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UNIQUENESS OF SOLUTIONS TO A MATHEMATICAL MODEL DESCRIBING MOISTURE TRANSPORT IN CONCRETE MATERIALS

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Abstract. When dealing with concrete materials it is always a big issue how to deal with the moisture transport. Here, we consider a mathematical model for moisture transport, which is given as a system consisting of the diffusion equation for moisture and of the ordinary differential equation which describes a hysteresis operator. In [3] we already proved the existence of a time global solution of an initial boundary value problem of this system, however, the uniqueness is obtained only for one dimensional domains. The main purpose of this paper is to establish the uniqueness of a solution of our problem in three dimensional domains under the assumption of the smooth boundary and initial data.

1. **Introduction.** While studying concrete carbonation it is a crucial step to investigate the mathematical model describing moisture transport part, for instance, see [4, 7]. We already proposed a model for moisture transport in [1, 2]. Here, we aim to prove the uniqueness of solutions to this model, which is the following initial boundary value problem (P) for a parabolic-type equation including a hysteresis operator:

$$\frac{\partial u}{\partial t} - \mathrm{div}(g(u)\nabla u) = wf \quad \text{in } Q(T) := (0,T)\times\Omega, \tag{1.1}$$

$$\frac{\partial w}{\partial t} + \partial I(u; w) \ni 0 \quad \text{in } Q(T), \tag{1.2}$$

$$u = u_b \quad \text{on } S(T) := (0,T)\times\partial\Omega, \tag{1.3}$$

$$u(0) = u_0, \quad w(0) = w_0 \text{ in } \Omega. \tag{1.4}$$

Here, $0 < T < \infty$, Ω is a bounded domain in $\mathbf{R}^3$ with the smooth boundary $\partial\Omega$, and g is a given function in $C^1((0,\infty))$, f and u_b are given functions on $Q(T)$ and u_0

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and w_0 are given initial functions defined in Ω. The function I denotes the indicator function of the closed interval $[f_*(u), f^*(u)]$, that is,

$$I(u;w) = \begin{cases} 0 & \text{if } f_*(u) \le w \le f^*(u), \\ +\infty & \text{otherwise,} \end{cases}$$

where f^* and f_* are given functions on $\mathbf{R}$ with $f_* \le f^*$ on $\mathbf{R}$. Moreover, ∂I represents the subdifferential of I, namely,

$$\partial I(u;w) = \begin{cases} [0,\infty) & \text{if } w = f^*(u), \\ \{0\} & \text{if } f_*(u) < w < f^*(u), \\ (-\infty, 0] & \text{if } w = f_*(u). \end{cases}$$

The system $\{(1.1)–(1.2)\}$ is a mathematical model of moisture transport for the concrete carbonation phenomenon in three dimensions and was proposed in Aiki-Kumazaki [1, 2, 3]. Physically, u and w indicate the relative humidity and the degree of saturation, respectively, and the diffusion equation (1.1) with the moisture conductivity g is derived from mass conservation law for the moisture. Here, as mentioned in [2], g satisfies

$$\lim_{r\to 0+} g(r) = +\infty, \ \lim_{r\to 1} g(r) = 1, \ g(r) \ge \kappa \text{ for } 0 \le r \le 1,$$

where κ is a positive constant. In the concrete carbonation process it is known that the relationship between u and w is given as a hysteresis with an anticlockwise trend. The functions f^* and f_* correspond to the upper and lower curves of the hysteresis loop, respectively. Therefore, we assume that it is described by a play operator with the input function u and the output function w. Accordingly, the relationship between u and w is represented the ordinary differential equation (1.2) (See, for instance, Brokate and Sprekels [6] and Visintin [13]). On problem (P), Aiki and Kumazaki already proved the existence of a time global solution in three dimensions and the uniqueness in only one dimension.

Considering the mathematical results for systems consisting of a parabolic-type equation and a hysteresis operator, Kenmochi, Koyama and Meyer [9] looked into the following system which is a mathematical model for a real time controled system:

$$u_t - \Delta u + w = f \quad \text{in } Q(T), \tag{1.5}$$

$$w_t - \nu\Delta w + \partial I(u;w) \ni 0 \quad \text{in } Q(T). \tag{1.6}$$

For the system $\{(1.5)-(1.6)\}$ with a boundary condition and a initial condition, they proved the existence and uniqueness of a global-in-time solution for $\nu \ge 0$. Also, Colli, Kenmochi and Kubo in [8] studied the following system which represents a solid-liquid phase transition with a hysteric effect in the kinetics of interface:

$$u_t + w_t - \Delta u = h \quad \text{in } Q(T), \tag{1.7}$$

$$w_t - \nu\Delta w + l(u,w) + \partial I(u;w) \ni 0 \quad \text{in } Q(T), \tag{1.8}$$

where h is a given function on $Q(T)$ and $l(\cdot,\cdot)$ is a smooth function on $\mathbf{R}\times\mathbf{R}$. On the initial and boundary value problem for this system $\{(1.7)-(1.8)\}$, they showed the existence of a time global solution for $\nu \ge 0$ and the uniqueness in the case $\nu = 0$.

The aim of this paper is to prove the uniqueness of solutions to (P) in three dimensions. To do so we faced the following two difficulties: The first difficulty is concerned with the estimate for ∇u, and the second one causes from the lack of continuity on (1.2) between the input function u and the output function w.

The first difficulty comes from the nonlinearity of $g(u)$. Precisely, a standard way to prove uniqueness is to estimate a difference of two solutions as follows. Let $\{u_i, w_i\}$ be a solution of (P) for $i = 1, 2$. Then, from the divergence term in (1.1) the following term appears:

$$\int_\Omega (g(u_1) - g(u_2))\nabla u_2(\nabla u_1 - \nabla u_2)dx. \tag{1.9}$$

If g is linear, this kind of terms never appears. Also, in one-dimensional case we can estimate ∇u_2 in (1.9) by applying the Sobolev embedding theorem from $H^1(\Omega)$ to $L^\infty(\Omega)$. Then, it is not hard to prove the uniqueness. However, in three dimensional case, this embedding is not valid. In this paper in order to overcome this difficulty we establish that $\nabla u \in L^\infty(Q(T))$ by applying the classical theory for quasi linear parabolic equations shown in Ladyženskaja-Solonikov-Uralćeva [10]. In Section 3 the boundedness ∇u will be proved under the smoothness assumption for boundary and initial data.

The detail of the second difficulty is as follows. When we consider the difference of two solutions, $w_1 - w_2$ appears in the right hand side of (1.1). To give an estimate for $w_1 - w_2$ we can obtain the following estimate for its L^∞-norm :

$$|w_1 - w_2|_{L^\infty(Q(s))} \le \max\{|f^*(u_1) - f^*(u_2)|_{L^\infty(Q(s))}, |f_*(u_1) - f_*(u_2)|_{L^\infty(Q(s))}\} \quad \text{for } 0 \le s \le T. \tag{1.10}$$

This kind of estimates for (1.6) with $\nu = 0$ was found in Visintin [13], and was proved by Kenmochi, Koyama and Meyer [9] in case $\nu > 0$. However, it is not easy to obtain the L^∞-norm of the difference of solutions to quasi-linear parabolic equations in three dimensions. In this paper, by applying the following inequalities (1.11) and (1.12) we have estimated the difference $w_1 - w_2$: The first inequality (See Visintin [13, Lemma 2.1 in Chapter 3]) is:

$$|w_1 - w_2|_{L^\infty(0,s)} \le \max\{|f^*(u_1) - f^*(u_2)|_{L^\infty(0,s)}, |f_*(u_1) - f_*(u_2)|_{L^\infty(0,s)}\} \text{ on } \Omega \quad \text{for } 0 \le s \le T. \tag{1.11}$$

The second one is concerned with the embedding between two spaces:

$$|z|_{L^{q_2}(\Omega, C([0,T]))} \le C\left(|z|_{L^{p_1}(0,T;W^{1,q_1}(\Omega))} + |z_t|_{L^{p_0}(0,T;L^{q_0}(\Omega))}\right) \quad \text{for } z \in L^{p_1}(0,T;W^{1,q_1}(\Omega)) \text{ with } z_t \in L^{p_0}(0,T;L^{q_0}(\Omega)), \tag{1.12}$$

where p_0, q_0, p_1, q_1 and q_2 are positive constants (for detail, see the end of Section 2). On account of these ideas we shall prove the uniqueness in three dimensional domain in Section 5.

2. **Notation and assumptions.** In this paper we use the following notations. In general, for a Banach space X we denote by $|\cdot|_X$ its norm. Particularly, we denote by $H = L^2(\Omega)$, and the norm and the inner product of H are simply denoted by $|\cdot|_H$ and $(\cdot,\cdot)_H$, respectively. Also, $H^1(\Omega)$, $H^1_0(\Omega)$ and $H^2(\Omega)$ are the usual Sobolev spaces.

Throughout this paper we assume the following (A1)–(A7):

(A1) Ω is a open bounded connected domain of $\mathbf{R}^3$ which has the boundary $\partial\Omega$ in the class of C^2.

(A2) T is a positive constant.

(A3) $G : (0,\infty) \to \mathbf{R}$ is continuous, $g(r) := G'(r)$ is continuous on $(0,\infty)$, $g \in C^2((0,\infty))$ and $g(r) \ge g_0$ for $r > 0$, where g_0 is a positive constant.

(A4) $f \in L^\infty(Q(T))$ and $f_t \in L^2(0,T;H)$ with $f \geq 0$ a.e. on $Q(T)$.

(A5) $f_*, f^* \in C^2(\mathbf{R}) \cap W^{2,\infty}(\mathbf{R})$ with $0 \leq f_* \leq f^* \leq w_*$ on $\mathbf{R}$, where w_* is a positive constant. We put $L_* = \max\{|f_*|_{W^{2,\infty}(\mathbf{R})}, |f^*|_{W^{2,\infty}(\mathbf{R})}\}$.

(A6) $u_b \in C^{2,1}(\overline{Q(T)})$ and $u_{bt} \in L^2(0,T;H^2(\Omega))$ with $u_b \geq \kappa_0$ for some positive constant κ_0. Then, there exists a constant $M_0 > 0$ such that

$$|\nabla u_b|_{L^\infty(Q(T))} \leq M_0. \tag{2.1}$$

(A7) $u_0 \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$ with $u_0 \geq \kappa_0$ and $w_0 \in L^\infty(\Omega)$ with $w_0 \geq 0$ a.e. on Ω. Also, there exists a constant $N_0 > 0$ such that

$$|\nabla u_0|_{L^\infty(\Omega)} + |\Delta u_0|_{L^\infty(\Omega)} \leq N_0. \tag{2.2}$$

Moreover, $u_0 = u_b(0)$ a.e. on $\partial\Omega$ and $f_*(u_0) \leq w_0 \leq f^*(u_0)$ a.e. on Ω.

Next, we define a solution of (P) on $[0,T]$ in the following way:

Definition 2.1. Let u and w be functions on $Q(T)$. We call that the pair $\{u, w\}$ is a solution of (P) on $[0,T]$ if the conditions (S1) $\sim$ (S4) hold:

(S1) $u \in W^{1,2}(0,T;H) \cap L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$, $u > 0$ a.e. on $Q(T)$ and $w \in W^{1,2}(0,T;H)$.

(S2) $u_t - \text{ div } (\nabla(G(u))) = wf \quad$ a.e. in $Q(T)$.

(S3) $w_t + \partial I(u;w) \ni 0 \quad$ a.e. in $Q(T)$.

(S4) $u = u_b$ a.e. on $S(T)$ and $u(0) = u_0, w(0) = w_0$ a.e. on Ω.

First, we recall the following theorem concerned with the existence of a solution of (P).

Theorem 2.2. *(Aiki-Kumazaki [3]) If (A1) $\sim$ (A7) hold, then (P) has at least one solution on $[0,T]$ such that*

$$\kappa_0 \leq u \leq u^* \text{ and } \quad 0 \leq w \leq w_* \quad \text{a.e. on } Q(T),$$

where $u^* = \max\{|u_0|_{L^\infty(\Omega)}, |u_b|_{L^\infty(Q(T))}, w_*|f|_{L^\infty(Q(T))}\}(T+1)$ and κ_0 and w_* are the same constants as in (A5) and (A6), respectively.

Here, we set

$$g^* := \max_{\kappa_0 \leq r \leq u^*} g(r). \tag{2.3}$$

The main theorem of this paper is the following:

Theorem 2.3. *Under (A1) $\sim$ (A7), let $\{u_i, w_i\}$ be solutions of (P) on $[0,T]$ for $i = 1,2$. If $\kappa_0 \leq u_i \leq u^*$ a.e. on $Q(T)$ for $i = 1,2$, then $u_1 = u_2$ and $w_1 = w_2$ a.e. on $Q(T)$.*

The proof of Theorem 2.3 is shown in Section 5. To conclude this section we state the following useful lemma.

Lemma 2.4. *If (A1) holds, then (1)$\sim$(3) hold:*

(1) (cf. [11, Lemma 3.7.1]*) There exists a positive constant C_* such that*

$$|u|_{H^2(\Omega)} \leq C_*(|\Delta u|_H + |u|_H) \quad \text{for } u \in H^2(\Omega).$$

(2) (cf. [10, Theorem 2.2 in Chapter 2]*) There exists a positive constant C^* such that*

$$|u|_{L^q(\Omega)} \leq C^*|\nabla u|^\alpha_{L^m(\Omega)}|u|^{1-\alpha}_{L^r(\Omega)} \text{ for } u \in W^{1,m}_0(\Omega),$$

where $\alpha = \left(\frac{1}{r} - \frac{1}{q}\right)\left(\frac{1}{n} - \frac{1}{m} + \frac{1}{r}\right)^{-1}$ and $m, r \geq 1$, q are positive constants satisfying the following condition: If $m < n$ and $r \leq nm/(n-m)$, then q is any number

from $[r, nm/(n-m)]$, *if* $m \geq n > 1$, *then* q *is any number in the interval* $[r, \infty)$ *and if* $m > n > 1$, *then* $q = \infty$ *is also valid.*
(3) (cf. [5]) For $p_0, q_0, p_1, q_1 \geq 1$, *let* $W^{p_0,q_0:p_1,q_1}((0,T),\Omega)$ *be the following set:*

$$W^{p_0,q_0:p_1,q_1}((0,T),\Omega) := \{z \in L^1(Q(T)) | \ \frac{\partial z}{\partial t} \in L^{p_0}(0,T,L^{q_0}(\Omega)),$$

$$\frac{\partial z}{\partial x_i} \in L^{p_1}(0,T,L^{q_1}(\Omega)) \quad \text{for } i = 1,2,3\}.$$

If $q_2 \geq \max\{q_0, q_1\}$ *and* $p_2 \geq \max\{p_0, p_1\}$, *and*

$$\Big(1 - \frac{1}{p_0} + \frac{1}{p_2}\Big)\Big(\frac{1}{N} - \frac{1}{q_1} + \frac{1}{q_2}\Big) > \Big(\frac{1}{p_1} - \frac{1}{p_2}\Big)\Big(\frac{1}{q_0} - \frac{1}{q_2}\Big), \tag{2.4}$$

then there exists a positive constant C_e *such that*

$$|z|_{L^{p_2}(0,T;L^{q_2})} \leq C_e\Big(|z|_{L^{p_1}(0,T;W^{1,q_1}(\Omega))} + |z_t|_{L^{p_0}(0,T;L^{q_0}(\Omega))}\Big) \tag{2.5}$$

$$\text{for } z \in W^{p_0,q_0:p_1,q_1}((0,T),\Omega).$$

In the case of $p_2 = \infty$, *if* $q_2 \geq \max\{q_0, q_1\}$ *and*

$$\frac{1}{p_0'}\Big(\frac{1}{N} - \frac{1}{q_1} + \frac{1}{q_2}\Big) > \frac{1}{p_1}\Big(\frac{1}{q_0} - \frac{1}{q_2}\Big),$$

where p_0' *is the dual index of* p_0, *then there exists a positive constant* C_e *such that*

$$|z|_{L^{q_2}(\Omega, C([0,T]))} \leq C_e\Big(|z|_{L^{p_1}(0,T;W^{1,q_1}(\Omega))} + |z_t|_{L^{p_0}(0,T;L^{q_0}(\Omega))}\Big) \tag{2.6}$$

$$\text{for } z \in W^{p_0,q_0:p_1,q_1}((0,T),\Omega).$$

Proof. The assertion (2) is a direct consequence of Gagliardo-Nirenberg's inequality. Also, by repeating the argument of [5, Chapter IV, Vol. II], we can derive the inequality in the assertion (3). Here, we give a proof of (2.5) with $N = 3$, $\Omega = \mathbf{R}^3$ and $\mathbf{R}$ in place of $[0,T]$. Similarly to the following proof, we can show that (2.6) holds. In this proof, we put $|\cdot|_{p,q} = |\cdot|_{L^p(\mathbf{R},L^q(\mathbf{R}^3))}$. First, for $\sigma \in (0,1]$ and $u \in W^{p_0,q_0,p_1,q_1}(\mathbf{R}^3,\mathbf{R})$, we define

$$u^\sigma(x,t) = \sigma^{-3-\lambda}\int_{\mathbf{R}}\int_{\mathbf{R}^3}\varphi\Big(\frac{x-y}{\sigma}, \frac{t-s}{\sigma^\lambda}\Big)u(y,s)dyds.$$

Here, φ is a smooth nonnegative function on $\mathbf{R}^3 \times \mathbf{R}$ and vanishes outside $B(1) \times (-1,1)$, where $B(r)$ is a ball in $\mathbf{R}^3$ at the origin with a radius r, and it satisfies that $\int_{-1}^1\int_{B(1)}\varphi(x,t)dxdt = 1$. Also, we choose

$$\lambda = \frac{1 + 3(\frac{1}{q_0} - \frac{1}{q_1})}{\frac{1}{p_0'} + \frac{1}{p_1}},$$

where p_0' is the dual index of p_0. By (2.4) we see that $\lambda > 0$. Easily, we have

$$\frac{\partial}{\partial\sigma}u^\sigma(x,t) = -\lambda\sigma^{-3-1}\int_{\mathbf{R}}\int_{\mathbf{R}^3}\frac{t-s}{\sigma^\lambda}\varphi\Big(\frac{x-y}{\sigma}, \frac{t-s}{\sigma^\lambda}\Big)\frac{\partial u}{\partial s}(y,s)dyds$$

$$-\sigma^{-3-\lambda}\int_{\mathbf{R}}\int_{\mathbf{R}^3}\sum_{i=1}^3\frac{x_i-y_i}{\sigma}\varphi\Big(\frac{x_i-y_i}{\sigma}, \frac{t-s}{\sigma^\lambda}\Big)\frac{\partial u}{\partial y_i}(y,s)dy_ids.$$

Now, for $0 < \alpha < \beta \leq 1$, let

$$\mathcal{L}_0(x,t) = \int_\alpha^\beta \sigma^{-3-1} \int_{\mathbf{R}} \int_{\mathbf{R}^3} \frac{t-s}{\sigma^\lambda} \varphi\Big(\frac{x-y}{\sigma}, \frac{t-s}{\sigma^\lambda}\Big) \frac{\partial u}{\partial s}(y,s) dy ds d\sigma,$$

$$\mathcal{L}_1(x,t) = \int_\alpha^\beta \sigma^{-3-\lambda} \int_{\mathbf{R}} \int_{\mathbf{R}^3} \sum_{i=1}^3 \frac{x_i - y_i}{\sigma^\lambda} \varphi\Big(\frac{x_i-y_i}{\sigma}, \frac{t-s}{\sigma^\lambda}\Big) \frac{\partial u}{\partial y_i}(y,s) dy_i ds d\sigma.$$

Then, we have

$$|u^\beta(x,t) - u^\alpha(x,t)| \leq \lambda |\mathcal{L}_0(x,t)| + |\mathcal{L}_1(x,t)| \text{ for } (x,t) \in \mathbf{R}^3 \times \mathbf{R}. \tag{2.7}$$

By using Minkowski's inequality and Young's inequality for convolutions, we can obtain

$$|\mathcal{L}_0|_{p_2,q_2} \leq |\Phi_0|_{s_0,r_0} \left|\frac{\partial u}{\partial s}\right|_{p_0,q_0} \int_\alpha^\beta \sigma^{-N-1+N/r_0+\lambda/s_0} d\sigma, \tag{2.8}$$

and

$$|\mathcal{L}_1|_{p_2,q_2} \leq |\Phi_1|_{s_1,r_1} |\nabla u|_{p_1,q_1} \int_\alpha^\beta \sigma^{-N-\lambda+N/r_1+\lambda/s_1} d\sigma, \tag{2.9}$$

where $\Phi_0(\xi,\tau) = \tau\varphi(\xi,\tau)$, $\Phi_1(\xi,\tau) = \xi\varphi(\xi,\tau)$ and

$$\frac{1}{r_0} = 1 - \frac{1}{q_0} + \frac{1}{q_2}, \frac{1}{s_0} = 1 - \frac{1}{p_0} + \frac{1}{p_2}, \frac{1}{r_1} = 1 - \frac{1}{q_1} + \frac{1}{q_2}, \frac{1}{s_1} = 1 - \frac{1}{p_1} + \frac{1}{p_2}.$$

Here, we set

$$\tilde{\kappa} = 3\Big(\frac{1}{p_0'} + \frac{1}{p_0}\Big)^{-1} \Big\{ \Big(1 - \frac{1}{p_0} + \frac{1}{p_2}\Big)\Big(\frac{1}{3} - \frac{1}{q_1} + \frac{1}{q_2}\Big) - \Big(\frac{1}{q_0} - \frac{1}{q_2}\Big)\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big) \Big\}.$$

By using the fact that $-4+3/r_0+\lambda/s_0 = -3-\lambda+3/r_1+\lambda/s_1 = \tilde{\kappa}-1$ and applying (2.8), (2,9) to (2.7), we have

$$|u^\beta - u^\alpha|_{p_2,q_2} \leq \mathcal{C}_1(\beta^{\tilde{\kappa}} - \alpha^{\tilde{\kappa}}) \bigg(\left|\frac{\partial u}{\partial s}\right|_{p_0,q_0} + |\nabla u|_{p_1,q_1} \bigg), \tag{2.10}$$

where $\mathcal{C}_1$ is a positive constant depending on $p_0, p_1, p_2, q_0, q_1, q_2$. This implies that u^σ is a Cauchy sequence in $L^{p_2}(\mathbf{R}, L^{q_2}(\mathbf{R}^3))$ so that there exists $u^* \in L^{p_2}(\mathbf{R}, L^{q_2}(\mathbf{R}^3))$ such that $u^\sigma \to u^*$ in $L^{p_2}(\mathbf{R}, L^{q_2}(\mathbf{R}^3))$ as $\sigma \to 0$. By the definition of u^σ, we see that $u^* = u$. Since there exists $\mathcal{C}_2 > 0$ such that $|u^\alpha|_{p_2,q_2} \leq \mathcal{C}_2 |u|_{p_1,q_1}$ for any $\alpha \in (0,1]$ by letting $\alpha \to 0$ in (2.10) we see that

$$\begin{aligned} |u|_{p_2,q_2} &\leq |u - u^\beta|_{p_2,q_2} + |u^\beta|_{p_2,q_2} \\ &\leq \mathcal{C}_1 \beta^{\tilde{\kappa}} \bigg(\left|\frac{\partial u}{\partial s}\right|_{p_0,q_0} + |\nabla u|_{p_1,q_1} \bigg) + \mathcal{C}_2 |u|_{p_1,q_1} \\ &\leq 2(\mathcal{C}_1 + \mathcal{C}_2) \bigg(\left|\frac{\partial u}{\partial s}\right|_{p_0,q_0} + |u|_{L^{p_1}(\mathbf{R}, W^{1,q_1}(\mathbf{R}^3))} \bigg). \end{aligned}$$

Therefore, we obtain the desired inequality. □

3. **Boundedness of ∇u.** In this section, we prove that $\nabla u \in L^\infty(Q(T))$ in a similar way to that of Ladyženskaja-Solonikov-Uralćeva [10, Section 10 in Chapter 3]. The proof is rather long so that we divide it several steps. As the first step, we show the boundedness of ∇u on the boundary in Lemma 3.1. Next, we give the Hölder continuity of u in Lemma 3.2, and by using this fact and the boundedness of the boundary we can obtain that $\nabla u \in L^p(Q(T))$ for any $p \geq 2$. Then we can get the boundedness of ∇u on the whole domain.

Lemma 3.1. *(cf. [10, Lemma 3.1 in Chapter 6]) Let $\{u, w\}$ be a solution of (P) with $\kappa_0 \leq u \leq u^*$ a.e. on $Q(T)$ under the assumptions (A1)–(A7). Then, there exists a positive constant N_1 such that $|\nabla u| \leq N_1$ on $S(T)$.*

Proof. First, we put $v = G(u)$, $\rho(r) = G^{-1}(r)$ for $r \in \mathbf{R}$ and $\tilde{v} = v - G(u_b)$. Then, we have

$$\rho'(v)\tilde{v}_t - \Delta\tilde{v} = wf - \rho'(v)(G(u_b))_t + \Delta G(u_b) \quad \text{a.e. in } Q(T),$$

$$\tilde{v} = 0 \quad \text{a.e. on } S(T),$$

$$\tilde{v}(0) = G(u_0) - G(u_b(0)) \quad \text{in } \Omega.$$

From the assumption for u on $Q(T)$, there exists positive constants δ_* and δ^* such that $\delta_* \leq \rho'(v) \leq \delta^*$ on $Q(T)$. By putting $\tilde{f} = wf - \rho'(v)(G(u_b))_t + \Delta G(u_b)$, we see from (A3) and (A6) that $\tilde{f} \in L^\infty(Q(T))$ and from (A7) that $\tilde{v}(0) \in H_0^1(\Omega) \cap W^{1,\infty}(\Omega)$. Here, we take sequences $\{a_\varepsilon\} \subset C^2(\overline{Q(T)})$, $\{\tilde{f}_\varepsilon\} \subset C^\infty(\overline{Q(T)})$ and $\{\tilde{v}_{0,\varepsilon}\} \subset C_0^\infty(\Omega)$ with $\delta_*/2 \leq a_\varepsilon \leq 2\delta^*$ on $Q(T)$, $|\tilde{f}_\varepsilon|_{L^\infty(Q(T))} \leq |\tilde{f}|_{L^\infty(Q(T))} + 1$ and $|\tilde{v}_{0,\varepsilon}|_{W^{1,\infty}(\Omega)} \leq |\tilde{v}_0|_{W^{1,\infty}(\Omega)} + 1$ such that $a_\varepsilon \to \rho'(v)$, $\tilde{f}_\varepsilon \to \tilde{f}$ strongly in $L^2(0,T;H)$ and $\tilde{v}_{0,\varepsilon} \to \tilde{v}_0$ in $H_0^1(\Omega)$ as $\varepsilon \to 0$. Then, the following problem (P)$_\varepsilon$

$$a_\varepsilon \tilde{v}_{\varepsilon t} - \Delta\tilde{v}_\varepsilon = \tilde{f}_\varepsilon \quad \text{a.e. in } Q(T),$$

$$\tilde{v}_\varepsilon = 0 \quad \text{a.e. on } S(T),$$

$$\tilde{v}_\varepsilon(0) = \tilde{v}(0) \quad \text{in } \Omega.$$

has a unique classical solution $\tilde{v}_\varepsilon$ on $Q(T)$ (see for instance [10, Chapter 4]). Easily, we get a constant $K > 0$ such that $|\tilde{v}_\varepsilon|_{L^\infty(Q(T))} \leq K$ for $\varepsilon > 0$.

In order to estimate the flux of a solution to (P) on the boundary we need to describe $\partial\Omega$ in the following exact form (see [12, Chapter 1 and 2]): For $\delta > 0$ we set

$$\Omega_\delta := \{x \in \Omega |\ d(x) < \delta\},$$

where $d(x) = \text{dist}(x, \partial\Omega)$ for $x \in \Omega$. Since $\partial\Omega$ is in the class of C^2, there exists $i^* \in N$, $\delta > 0$, a disc $\Delta_i \subset \mathbf{R}^2$, $a_i \in C^2(\overline{\Delta_i})$, $\Omega_i \subset \Omega_\delta$ and the local coordinate $y_i' \in \mathbf{R}^2$ on Δ_i for $1 \leq i \leq i^*$ satisfying the following (i), (ii) and (iii):

(i) $$\Omega_\delta = \cup_{i=1}^{i^*}\Omega_i, \quad \Delta_i = \{y_i' \in \mathbf{R}^2 |\ |y_i'| < \sigma_i\},$$

where $\sigma_i > 0$,

$$\partial\Omega = \cup_{i=1}^{i^*}\{(y_i', a_i(y_i')) |\ y_i' \in \Delta_i\}.$$

(ii) For $1 \leq i \leq i^*$ we define an operator $\mathcal{T}_i : \Delta_i \times (0, \delta) \to \Omega_i$ by

$$\mathcal{T}_i(y_i', \tau) = (y_i', a_i(y_i')) - \tau\nu(y_i', a_i(y_i')) \text{ for } y_i' \in \Delta_i \text{ and } 0 < \tau < \delta,$$

where ν is the outward normal vector on $\partial\Omega$, and $\mathcal{T}_i$ is a bijective.

(iii) By putting $\Gamma_i = \mathcal{T}_i^{-1}$ and $b_{kl}^{(i)} = \sum_{j=1}^{3} \frac{\partial \Gamma_{ik}}{\partial x_j} \frac{\partial \Gamma_{il}}{\partial x_j}$ it holds that

$$\sum_{k,l=1}^{3} b_{kl}^{(i)} \xi_k \xi_l \geq \mu \sum_{k=1}^{3} \xi_k^2 \quad \text{for } \xi = (\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3 \text{ and } 1 \leq i \leq i^*, \tag{3.1}$$

where μ is a positive constant.

Here, for any $\alpha > 0$ and $\beta > 0$, let

$$z(x) = \alpha e^{-\beta d(x)} \text{ for } x \in \overline{\Omega_\delta}, \tag{3.2}$$

and

$$\hat{z}(y_i', \tau) = z(\mathcal{T}_i(y_i', \tau)) \text{ for } (y_i', \tau) \in \Delta_i \times (0, \delta).$$

Then it is easy to see that $\hat{z}(y_i', \tau) = \alpha e^{-\beta\tau}$ for $(y_i', \tau) \in \Delta_i \times (0, \delta)$ and $z \in C^2(\overline{\Omega_\delta})$. Also, we have

$$\begin{aligned} \Delta z &= \sum_{k=1}^{3} \frac{\partial^2 z}{\partial x_k^2} \\ &= \alpha\beta^2 e^{-\beta\tau} \sum_{k=1}^{3} (\frac{\partial \Gamma_{i3}}{\partial x_k})^2 - \alpha\beta e^{-\beta\tau} \sum_{k=1}^{3} \frac{\partial^2 \Gamma_{i3}}{\partial x_k^2} \\ &\geq \alpha\beta^2 e^{-\beta\tau} \mu - \alpha\beta e^{-\beta\tau} \sum_{k=1}^{3} \frac{\partial^2 \Gamma_{i3}}{\partial x_k^2} \\ &\geq \alpha\beta e^{-\beta\tau} (\beta\mu - C_\Omega) \quad \text{ on } \overline{\Omega}_i, \end{aligned}$$

where C_Ω is a positive constant satisfying

$$\sum_{k=1}^{3} \left| \frac{\partial^2 \Gamma_{ik}}{\partial x_k^2} \right| \leq C_\Omega \text{ on } \Omega_i \text{ for } 1 \leq i \leq i^*.$$

Accordingly, by taking $\beta > 0$ such that $\beta \geq \frac{2C_\Omega}{\mu}$ we have

$$\Delta z \geq \alpha\beta e^{-\beta\delta} C_\Omega \quad \text{ on } \overline{\Omega_\delta}. \tag{3.3}$$

Next, we can take $\alpha_1 > 0$ such that $\tilde{v}_\varepsilon(0) + z \leq \alpha_1$ in Ω_δ. In fact, for $x \in \Omega_\delta$ there exists $\tau > 0$ and $x_0 \in \partial\Omega$ such that $x = x_0 - \tau\nu(x_0)$. Since $\tilde{v}_{0,\varepsilon}(x_0) = 0$ and

$$\begin{aligned} \tilde{v}_\varepsilon(0, x) + z(x) &= \tilde{v}_{0,\varepsilon}(x) - \tilde{v}_{0,\varepsilon}(x_0) + \alpha e^{-\beta\tau} \\ &\leq |\nabla \tilde{v}_{0,\varepsilon}|_{L^\infty(\Omega)} |x - x_0| + \alpha \\ &= |\nabla \tilde{v}_{0,\varepsilon}|_{L^\infty(\Omega)} \tau + \alpha, \end{aligned}$$

elementary calculations implies

$$\tilde{v}_\varepsilon(0, x) + z(x) \leq \alpha \text{ on } \Omega_\delta \text{ for } \varepsilon > 0 \text{ and } \alpha \geq \alpha_1,$$

where α_1 is some positive constant. Also, since $|\tilde{v}_\varepsilon|_{L^\infty(Q(T))} \leq K$ for $\varepsilon > 0$, we see that for $\alpha \geq K/(1 - e^{-\beta\delta})$,

$$\tilde{v}_\varepsilon + z \leq K + \alpha e^{-\beta\delta} \leq \alpha(1 - e^{-\beta\delta}) + \alpha e^{-\beta\delta} = \alpha \text{ on } \partial\Omega_\delta.$$

Furthermore, let

$$\alpha = \max\left\{ \frac{|\tilde{f}|_{L^\infty(Q(T))} + 1}{\beta e^{-\beta\delta} C_\Omega}, \frac{K}{1 - e^{-\beta\delta}}, \alpha_1 \right\}.$$

Then, it holds that for $\varepsilon > 0$

$$a_\varepsilon(\tilde{v}_\varepsilon + z)_t - \Delta(\tilde{v}_\varepsilon + z) \leq 0 \quad \text{ in } (0, T) \times \Omega_\delta, \tag{3.4}$$

$$\tilde{v}_\varepsilon + z \le \alpha \quad \text{on } (0,T)\times\partial\Omega_\delta,$$
$$\tilde{v}_\varepsilon(0) + z(0) \le \alpha \quad \text{in } \Omega_\delta.$$

Now, we multiply (3.4) by $[\tilde{v}_\varepsilon + z - \alpha]^+$. Then, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_\delta} a_\varepsilon |[\tilde{v}_\varepsilon + z - \alpha]^+|^2 dx \le \frac{1}{2}\int_{\Omega_\delta} (a_\varepsilon)_t |[\tilde{v}_\varepsilon + z - \alpha]^+|^2 dx.$$

Since

$$\int_{\Omega_\delta} (a_\varepsilon)_t |[\tilde{v}_\varepsilon + z - \alpha]^+|^2 dx \le \left|\frac{(a_\varepsilon)_t}{a_\varepsilon}\right|_{L^\infty(Q(T))} \int_{\Omega_\delta} a_\varepsilon |[\tilde{v}_\varepsilon + z - \alpha]^+|^2 dx,$$

by Gronwall's inequality we can show that $\tilde{v}_\varepsilon + z \le \alpha$ a.e. on $(0,T)\times\Omega_\delta$ for $\varepsilon > 0$. Therefore, for $x \in \partial\Omega$ and $0 < r < \delta$, we have

$$\frac{\partial \tilde{v}_\varepsilon}{\partial \nu} = \lim_{r\to 0} \frac{\tilde{v}_\varepsilon(x - r\nu(x)) - \tilde{v}_\varepsilon(x)}{r} \le \lim_{r\to 0} \alpha \frac{(1 - e^{-\beta r})}{r} \le \alpha\beta. \tag{3.5}$$

Since $\tilde{v}_\varepsilon$ is bounded in $W^{1,2}(0,T;H) \cap L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$, we can take a subsequence $\{\tilde{v}_{\varepsilon,m}\}$ such that $\tilde{v}_{\varepsilon,m} \to v^*$ strongly in $C([0,T];H) \cap L^2(0,T;H^1(\Omega))$, weakly in $W^{1,2}(0,T;H)$ and $L^2(0,T;H^2(\Omega))$, weakly star in $L^\infty(0,T;H^1(\Omega))$ and $\frac{\partial \tilde{v}_{\varepsilon,m}}{\partial\nu} \to \frac{\partial v^*}{\partial \nu}$ weakly in $L^2(0,T;L^2(\partial\Omega))$ as $m \to \infty$. Therefore, by letting $m \to \infty$ we obtain

$$\begin{cases} \rho'(v)v^*_t - \Delta v^* = \tilde{f} & \text{a.e. in } Q(T), \\ v^* = 0 & \text{a.e. on } S(T), \\ v^*(0) = \tilde{v}(0) & \text{on } \Omega, \end{cases} \tag{3.6}$$

and by the lower semi continuity of weak convergences, we derive

$$\frac{\partial v^*}{\partial \nu} \le \alpha\beta \text{ on } \partial\Omega. \tag{3.7}$$

Now, by the uniqueness of (3.6) we see that $v^* = \tilde{v} = v - G(u_b)$. Therefore, from (3.6), we see that $\frac{\partial\tilde{v}}{\partial\nu} \le \alpha\beta$ a.e. on $S(T)$. Similarly, we can show that $\frac{\partial(-\tilde{v})}{\partial\nu} \le \alpha\beta$ a.e. on $S(T)$. Finally, since $\tilde{v} = 0$ a.e. on $S(T)$, we have $|\nabla\tilde{v}| \le \alpha\beta$ a.e. on $S(T)$ so that (A3) and (A6) implies the conclusion of this lemma. □

Next, we prove the Hölder continuity of u. The proof is quite similar to that of [10] so that we may skip the proof. Since the equation (1.1) does not contain the class dealt in [10], we give a part of the proof. Precisely, the goal of the following proof of Lemma 3.2 is to show $u \in \mathcal{B}$, where the set $\mathcal{B}$ will be defined in the proof. This set was already considered in [10]. Here, we note that the range of the parameter k is little bit difference from that of [10]. However, we can prove Lemma 3.2 in a similar way to that of [10, Theorems 7.1 and 8.1 in Chapter 2]. Thus we finish the proof when we get $u \in \mathcal{B}$.

Lemma 3.2. *(cf.* [10, Theorem 10.1 in Chapter 3]*) Under (A1)* $\sim$ *(A7) hold, let* $\{u, w\}$ *be a solution of (P) with* $\kappa_0 \le u \le u^*$ *a.e on* $Q(T)$*. Then* u *is Hölder continuous on* $\overline{Q(T)}$*.*

Proof. First, for $x_0 \in \overline{\Omega}$ and $\rho > 0$ let $K_\rho = K_\rho(x_0)$ is a ball in $\mathbf{R}^3$ at a center x_0 with a radius ρ and $\Omega_\rho(x_0) := K_\rho(x_0) \cap \Omega$. Then by (A1) there exists $0 < \theta_0 < 1$ such that

$$\text{mes } \Omega_\rho(x_0) \le (1 - \theta_0)\text{mes } K_\rho(x_0) \text{ for any } x_0 \in \partial\Omega \text{ and } \rho > 0, \tag{3.8}$$

where mes A is the Lebesgue measure of A for a measurable subset $A \subset \mathbf{R}^3$.

Next, let M, γ and κ be positive numbers. Here, in order to define a set $\mathcal{B} := \mathcal{B}(M,\gamma,r,\kappa)$ of functions we introduce the following notations:

$$w^{(k)}(t,x) = \max\{w(t,x)-k,0\} \text{ for } k \in \mathbf{R};$$

$$Q(\rho,\tau) = (t_0,t_0+\tau)\times\Omega_\rho(x_0) \text{ for } \rho>0,\tau>0,t_0\in\mathbf{R} \text{ with } 0\le t_0\le t_0+\tau\le T;$$

$$|z|^2_{V(Q(\rho,\tau))} = \text{ess sup}_{t_0\le t\le t_0+\tau}|z(t)|^2_{L^2(\Omega_\rho(x_0))} + \int_{t_0}^{t_0+\tau}|\nabla z(t)|^2_{L^2(\Omega_\rho(x_0))}dt;$$

$$A_{k,\rho}(t) = \{x\in\Omega_\rho(x_0)|\ w(t,x)>k\} \text{ for } k\in\mathbf{R}, 0\le t\le T \text{ and } \rho>0;$$

q and r are positive constants satisfying

$$\frac{1}{r}+\frac{3}{2q}=\frac{3}{4}$$

with $q\in(2,6]$ and $r\in[2,\infty)$. By using these notations we define the set $\mathcal{B}$ as follows: we say that $u\in\mathcal{B}$ if $u\in V(T)$, $|u|_{L^\infty(Q(T))}\le M$ and the function $w(t,x)=\pm u(t,x)$ satisfies the following inequalities (3.10) and (3.11) for $0\le t_0\le t_0+\tau\le T$, $\rho>0$, σ_1, $\sigma_2\in(0,1)$ and k with

$$\begin{cases} k\in[-M,M] & \text{if } K_\rho(x_0)\subset\Omega,\\ k\in[0,M] & \text{otherwise,}\end{cases} \tag{3.9}$$

$$\max_{t_0\le t\le t_0+\tau}|w^{(k)}(t,x)|^2_{L^2(\Omega_{\rho-\sigma_1\rho})} \le |w^{(k)}(t_0)|^2_{L^2(\Omega_\rho)}$$
$$+\gamma\left[(\sigma_1\rho)^{-2}|w^{(k)}|^2_{L^2(Q(\rho,\tau))} + \left(\int_{t_0}^{t_0+\tau}(\text{mes}A_{k,\rho}(t))^{\frac{r}{q}}dt\right)^{\frac{2}{r}(1+\kappa)}\right], \tag{3.10}$$

$$|w^{(k)}|^2_{V(Q(\rho-\sigma_1\rho,\ \tau-\sigma_2\tau))}$$
$$\le\gamma\left\{[(\sigma_1\rho)^{-2}+(\sigma_2\rho)^{-1}]|w^{(k)}|^2_{L^2(Q(\rho,\tau))} + \left(\int_{t_0}^{t_0+\tau}(\text{mes}A_{k,\rho}(t))^{\frac{r}{q}}dt\right)^{\frac{2}{r}(1+\kappa)}\right\}, \tag{3.11}$$

Although our definition of $\mathcal{B}$ is little bit different from one in [10, Section 7 in Chapter 2], we can prove that $u\in\mathcal{B}$ implies the Hölder continuity of u in a similar way to that of [10].

From now on, we shall show that $u\in\mathcal{B}$ for some positive numbers M, γ, r, δ and κ. Let $x_0\in\overline{\Omega}$, $M=u^*$, where u^* is the same positive constant as in Theorem 2.2, and $\xi\in C^\infty([0,T]\times\overline{\Omega})$ with supp $\xi(t)\subset K_\rho(x_0)$ for $0\le t\le T$ and $0\le\xi\le1$ a.e. on $[0,T]\times\Omega$, and k be a number satisfying (3.7), and $\rho>0$, $0\le t_0\le t_0+\tau\le T$, and σ_1, $\sigma_2\in(0,1)$. Also, we put $v=u-u_b$, $f_1=wf-u_{bt}$ and $f_2=-g(u)\nabla u_b$. Then it holds that

$$v_t-\text{div}(g(u)\nabla v)=f_1-\text{div}f_2 \text{ in } Q(T). \tag{3.12}$$

By testing $[v-k]^+\xi^2\in H^1_0(\Omega)$ to (3.12), we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega|[v-k]^+|^2\xi^2dx+g_0\int_\Omega|\nabla[v-k]^+|^2\xi^2dx$$
$$\le\int_\Omega f_1[v-k]^+\xi^2dx+\int_\Omega f_2\nabla([v-k]^+\xi^2)dx$$
$$+\int_\Omega|[v-k]^+|^2\xi_t\xi dx+2\int_\Omega g(u)\nabla[v-k]^+[v-k]^+\xi\nabla\xi dx \text{ a.e. on } [0,T]. \tag{3.13}$$

It is easy to see that

$$\int_\Omega f_1[v-k]^+\xi^2(t)dx \le 2M|f_1|_{L^\infty(Q(T))}\text{mes}A_{k,\rho},$$

$$\begin{aligned}
&\int_\Omega f_2\nabla([v-k]^+\xi^2)dx\\
= &\int_\Omega f_2(\nabla[v-k]^+)\xi^2dx + 2\int_\Omega f_2[v-k]^+\xi\nabla\xi dx\\
\le &\frac{g_0}{4}\int_\Omega|\nabla[v-k]^+|^2\xi^2dx + (1+\frac{1}{g_0})|f_2|^2_{L^\infty(Q(T))}\text{mes}A_{k,\rho}\\
&+\int_{A_{k,\rho}}|[v-k]^+|^2|\nabla\xi|^2dx,
\end{aligned}$$

$$2\int_\Omega g(u)\nabla[v-k]^+[v-k]^+\xi\nabla\xi dx,$$

$$\le \frac{g_0}{4}\int_{A_{k,\rho}}|\nabla[v-k]^+|^2\xi^2dx + \frac{4(g^*)^2}{g_0}\int_{A_{k,\rho}}|[v-k]^+|^2|\nabla\xi|^2dx \text{ a.e. on } [0,T].$$

From these inequalities it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega|[v-k]^+|^2\xi^2dx + \frac{g_0}{2}\int_\Omega|\nabla[v-k]^+|^2\xi^2dx\\
&\le \gamma_1\text{mes}A_{k,\rho} + \gamma_1\int_{A_{k,\rho}}(|\nabla\xi|^2+|\xi_t|\xi)|[v-k]^+|^2dx,
\end{aligned}$$

where $\gamma_1 = 2M|f_1|_{L^\infty(Q(T))} + (1+\frac{1}{g_0})|f_2|^2_{L^\infty(\Omega)} + 1 + \frac{4(g^*)^2}{g_0}$. By integrating this inequality over $[t_0,t_1]$ for $0\le t_0\le t_1\le T$, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Omega|[v(t_1)-k]^+|^2\xi^2(t_1)dx + \frac{g_0}{2}\int_{t_0}^{t_1}\int_\Omega|\nabla[v-k]^+|^2\xi^2dxdt\\
&\le \frac{1}{2}\int_\Omega|[v(t_0)-k]^+|^2\xi^2(t_0)dx + \gamma_1\int_{t_0}^{t_1}\text{mes}A_{k,\rho}dxdt\\
&\quad+\int_{t_0}^{t_1}\int_{A_{k,\rho}}(|\nabla\xi|^2+\xi_t\xi)|[v-k]^+|^2dxdt.
\end{aligned}$$

Here, let r and q be two positive numbers satisfying $1/r+3/2q = 1+\kappa_*$ and $\kappa_* = 3/8$, for instance $q=r=4/3$. Clearly, we have

$$\int_{t_0}^{t_1}\text{mes}A_{k,\rho}(t)dxdt \le T^{\frac{1}{r'}}|\Omega|^{\frac{1}{q'}}\left(\int_{t_0}^{t_1}(\text{mes }A_{k,\rho}(t))^{\frac{r}{q}}dt\right)^{\frac{1}{r}},$$

where r' and q' are the dual indexes of r and q, respectively. Moreover, by setting $k := 2\kappa_*/3 = 1/4$ and taking $\hat{r}$ and $\hat{q}$ such that $\hat{r} = 2(1+k)r$ and $\hat{q} = 2(1+k)q$, we see that

$$\begin{aligned}
&\frac{1}{2}\int_\Omega|[v(t_1)-k]^+|^2\xi^2(t_1)dx + \frac{g_0}{2}\int_{t_0}^{t_1}\int_\Omega|\nabla[v-k]^+|^2\xi^2dxdt\\
&\le \frac{1}{2}\int_\Omega|[v(t_0)-k]^+|^2\xi^2(t_0)dx + \gamma_1T^{\frac{1}{r'}}|\Omega|^{\frac{1}{q'}}\left(\int_{t_0}^{t_1}(\text{mes }A_{k,\rho}(t))^{\frac{\hat{r}}{\hat{q}}}dt\right)^{\frac{2(1+r)}{\hat{r}}}\\
&\quad+\gamma_1\int_{t_0}^{t_1}\int_{A_{k,\rho}}(|\nabla\xi|^2+|\xi_t|\xi)|[v-k]^+|^2dxdt
\end{aligned}$$

$$\text{for } 0 \le t_0 \le t_1 \le T \text{ and } \frac{1}{\hat r} + \frac{3}{2\hat q} = \frac{3}{4}. \tag{3.14}$$

Then by using [10, Remark 7.2 in Chapter 2] (3.14) implies that $u \in \mathcal{B}(M, \gamma, \hat r, \kappa)$ for some $\gamma > 0$. Also, by (3.8), (3.14) and the Hölder continuity of u_b up to the boundary, we see that u satisfies the assumption of [10, Theorem 8.1 in Chapter 2]. Therefore, on account of [10, Theorems 7.1 and 8.1 in Chapter 2] we can show that v is Hölder continuous on $\overline{Q(T)}$. Finally, since u_b is Hölder continuous on $\overline{Q(T)}$, $u = v + u_b$ is Hölder continuous on $\overline{Q(T)}$, too. □

By using the estimate of ∇u on the boundary (Lemma 3.1) and the Hölder continuity on $\overline{Q(T)}$ (Lemma 3.2), we prove that $\nabla u \in L^p(Q(T))$ for any $p \ge 2$.

Lemma 3.3. *Under the same assumption as in Lemma 3.1, $\nabla u \in L^p(Q(T))$ for $p \ge 2$.*

Proof. For $p \ge 2$ and $M > 1$, we put

$$\varphi_M(r) := \begin{cases} r^p & \text{if } r < M, \\ M^p + \dfrac{(r-M)pM^{p-1}}{2} & \text{if } M \le r \le M+1, \\ M^p + \dfrac{pM^{p-1}}{2} & \text{if } r > M+1, \end{cases},$$

$$\widehat\varphi_M(r) = \int_0^r \varphi_M(s)ds,$$

$v(t) = |\nabla u(t)|^2$ and $M_1 := \max\{N_1^2, N_0\}$ where N_1 and N_0 are the same constants as in Lemma 3.1 and as in (2.2), respectively. Then, because of $u \in L^2(0,T;H^2(\Omega))$, we can see that $\frac{\partial}{\partial x_i}(\varphi_M([v(t)-M_1]^+)\frac{\partial u(t)}{\partial x_i}\xi^2) \in H$ for $1 \le i \le 3$ and a.e. $t \in [0,T]$, and $\xi \in C^\infty(\overline\Omega)$. By multiplying (1.1) by $\frac{\partial}{\partial x_i}(\varphi_M([v(t)-M_1]^+)\frac{\partial u(t)}{\partial x_i}\xi^2)$ and summing up from $i=1$ to 3, we have by partial integration and using Lemma 3.1

$$\frac12 \frac{d}{dt}\int_\Omega \widehat\varphi_M([v-M_1]^+)\xi^2 dx$$
$$+\sum_{k=1}^3\sum_{i=1}^3 \int_\Omega \frac{\partial}{\partial x_i}(g(u)\frac{\partial u}{\partial x_k}) \times \frac{\partial}{\partial x_k}\left[\varphi_M([v-M_1]^+)\frac{\partial u}{\partial x_i}\xi^2\right]dx$$
$$= \sum_{i=1}^3 \int_\Omega wf \times \left[\frac{\partial}{\partial x_i}(\varphi_M([v-M_1]^+))\frac{\partial u}{\partial x_i}\xi^2\right.$$
$$\left. +\varphi_M([v-M_1]^+)\frac{\partial^2 u}{\partial x_i^2}\xi^2 + \varphi_M([v-M_1]^+)\frac{\partial^2 u}{\partial x_i^2}2\xi\frac{\partial \xi}{\partial x_i}\right]dx$$
$$\text{a.e. on } Q(T).$$

Here, we note that the following properties hold:

(i) $r\varphi_M(r) \le (p+1)\widehat\varphi_M(r)$ for $r \ge 0$,
(ii) $\varphi_M(r) \le (p+1)\widehat\varphi_M(r) + 1$ for $r \ge 0$,
(iii) $\dfrac{\varphi_M(r)}{r^2} \le (p+1)\widehat\varphi_M(r) + 1$ for ≥ 0,
(iv) $\varphi_M(r)r^{\frac12} \le (p+1)\widehat\varphi_M(r) + 1$ for $r \ge 0$,
(v) $\varphi_M'(r)r \le p\varphi_M(r)$ for $r \ge 0$,
(vi) $\varphi_M'(r) \le p(\varphi_M(r)+1)$ for $r \ge 0$.

By using the above properties and Young's inequality, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \widehat{\varphi}_M([v(t)-M_1]^+)\xi^2dx + C_1\int_\Omega g(u(t))\varphi_M([v(t)-M_1]^+)|u_{xx}(t)|^2\xi^2$$

$$\le C_2\int_\Omega \varphi_M([v(t)-M_1]^+)v^2(t)\xi^2dx \tag{3.15}$$

$$+C_3\int_\Omega \widehat{\varphi}_M([v(t)-M_1]^+)\xi^2dx + C_4\int_\Omega \xi^2dx$$

$$+C_5\int_\Omega \widehat{\varphi}_M([v(t)-M_1]^+)|\nabla\xi|^2dx + C_6\int_\Omega |\nabla\xi|^2dx,$$

where $|u_{xx}|^2 = \sum_{i,j=1}^3 |\frac{\partial^2 u}{\partial x_i \partial x_j}|^2$ and $C_i(1\le i\le 6)$ is a positive constant depending on p, M_1, and $|wf|_{L^\infty(Q(T))}$.

Here, we estimate the term of $\int_\Omega \varphi_M([v-M_1]^+)v^2\xi^2dx$ in the following way. For $\varepsilon>0$ we take $\xi_l\in C^\infty(\overline{\Omega})$ for $l=1,\cdots l_\varepsilon$ where l_ε is a positive number determined later by

$$\begin{cases}\text{supp } \xi_l \subset B_{l,\varepsilon}(x_l) := \{x\in \mathbf{R}^n |\ |x_l - x| < \varepsilon\}, \quad 0\le \xi_l\le 1 \text{ on } \overline{\Omega}\\ \sum_{l=1}^{l_\varepsilon}\xi_l(x) = 1 \text{ on } \overline{\Omega}, \quad \Omega_l = \Omega\cap B_{l,\varepsilon}(x_l).\end{cases}$$

Then, for each $l\le l_\varepsilon$ and $x_l\in\Omega_l$, by taking $\xi=\xi_l$ and using $v\varphi_M([v-M_1]^+)\xi_l^2\nabla u = 0$ on $\partial\Omega_l$ we have

$$\int_\Omega \varphi_M([v(t)-M_1]^+)v^2(t)\xi_l^2dx = \int_{\Omega_l}\varphi_M([v(t)-M_1]^+)v^2(t)\xi_l^2dx$$

$$= \int_{\Omega_l}\nabla u(t)(\nabla u(t)\varphi_M([v(t)-M_1]^+)v(t)\xi_l^2dx$$

$$= \int_{\Omega_l}\nabla(u(t,x)-u(t,x_l))(\nabla u(t)\varphi_M([v(t)-M_1]^+)v(t)\xi_l^2)dx$$

$$= -\int_{\Omega_l}(u(t,x)-u(t,x_l))\text{div}(\nabla u(t)\varphi_M([v(t)-M_1]^+)v(t)\xi_l^2)dx$$

$$= -\int_{\Omega_l}(u(t,x)-u(t,x_l))\Big[\Delta u(t)\varphi_M([v(t)-M_1]^+)v(t)\xi_l^2$$

$$+\nabla u(t)\varphi_M'([v(t)-M_1]^+)\nabla v(t)v(t)\xi_l^2 + \nabla u(t)\varphi_M([v(t)-M_1]^+)v(t)\times 2\xi_l(t)\nabla\xi_l$$

$$+\nabla u(t)\varphi_M([v(t)-M_1]^+)\xi_l^2(t)\nabla v(t)\Big]dx =: \sum_{j=1}^4 I_j(t) \text{ for } t\in[0,T]. \tag{3.16}$$

Here, we put $\text{osc}\{u(t);\Omega_l\} := \max_{\Omega_l}u(t) - \min_{\Omega_l}u(t)$. By using the properties of φ_M and $\widehat{\varphi}_M$ and Young's inequality for any $\delta>0$, we have

$$I_1(t)\le \sqrt{3}\text{osc}\{u(t);\Omega_l\}\int_{\Omega_l}|u_{xx}(t)|\varphi_M([v(t)-M_1]^+)v(t)\xi_l^2dx$$

$$\le \frac{3}{2\delta}\text{osc}\{u(t);\Omega_l\}^2\int_{\Omega_l}g(u(t))|u_{xx}(t)|^2\varphi_M([v(t)-M_1]^+)\xi_l^2dx \tag{3.17}$$

$$+\frac{\delta}{2g^*}\int_{\Omega_l}\varphi_M([v(t)-M_1]^+)v^2(t)\xi_l^2dx,$$

$$I_2(t)\le \text{osc}\{u(t);\Omega_l\}\int_{\Omega_l}\varphi_M'([v(t)-M_1]^+)\times 2|u_{xx}(t)|v^2(t)\xi_l^2dx$$

$$= \mathrm{osc}\{u(t);\Omega_l\}\int_{\Omega_l}\varphi'_M([v(t)-M_1]^+)\times 2|u_{xx}(t)|[v(t)-M_1]v(t)\xi_l^2dx$$
$$+\mathrm{osc}\{u(t);\Omega_l\}\int_{\Omega_l}\varphi'_M([v(t)-M_1]^+)\times 2|u_{xx}(t)|M_1[v(t)-M_1]\xi_l^2dx$$
$$+\mathrm{osc}\{u(t);\Omega_l\}\int_{\Omega_l}\varphi'_M([v(t)-M_1]^+)\times 2|u_{xx}(t)|M_1^2\xi_l^2dx$$
$$\le 2\mathrm{osc}\{u(t);\Omega_l\}\int_{\Omega_l}(p+1)\varphi_M([v(t)-M_1]^+)|u_{xx}(t)|v(t)\xi_l^2dx$$
$$+2\mathrm{osc}\{u(t);\Omega_l\}\int_{\Omega_l}(p+1)\varphi_M([v(t)-M_1]^+)M_1|u_{xx}(t)|\xi_l^2dx$$
$$+2\mathrm{osc}\{u(t);\Omega_l\}\int_{\Omega_l}\frac{\varphi_M([v(t)-M_1]^+)}{|[v(t)-M_1]^+|}|u_{xx}(t)|M_1^2\xi_l^2dx$$
$$\le \frac{3}{2\delta}\mathrm{osc}\{u(t);\Omega_l\}^2\int_{\Omega_l}g(u(t))|u_{xx}(t)|^2\varphi_M([v(t)-M_1]^+)\xi_l^2dx \tag{3.18}$$
$$+\frac{2\delta(p+1)^2}{g^*}\int_{\Omega_l}\varphi_M([v(t)-M_1]^+)v^2(t)\xi_l^2dx$$
$$+\frac{\delta}{2}\left(\frac{4(p+1)^2M_1^2}{g^*}+\frac{4M_1^4}{g^*}\right)\int_{\Omega_l}((p+1)\widehat{\varphi}_M([v(t)-M_1]^+)+1)\xi_l^2dx,$$
$$I_3(t)\le \mathrm{osc}\{u(t);\Omega_l\}\int_{\Omega_l}\varphi_M([v(t)-M_1]^+)v^{\frac12}(t)v(t)\times 2\xi_l|\nabla\xi_l|dx$$
$$\le \frac{2}{\delta}\mathrm{osc}\{u(t);\Omega_l\}^2\int_{\Omega_l}\varphi_M([v(t)-M_1]^+)v(t)|\nabla\xi_l|^2dx$$
$$+\frac{\delta}{2}\int_{\Omega_l}\varphi_M([v(t)-M_1]^+)v^2(t)\xi_l^2(t)dx$$
$$\le \frac{2}{\delta}\mathrm{osc}\{u(t);\Omega_l\}^2\left[\int_{\Omega_l}(p+1)\widehat{\varphi}_M([v(t)-M_1]^+)|\nabla\xi_l|^2dx\right.$$
$$\left.+\int_{\Omega_l}M_1[(p+1)\widehat{\varphi}_M([v(t)-M_1]^+)+1]|\nabla\xi_l|^2dx\right]$$
$$+\frac{\delta}{2}\int_{\Omega_l}\varphi_M([v(t)-M_1]^+)v^2(t)\xi_l^2(t)dx, \tag{3.19}$$

and

$$I_4(t)\le \mathrm{osc}\{u(t);\Omega_l\}\int_{\Omega_l}\varphi_M([v(t)-M_1]^+)\times 2|u_{xx}(t)|v(t)\xi_l^2dx$$
$$\le \frac{1}{2\delta}\mathrm{osc}\{u(t);\Omega_l\}^2\int_{\Omega_l}g(u(t))\varphi_M([v(t)-M_1]^+)|u_{xx}(t)|^2\xi_l^2dx$$
$$+\frac{2\delta}{g^*}\int_{\Omega_l}\varphi_M([v(t)-M_1]^+)v^2(t)\xi_l^2dx \text{ for a.e. } t\in[0,T]. \tag{3.20}$$

By taking a suitable number $\delta>0$ in (3.17)–(3.20), from (3.16), it follows that

$$\frac12\int_{\Omega_l}\varphi_M([v(t)-M_1]^+)v^2(t)\xi_l^2dx\le$$
$$+C_7\mathrm{osc}\{u(t);\Omega_l\}^2\int_{\Omega_l}g(u(t))\varphi_M([v(t)-M_1]^+)|u_{xx}(t)|^2\xi_l^2dx$$
$$+C_8\int_{\Omega_l}(\widehat{\varphi}_M([v(t)-M_1]^+)+1)\xi_l^2dx$$

$$+C_9\text{osc}\{u(t);\Omega_l\}^2\int_{\Omega_l}(\widehat{\varphi}_M([v(t)-M_1]^+)+1)|\nabla\xi_l|^2dx \text{ for a.e. } t\in[0,T], \quad (3.21)$$

where $C_i (i=7,8,9)$ is a positive constant. By (3.15) replaced ξ and Ω by ξ_l and Ω_l we have

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_l}\widehat{\varphi}_M([v(t)-M_1]^+)\xi_l^2dx$$
$$+\Big(C_1-2C_2C_7\text{osc}\{u(t);\Omega_l\}^2\Big)\int_{\Omega_l}g(u(t))\varphi_M([v(t)-M_1]^+)|u_{xx}(t)|^2\xi_l^2dx$$
$$\le\Big(C_3+2C_2C_8\Big)\int_{\Omega_l}\widehat{\varphi}_M([v(t)-M_1]^+)\xi_l^2dx+\Big(C_4+2C_2C_8\Big)\int_{\Omega_l}\xi_l^2dx$$
$$+\Big(C_5+2C_2C_9\text{osc}\{u(t);\Omega_l\}^2\Big)\int_{\Omega}\widehat{\varphi}_M([v(t)-M_1]^+)|\nabla\xi_l|^2dx$$
$$+\Big(C_6+2C_2C_9\text{osc}\{u(t);\Omega_l\}^2\Big)\int_{\Omega_l}|\nabla\xi_l|^2dx. \quad (3.22)$$

Here, by Lemma 3.2 since the solution u is Hölder continuous with exponent α on $\overline{Q(T)}$ and the points which takes the maximum and minimum value of u on Ω_l belongs to $\Omega_l\subset B_{l,\varepsilon}$, there exists $\tilde{C}>0$ such that

$$\text{osc}\{u(t);\Omega_l\}\le|\text{max}_{\Omega_l}u(t)-\text{min}_{\Omega_l}u(t)|\le\tilde{C}\varepsilon^\alpha \text{ for } t\in[0,T].$$

Therefore, by taking $\varepsilon_0>0$ such that $\mu_0=C_1-2C_2C_7\tilde{C}^2\varepsilon_0^{2\alpha}>0$ from (3.22) there exists $C_i>0$ $(10\le i\le 13)$ such that

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_l}\widehat{\varphi}_M([v(t)-M_1]^+)\xi_l^2dx+\mu_0\int_{\Omega_l}g(u(t))\varphi_M([v(t)-M_1]^+)|u_{xx}(t)|^2\xi_l^2dx$$
$$\le C_{10}\int_{\Omega_l}\widehat{\varphi}_M([v(t)-M_1]^+)\xi_l^2dx+C_{11}\int_{\Omega_l}\xi_l^2dx$$
$$+C_{12}\int_{\Omega_l}\widehat{\varphi}_M([v(t)-M_1]^+)|\nabla\xi_l|^2dx+C_{13}\int_{\Omega_l}|\nabla\xi_l|^2dx \text{ for a.e. } t\in[0,T]. \quad (3.23)$$

Now, we take l_{ε_0} for ε_0 in (3.23) and set $\hat{K}=\max_{1\le l\le l_{\varepsilon_0}}|\nabla\xi_l|$. Then from (3.23), we have

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_l}\widehat{\varphi}_M([v(t)-M_1]^+)\xi_l^2dx+\mu_0\int_{\Omega_l}g(u(t))\varphi_M([v(t)-M_1]^+)|u_{xx}(t)|^2\xi_l^2dx$$
$$\le C_{10}\int_{\Omega_l}\widehat{\varphi}_M([v(t)-M_1]^+)\xi_l^2dx+C_{11}\int_{\Omega_l}\xi_l^2dx.$$
$$+C_{12}\hat{K}^2\int_{\Omega_l}\widehat{\varphi}_M([v(t)-M_1]^+)dx+C_{13}\hat{K}^2|\Omega| \text{ for a.e. } t\in[0,T]. \quad (3.24)$$

By the summation from $l=1$ to l_{ε_0} in (3.23) we have

$$\frac{1}{2}\frac{d}{dt}\sum_{l=1}^{l_{\varepsilon_0}}\int_{\Omega_l}\widehat{\varphi}_M([v(t)-M_1]^+)\xi_l^2dx$$
$$\le C_{10}\sum_{l=1}^{l_{\varepsilon_0}}\int_{\Omega_l}\widehat{\varphi}_M([v(t)-M_1]^+)\xi_l^2dx+C_{11}l_{\varepsilon_0}|\Omega|$$
$$+C_{12}l_{\varepsilon_0}\hat{K}^2\int_{\Omega}\widehat{\varphi}_M([v(t)-M_1]^+)dx+C_{13}l_{\varepsilon_0}\hat{K}^2|\Omega| \text{ for a.e. } t\in[0,T]. \quad (3.25)$$

We note that there exists $\hat{C} > 0$ such that $1 \leq \hat{C} \sum_{l=1}^{l_{\varepsilon_0}} \xi_l^2$. Then, it holds that

$$\int_\Omega \widehat{\varphi}_M([v(t) - M_1]^+)dx \leq \hat{C} \sum_{l=1}^{l_{\varepsilon_0}} \int_{\Omega_l} \widehat{\varphi}_M([v(t) - M_1]^+)\xi_l^2 dx \text{ for } t \in [0, T].$$

Therefore, from (3.25), we have

$$\frac{1}{2}\frac{d}{dt} \sum_{l=1}^{l_{\varepsilon_0}} \int_{\Omega_l} \widehat{\varphi}_M([v(t) - M_1])\xi_l^2 dx$$

$$\leq (C_{10} + C_{12}\hat{C} l_{\varepsilon_0} \hat{K}^2) \sum_{l=1}^{l_{\varepsilon_0}} \int_{\Omega_l} \widehat{\varphi}_M([v(t) - M_1]^+)\xi_l^2 dx + (C_{11} l_{\varepsilon_0} + C_{13} l_{\varepsilon_0} \hat{K}^2)|\Omega|$$

$$\text{for a.e. } t \in [0, T].$$

By setting $C_{14} = C_{10} + C_{12}\hat{C} l_{\varepsilon_0} \hat{K}^2$ and $C_{15} = C_{11} l_{\varepsilon_0} + C_{13} l_{\varepsilon_0} \hat{K}^2$ and by Gronwall's inequality, we have

$$\sum_{l=1}^{l_{\varepsilon_0}} \int_{\Omega_l} \widehat{\varphi}_M([v(t) - M_1]^+)\xi_l^2 dx \leq 2C_{15}|\Omega| e^{2C_{14}T} T \text{ for any } t \in [0, T]. \quad (3.26)$$

By the properties (ii) between φ and $\widehat{\varphi}_M$, from (3.25),

$$\int_\Omega \varphi_M([v(t) - M_1]^+) \leq (2(p+1)\hat{C} C_{15} e^{2C_{14}T} T + 1)|\Omega| \text{ for any } t \in [0, T]. \quad (3.27)$$

Since $\varphi_M(r) \to r^p$ as $M \to \infty$ for each $r > 0$, by Fatou's Lemma and (3.27), we obtain

$$\int_\Omega |[v(t) - M_1]^+|^p dx \leq \operatorname{liminf}_{M\to\infty} \int_\Omega \varphi_M([v(t) - M_1]^+)dx$$
$$\leq (2(p+1)\hat{C} C_{15} e^{2C_{14}T} T + 1)|\Omega|.$$

This implies that Lemma 3.3 holds. □

On account of Lemma 3.3, we see that various L^p-norm of ∇u are finite. By using this result, we prove the boundedness of ∇u.

Lemma 3.4. *Let $\{u, w\}$ be a solution of (P) under the assumption (A1)–(A7). Then, there exists $M > 0$ such that $|\nabla u| \leq M$ a.e. on $Q(T)$.*

Proof. First, for the number N_1 as in Lemma 3.1 we set $\tilde{M} := \{1, N_1, N_0^2\}$ where N_0 is the same positive constant as in (2.2) and consider $\frac{\partial}{\partial x_i}([v(t)-k]^+ \frac{\partial u(t)}{\partial x_i})$ for $k \geq \tilde{M}$ and $t \in [0, T]$, where $v(t) = |\nabla u(t)|^2$. Similarly to the proof of Lemma 3.3, we note that $[v(t) - k]^+ \frac{\partial u(t)}{\partial x_i} \in H_0^1(\Omega)$ for a.e. $t \in [0, T]$ and $\frac{\partial}{\partial x_i}([v(t) - k]^+ \frac{\partial u(t)}{\partial x_i}) \in H$ for a.e. $t \in [0, T]$ and $i = 1, 2, 3$. Now, we multiply $\frac{\partial}{\partial x_i}([v(t) - k]^+ \frac{\partial u(t)}{\partial x_i})$ to (1.1) and have

$$\frac{1}{2}\frac{d}{dt} \int_\Omega |[v(t) - k]^+|^2 dx$$

$$+ \sum_{k=1}^{3} \sum_{i=1}^{3} \int_\Omega \frac{\partial}{\partial x_i}(g(u)\frac{\partial u}{\partial x_k}) \times \frac{\partial}{\partial x_i}([v(t) - k]^+ \frac{\partial u}{\partial x_k})dx$$

$$= \sum_{i=1}^{3} \int_\Omega w(t) f(t) \times \left[\frac{\partial}{\partial x_i}([v(t) - k]^+)\frac{\partial u}{\partial x_i} + [v(t) - k]^+ \frac{\partial^2 u}{\partial x_i^2}\right] dx$$

$$\text{for a.e. } t \in [0, T].$$

Now, we set $A_k(t) = \{x \in \Omega | \ v(t,x) > k\}$ for $k \in \mathbf{R}$ and $t \in [0,T]$. By using Young's inequality to each term of the above identity for $\varepsilon > 0$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |[v(t)-k]^+|^2 dx \\
&+\left(\frac{1}{2} - |wf|_{L^\infty(Q(T))}\frac{\varepsilon L_g}{2}\right)\int_\Omega g(u)[v(t)-k]^+|u_{xx}|^2 dx + \frac{3}{8}\int_\Omega g(u)|\nabla v(t)|^2 dx \\
&\le \frac{L_g^2}{2g_0}\int_{A_k(t)} [v(t)-k]^+ v^2(t) dx + \frac{L_g^2}{g_0}\int_{A_k(t)} v^3(t) dx \\
&+\frac{2|wf|^2_{L^\infty(Q(T))}}{g_0}\int_{A_k(t)} v(t) dx + \frac{1}{2\varepsilon}\frac{1}{g_0}\int_{A_k(t)} [v(t)-k]^+ dx, \qquad (3.28)
\end{aligned}$$

where L_g is a positive constant given by (A2). By taking $\varepsilon_0 > 0$ such that $1/2 - 2(\varepsilon_0 |wf|_{L^\infty(Q(T))} L_g) > 0$ and integrating it over $[0,t]$ for $0 \le t \le T$, we have

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |[v(t)-k]^+|^2 dx + \frac{3}{8}\int_\Omega g(u)|\nabla v(t)|^2 dx \\
&\le \frac{L_g^2}{2g_0}\int_0^t\int_{A_k(\tau)} [v(\tau)-k]^+ v^2(\tau) dx d\tau + \frac{L_g^2}{g_0}\int_0^t\int_{A_k(\tau)} v^3(\tau) dx d\tau \\
&+\frac{2|wf|^2_{L^\infty(Q(T))}}{g_0}\int_0^t\int_{A_k(\tau)} v(\tau) dx d\tau + \frac{1}{2\varepsilon_0}\frac{1}{g_0}\int_0^t\int_{A_k(\tau)} [v(\tau)-k]^+ dx d\tau \\
&\text{for } t \in [0,T]. \qquad (3.29)
\end{aligned}$$

Let $q = 10$, $r = 2$, $\kappa = 1/2$ and q' and r' be the dual indexes of q and r, respectively. Then, it is easy to see that

$$\int_{A_k(t)} [v(t)-k]^+ v^2(t) \le \left(\int_{A_k(t)} v^{2q}(t)\right)^{\frac{1}{q}} \left(\int_{A_k(t)} ||[v(t)-k]^+|^2 + k^2|^{q'}\right)^{\frac{1}{q'}}, \quad (3.30)$$

$$\begin{aligned}
&\int_{A_k(t)} v^3(t) dx \le 2\int_{A_k(t)} v(t)(|[v(t)-k]^+|^2 + k^2) dx \\
&\le 2\left(\int_{A_k(t)} v^q(t)\right)^{\frac{1}{q}} \left(\int_{A_k(t)} ||[v(t)-k]^+|^2 + k^2|^{q'}\right)^{\frac{1}{q'}}, \qquad (3.31)
\end{aligned}$$

$$\begin{aligned}
&\int_{A_k(t)} v(t) dx = \int_{A_k(t)} ([v(t)-k]^+ + k) dx \\
&\le 2\int_{A_k(t)} (|[v(t)-k]^+|^2 + k^2| dx \\
&\le |\Omega|^{\frac{1}{q}} \left(\int_{A_k(t)} ||[v(t)-k]^+|^2 + k^2|^{q'}\right)^{\frac{1}{q'}}, \qquad (3.32)
\end{aligned}$$

$$\begin{aligned}
&\int_{A_k(t)} [v(t)-k]^+ dx \le \int_{A_k(t)} ||[v(t)-k]^+|^2 + k^2| dx \\
&\le |\Omega|^{\frac{1}{q}} \left(\int_{A_k(t)} ||[v(t)-k]^+|^2 + k^2|^{q'} dx\right)^{\frac{1}{q'}} \text{ for } t \in [0,T]. \qquad (3.33)
\end{aligned}$$

By (3.29)–(3.33), we have

$$\frac{1}{2}\int_\Omega |[v(t)-k]^+|^2 dx + \frac{3}{8}\int_\Omega g(u)|\nabla v(t)|^2 dx$$

$$\leq C_0\Big[\Big(\int_0^t\int_{A_k(\tau)} v^{2q}(\tau)dxd\tau\Big)^{\frac{1}{q}}+\Big(\int_0^t\int_{A_k(\tau)} v^{q}(\tau)dxd\tau\Big)^{\frac{1}{q}}+2t^{\frac{1}{p}}|\Omega|^{\frac{1}{q}}\Big]$$
$$\times\Big[\Big(\int_0^t\int_{A_k(\tau)} |[v(\tau)-k]^+|^{2q'}dxd\tau\Big)^{\frac{1}{q'}}+k\Big(\int_0^t \mathrm{mes}A_k(\tau)d\tau\Big)^{\frac{1}{q'}}\Big] \text{ for } 0\leq t\leq T, \tag{3.34}$$

where C_0 is a positive constant. Here, we note that

$$\Big(\int_0^t\int_{A_k(\tau)} |[v(\tau)-k]^+|^{2q'}dxd\tau\Big)^{\frac{1}{q'}}$$
$$\leq |\Omega|^{\frac{\kappa}{q'(1+\kappa)}}t^{\frac{\kappa}{q'(1+\kappa)}}\Big(\int_0^t\int_{A_k(\tau)} |[v(\tau)-k]^+|^{2q'(1+\kappa)}dxd\tau\Big)^{\frac{1}{q'(1+\kappa)}} \text{ for } 0\leq t\leq T. \tag{3.35}$$

Therefore, by (3.34) with (3.35), we obtain

$$\frac{1}{2}\int_\Omega |[v(t)-k]^+|^2dx+\frac{3}{8}\int_\Omega g(u)|\nabla v(t)|^2dx$$
$$\leq C_0\Big[\Big(\int_0^t\int_{A_k(\tau)} v^{20}(\tau)dxd\tau\Big)^{\frac{1}{10}}+\Big(\int_0^t\int_{A_k(\tau)} v^{10}(\tau)dxd\tau\Big)^{\frac{1}{10}}+2t^{\frac{1}{10}}|\Omega|^{\frac{1}{10}}\Big]$$
$$\times\Big[|\Omega|^{\frac{3}{10}}t^{\frac{3}{10}}\Big(\int_0^t\int_{A_k(\tau)} |[v(\tau)-k]^+|^{\frac{10}{3}}dxd\tau\Big)^{\frac{3}{5}}+k^2\Big(\int_0^t \mathrm{mes}A_k(\tau)d\tau\Big)^{\frac{9}{10}}\Big]$$
$$\text{for } 0\leq t\leq T. \tag{3.36}$$

Here, Lemma 3.3 guarantees that $\int_0^T\int_\Omega v^{20}(t)dxdt$ and $\int_0^T\int_\Omega v^{10}(t)dxdt$ are finite. Also, by (2) of Lemma 2.4 in Section 2 with $q=10/3$, $m=r=2$ and $n=3$, we have for $z\in C([0,T];H)\cap L^2(0,T;H^1(\Omega))$,

$$\begin{aligned}\Big(\int_0^t |z|^{\frac{10}{3}}_{L^{\frac{10}{3}}(\Omega)}d\tau\Big)^{\frac{3}{5}} &\leq \Big(\int_0^t C^*|\nabla z|_H^2|z|_H^{\frac{4}{3}}d\tau\Big)^{\frac{3}{5}}\\ &\leq C^*\max_{0\leq s\leq t}|z(s)|_H^{\frac{4}{5}}|\nabla z|^{\frac{6}{5}}_{L^2(0,t;H)}\\ &\leq C_*(|z|_{C([0,t];H)}+|\nabla z|_{L^2(0,t;H)})^2.\end{aligned} \tag{3.37}$$

Here, for any $t\in[0,T]$, we denote by $V(t)=C([0,t];H)\cap L^2(0,t;H_0^1(\Omega))$ with the norm

$$|z|_{V(t)}=\max_{0\leq s\leq t}|z(s)|_H+|z|_{L^2(0,t;H_0^1(\Omega))} \text{ for } z\in V(t).$$

By putting $\tilde{C_0}=C_0\Big[\Big(\int_0^T\int_\Omega v^{20}(t)dxdt\Big)^{\frac{1}{10}}+\Big(\int_0^T\int_\Omega v^{10}(t)dxdt\Big)^{\frac{1}{10}}+2T^{\frac{1}{10}}|\Omega|^{\frac{1}{10}}\Big]$ and using (3.37) from (3.36) it follows that

$$\Big(\min\Big\{\frac{1}{2},\frac{3}{8}g_0\Big\}-\tilde{C_0}|\Omega|^{\frac{3}{10}}t^{\frac{3}{10}}\Big)|[v-k]^+|^2_{V(t)}\leq \tilde{C_0}k^2\Big(\int_0^t \mathrm{mes}A_k(\tau)d\tau\Big)^{\frac{9}{10}}$$
$$\text{for } 0\leq t\leq T.$$

By taking $t^*>0$ such that $m_0:=\min\{\frac{1}{2},\frac{3}{8}g_0\}-\tilde{C_0}|\Omega|^{\frac{3}{10}}(t_*)^{\frac{3}{10}}>0$, we obtain

$$|[v-k]^+|_{V(t)}\leq 2\sqrt{\frac{\tilde{C_0}}{m_0}}k\Big(\int_0^t \mathrm{mes}A_k(\tau)d\tau\Big)^{\frac{3}{10}(1+\frac{1}{2})} \quad \text{for } t\leq t^*. \tag{3.38}$$

Therefore, from Theorem 6.1 in Section 2 of [10], we see that there exists $\tilde{k} > 1$ such that $v \leq 2M_1\tilde{k}$ for a.e. on $Q(t)$ for $t < t^*$. Since the choice of t^* does not depend on the initial data, by repeating this argument we conclude that $v \leq 2M_1\tilde{k}$ for a.e. on $Q(T)$. Thus, Lemma 3.4 is proved. □

4. **Boundedness of** $|u_t|_H$**.** In the previous section, we showed that $\nabla u \in L^\infty(Q(T))$. The aim of this section is to establish $u_t \in L^\infty(0,T;H)$.

Lemma 4.1. *If* $\{u, w\}$ *is a solution of (P) with* $\kappa_0 \leq u \leq u^*$ *a.e. on* $Q(T)$*. Then* $u_t \in L^\infty(0,T;H)$

Proof. First, we consider the following problem (AP):

$$\begin{cases} v_t - \Delta(g(u)v) = F & \text{in } Q(T), \\ v = 0 & \text{on } S(T), \\ v(0) = v_0 & \text{in } \Omega, \end{cases}$$

where $F = (wf)_t + \Delta(g(u)u_{bt}) - u_{btt}$ and $v_0 = w(0)f(0) + \Delta G(u(0)) - u_{bt}(0)$. Here, from (A4) and (A6) we note that $F \in L^2(0,T;H)$ and $v_0 \in H$.

First, we show that (AP) has a weak solution $v \in V(T)$ in the following sense:

$$-\int_{Q(T)} v(t)\eta_t(t)dxdt - \int_\Omega v(0)\eta(0)dx$$
$$+\int_{Q(T)} \nabla(g(u(t))v(t))\nabla\eta(t)dxdt = \int_{Q(T)} F(t)\eta(t)dxdt,$$
$$\text{for } \eta \in W^{1,2}(0,T;H) \cap L^2(0,T;H^1_0(\Omega)) \text{ with } \eta(T) = 0. \tag{4.1}$$

Indeed, we take sequences $\{\mathcal{H}_n\} \subset C^\infty(\overline{Q(T)})$ such that $g_0 \leq \mathcal{H}_n \leq g^*$ on $Q(T)$ and $|\nabla\mathcal{H}_n| \leq M$ on $Q(T)$, where g^* and M are the same positive constants as in (2.3) and Lemma 3.4, and $\mathcal{H}_n \to g(u)$ in $L^2(0,T;H^1(\Omega))$ as $n \to \infty$, and $\{v_{0,n}\} \subset C^\infty(\Omega)$ such that $v_{0,n} \to v_0$ in H as $n \to \infty$. Since $\mathcal{H}_n$, $\nabla\mathcal{H}_n$ and $\Delta\mathcal{H}_n$ belong to $L^\infty(Q(T))$ for each n, the classical theory for parabolic equations leads to the following problem

$$\begin{cases} v_t - \Delta(\mathcal{H}_n v) = F & \text{in } Q(T), \\ v = 0 & \text{on } S(T), \\ v(0) = v_{0,n} & \text{in } \Omega, \end{cases}$$

has a solution $v = v_n \in W^{1,2}(0,T;H) \cap L^\infty(0,T;H^1_0(\Omega))$. Then, by multiplying v_n to the equation we can have the following estimate:

$$|v_n|_{L^\infty(0,T;H)} + |v_n|_{L^2(0,T;H^1(\Omega))} \leq M_2,$$

where M_2 is a positive constant independent of n. Therefore, we can take a subsequence $\{v_{nk}\} \subset \{v_n\}$ such that for some $v \in L^\infty(0,T;H) \cap L^2(0,T;H^1(\Omega))$, $v_{nk} \to v$ weakly in $L^2(0,T;H^1(\Omega))$ and weakly star in $L^\infty(0,T;H)$ as $k \to \infty$.

Let $\eta \in W^{1,2}(0,T;H) \cap L^2(0,T;H^1_0(\Omega))$ with $\eta(T) = 0$. For any $\delta > 0$, there exists $\eta_\delta \in C^\infty(\overline{Q(T)})$ such that $|\nabla\eta - \nabla\eta_\delta|_{L^2(Q(T))} < \delta$. Therefore, we see that

$$\begin{aligned} & \left|\int_{Q(T)} (\nabla(\mathcal{H}_{nk}v_{nk}) - \nabla(g(u)v))\nabla\eta dxdt\right| \\ = & \left|\int_{Q(T)} (\nabla\mathcal{H}_{nk}v_{nk} - \nabla g(u)v)\nabla\eta dxdt + \int_{Q(T)} (\mathcal{H}_{nk}\nabla v_{nk} - g(u)\nabla v)\nabla\eta dxdt\right| \end{aligned}$$

$$\begin{aligned}
&\le \left|\int_{Q(T)} \nabla \mathcal{H}_{nk}(v_{nk}-v)\nabla \eta dxdt + \int_{Q(T)} (\nabla \mathcal{H}_{nk} - \nabla g(u)) v \nabla \eta dxdt\right| \\
&\quad + \left|\int_{Q(T)} (\mathcal{H}_{nk} - g(u)) \nabla v_{nk} \nabla \eta dxdt + \int_{Q(T)} g(u)(\nabla v_{nk} - \nabla v)\nabla \eta dxdt\right| \\
&\to 0 \text{ as } k \to \infty.
\end{aligned} \tag{4.2}$$

Also, by integration by parts we have

$$-\int_{Q(T)} v_{nk}(t)\eta_t(t)dxdt - \int_\Omega v(0)\eta(0)dx$$
$$+\int_{Q(T)} \nabla(H_{nk}(t)v_{nk}(t))\nabla\eta(t)dxdt = \int_{Q(T)} F(t)\eta(t)dxdt. \tag{4.3}$$

by letting $k \to \infty$ in (4.2) on account of (4.3) we obtain (4.1).

To accomplish the proof of this lemma it is sufficient to show that $v = u_t - u_{bt}$. To do so, let $\tilde\eta$ be any function in $W^{1,2}(0,T;H) \cap L^2(0,T;H_0^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$ with $\tilde\eta(T) = 0$. Then, we can take sequence $\{\eta_n\} \subset C^\infty(\overline{Q(T)})$ such that $\tilde\eta_n \to \tilde\eta$ in $W^{1,2}(0,T;H) \cap L^2(0,T;H_0^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$ as $n \to \infty$. Then, by multiplying $\tilde\eta_{nt}$ to (1.1) we have

$$\int_{Q(T)} (u_t - u_{bt})\tilde\eta_{nt}dxdt + \int_{Q(T)} \nabla G(u)\nabla\tilde\eta_{nt}dxdt$$
$$= \int_{Q(T)} (wf - u_{bt})\tilde\eta_{nt}dxdt \text{ for each } n. \tag{4.4}$$

Here, by using integration by parts it holds that

$$\begin{aligned}
&\int_{Q(T)} \nabla G(u)\nabla\tilde\eta_{nt}dxdt \\
=\ & -\int_{Q(T)} G(u)\Delta\tilde\eta_{nt}dxdt + \int_{S(T)} G(u_b)\frac{\partial\tilde\eta_{nt}}{\partial\nu}dxdt \\
=\ & \int_{Q(T)} g(u)u_t\Delta\tilde\eta_n dxdt + \int_\Omega G(u_0)\Delta\tilde\eta_n(0)dx \\
&+\int_{S(T)} G(u_b)\frac{\partial\tilde\eta_{nt}}{\partial\nu}dxdt \text{ for each } n.
\end{aligned}$$

Therefore, from (4.4) we obtain

$$\int_{Q(T)} (u_t - u_{bt})\tilde\eta_{nt}dxdt + \int_{Q(T)} g(u)(u_t - u_{bt})\Delta\tilde\eta_n dxdt$$
$$= \int_{Q(T)} (wf - u_{bt})\tilde\eta_{nt}dxdt - \int_{Q(T)} g(u)u_{bt}\Delta\tilde\eta_n dxdt$$
$$-\int_\Omega G(u_0)\Delta\tilde\eta_n(0)dx - \int_{S(T)} G(u_b)\frac{\partial\tilde\eta_{nt}}{\partial\nu}dxdt. \tag{4.5}$$

We note that

$$\int_{Q(T)} (wf - u_{bt})\tilde\eta_{nt}dxdt$$
$$= -\int_{Q(T)} ((wf)_t - u_{btt})\tilde\eta_n dxdt - \int_\Omega (w(0)f(0) - u_{bt}(0))\tilde\eta_n(0)dxdt, \tag{4.6}$$
$$-\int_{Q(T)} g(u)u_{bt}\Delta\tilde\eta_n dxdt$$

$$= -\int_{Q(T)} \Delta(g(u)u_{bt})\tilde{\eta}_n dxdt - \int_{S(T)} g(u_b)u_{bt}\frac{\partial \tilde{\eta}_n}{\partial \nu}dxdt, \tag{4.7}$$

$$\int_{\Omega} G(u_0)\Delta\tilde{\eta}_n(0)dx = \int_{\Omega} \Delta G(u_0)\tilde{\eta}_n(0)dx + \int_{\partial\Omega} G(u_b(0))\frac{\partial \tilde{\eta}_n(0)}{\partial \nu}dx, \tag{4.8}$$

and

$$\begin{aligned}
&\int_{S(T)} G(u_b)\frac{\partial \tilde{\eta}_{nt}}{\partial \nu}dxdt \\
= &\int_{Q(T)} \nabla G(u)\nabla\tilde{\eta}_{nt}dxdt + \int_{Q(T)} G(u)\Delta\tilde{\eta}_{nt}dxdt \\
= &\int_{Q(T)} (G(u)-G(u_b))_t\Delta\tilde{\eta}_n dxdt + \int_{\Omega}(G(u(0))-G(u_b(0)))\Delta\tilde{\eta}_n(0)dx \\
&-\int_{Q(T)} (\nabla G(u_b))_t\nabla\tilde{\eta}_n dxdt - \int_{\Omega}\nabla G(u_b(0))\nabla\tilde{\eta}_n(0)dx \\
&-\int_{Q(T)} (G(u))_t\Delta\tilde{\eta}_n dxdt - \int_{\Omega} G(u_0)\Delta\tilde{\eta}_n(0)dx \\
= &-\int_{Q(T)} (G(u_b))_t\Delta\tilde{\eta}_n dxdt - \int_{\Omega} G(u_b(0))\Delta\tilde{\eta}_n(0)dx \\
&-\int_{Q(T)} (\nabla G(u_b))_t\nabla\tilde{\eta}_n dxdt - \int_{\Omega}\nabla G(u_b(0))\nabla\tilde{\eta}_n(0)dx \\
&= -\int_{S(T)} g(u_b)u_{bt}\frac{\partial \tilde{\eta}_n}{\partial \nu}dxdt - \int_{\partial\Omega} G(u_b(0))\frac{\partial \tilde{\eta}_n(0)}{\partial \nu}dx.
\end{aligned} \tag{4.9}$$

Therefore, by (4.6)–(4.9), we have

$$\begin{aligned}
&\int_{Q(T)} (u_t-u_{bt})\tilde{\eta}_{nt}dxdt + \int_{Q(T)} g(u)(u_t-u_{bt})\Delta\tilde{\eta}_n dxdt \\
&\qquad = -\int_{Q(T)} F\tilde{\eta}_n dxdt - \int_{\Omega} v(0)\tilde{\eta}_n(0)dx.
\end{aligned} \tag{4.10}$$

By $n \to \infty$ in (4.10) we get

$$\begin{aligned}
&\int_{Q(T)} (u_t-u_{bt})\tilde{\eta}_t dxdt + \int_{Q(T)} g(u)(u_t-u_{bt})\Delta\tilde{\eta} dxdt \\
&\qquad = -\int_{Q(T)} F\tilde{\eta} dxdt - \int_{\Omega} v(0)\tilde{\eta}(0)dx.
\end{aligned} \tag{4.11}$$

Therefore, by (4.1) and (4.11) we have

$$\begin{aligned}
&\int_{Q(T)} (v(t)-u_t(t)+u_{bt}(t))\tilde{\eta}_t(t)dxdt \\
&+\int_{Q(T)} g(u(t))(v(t)-u_t(t)+u_{bt}(t))\Delta\tilde{\eta}(t)dxdt = 0.
\end{aligned} \tag{4.12}$$

Here, we set $\tilde{u}(t) = u(T-t)$ and $\tilde{\varphi}(t) = \varphi(T-t)$ for any $\varphi \in C^\infty(\overline{Q(T)})$. Then, by the result of [10], we see that the following problem

$$\begin{cases} \bar{\eta}_t - g(\tilde{u})\Delta\bar{\eta} = \tilde{\varphi} \text{ in } Q(T), \\ \bar{\eta} = 0 \text{ on } S(T), \\ \bar{\eta}(0) = 0 \text{ in } \Omega \end{cases}$$

has a unique solution $\bar\eta \in W^{1,2}(0,T;H) \cap L^2(0,T;H_0^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$. Put $\eta^*(t) = \bar\eta(T-t)$. Then we have $\eta^*_t + g(u)\Delta\eta^* = -\varphi$ in $Q(T)$, $\eta^* = 0$ on $S(T)$ and $\eta^*(T) = 0$ in Ω. Accordingly, by taking η^* as $\tilde\eta$ in (4.12) we have

$$\int_{Q(T)} (v - u_t + u_{bt})\varphi dxdt = 0 \text{ for any } \varphi \in C^\infty(\overline{Q(T)}).$$

This implies that $v = u_t - u_{bt}$ a.e. in $Q(T)$. Finally, since $v \in L^\infty(0,T;H)$, we have $u_t \in L^\infty(0,T;H)$. Thus, Lemma 4.1 is proved. □

5. **Proof of Theorem 2.3.** In this Section, by using the properties (I) and (II) we prove the uniqueness of a solution of (P). Let $\{u_1, w_1\}$ and $\{u_2, w_2\}$ be solutions of (P) on $[0,T]$, $\xi_1, \xi_2 \in L^2(0,T;H)$ defined by (S3), that is, $\xi_i(t) \in \partial I(u_i(t); w_i(t))$ for a.e. $t \in [0,T]$, and $u = u_1 - u_2$ and $w = w_1 - w_2$ on $Q(T)$. Then it follows that

$$u_t - \text{div}(g(u_1)\nabla u_1 - g(u_2)\nabla u_2) = wf \quad \text{in } Q(T), \tag{5.1}$$

$$w_{it} - \nu\Delta w_i + \xi_i = 0 \quad \text{in } Q(T) \text{ for } i = 1,2, \tag{5.2}$$

$$u = 0, w_i = w_b \quad \text{on } S(T), \text{ for } i = 1,2,$$

$$u(0) = 0, w_i(0) = w_0 \quad \text{on } \Omega, \text{ for } i = 1,2.$$

For simplicity, for $0 < s \le T$ and $x \in \Omega$ we put

$$M(s,x) = \max\{|f_*(u_1(x)) - f_*(u_2(x))|_{L^\infty(0,T)}, |f^*(u_1(x)) - f^*(u_2(x))|_{L^\infty(0,T)}\}.$$

Here, we recall the following lemma.

Lemma 5.1.

$$|w|_{L^\infty(0,s)} \le M(s) \quad \textit{a.e. on } Q(s) \textit{ for } 0 < s \le T.$$

This Lemma is already proved by [9, Lemma 3.1] so that we omit this proof.

Proof of Theorem 2.3. First, from Lemma 4.1, $w_i f \in L^\infty(Q(T))$ and $u_{it} \in L^\infty(0,T;H)$ for $i = 1,2$, we see that

$$|\Delta G(u_i)|_{L^\infty(0,T;H)} \le |w_i f - u_{it}|_{L^\infty(0,T;H)} < +\infty.$$

Therefore, by Lemma 3.4, we see that there exists $R > 0$ such that

$$|\Delta u_i|_{L^\infty(0,T;H)} = \left|\frac{\Delta G(u_i) - g'(u_i)|\nabla u_i|^2}{g(u_i)}\right|_{L^\infty(0,T;H)} \le R. \tag{5.3}$$

(5.1) implies that

$$u_t - [g'(u_1)|\nabla u_1|^2 - g'(u_2)|\nabla u_2|^2 + g(u_1)\Delta u_1 - g(u_2)\Delta u_2] = wf \quad \text{in } Q(T),$$

we multiply it by $(-\Delta u)$ we obtain

$$\frac{1}{2}\frac{d}{dt}|\nabla u|_H^2 + g_0|\Delta u|_H^2 \le (wf, -\Delta u)_H$$
$$+([g'(u_1)|\nabla u_1|^2 - g'(u_2)|\nabla u_2|^2, -\Delta u)_H + ((g(u_1) - g(u_2))\Delta u_2, -\Delta u)_H$$
$$\text{a.e. on } [0,T]. \tag{5.4}$$

First, it is easy to see that

$$(wf, -\Delta u)_H \le \frac{2|f|^2_{L^\infty(Q(T))}}{g_0}|w|_H^2 + \frac{g_0}{8}|\Delta u|_H^2. \tag{5.5}$$

Next, it holds that

$$([g'(u_1)|\nabla u_1|^2 - g'(u_2)|\nabla u_2|^2, -\Delta u)_H$$

$$= ((g'(u_1) - g'(u_2))|\nabla u_1|^2 + g'(u_2)\nabla u \nabla(u_1 + u_2), -\Delta u)_H,$$

$$((g'(u_1) - g'(u_2))|\nabla u_1|^2, -\Delta u)_H \le \frac{2(M^2 L_g C_P)^2}{g_0}|\nabla u|_H^2 + \frac{g_0}{8}|\Delta u|_H^2, \tag{5.6}$$

where M is the same positive constant as in Lemma 3.4, L_g is a Lipschitz constant of g and C_P is a positive constant by Poincare's inequality, and

$$(g'(u_2)\nabla u \nabla(u_1 + u_2), -\Delta u)_H \le \frac{2(2L_g M)^2}{g_0}|\nabla u|_H^2 + \frac{g_0}{8}|\Delta u|_H^2 \text{ a.e. on } [0,T]. \tag{5.7}$$

For the last term of the right hand side in (5.4), from (5.3) we have

$$((g(u_1) - g(u_2))\Delta u_2, -\Delta u)_H \le \frac{2L_g^2}{g_0}|u|_{L^\infty(\Omega)}^2 R + \frac{g_0}{8}|\Delta u|_H^2. \tag{5.8}$$

Here, by (1) and (2) of Lemma 2.4 in Section 2 with $q = \infty$, $n = 3$, $r = 2$ and $m = 6$ for any $\eta > 0$ we obtain

$$\begin{aligned}
|z|_{L^\infty(\Omega)}^2 &\le C^*|\nabla z|_{L^6(\Omega)}^{\frac{3}{2}}|z|_H^{\frac{1}{2}} \le C^*|z|_{H^2(\Omega)}^{\frac{3}{2}}|z|_H^{\frac{1}{2}} \\
&\le C^* C_*^{\frac{3}{2}}(|\Delta z|_H + |z|_H)^{\frac{3}{2}}|z|_H^{\frac{1}{2}} \\
&\le C^* C_*^{\frac{3}{2}}\left(\frac{3\eta^{\frac{4}{3}}}{4}(|\Delta z|_H^2 + |z|_H^2) + \frac{1}{4\eta^4}|z|_H^2\right) \\
&= \frac{3}{4}C^* C_*^{\frac{3}{2}}\eta^{\frac{4}{3}}|\Delta z|_H^2 + C^* C_*^{\frac{3}{2}}\left(\frac{3}{4}\eta^{\frac{4}{3}} + \frac{1}{4\eta^4}\right)C_P|\nabla z|_H^2 \text{ for } z \in H^2(\Omega).
\end{aligned} \tag{5.9}$$

Therefore, by adding (5.4)–(5.9), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\nabla u|_H^2 + \left(\frac{g_0}{2} - \frac{3}{2}\frac{L_g^2}{g_0}RC^* C_*^{\frac{3}{2}}(\eta)^{\frac{4}{3}}\right)|\Delta u|_H^2 \le \frac{2|f|_{L^\infty(Q(T))}^2}{g_0}|w|_H^2 \\
&+\left(\frac{2(M^2 L_g C_P)^2}{g_0} + \frac{2(2L_g M)^2}{g_0} + \frac{2L_g^2}{g_0}RC^* C_*^{\frac{3}{2}}C_P\left(\frac{3}{4}\eta^{\frac{4}{3}} + \frac{1}{4\eta^4}\right)\right)|\nabla u|_H^2 \\
&\text{a.e. on } [0,T].
\end{aligned} \tag{5.10}$$

Therefore, by taking η_0 such that $\tilde{m} = g_0/2 - 3/2(L_g^2 RC^* C_*^{\frac{3}{2}}(\eta)^{\frac{4}{3}})/g_0 > 0$ and putting the coefficient of $|\nabla u|_H^2$ in (5.10) by $\tilde{C}_1 = \tilde{C}_1(\eta_0)$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\nabla u(t)|_H^2 + \tilde{m}|\Delta u(t)|_H^2 \le \frac{2|f|_{L^\infty(Q(T))}^2}{g_0}|w(t)|_H^2 + \tilde{C}_1|\nabla u(t)|_H^2 \\
&\text{for a.e. } t \in [0,T].
\end{aligned} \tag{5.11}$$

By setting $I(t) = 1/2|\nabla u(t)|_H^2 + \tilde{m}\int_0^t |\Delta u(\tau)|_H^2 d\tau$ for $t \in [0,T]$ and using the Gronwall's lemma, we get

$$I(t) \le \left(\frac{2|f|_{L^\infty(Q(T))}^2}{g_0}\int_0^t |w(\tau)|_H^2 d\tau\right)e^{\tilde{C}_1 t} \text{ for } t \in [0,T]. \tag{5.12}$$

On the other hand, since $u_t = wf - \operatorname{div}(g(u_1)\nabla u_1 - g(u_2)\nabla u_2)$, by (5.1)

$$\begin{aligned}
|u_t|_H^2 &\le \tilde{C}_2(|f|_{L^\infty(Q(T))}^2|w|_H^2 + [(M^2 L_g C_P)^2 \\
&+(2L_g M)^2]|\nabla u|_H^2 + (RL_g)^2|u|_{L^\infty(\Omega)}^2 + (g^*)^2|\Delta u|_H^2 \text{ a.e. on } [0,T].
\end{aligned}$$

By using (5.9) with $\eta = 1$ we have

$$|u_t(t)|_H^2 \le \tilde{C}_2|f|_{L^\infty(Q(T))}^2|w(t)|_H^2 + \tilde{C}_3|\nabla u(t)|_H^2 + \tilde{C}_4|\Delta u(t)|_H^2) \text{ for } t \in [0,T], \tag{5.13}$$

where $\tilde{C}_3 = \tilde{C}_2((M^2 L_g C_P)^2 + (2L_g M)^2 + R^2 L_g C^* C_*^{\frac{3}{2}} C_P)$ and $\tilde{C}_4 = \tilde{C}_2((g^*)^2 + R^2 L_g 3/4 C^* C_*^{\frac{3}{2}})$. Therefore, by integrating (5.13) over $[0,t]$ we obtain

$$\int_0^t |u_t(\tau)|_H^2 d\tau \le \tilde{C}_2 |f|^2_{L^\infty(Q(T))} \int_0^t |w(\tau)|_H^2 d\tau$$
$$+\tilde{C}_3 t \max_{0\le t\le \tau} |\nabla u(\tau)|_H^2 + \tilde{C}_4 \int_0^t |\Delta u(\tau)|_H^2 d\tau. \tag{5.14}$$

By using the estimate (5.12) to (5.14), we get

$$\int_0^t |u_t(\tau)|_H^2 d\tau \le \tilde{C}_2 |f|^2_{L^\infty(Q(T))} \int_0^t |w(\tau)|_H^2 d\tau$$
$$+\max(\tilde{C}_3, \tilde{C}_4)(t+1)\left(2 + \frac{1}{\tilde{m}}\right)\left(\frac{2|f|^2_{L^\infty(Q(T))}}{g_0} \int_0^t |w(\tau)|_H^2 d\tau\right) e^{\tilde{C}_1 t} \text{ for } t \in [0,T]. \tag{5.15}$$

Therefore, by (5.12) and (5.15), we see that the following inequality holds:

$$\int_0^t |u_t(\tau)|_H^2 d\tau + \max_{0\le s\le t} |\nabla u(s)|_H^2 + \int_0^t |\Delta u(\tau)|_H^2 d\tau \le \tilde{C}_6 \int_0^t |w(\tau)|_H^2 d\tau$$
$$\text{for } t \in [0,T], \tag{5.16}$$

where $\tilde{C}_6$ is a positive constant. Here, by putting $p_0 = q_0 = 2$, $p_1 = q_1 = 10/3$ and $q_2 = 5$ in (2.6) of Lemma 2.4 in Section 2, we see that there exists $C_e > 0$ such that

$$|u|_{L^5(\Omega, C([0,T]))} \le C_e(|u_t|_{L^2(0,T;H)} + |u|_{L^{\frac{10}{3}}(0,T;W^{1,\frac{10}{3}}(\Omega))}). \tag{5.17}$$

By using (5.11), we obtain

$$\left(\int_0^t |u(\tau)|^{\frac{10}{3}}_{L^{\frac{10}{3}}(\Omega)} d\tau\right)^{\frac{3}{10}} \le \mathcal{C}\left(\int_0^t |\nabla u(\tau)|_H^{\frac{10}{3}} d\tau\right)^{\frac{3}{10}}$$
$$\le \mathcal{C} T^{\frac{3}{10}} \max_{0\le s\le t} |\nabla u(s)|_H \le \mathcal{C} T^{\frac{3}{10}} \sqrt{\tilde{C}_6} |w|_{L^2(0,t;H)} \text{ for } t \in [0,T], \tag{5.18}$$

where $\mathcal{C}$ is a positive constant. Similarly to the proof of (3.26) in Section 3, by using (1) of Lemma 2.4 in Section 2 and Poincare's inequality, we have

$$\left(\int_0^t |\nabla u(\tau)|^{\frac{10}{3}}_{L^{\frac{10}{3}}(\Omega)} d\tau\right)^{\frac{3}{10}} \le (C^*)^{\frac{3}{10}} \max_{0\le s\le t} |\nabla u(s)|_H^{\frac{4}{10}} \left(\int_0^t |u(\tau)|^2_{H^2(\Omega)} d\tau\right)^{\frac{3}{10}}$$
$$\le (C^*)^{\frac{3}{10}} \max_{0\le s\le t} |\nabla u(s)|_H^{\frac{4}{10}} [C_*^2(C_P^2+1)]^{\frac{3}{10}} \left(\int_0^t |\nabla u(\tau)|_H^2 d\tau + \int_0^t |\Delta u(\tau)|_H^2 d\tau\right)^{\frac{3}{10}}$$
$$\le (C^*)^{\frac{3}{10}} [C_*^2(C_P^2+1)]^{\frac{3}{10}} \Bigg(T \max_{0\le s\le t} |\nabla u(s)|_H$$
$$+ \max_{0\le s\le t} |\nabla u(s)|_H^{\frac{4}{10}} \left(\int_0^t |\Delta u(\tau)|_H^2 d\tau\right)^{\frac{3}{10}}\Bigg) \text{ for } t \in [0,T],$$

so that on account of (5.16) we get

$$\left(\int_0^t |\nabla u|^{\frac{10}{3}}_{L^{\frac{10}{3}}(\Omega)} dt\right)^{\frac{3}{10}} \le 2(C^*)^{\frac{3}{10}} [C_*^2(C_P^2+1)]^{\frac{3}{10}} (T+1) \sqrt{\tilde{C}_6} |w|_{L^2(0,t;H)}$$
$$\text{for } t \in [0,T]. \tag{5.19}$$

By putting $\tilde{C}_7 = 2(C^*)^{\frac{3}{10}}[C_*^2(C_P^2+1)]^{\frac{3}{10}}(T+1)\sqrt{\tilde{C}_6} + \mathcal{C}T^{\frac{3}{10}}\sqrt{\tilde{C}_6}$. Then, by (5.16), (5.17), (5.18) and (5.19) we obtain that

$$|u|_{L^5(\Omega,C([0,t]))} \leq C_e(\tilde{C}_6 + \tilde{C}_7)|w|_{L^2(0,t;H)}. \tag{5.20}$$

Therefore, by Hölder's inequality, Lemma 5.1 and (5.20) we have

$$\begin{aligned} |w|_{L^2(0,t;H)} &\leq \sqrt{t}|\Omega|^{\frac{3}{10}}|w|_{L^5(\Omega,C([0,t]))} \\ &\leq L_*\sqrt{t}|\Omega|^{\frac{3}{10}}|u|_{L^5(\Omega,C([0,t]))} \\ &\leq L_*\sqrt{t}|\Omega|^{\frac{3}{10}}C_e(\tilde{C}_6+\tilde{C}_7)|w|_{L^2(0,t;H)}. \end{aligned}$$

Finally, by taking $t^* > 0$ such that $1 - L_*\sqrt{t^*}|\Omega|^{\frac{3}{10}}C_e(\tilde{C}_6+\tilde{C}_7) > 0$ we see that $|w|_{L^2(0,t;H)} = 0$ for $0 \leq t \leq t^*$ so that $w_1 = w_2$ for $0 \leq t \leq t^*$. Then, from (5.20), we also see that $u_1 = u_2$ for $0 \leq t \leq t^*$. By repeating this argument for $t \geq t^*$ we see that $u_1 = u_2$ and $w_1 = w_2$ for $0 \leq t \leq T$. Thus, Theorem 2.3 is proved.

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