation $\Delta v + u = 0$. Then*

$$\nabla v(x) \cdot x = -\frac{1}{\sigma_d}|x|^{2-d} \int\limits_{\{|y| \leq |x|\}} u(y)\,\mathrm{d}y.$$

Proof. By the Gauss formula, we have for the distribution function M of u

$$M(R) \equiv \int\limits_{\{|y| \leq R\}} u(y)\,\mathrm{d}y = -\int\limits_{\{|y|=R\}} \nabla v(y) \cdot \frac{y}{|y|}\,\mathrm{d}S.$$

Thus, for the radial function $\nabla v(x) \cdot \frac{x}{|x|}$ and $|x| = R$, we obtain the identity

$$\nabla v(x) \cdot x = \frac{1}{\sigma_d}R^{2-d} \int\limits_{\{|y|=R\}} \nabla v(y) \cdot \frac{y}{|y|}\,\mathrm{d}S = -\frac{1}{\sigma_d}R^{2-d}M(R).$$

$\square$

Proof of Theorem 1.1. We will derive a differential inequality for a local moment of the solution

$$w_R(t) = \int \psi_R(x)u(x,t)\,\mathrm{d}x \tag{9}$$

with the scaled weight function ψ_R supported on the ball $\{|x| \leq R\}$

$$\psi(x) = (1 - |x|^2)^2 \mathbb{1}_{\{|x| \leq 1\}} \quad \text{and} \quad \psi_R(x) = \psi\left(\frac{x}{R}\right) \quad \text{with} \quad R > 0. \tag{10}$$

The function $\psi \in \mathcal{C}^1(\mathbb{R}^d)$ has piecewise continuous and bounded second derivatives:

$$\nabla \psi(x) = -4x(1 - |x|^2)\mathbb{1}_{\{|x| \leq 1\}}, \tag{11}$$

$$\Delta \psi(x) = (-4d + 4(d+2)|x|^2)\mathbb{1}_{\{|x| \leq 1\}}. \tag{12}$$

Now, using equation (1), integrations by parts and applying relations (10)–(12), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}w_R(t) = \int \Delta\psi_R(x)u(x,t)\,\mathrm{d}x + \int u(x,t)\nabla v(x,t) \cdot \nabla\psi_R(x)\,\mathrm{d}x$$

$$= R^{-2}\left(\int\limits_{\{|x| \leq R\}} \left(-4d + 4(d+2)\frac{|x|^2}{R^2}\right)u(x,t)\,\mathrm{d}x \right. \tag{13}$$

$$-4 \int\limits_{\{|x|\leq R\}} u(x,t)\big(\nabla v(x,t)\cdot x\big)\left(1-\frac{|x|^2}{R^2}\right)\mathrm{d}x\bigg).$$

Thus, using the assumption on radial symmetry of u, the useful relation

$$u(x) = \frac{1}{\sigma_d}R^{1-d}M'(R) \quad \text{for} \quad |x| = R,$$

and Lemma 2.1 we get

$$
\frac{1}{4}R^2\frac{\mathrm{d}}{\mathrm{d}t}w_R(t) \;=\; R\int\limits_0^1 M'(Rr,t)(-d+(d+2)r^2)\,\mathrm{d}r
$$

$$
+\frac{R^{2-d}}{\sigma_d}R\int\limits_0^1 M'(Rr,t)M(Rr,t)r^{2-d}(1-r^2)\,\mathrm{d}r. \qquad (14)
$$

Notice also the following (obvious) relations

$$
M(R,t) = \int\limits_{\{|y|\leq R\}} u(y,t)\,\mathrm{d}y = R\int\limits_0^1 M'(Rr,t)\,\mathrm{d}r
$$

$$
\geq \quad w_R(t) = R\int\limits_0^1 M'(Rr,t)(1-r^2)^2\,\mathrm{d}r = 4\int\limits_0^1 M(Rr,t)r(1-r^2)\,\mathrm{d}r. \quad (15)
$$

Now, let $\lambda \in \mathbb{R}$ be a number to be fixed further. Since

$$
w_R(t) = 4R\int\limits_0^1 M(Rr,t)r(1-r^2)\,\mathrm{d}r,
$$

we may rewrite equation (14) after the integration by parts and use (15) as follows

$$
\frac{1}{4}R^2\frac{\mathrm{d}}{\mathrm{d}t}w_R(t) = 2\int\limits_0^1 M'(Rr,t)r^2\,\mathrm{d}r - 2d\int\limits_0^1 M(Rr,t)r\,\mathrm{d}r
$$

$$
-\lambda\left(4\int\limits_0^1 M(Rr,t)r(1-r^2)\,\mathrm{d}r - w_R(t)\right)
$$

$$
+\frac{R^{2-d}}{2\sigma_d}\int\limits_0^1 M(Rr,t)^2 r^{1-d}\big((d-2)-(d-4)r^2\big)\,\mathrm{d}r
$$

$$
\geq \lambda w_R(t)
$$

$$
+\frac{R^{2-d}}{2\sigma_d}\int\limits_0^1 \big((d-2)-(d-4)r^2\big)
$$

$$
\times\left(M(Rr,t)r^{\frac{1-d}{2}} - \frac{2\sigma_d}{R^{2-d}}\frac{d+2\lambda(1-r^2)}{(d-2)-(d-4)r^2}r^{\frac{1+d}{2}}\right)^2\mathrm{d}r
$$

$$
-\frac{2\sigma_d}{R^{2-d}}\int\limits_0^1 \frac{(d+2\lambda(1-r^2))^2}{(d-2)-(d-4)r^2}r^{d+1}\,\mathrm{d}r.
$$

Thus, we may estimate the right-hand side as

$$\frac{1}{4}R^2\frac{\mathrm{d}}{\mathrm{d}t}w_R(t) \geq \lambda w_R(t) - \frac{2\sigma_d}{R^{2-d}}\int_0^1 \frac{(d+2\lambda(1-r^2))^2}{(d-2)-(d-4)r^2}r^{d+1}\,\mathrm{d}r. \qquad (16)$$

By direct calculations, the function

$$F(\lambda) \equiv \frac{1}{\lambda}\int_0^1 \frac{(d+2\lambda(1-r^2))^2}{(d-2)-(d-4)r^2}r^{d+1}\,\mathrm{d}r,$$

satisfies

$$\min_{\lambda>0} F(\lambda) = F\left(\sqrt{\frac{C}{A}}\right) = 2\sqrt{AC} + B, \quad \text{where} \quad A = 4\int_0^1 \frac{(1-r^2)^2}{(d-2)-(d-4)r^2}r^{d+1}\,\mathrm{d}r,$$

$$B = 4d\int_0^1 \frac{(1-r^2)}{(d-2)-(d-4)r^2}r^{d+1}\,\mathrm{d}r, \quad C = d^2\int_0^1 \frac{r^{d+1}}{(d-2)-(d-4)r^2}\,\mathrm{d}r.$$

Thus, it is clear from (16) that if

$$R^{2-d}w_R(0) > 2\sigma_d\left(2\sqrt{AC} + B\right), \qquad (17)$$

then $\frac{1}{4}R^2\frac{\mathrm{d}}{\mathrm{d}t}w_R(t) \geq \delta$ for some $\delta > 0$ since $w_R(t)$ increases. As a consequence, the function $w_R(t)$ becomes greater than $M = \int u(x,t)\,\mathrm{d}x$ in a finite time which is a contradiction with the existence of nonnegative, mass conserving solutions.

Finally, let us express condition (17) in a more explicit way. By the Cauchy–Schwarz inequality we see that $B \leq 2\sqrt{AC}$. Evidently, for $d \geq 3$ we have

$$A \leq \frac{4}{d-2}\int_0^1 (1-r^2)r^{d+1}\,\mathrm{d}r = \frac{8}{(d-2)(d+2)(d+4)}$$

and

$$C \leq d^2\frac{1}{2}\int_0^1 r^{d+1}\,\mathrm{d}r = \frac{d^2}{2(d+2)}.$$

For an estimate of the constant C_d for $d \geq 3$ note that (17) is satisfied if, e.g.,

$$R^{2-d}w_R(0) > 16\frac{d}{d+2}\sqrt{\frac{1}{(d-2)(d+4)}}\sigma_d. \qquad (18)$$

Thus, the condition (18) follows for large d if $\|u_0\| > \frac{C_d}{d}\sigma_d$ with a $C_d = 16$, since the conditions $\|u_0\| > \frac{C_d}{d}\sigma_d$ and $R^{2-d}w_R(0) > \frac{C_d}{d}\sigma_d$ for a particular $R > 0$ are equivalent. $\qquad\square$

Remark 4. Note that for nonnegative locally integrable functions ω, each $R > 0$ and $s \in (0,1)$ we have

$$\int \psi_R(x)\omega(x)\,\mathrm{d}x \geq \int_{|x|\leq R\sqrt{1-s}} (1-(1-s))^2\omega(x)\,\mathrm{d}x = s^2\int_{|x|\leq R\sqrt{1-s}} \omega(x)\,\mathrm{d}x,$$

and

$$\max_{s\in[0,1]} s^2(1-s)^{d/2-1} = \left(\frac{4}{d+2}\right)^2 \left(\frac{d-2}{d+2}\right)^{d/2-1} \equiv H_d,$$

so that $\|\omega\| \geq H_d L$ if $|\omega|_{M^{d/2}} > L$ because

$$(R\sqrt{1-s})^{2-d} \int_{|x|\leq R\sqrt{1-s}} \omega(x)\,\mathrm{d}x > L$$

holds for some $R > 0$ and $s = \frac{4}{d+2}$. Therefore,

$$|u_0|_{M^{d/2}} > \frac{C_d}{d}\sigma_d H_d^{-1}$$

is a sufficient condition for the occurrence of the blowup for radial initial data $u_0 \geq 0$ expressed in terms of the Morrey norms. Note that

$$\limsup_{d\to\infty} \left(\frac{C_d}{d}\sigma_d H_d^{-1}\right) / \left(\mathrm{e}^2 d\sigma_d\right) \leq 1 \qquad \text{as} \quad d\to\infty.$$

Remark 5. One may improve the conditions (17) and (18) for blowup (i.e. diminish the numerical value of C_d) by solving the following rescaled variational problem

$$\inf \mathcal{F}[\varphi], \quad \text{where}$$

$$\mathcal{F}[\varphi] = 2\int_0^1 \varphi'(r)r^2\,\mathrm{d}r - 2d\int_0^1 \varphi(r)r\,\mathrm{d}r + \frac{1}{2}\int_0^1 \varphi(r)^2 r^{1-d}((d-2)-(d-4)r^2)\,\mathrm{d}r \tag{19}$$

with the constraints

$$\varphi(0) = 0, \quad \varphi(1) = \widetilde{A}, \quad \int_0^1 \varphi(r)r(1-r^2)\,\mathrm{d}r = \widetilde{B}, \quad \varphi'(r) \geq 0. \tag{20}$$

Indeed, by relations (14)–(15), we have got

$$\frac{1}{4}R^2\frac{\mathrm{d}}{\mathrm{d}t}w_R(t) \geq \sigma_d R^{d-2}\inf \mathcal{F}[\varphi],$$

where φ is is chosen such a way to have relations (20), namely, $R^{d-2}\sigma_d \widetilde{A} = M$ and $R^{d-2}\sigma_d \widetilde{B} = w_R(t)$. The monotonicity of φ reflects the increasing property of $M(.,t)$, and $\widetilde{B} \in \left(\frac{1}{6\sqrt{3}}\widetilde{A}, \frac{1}{4}\widetilde{A}\right)$ is here an obvious restriction.

To solve the variational problem (19)–(20), let $E_\lambda(r) = \frac{d+2\lambda(1-r^2)}{(d-2)-(d-4)r^2}r^d$ with a parameter $\lambda \in \mathbb{R}$. Whenever E_λ is not an increasing function, the constraint $\varphi'(r) \geq 0$ leads to the consideration of "broken" extremals, i.e.

$$\varphi_\lambda(r) = \begin{cases} 0 & \text{if } 0 \leq r \leq r_1, \\ E_\lambda(r-r_1) & \text{if } r_1 < r < r_2, \\ \widetilde{A} & \text{if } r_2 \leq r \leq 1, \end{cases}$$

where E_λ is nondecreasing on $[r_1, r_2]$, $\widetilde{A} = E_\lambda(r_2 - r_1)$, and $0 \leq r_1 < r_2 \leq 1$ are chosen so that $\int_0^1 \varphi_\lambda(r)r(1-r^2)\,\mathrm{d}r = \widetilde{B}$. One can show, that the functional $\mathcal{F}[\varphi]$ attains its minimum for those φ_λ.

3. **Blowup of solutions for fractional diffusion Keller–Segel model.** In this section we prove Theorem 1.2 using the method of truncated moments which is reminiscent of that in the papers [12, 10]. The "bump" function ψ defined in (10) satisfies (cf. (12))

$$\Delta\psi(x) = (-8 + 16|x|^2) \geq -8\psi(x) \geq -8 \text{ for } |x| < 1, \tag{21}$$

and ψ is strictly concave in a neighbourhood of $x = 0$. For the readers' convenience, we recall now auxiliary lemmata from [5].

Lemma 3.1. *For each $\varepsilon \in \left(0, \frac{1}{\sqrt{3}}\right)$, the function ψ defined in (10) is strictly concave for all $|x| \leq \varepsilon$. More precisely, ψ satisfies*

$$H\psi \leq -\theta(\varepsilon)I \tag{22}$$

for all $|x| \leq \varepsilon$, where $H\psi$ is the Hessian matrix of second derivatives of ψ, $\theta(\varepsilon) = 4\left(1 - 3\varepsilon^2\right)$, and I is the identity matrix. In particular, we have

$$\theta(\varepsilon) \nearrow 4 \quad as \quad \varepsilon \searrow 0. \tag{23}$$

Proof. For every $\xi \in \mathbb{R}^2$ the following identity holds

$$\xi \cdot H\psi\,\xi = 4\left(-|\xi|^2\left(1 - |x|^2\right) + 2(x \cdot \xi)^2\right).$$

Thus, by the Cauchy–Schwarz inequality, we have $\xi \cdot H\psi\,\xi \leq 4|\xi|^2\left(3|x|^2 - 1\right)$. $\quad\square$

Next, we recall a well-known property of concave functions and we formulate a crucial inequality in our proof of the blowup result.

Lemma 3.2. *For every function $\Psi : \mathbb{R}^2 \to \mathbb{R}$ which is strictly concave on a convex domain $\Omega \subset \mathbb{R}^2$ we have for all $x, y \in \Omega$*

$$(x - y) \cdot (\nabla\Psi(x) - \nabla\Psi(y)) \leq -\theta|x - y|^2, \tag{24}$$

where $\theta > 0$ is the constant of strict concavity of Ψ on Ω, i.e. satisfying $H\Psi \leq -\theta\,I$.

In particular, for the fundamental solution $E_2(x)$ of the Laplacian on $\mathbb{R}^2$ which satisfies $\nabla E_2(x) = -\frac{1}{2\pi}\frac{x}{|x|^2}$, and a strictly concave function Ψ we have for all x, y on the domain of the strict concavity of Ψ

$$\nabla E_2(x - y) \cdot (\nabla\Psi(x) - \nabla\Psi(y)) \geq \frac{\theta}{2\pi}. \tag{25}$$

Proof. By the concavity, we obtain

$$\Psi(x) \leq \Psi(y) + \nabla\Psi(y) \cdot (x - y) - \frac{\theta}{2!}|x - y|^2.$$

Summing this inequality with its symmetrized version (with x, y interchanged) leads to the claim. $\quad\square$

We have the following scaling property of the fractional Laplacian

$$(-\Delta)^{\alpha/2}\psi_R(x) = R^{-\alpha}\big((-\Delta)^{\alpha/2}\psi\big)_R, \tag{26}$$

and we notice the following boundedness property of $(-\Delta)^{\alpha/2}\psi$.

Lemma 3.3. *For every $\alpha \in (0, 2]$ there exists a constant $k_\alpha > 0$ such that*

$$\left|(-\Delta)^{\alpha/2}\psi(x)\right| \leq k_\alpha. \tag{27}$$

In particular, $k_2 = 8$ by (21). Moreover, $(-\Delta)^{\alpha/2}\psi(x) \leq 0$ for $|x| \geq 1$.

Proof. For $\alpha = 2$, this is an obvious consequence of the explicit form of ψ, hence we assume $\alpha \in (0, 2)$.

To show estimate (27) for $\alpha \in (0, 2)$, it suffices to use the following well-known representation of the fractional Laplacian with $\alpha \in (0, 2)$

$$(-\Delta)^{\alpha/2} \psi(x) = -c_\alpha \, \mathrm{P.V.} \int \frac{\psi(x+y) - \psi(x)}{|y|^{2+\alpha}} \, \mathrm{d}y$$

for certain explicit constant $c_\alpha > 0$. Now, using the Taylor formula together with the fact that $\psi, D^2\psi \in L^\infty(\mathbb{R}^2)$, we immediately obtain that the integral on the right-hand side is finite and uniformly bounded in $x \in \mathbb{R}^2$. Since $\psi(x) \geq 0$, $\psi \not\equiv 0$ and $\psi(x) = 0$ for $|x| \geq 1$, we have for all $|x| \geq 1$

$$(-\Delta)^{\alpha/2} \psi(x) = -c_\alpha \, \mathrm{P.V.} \int \frac{\psi(x+y)}{|y|^{2+\alpha}} \, \mathrm{d}y < 0. \tag{28}$$

$\square$

Now we prove our second blowup result.

Proof of Theorem 1.2. We consider again the quantity (9). Let

$$M_R(t) \equiv \int_{\{|x| \leq R\}} u(x, t) \, \mathrm{d}x \geq w_R(t) \tag{29}$$

denote mass of the distribution u contained in the ball $\{|x| \leq R\}$ at time t. Now, using equation (1) we determine the evolution of $w_R(t)$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} w_R(t) &= -\int (-\Delta)^{\alpha/2} u(x, t) \psi_R(x) \, \mathrm{d}x + \int u(x, t) \nabla v(x, t) \cdot \nabla \psi_R(x) \, \mathrm{d}x \\
&= -\int u(x, t)(-\Delta)^{\alpha/2} \psi_R(x) \, \mathrm{d}x \\
&\quad + \frac{1}{2} \iint u(x, t) u(y, t) \nabla E_2(x - y) \cdot \big(\nabla \psi_R(x) - \nabla \psi_R(y)\big) \, \mathrm{d}y \, \mathrm{d}x,
\end{aligned} \tag{30}$$

where we applied the formula $v = E_2 * u$, and the last expression follows by the symmetrization of the double integral: $x \mapsto y$, $y \mapsto x$. Since $u(x, t) \geq 0$, by the scaling relation (26) and Lemma 3.3, we obtain

$$-\int u(x, t)(-\Delta)^{\alpha/2} \psi_R(x) \, \mathrm{d}x \geq -R^{-\alpha} k_\alpha \int_{\{|x| \leq R\}} u(x, t) \, \mathrm{d}x. \tag{31}$$

Now, let $\varepsilon \in \left(0, \frac{1}{\sqrt{3}}\right)$. By Lemma 3.1, the weight function ψ_R in (10) is concave for $|x| \leq \varepsilon R$ with a concavity constant $\theta = R^{-2}\theta(\varepsilon)$. Thus, by Lemma 3.2, we have

$$\nabla E_2(x - y) \cdot (\nabla \psi_R(x) - \nabla \psi_R(y)) \geq R^{-2} \frac{\theta(\varepsilon)}{2\pi}$$

for $|x|, |y| < \varepsilon R$. Hence, the bilinear term on the right-hand side of (30) satisfies

$$\begin{aligned}
\frac{1}{2} \iint u(x, t) u(y, t) \nabla E_2(x - y) \cdot &\big(\nabla \psi_R(x) - \nabla \psi_R(y)\big) \, \mathrm{d}y \, \mathrm{d}x \\
&\geq R^{-2} \frac{\theta(\varepsilon)}{4\pi} \int_{\{|x| \leq \varepsilon R\}} \int_{\{|y| \leq \varepsilon R\}} u(x, t) u(y, t) \, \mathrm{d}y \, \mathrm{d}x + \frac{1}{2} J,
\end{aligned} \tag{32}$$

where the letter J denotes the integral

$$J = \iint_{\mathbb{R}^2 \times \mathbb{R}^2 \setminus (\{|x| < \varepsilon R\} \times \{|y| < \varepsilon R\})} u(x,t) u(y,t) \nabla E_2(x-y) \cdot (\nabla \psi_R(x) - \nabla \psi_R(y)) \, \mathrm{d}y \, \mathrm{d}x.$$

We estimate the first integral on the right-hand side of (32) in the following way

$$\int_{\{|x| \leq \varepsilon R\}} \int_{\{|y| \leq \varepsilon R\}} u(x,t) u(y,t) \, \mathrm{d}y \, \mathrm{d}x = \left(M_R(t) - \int_{\{\varepsilon R \leq |x| \leq R\}} u(x,t) \, \mathrm{d}x \right)^2$$

$$\geq M_R^2(t) - 2 M_R(t) \int_{\{\varepsilon R \leq |x| \leq R\}} u(x,t) \frac{1 - \psi_R(x)}{\inf_{\{|x| \geq \varepsilon R\}} \left(1 - \psi_R(x) \right)} \, \mathrm{d}x \tag{33}$$

$$\geq M_R(t)^2 - 2 C_\varepsilon M_R(t)(M - w_R(t)),$$

where $C_\varepsilon = \left(\inf_{\{|x| \geq \varepsilon R\}} \left(1 - \psi_R(x) \right) \right)^{-1} = \left(1 - (1 - \varepsilon^2)^2 \right)^{-1}$. Next, since we have the inclusion

$$\mathbb{R}^2 \times \mathbb{R}^2 \setminus \left(\{|x| < \varepsilon R\} \times \{|y| < \varepsilon R\} \right) \subset$$

$$\left(\{|x| < R\} \times \{|y| \geq \varepsilon R\} \right) \cup \left(\{|x| \geq \varepsilon R\} \times \{|y| < R\} \right) \cup \left(\{|x| \geq R\} \times \{|y| \geq R\} \right)$$

and the factor with $\nabla \psi_R$ vanishes on the set $\{|x| \geq R\} \times \{|y| \geq R\}$, we obtain immediately the estimate

$$|J| \leq 2CR^{-2} \int_{\{|x| \leq R\}} \int_{\{|y| \geq \varepsilon R\}} u(x,t) u(y,t) \frac{1 - \psi_R(y)}{\inf_{\{|y| \geq \varepsilon R\}} \left(1 - \psi_R(y) \right)} \, \mathrm{d}x \, \mathrm{d}y$$

$$\leq 2R^{-2} C C_\varepsilon M_R(t) \int u(y,t) \left(1 - \psi_R(y) \right) \mathrm{d}y$$

$$\leq 2R^{-2} C C_\varepsilon M_R(t)(M - w_R(t)), \tag{34}$$

where $C = \sup |z \cdot \nabla E_2(z)| \, \|D^2 \psi\|_\infty = \frac{1}{2\pi} \|D^2 \psi\|_\infty$. Finally, estimates (31)–(34) as well as inequality (29) applied to equation (30) lead to the inequalities

$$\frac{\mathrm{d}}{\mathrm{d}t} w_R(t) \geq R^{-\alpha} M_R(t) \left(-k_\alpha + \frac{\theta(\varepsilon)}{4\pi} R^{\alpha-2} M_R(t) + C(\varepsilon) R^{\alpha-2}(w_R(t) - M) \right)$$

$$\geq R^{-\alpha} w_R(t) \left(-k_\alpha + \frac{\theta(\varepsilon)}{4\pi} R^{\alpha-2} w_R(t) + C(\varepsilon) R^{\alpha-2}(w_R(t) - M) \right), \tag{35}$$

whenever the expression in the parentheses is nonnegative, with $C(\varepsilon) = 3CC_\varepsilon = 3C \left(1 - (1 - \varepsilon^2)^2 \right)^{-1}$.

Now, notice that the linear function of w_R in the parentheses on the right-hand side of (35) is monotone increasing. Thus, if at time $t = 0$ the right-hand side of (35) is positive, then $w_R(t)$ will increase indefinitely in time. Consequently, after time $T = \mathcal{O}\left(R^\alpha \left(\frac{M}{w_R(0)} - 1 \right) \right)$ the function $w_R(t)$ will become larger than the total mass M. This is a contradiction with the global-in-time existence of a nonnegative solution u since it conserves mass (4).

Now, let us analyze the cases when the right-hand side of inequality (35) is strictly positive.

Case (i). $\alpha = 2$. We recall that, by Lemma 3.3, $k_2 = 8$ holds. Thus, in view of (23), the quantity $\frac{\theta(\varepsilon)}{4\pi}$ is close to $\frac{1}{\pi}$ at the expense of taking sufficiently small $\varepsilon > 0$. Choosing $\varepsilon > 0$ small enough, we get blowup in the optimal range $M > 8\pi$. Indeed,

if $M > 8\pi$, then there exists $\varepsilon > 0$ small, and $R \geq R(\varepsilon) > 0$ sufficiently large so that $w_R(0)$ is sufficiently close to M and we have

$$-8 + \frac{1}{\pi} w_R(0) + C(\varepsilon)(w_R(0) - M) > 0.$$

Case (ii). In the case $\alpha < 2$, the blowup occurs if for some $R > 0$ the quantity

$$R^{\alpha-2} \int\limits_{\{|x| \leq R\}} u_0(x)\, \mathrm{d}x$$

is large enough, and simultaneously u_0 is well concentrated, i.e. $C(\varepsilon)(M - w_R(0))$ is small.

Case (iii). First, observe that for each $\alpha \in (0, 2]$ there exists $\ell_\alpha > 0$ such that

$$(-\Delta)^{\alpha/2}\psi \leq \ell_\alpha \psi.$$

Indeed, this is a consequence of the fact that $\sup_{|x|=1}(-\Delta)^{\alpha/2}\psi(x) < 0$ (see equation (28) and notice that $\psi \not\equiv 0$) and the continuity of $(-\Delta)^{\alpha/2}\psi$ on $\mathbb{R}^2$. Note also that $\ell_2 = 8$ is the optimal constant.

We infer from the computations in (13)–(14) that

$$\frac{\mathrm{d}}{\mathrm{d}t} w_R(t) \geq -\ell_\alpha R^{-\alpha} w_R(t) + 4 \frac{R^{-2}}{2\pi} \int\limits_0^1 M(Rr, t)^2 r\, \mathrm{d}r \tag{36}$$

$$\geq -\ell_\alpha R^{-\alpha} w_R(t) + \frac{3R^{-2}}{4\pi} w_R(t)^2$$

since by (15) we have

$$\left(4\int\limits_0^1 M(Rr, t) r(1 - r^2)\, \mathrm{d}r\right)^2 \leq 16 \int\limits_0^1 M(Rr, t)^2 r\, \mathrm{d}r \int\limits_0^1 r(1 - r^2)^2\, \mathrm{d}r$$

$$= \frac{8}{3} \int\limits_0^1 M(Rr, t)^2 r\, \mathrm{d}r.$$

Therefore, if $R^{\alpha-2} w_R(0) > \frac{4\pi}{3} \ell_\alpha$, then we have

$$\frac{\mathrm{d}}{\mathrm{d}t} w_R(t) \geq \delta > 0$$

for some δ. We arrive at a contradiction with existence of global nonnegative solutions as was in the proof of Theorem 1.1. It is worth noting that the blowup time obtained from the argument in (36) depends only on R and $w_R(0)$ unlike in the case (ii) of not necessarily radial solutions. $\square$

Remark 6. Results concerning the system (1)–(2) are generalized in [5] to the case of the parabolic-elliptic Keller–Segel model with the consumption of the chemoattractant effect, i.e. with (2) replaced by the equation

$$\Delta v - \gamma v + u = 0.$$

In such a case the Newtonian potential E_2 is replaced by the Bessel kernel K_γ of the operator $(-\Delta + \gamma)^{-1}$ on $\mathbb{R}^2$ satisfying the relation

$$\nabla K_\gamma(x) = -\frac{1}{2\pi} \frac{x}{|x|^2} g_\gamma(|x|).$$

with a decreasing smooth function g_γ such that $g_\gamma(0) = 1$, and $g_\gamma(|x|) \le Ce^{-\gamma^{1/2}|x|}$. For more general kernels leading to blowup of solutions, see also [9].

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