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# SINGULAR LIMIT ANALYSIS OF A REACTION-DIFFUSION SYSTEM WITH PRECIPITATION AND DISSOLUTION IN A POROUS MEDIUM

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Abstract. In this paper we consider a three-component reaction-diffusion system with a fast precipitation and dissolution reaction term. We investigate its singular limit as the reaction rate tends to infinity. The limit problem is described by a combination of a Stefan problem and a linear heat equation. The rate of convergence with respect to the reaction rate is established in a specific case.

1. **Introduction.** Deep geological repositories are one of the possibilities for the storage of radioactive waste. The cement-based materials of a repository are subject to chemical degradation caused for instance by sulfate attacks or leaching. These chemical attacks are mainly linked with dissolution-precipitation processes of the solid constituents of the cement matrix. The model which we study deals with the diffusion of chemical species transported by water, with possible dissolution or precipitation and for a rather general kinetics law.

In this paper, we consider a reaction-diffusion system composed of two parabolic equations and an ordinary differential equation which are coupled by a reaction term. This reaction term, which is nonlinear and discontinuous, may change sign. More precisely, we study the problem

$$
(\mathrm{P}^\lambda)\qquad
\begin{cases}
\dfrac{\partial u}{\partial t} = \Delta u - \alpha\lambda G(u,v,w) & \text{in} \quad Q_T := \Omega\times(0,T),\\[2mm]
\dfrac{\partial v}{\partial t} = \Delta v - \beta\lambda G(u,v,w) & \text{in} \quad Q_T,\\[2mm]
\dfrac{\partial w}{\partial t} = \lambda G(u,v,w) & \text{in} \quad Q_T,\\[2mm]
\dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on} \quad \partial\Omega\times(0,T),\\[2mm]
u(\cdot,0)=u_0,\ v(\cdot,0)=v_0,\ w(\cdot,0)=w_0 & \text{in} \quad \Omega,
\end{cases}
$$

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where $\Omega$ is a bounded domain in $\mathbb{R}^N$ ($N \in \mathbb{N}$) with smooth boundary $\partial\Omega$, $T$, $\lambda$, $\alpha$ are positive constants and $\beta \in \mathbb{R}$ is a constant, and $n$ is the outward normal unit vector to the boundary $\partial\Omega$. The function $G$ is given by

$$G(u,v,w) = F(u,v)^+ - \mathrm{sign}^+(w)F(u,v)^-. \tag{1}$$

Here,

$$s^+ = \max(0,s), \quad s^- = \max(0,-s), \quad \mathrm{sign}(s) = \begin{cases} 1 & \text{if} \quad s>0, \\ 0 & \text{if} \quad s=0, \\ -1 & \text{if} \quad s<0 \end{cases}$$

for arbitrary $s \in \mathbb{R}$, and $F$ is a given function. We remark that the function $G$ can also be written as

$$G(u,v,w) = \begin{cases} 0 & \text{if} \qquad w \le 0, F(u,v) \le 0, \\ F(u,v) & \text{else.} \end{cases}$$

Problem ($\mathrm{P}^\lambda$) models reactive transport in a cement based material where one mineral species ($W$) and two aqueous species ($U$ and $V$) react according to a kinetic law. The functions $u$, $v$ and $w$ stand for concentrations of $U$, $V$ and $W$, respectively. The function $F$ denotes the thermodynamical equilibrium gap, that is,

$$\begin{cases} F>0 & \text{involves regions where the mineral precipitates,} \\ F=0 & \text{involves regions with chemical equilibrium,} \\ F<0 & \text{involves regions where the mineral dissolves.} \end{cases}$$

The term $\mathrm{sign}^+(w)$ expresses the fact that mineral dissolution stops once the mineral has disappeared. Let us consider chemical reactions of the form

$$W \Longleftrightarrow \alpha U + \beta V,$$

where $\alpha > 0$ and $\beta$ are algebraic stoichiometric coefficients. More precisely, if $\alpha > 0$ and $\beta > 0$, this models the chemical reaction

$$\alpha U + \beta V \Longleftrightarrow W,$$

whereas if $\alpha > 0$ and $\beta < 0$, we have the chemical reaction

$$\alpha U \Longleftrightarrow W + \{-\beta\} V.$$

For such reactions, we suppose as in [2] that the function $F$ is given by

$$F(u,v) = u^\alpha v^{\beta^+} - K v^{\beta^-} \text{ for } (u,v) \in \mathbb{R}^+ \times \mathbb{R}^+, \tag{2}$$

where $K$ is a positive constant. For the derivation of Problem ($\mathrm{P}^\lambda$) with (2), see [2] and references therein. The function $F$ is extended according to

- $F(u,v) = F(u,0)$ for all $u \ge 0$ and $v < 0$,
- $F(u,v) = F(0,v)$ for all $v \ge 0$ and $u < 0$,
- $F(u,v) = F(0,0)$ for all $u, v < 0$.

Geologists observe in experiments or numerical simulations patterns which seem to correspond to solutions of moving boundary problems (see [2]). Our purpose here is to give a theoretical justification of these facts. We focus on reactions which are very fast compared with the diffusion process so that $\lambda$ is a large parameter. In the special case where $\alpha = 1$ and $\beta = 0$, Problem ($\mathrm{P}^\lambda$) reduces to a system of two equations for $u$ and $w$. For such a reaction, the singular limit of the solution $(u^\lambda, v^\lambda)$ as $\lambda$ tends to infinity has been studied by Pousin [5] and by Bouillard et al. [1]. They have shown that the limit equation takes the form of a Stefan problem.

In this paper, we search for the fast reaction limit of the general problem ($P^\lambda$); it turns out that the limit problem is given by the combination of a heat equation and a Stefan problem.

This paper is organized as follows. In the next section, we precisely state the assumptions and the problems. Then, we present our main result, namely the convergence of the solution of ($P^\lambda$) to its limit as $\lambda$ tends to infinity. Section 3 is devoted to establishing a priori estimates. The main theorem is proved in Section 4. In Section 5, we discuss the convergence rate with respect to $\lambda$.

2. **Notations and main results.** In this section we present the assumptions, precisely state the problem and give our main results.

2.1. **Notations and assumptions.** From the equations for $u$ and $v$ in ($P^\lambda$), we obtain the following relation between $u$ and $v$:

$$v = h + \frac{\beta}{\alpha} u. \tag{3}$$

Here, $h = e^{t\Delta}\left(v_0 - \frac{\beta}{\alpha}u_0\right)$ denotes the solution of the linear heat equation with the homogeneous Neumann boundary condition and with the initial function $v_0 - \dfrac{\beta}{\alpha}u_0$.

In this paper, we suppose that the following hypotheses are satisfied.

(H1) The initial functions $u_0, v_0$ and $w_0$ are smooth and satisfy

$$0 \le u_0, v_0, w_0 \le M$$

for some positive constant $M$.

(H2) There exist a nonnegative function $f$ only depending on $h$, $\alpha$, $\beta$ and $K$ and a positive function $g$ only depending on $u$, $h$, $\alpha$, $\beta$, $K$ such that $f \in L^\infty(Q_T) \cap W_2^{2,1}(Q_T)$, $\dfrac{\partial f}{\partial n} = 0$ on $\partial\Omega \times (0,T)$ and

$$F\left(u, h + \frac{\beta}{\alpha}u\right) = (u - f)g \tag{4}$$

for all $u \ge 0$ and a.e. in $Q_T$. If $\beta < 0$, we suppose in addition to the above that $\dfrac{\partial f}{\partial t} - \Delta f \in L^\infty(Q_T)$.

Many functions $F$ of the form (2) (see examples below) can be factorized in the form (4) but it is not the case if the coefficient of the highest order of the left hand side of (4) is negative. However, we can also deal with the following case.

(H2)* $\beta \ne 0$ and there exists a nonnegative function $f^*$ only depending on $h$, $\alpha$, $\beta$ and $K$ and a negative function $g^*$ only depending on $v$, $h$, $\alpha$, $\beta$, $K$ such that $f^* \in L^\infty(Q_T) \cap W_2^{2,1}(Q_T)$, $\dfrac{\partial f^*}{\partial t} - \Delta f^* \in L^\infty(Q_T)$, $\dfrac{\partial f^*}{\partial n} = 0$ on $\partial\Omega \times (0,T)$ and

$$F\left(\frac{\alpha}{\beta}(v - h), v\right) = (v - f^*)g^* \tag{5}$$

for all $v \ge 0$ and a.e. $(x,t) \in Q_T$.

**Remark 1.** If the function $F$ can be factorized as in (4), the function $f$ is uniquely determined. Indeed, if there are two couples of functions $(f_1, g_1)$ and $(f_2, g_2)$, the following relation holds

$$(u - f_1(x,t))g_1(u,x,t) = (u - f_2(x,t))g_2(u,x,t)$$

for all $u \geq 0$ and for a.e. $(x,t) \in Q_T$. Since $f_1$ is non-negative, we can take $u = f_1(x,t)$. Then we have

$$(f_1(x,t) - f_2(x,t))g_2(f_1(x,t),x,t) = 0.$$

In view of the positivity of the function $g_2$, we obtain the $f_1 = f_2$ a.e. Similarly, the function $f^*$ is uniquely defined by the relation (5).

In this paper, we show that as $\lambda$ tends to infinity,

$$u^\lambda \to f - Z^+, \quad v^\lambda \to h + \frac{\beta}{\alpha} f - \frac{\beta}{\alpha} Z^+, \quad w^\lambda \to \frac{1}{\alpha} Z^-$$

in a certain sense, where $Z$ is the unique weak solution of the following one-phase Stefan problem:

$$\text{(SP)} \qquad \begin{cases} \dfrac{\partial Z}{\partial t} = \Delta Z^+ + \dfrac{\partial f}{\partial t} - \Delta f & \text{in} \quad Q_T, \\ \dfrac{\partial Z^+}{\partial n} = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ Z(\cdot,0) = f(0) - u_0 - \alpha w_0 & \text{in} \quad \Omega. \end{cases}$$

**Examples of $f$ and $g$ satisfying the assumption (H2).**

1. Case $\alpha \in \mathbb{N}$, $\beta = 0$. $F(u,v) = u^\alpha - K$.

$$f = K^{1/\alpha} > 0, \quad g = \sum_{k=1}^{\alpha} u^{\alpha-k} K^{\frac{k-1}{\alpha}} \geq K^{\frac{\alpha-1}{\alpha}} > 0, \quad f_t - \Delta f = 0.$$

2. Case $\alpha = 1$, $\beta = -1$. $F(u,v) = u - Kv$.

$$f = \frac{K}{1+K} h \geq 0, \quad g = 1 + K > 0, \; f_t - \Delta f = 0.$$

3. Case $\alpha = 1$, $\beta = 1$. $F(u,v) = uv - K$.

$$f = \frac{1}{2}(-h + \sqrt{h^2 + 4K}) > 0, \quad g = u - \frac{1}{2}(-h - \sqrt{h^2+4K}) \geq \sqrt{M^2 + K} - M > 0,$$

and $f$ is such that

$$f_t - \Delta f \in L^\infty(Q_T).$$

4. Case $\alpha = 2$, $\beta = -1$. $F(u,v) = u^2 - Kv$.

$$f = \sqrt{\frac{K^2}{16} + Kh} - \frac{K}{4} \geq 0, \quad g = u + \sqrt{\frac{K^2}{16} + Kh} + \frac{K}{4} \geq \frac{K}{4} > 0,$$

and $f$ is such that

$$f_t - \Delta f \in L^\infty(Q_T).$$

**Example of functions $f^*$ and $g^*$ satisfying the assumption (H2)$^*$.**

5. Case $\alpha = 1$, $\beta = -2$ with (2), that is, $F(u,v) = u - Kv^2$. We can not factorize $F(u,v)$ in the form (4), but it can be factorized as in (5):

$$f^* = \sqrt{\frac{1}{16K^2} + \frac{1}{2K} h} - \frac{1}{4K} \geq 0, \quad g^* = -K\left(v + \sqrt{\frac{1}{16K^2} + \frac{1}{2K} h} + \frac{1}{4K}\right) < 0,$$

and $f^*$ is such that

$$f^*_t - \Delta f^* \in L^\infty(Q_T).$$

2.2. **Definitions.** Let $\langle \cdot, \cdot \rangle$ denote both the inner product in $L^2(\Omega)$ and the duality pairing between $H^1(\Omega)^*$ and $H^1(\Omega)$. Solutions of Problem ($P^\lambda$) are defined in the following weak sense.

**Definition 2.1.** A triple $(u, v, w)$ is a weak solution of Problem ($\mathrm{P}^\lambda$) if it satisfies $u, v \in L^\infty(Q_T) \cap L^2(0, T; H^1(\Omega)) \cap H^1(0, T; H^1(\Omega)^*)$, $w \in L^\infty(Q_T) \cap H^1(0, T; L^2(\Omega))$ and

$$\int_0^T \left\langle \frac{\partial u}{\partial t}, \varphi \right\rangle + \int_0^T \langle \nabla u, \nabla \varphi \rangle + \alpha\lambda \int_0^T \langle G(u, v, w), \varphi \rangle = 0, \tag{6}$$

$$\int_0^T \left\langle \frac{\partial v}{\partial t}, \varphi \right\rangle + \int_0^T \langle \nabla v, \nabla \varphi \rangle + \beta\lambda \int_0^T \langle G(u, v, w), \varphi \rangle = 0 \tag{7}$$

for all functions $\varphi \in L^2(0, T; H^1(\Omega))$,

$$\frac{\partial w}{\partial t} = \lambda G(u, v, w) \qquad \text{a.e. in } Q_T \tag{8}$$

and

$$u(\cdot, 0) = u_0, \quad v(\cdot, 0) = v_0, \quad w(\cdot, 0) = w_0, \quad \text{a.e. in } \Omega.$$

Under the assumption (H1), we can prove the existence of a unique weak solution of ($\mathrm{P}^\lambda$) following the proof of Bouillard et al. [2]. Moreover, we can obtain the following property.

**Lemma 2.2.** *Assume* (H1) *is satisfied. Then, there exists a positive constant $C(\lambda)$ only depending on $T$, $\alpha$, $\beta$, $\lambda$, $F$, $u_0$, $v_0$, $w_0$ such that*

$$0 \le u^\lambda, v^\lambda, w^\lambda \le C(\lambda) \quad \textit{for a.e. in } Q_T.$$

Bouillard et al. [2] used a fully discrete finite volume approximation to prove these properties, so that the dimension and the shape of domain were restricted. We can remove those restrictions by using a time-discrete scheme.

Next we define a weak solution of Problem (SP).

**Definition 2.3.** A function $Z$ is a weak solution of Problem (SP) if it satisfies $Z \in H^1(0, T; H^1(\Omega)^*)$, $Z^+ \in L^2(0, T; H^1(\Omega))$ and

$$\int_0^T \left\langle \frac{\partial Z}{\partial t}, \varphi \right\rangle + \int_0^T \langle \nabla Z^+, \nabla \varphi \rangle = \int_0^T \left\langle \frac{\partial f}{\partial t} - \Delta f, \varphi \right\rangle \tag{9}$$

for all functions $\varphi \in L^2(0, T; H^1(\Omega))$, and

$$Z(x, 0) = f(x, 0) - u_0(x) - \alpha w_0(x), \quad \text{for a.e. } x \in \Omega.$$

The existence of a unique weak solution of Problem (SP) follows from [4]; whereas the existence is the consequence of the convergence of a finite volume scheme, its uniqueness follows from Theorem 4.4 in Section 4.3.6.

2.3. **Main results.** We are now in a position to state our results.

**Theorem 2.4.** *Assume that* (H1)–(H2) *hold. Let $(u^\lambda, v^\lambda, w^\lambda)$ be the weak solution of* ($\mathrm{P}^\lambda$)*. Then, as $\lambda$ tends to infinity,*

$$\begin{aligned}
u^\lambda &\to f - Z^+ && \textit{strongly in } L^p(Q_T)\ (\forall p \ge 1),\ \textit{a.e. in } Q_T \\
&&& \textit{and weakly in } L^2(0, T; H^1(\Omega)), \\
v^\lambda &\to h + \frac{\beta}{\alpha} f - \frac{\beta}{\alpha} Z^+ && \textit{strongly in } L^p(Q_T)\ (\forall p \ge 1),\ \textit{a.e. in } Q_T \\
&&& \textit{and weakly in } L^2(0, T; H^1(\Omega)), \\
w^\lambda &\to \frac{1}{\alpha} Z^- && \textit{strongly in } L^1(0, T; L^1(\tilde{\Omega}))\ (\forall \tilde{\Omega} \Subset \Omega)\ \textit{and a.e. in } Q_T,
\end{aligned}$$

*where $h$ and $f$ are the functions defined in* (3) *and* (H2) *respectively, and $Z$ is the weak solution of* (SP).

**Theorem 2.5.** *Assume that* (H1) *and* (H2)$^*$ *hold. Let $(u^\lambda, v^\lambda, w^\lambda)$ be the weak solution of* (P)$^\lambda$. *Then,*

$$\begin{aligned} u^\lambda &\to \frac{\alpha}{\beta}(f^* - h + Z^+) && \textit{strongly in } L^p(Q_T)\ (\forall p \geq 1),\ \textit{a.e. in } Q_T \\ & && \textit{and weakly in } L^2(0,T;H^1(\Omega)), \\ v^\lambda &\to f^* + Z^+ && \textit{strongly in } L^p(Q_T)\ (\forall p \geq 1),\ \textit{a.e. in } Q_T \\ & && \textit{and weakly in } L^2(0,T;H^1(\Omega)), \\ w^\lambda &\to -\tfrac{1}{\beta}Z^- && \textit{strongly in } L^1(0,T;L^1(\tilde{\Omega}))\ (\forall \tilde{\Omega} \Subset \Omega) \textit{ and a.e. in } Q_T \end{aligned}$$

*as $\lambda$ tends to infinity. Here, $h$ and $f^*$ are the functions defined in* (3) *and* (H2)$^*$, *respectively, and $Z$ is the weak solution of the Stefan problem* (SP) *in which the function $f$ is replaced by $-f^*$ with the initial datum $Z(\cdot,0) = v_0 - f^*(\cdot,0) + \beta w_0$.*

In this paper, we give all the proof of Theorem 2.4. Theorem 2.5 is obtained in the same fashion.

## 3. A priori estimates.

In this section, we establish a priori estimates. We deduce from (6) that

$$\int_0^T \left\langle \frac{\partial}{\partial t}(u^\lambda - f), \varphi \right\rangle + \int_0^T \langle \nabla(u^\lambda - f), \nabla\varphi\rangle + \alpha\lambda \int_0^T \langle G(u^\lambda, v^\lambda, w^\lambda), \varphi\rangle = -\int_0^T \left\langle \frac{\partial f}{\partial t} - \Delta f, \varphi \right\rangle \quad (10)$$

for all functions $\varphi \in L^2(0,T;H^1(\Omega))$. We define $z^\lambda := f - u^\lambda - \alpha w^\lambda$. Then,

$$\int_0^T \left\langle \frac{\partial z^\lambda}{\partial t}, \varphi \right\rangle + \int_0^T \langle \nabla(f - u^\lambda), \nabla\varphi\rangle = \int_0^T \left\langle \frac{\partial f}{\partial t} - \Delta f, \varphi \right\rangle \quad (11)$$

for all functions $\varphi \in L^2(0,T;H^1(\Omega))$.

**Lemma 3.1.** *Suppose that* (H1) *and* (H2) *hold. Then, $u^\lambda$ and $v^\lambda$ are uniformly bounded in $L^\infty(Q_T)$ with respect to $\lambda$.*

*Proof.* If $\beta < 0$, we can easily obtain the result. In fact, $v^\lambda - \frac{\beta}{\alpha}u^\lambda$ satisfies the linear heat equation with initial function $v_0 - \dfrac{\beta}{\alpha}u_0$. The maximum principle implies that $0 \leq v^\lambda - \dfrac{\beta}{\alpha}u^\lambda \leq \left(1 - \dfrac{\beta}{\alpha}\right)M$. We deduce from Lemma 2.2 that

$$0 \leq u^\lambda \leq \left(1 - \frac{\alpha}{\beta}\right)M, \quad 0 \leq v^\lambda \leq \left(1 - \frac{\beta}{\alpha}\right)M.$$

Next we prove the result in case where $\beta \geq 0$. First we show that $u^\lambda$ and $v^\lambda$ are uniformly bounded in $L^\infty(0,T;L^p(\Omega))$ with respect to $\lambda$ for all $p \geq 2$. Then, we pass to the limit in $p$. We set $\psi_p(s) = |s|^{p-2}s$ for $s \in \mathbb{R}$. For an arbitrary point $t_0 \in (0,T)$, we define

$$\varphi(x,t) = \begin{cases} \psi_p(u^\lambda(x,t) - f(x,t)) & \text{if } 0 \leq t < t_0, \\ 0 & \text{otherwise} \end{cases}$$

in (10) to obtain

$$\begin{aligned}&\frac{1}{p}\int_\Omega |u^\lambda(t_0)-f(t_0)|^p+\iint_{Q_{t_0}}\left\langle \nabla(u^\lambda-f),\nabla\psi_p(u^\lambda-f)\right\rangle\\&\quad+\alpha\lambda\int_0^{t_0}\left\langle G(u^\lambda,v^\lambda,w^\lambda),\psi_p(u^\lambda-f)\right\rangle\\&=\frac{1}{p}\int_\Omega |u_0-f(0)|^p-\int_0^{t_0}\left\langle\frac{\partial f}{\partial t}-\Delta f,\psi_p(u^\lambda-f)\right\rangle.\end{aligned}\tag{12}$$

Because of the monotonicity of $\psi_p$, the second term of the left hand side is nonnegative. Note that

$$G\left(u,h(x,t)+\frac{\beta}{\alpha}u,w\right)=(u-f(x,t))^+g(x,t)-\text{sign}^+(w)(u-f(x,t))^-g(x,t)$$

for all $u,w\geq 0$ and a.e. $(x,t)\in Q_T$ while (H2) holds. Therefore,

$$G(u^\lambda,v^\lambda,w^\lambda)\psi_p(u^\lambda-f)=|u^\lambda-f|^{p-2}g\left(((u^\lambda-f)^+)^2+\text{sign}^+(w^\lambda)((u^\lambda-f)^-)^2\right)\geq 0$$

a.e. in $Q_T$. Hence, the last term of the left hand side of (12) is nonnegative. The last term of the right hand side of (12) is estimated as follows:

$$\begin{aligned}\left|\int_0^{t_0}\left\langle\frac{\partial f}{\partial t}-\Delta f,\psi_p(u^\lambda-f)\right\rangle\right|&\leq\left\|\frac{\partial f}{\partial t}-\Delta f\right\|_{L^p(Q_{t_0})}\left\|\psi_p(u^\lambda-f)\right\|_{L^{p/(p-1)}(Q_{t_0})}\\&\leq M_f\,\text{meas}(Q_T)^{\frac{1}{p}}\left(\iint_{Q_{t_0}}|u^\lambda-f|^p\right)^{\frac{p-1}{p}}\\&\leq M_f\,\text{meas}(Q_T)^{\frac{1}{p}}\left(1+\iint_{Q_{t_0}}|u^\lambda-f|^p\right),\end{aligned}$$

where $M_f=\operatorname{ess\,sup}_{Q_T}\left(\frac{\partial f}{\partial t}-\Delta f\right)$. Collecting the previous bounds yields

$$\frac{1}{p}\int_\Omega|u^\lambda(t_0)-f(t_0)|^p\leq\frac{1}{p}\int_\Omega|u_0-f(0))|^p+M_f\,\text{meas}(Q_T)^{\frac{1}{p}}\left(1+\iint_{Q_{t_0}}|u^\lambda-f|^p\right).$$

It follows from the Gronwall inequality that

$$\begin{aligned}&\left\|u^\lambda-f\right\|_{L^\infty(0,T;L^p(\Omega))}\\&\quad\leq\left(\|u_0-f(0)\|_{L^p(\Omega)}+\left(M_f\,\text{meas}(Q_T)^{\frac{1}{p}}p\right)^{\frac{1}{p}}\right)\exp\left(M_f\,\text{meas}(Q_T)^{\frac{1}{p}}T\right)\end{aligned}$$

for all $p\geq 2$. Letting $p$ tend to infinity, we obtain

$$\left\|u^\lambda-f\right\|_{L^\infty(Q_T)}\leq\left(\|u_0-f(0)\|_{L^\infty(\Omega)}+1\right)\exp\left(M_fT\right),$$

which together with (3) completes the proof. □

In special cases, we obtain a uniform bound for $w$.

**Lemma 3.2.** *Suppose that* (H1) *holds. The function $F$ is defined as in* (2) *with $\alpha\in\mathbb{N}$, $\beta=0$ or with $\alpha=1$, $\beta=-1$. Then, $w^\lambda$ is uniformly bounded in $L^\infty(Q_T)$ with respect to $\lambda$.*

*Proof.* In the case where $\alpha \in \mathbb{N}$ and $\beta = 0$, the proof is straightforward because the system $(P^\lambda)$ reduces to the following two-component system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \alpha\lambda G(u,w) & \text{in} \quad Q_T, \\ \dfrac{\partial w}{\partial t} = \lambda G(u,w) & \text{in} \quad Q_T, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0, \;\; w(\cdot,0) = w_0 & \text{in} \quad \Omega, \end{cases} \tag{13}$$

where $G(u,w) = (u^\alpha - K)^+ - \text{sign}^+(w)(u^\alpha - K)^-$. The solution $(\bar{u},\bar{w})$ of the following system of ordinary differential equations

$$\begin{cases} \dfrac{\partial \bar{u}}{\partial t} = -\alpha\lambda G(\bar{u},\bar{w}) & \text{in} \quad (0,T), \\ \dfrac{\partial \bar{w}}{\partial t} = \lambda G(\bar{u},\bar{w}) & \text{in} \quad (0,T), \\ \bar{u}(0) = M, \;\; \bar{w}(0) = M \end{cases}$$

satisfies

$$0 \le \bar{u}(t) \le M(1+\alpha), \;\; 0 \le \bar{w}(t) \le \frac{M(1+\alpha)}{\alpha}.$$

The pair of functions $(\bar{u},\bar{w})$ is the constant in space solution of (13) with $u_0 = M$ and $w_0 = M$. In view of the comparison principle (see Theorem 2 in [1]), the solution $(u^\lambda, w^\lambda)$ of (13) satisfies

$$0 \le u^\lambda(x,t) \le \bar{u}(t) \le M(1+\alpha),$$
$$0 \le w^\lambda(x,t) \le \bar{w}(t) \le \frac{M(1+\alpha)}{\alpha}.$$

This concludes the proof for the case that $\beta = 0$.

Next we prove the result in the case that $\alpha = 1$, $\beta = -1$. Consider the auxiliary problem

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial t} = \Delta \tilde{u} - \lambda \tilde{G}(\tilde{u},\tilde{w}) & \text{in} \quad Q_T, \\ \dfrac{\partial \tilde{w}}{\partial t} = \lambda \tilde{G}(\tilde{u},\tilde{w}) & \text{in} \quad Q_T, \\ \dfrac{\partial \tilde{u}}{\partial n} = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ \tilde{u}(\cdot,0) = \dfrac{1}{1+K}u_0 - \dfrac{K}{1+K}v_0, & \text{in} \quad \Omega, \\ \tilde{w}(\cdot,0) = w_0 & \text{in} \quad \Omega, \end{cases} \tag{14}$$

where $\tilde{G}(u,w) = (1+K)(u^+ - \text{sign}^+(w)u^-)$. By the above argument, the solution $(\tilde{u}^\lambda, \tilde{w}^\lambda)$ of (14) is bounded from above, namely,

$$\tilde{u}^\lambda, \tilde{w}^\lambda \le \frac{2+K}{1+K}M.$$

We define $u^\lambda := \tilde{u}^\lambda + f$ and $w^\lambda := \tilde{w}^\lambda$. Here $f = \frac{K}{1+K}h$, $h = e^{t\Delta}(u_0 + v_0)$. Then the functions $u^\lambda, v^\lambda = h - u^\lambda, w^\lambda$ satisfy $(P^\lambda)$ with $\alpha = 1$, $\beta = -1$. Hence the solution $(u^\lambda, v^\lambda, w^\lambda)$ of $(P^\lambda)$ satisfies

$$w^\lambda \le \frac{2+K}{1+K}M,$$

which completes the proof. □

**Lemma 3.3.** *Assume that* (H1) *and* (H2) *are satisfied. Then, there exists a positive constant $C$ independent of $\lambda$ such that*

$$\left\|G(u^\lambda, v^\lambda, w^\lambda)\right\|_{L^1(Q_T)} \leq C/\lambda, \tag{15}$$

$$\left\|w^\lambda\right\|_{W^{1,1}(0,T;L^1(\Omega))} \leq C, \tag{16}$$

$$\left\|u^\lambda\right\|_{L^2(0,T;H^1(\Omega))} + \left\|v^\lambda\right\|_{L^2(0,T;H^1(\Omega))} + \left\|z^\lambda\right\|_{H^1(0,T;H^1(\Omega)^*)} \leq C. \tag{17}$$

*Proof.* We denote by $\text{sign}^\delta$ a smooth nondecreasing approximation of the sign function with $|\text{sign}^\delta| \leq 1$ which converges pointwise to its limit and define $\mathcal{H}^\delta(s) := \int_0^s \text{sign}^\delta(\tau)d\tau$. Take $\varphi = \text{sign}^\delta(u^\lambda - f) \in L^2(0,T;H^1(\Omega))$ in (10) to obtain

$$\begin{aligned}\int_\Omega \mathcal{H}^\delta(u^\lambda(T) - f(T)) &+ \iint_{Q_T} |\nabla(u^\lambda - f)|^2 (\text{sign}^\delta)'(u^\lambda - f)\\ &+ \alpha\lambda \int_0^T \left\langle G(u^\lambda, v^\lambda, w^\lambda), \text{sign}^\delta(u^\lambda - f)\right\rangle\\ &= \int_\Omega \mathcal{H}^\delta(u_0 - f(0)) - \int_0^T \left\langle \frac{\partial f}{\partial t} - \Delta f, \text{sign}^\delta(u^\lambda - f)\right\rangle.\end{aligned}$$

Since the first and the second terms are positive, we deduce from the property of $\text{sign}^\delta$ that

$$\alpha\lambda \int_0^T \left\langle G(u^\lambda, v^\lambda, w^\lambda), \text{sign}^\delta(u^\lambda - f)\right\rangle \leq \int_\Omega \mathcal{H}^\delta(u_0 - f(0)) + \left\|\frac{\partial f}{\partial t} - \Delta f\right\|_{L^1(Q_T)}.$$

Letting $\delta$ tend zero and using the boundedness of $u_0$ and $f$, we obtain

$$\alpha\lambda \int_0^T \left\langle G(u^\lambda, v^\lambda, w^\lambda), \text{sign}(u^\lambda - f)\right\rangle \leq C.$$

Since

$$G\left(u, h(x,t) + \frac{\beta}{\alpha}u, w\right) \text{sign}(u - f(x,t)) = \left|G\left(u, h(x,t) + \frac{\beta}{\alpha}u, w\right)\right|$$

for all $u, w \geq 0$ and a.e. $(x,t) \in Q_T$, we deduce the estimate (15). Moreover, we deduce (16) from the equation (8). Note that

$$G\left(u, h(x,t) + \frac{\beta}{\alpha}u, w\right)(u - f(x,t)) \geq 0$$

for all $u, w \geq 0$ and a.e. $(x,t) \in Q_T$. Choose $\varphi = u^\lambda - f \in L^2(0,T;H^1(\Omega))$ in (10) to get

$$\begin{aligned}&\frac{1}{2}\|u^\lambda(T) - f(T)\|^2_{L^2(\Omega)} + \left\|\nabla(u^\lambda - f)\right\|^2_{L^2(Q_T)}\\ &\quad\leq \frac{1}{2}\|u_0 - f(0)\|^2_{L^2(\Omega)} - \int_0^T \left\langle \frac{\partial f}{\partial t} - \Delta f, u^\lambda - f\right\rangle\\ &\quad\leq \frac{1}{2}\|u_0 - f(0)\|^2_{L^2(\Omega)} + \left\|\frac{\partial f}{\partial t} - \Delta f\right\|_{L^2(Q_T)} \left\|u^\lambda - f\right\|_{L^2(Q_T)}.\end{aligned}$$

The boundedness of $u^\lambda$ and $f$ implies that

$$\left\|u^\lambda\right\|_{L^2(0,T;H^1(\Omega))} \leq C.$$

Using the relation (3), we obtain

$$\left\|v^\lambda\right\|_{L^2(0,T;H^1(\Omega))} \le C.$$

It follows from (11) that

$$\begin{aligned}\left|\int_0^T \left\langle \frac{\partial z^\lambda}{\partial t}, \varphi \right\rangle\right| &\le \left( \left\|\nabla (f - u^\lambda)\right\|_{L^2(Q_T)} + \left\|\frac{\partial f}{\partial t} - \Delta f\right\|_{L^2(Q_T)} \right) \|\varphi\|_{L^2(0,T;H^1(\Omega))} \\ &\le C \|\varphi\|_{L^2(0,T;H^1(\Omega))},\end{aligned}$$

which implies the estimate (17). □

**Lemma 3.4.** *Assume that* (H1) *and* (H2) *hold. Then, there exists a positive constant* $C$ *and a positive function* $\sigma$ *independent of* $\lambda$ *such that* $\sigma(\xi) \to 0$ *as* $\xi \to 0$ *and*

$$\begin{aligned}&\int_0^T \int_{\Omega_r} \left|u^\lambda(x+\xi,t) - u^\lambda(x,t)\right|^p dxdt \\ &\quad + \int_0^T \int_{\Omega_r} \left|v^\lambda(x+\xi,t) - v^\lambda(x,t)\right|^p dxdt \le C|\xi|^2, \end{aligned} \tag{18}$$

$$\int_0^T \int_{\Omega_r} \left|w^\lambda(x+\xi,t) - w^\lambda(x,t)\right| dxdt \le \sigma(\xi), \tag{19}$$

$$\begin{aligned}&\int_0^{T-\tau} \int_{\Omega} \left|u^\lambda(x,t+\tau) - u^\lambda(x,t)\right|^p dxdt + \int_0^{T-\tau} \int_{\Omega} \left|v^\lambda(x,t+\tau) - v^\lambda(x,t)\right|^p dxdt \\ &\quad + \int_0^{T-\tau} \int_{\Omega} \left|w^\lambda(x,t+\tau) - w^\lambda(x,t)\right| dxdt \le C\tau \end{aligned} \tag{20}$$

*for all* $p \ge 2$, $\xi \in \mathbb{R}^N$, $|\xi| \le r$ *and* $\tau \in (0,T)$. *Here,* $r > 0$ *and* $\Omega_r = \{x \in \Omega : B(x,r) \subset \Omega\}$.

*Proof.* As in the proof by Bouillard et al [1], the following estimate holds.

$$\int_0^T \int_{\Omega_r} \left|u^\lambda(x+\xi,t) - u^\lambda(x,t)\right|^2 dxdt \le C|\xi|^2$$

for all $\xi \in \mathbb{R}^N$, $|\xi| \le r$. Since $u^\lambda$ is uniformly bounded in $L^\infty(Q_T)$ with respect to $\lambda$, we have

$$\int_0^T \int_{\Omega_r} \left|u^\lambda(x+\xi,t) - u^\lambda(x,t)\right|^p dxdt \le C|\xi|^2$$

for all $p \ge 2$. Thus, the proofs of (18) and (20) are analogous to those by Bouillard et al [1]. We can also prove (19) in a similar way as Bouillard et al since the function $G\left(u, h + \frac{\beta}{\alpha}u, w\right)$ is nondecreasing in $u$ and nonincreasing in $w$. □

## 4. **Proof of Theorem 2.4.** This section is devoted to the proof of Theorem 2.4.

*Proof of Theorem 2.4.* **Step 1. Convergence of the sequences.**

By the lemmas 3.1–3.4 and the Riesz-Fréchet-Kolmogorov Theorem [3], there exist subsequences, which are denoted by $\{u^\lambda\}$, $\{v^\lambda\}$, $\{w^\lambda\}$ and $\{z^\lambda\}$ again, and functions $u^*, v^* = h + \frac{\beta}{\alpha}u^* \in L^\infty(Q_T) \cap L^2(0,T;H^1(\Omega))$, $w^* \in L^1(0,T;L^1(\tilde\Omega))$ $(\forall \tilde\Omega \Subset \Omega)$ and $z^* = f - u^* - \alpha w^* \in H^1(0,T;H^1(\Omega)^*)$ such that $u^*, v^*, w^* \geq 0$ and

$$\begin{cases} u^\lambda \to u^* \text{ and } v^\lambda \to v^* & \text{strongly in } L^p(Q_T)\ (\forall p \geq 1), \text{ a.e. in } Q_T \\ & \text{and weakly in } L^2(0,T;H^1(\Omega)), \\ w^\lambda \to w^* & \text{strongly in } L^1(0,T;L^1(\tilde\Omega))\ (\forall \tilde\Omega \Subset \Omega) \text{ and a.e. in } Q_T, \\ z^\lambda \to z^* & \text{strongly in } L^1(0,T;L^1(\tilde\Omega))\ (\forall \tilde\Omega \Subset \Omega) \text{ and a.e. in } Q_T, \\ & \text{weakly in } H^1(0,T;H^1(\Omega)^*) \end{cases} \tag{21}$$

as $\lambda$ tends to infinity.

**Step 2. The relation $G(u^*, v^*, w^*) = 0$ holds.**

Indeed set

$$G_\varepsilon(u,v,w) = F(u,v)^+ - \text{sign}_\varepsilon^+(w)F(u,v)^-,$$

and define $\text{sign}_\varepsilon^+$ as a nonincreasing approximation of the $\text{sign}^+$ such that

$$\text{sign}_\varepsilon^+(x) = \begin{cases} 1 & \text{if} \quad x \geq \varepsilon, \\ x/\varepsilon & \text{if} \quad 0 \leq x < \varepsilon, \\ 0 & \text{if} \quad x < 0. \end{cases}$$

Recall that

$$G_\varepsilon(u, h(x,t) + \tfrac{\beta}{\alpha}u, w) = (u - f(x,t))^+ g(x,t) - \text{sign}_\varepsilon^+(w)(u - f(x,t))^- g(x,t)$$

for all $u, w \geq 0$ and a.e. $(x,t) \in Q_T$. We can easily check that

$$0 \leq G_\varepsilon(u, h(x,t) + \tfrac{\beta}{\alpha}u, w)(u - f(x,t)) \leq G(u, h(x,t) + \tfrac{\beta}{\alpha}u, w)(u - f(x,t)) \tag{22}$$

for all $u, w \geq 0$ and a.e. $(x,t) \in Q_T$. We deduce that

$$0 \leq \int_0^T \left\langle u^\lambda - f, G_\varepsilon(u^\lambda, v^\lambda, w^\lambda) \right\rangle \leq \int_0^T \left\langle u^\lambda - f, G(u^\lambda, v^\lambda, w^\lambda) \right\rangle.$$

It follows from (15) and (21) that

$$\int_0^T \langle u^* - f, G_\varepsilon(u^*, v^*, w^*) \rangle = 0.$$

Therefore, $G_\varepsilon(u^*, v^*, w^*) = 0$ or $u^* = f$ a.e. Passing to the limit in $\varepsilon$, we obtain $G(u^*, v^*, w^*) = 0$.

**Step 3. Relationships between $u^*$, $v^*$, $w^*$ and $z^*$.**

If $u^* \geq f$, then $G(u^*, v^*, w^*) = (u^* - f)g = 0$. Since $g$ is positive, we have $u^* = f$, that is, $z^* = -\alpha w^* \leq 0$. If $u^* < f$, then $G(u^*, v^*, w^*) = \text{sign}^+(w^*)(u^* - f)g = 0$. Because $u^* \neq f$ and $g \neq 0$, we obtain $\text{sign}^+(w^*) = 0$, which implies $w^* = 0$. Thus, $z^* = f - u^* > 0$. This yields the relations

$$u^* = f - z^{*+}, \quad v^* = h + \frac{\beta}{\alpha}f - \frac{\beta}{\alpha}z^{*+} \quad \text{and} \quad w^* = \frac{1}{\alpha}z^{*-}.$$

**Step 4. Characterization of $z^*$.**

If=t follows from the form of the initial condition for $z^\lambda$ that

$$\int_0^T \left\langle \frac{\partial z^\lambda}{\partial t}, \varphi \right\rangle + \int_0^T \left\langle z^\lambda - (f(0) - u_0 - \alpha w_0), \frac{\partial \varphi}{\partial t} \right\rangle = 0 \tag{23}$$

for all functions $\varphi \in H^1(Q)$ with $\varphi(\cdot, T) = 0$. Passing to the limit along subsequences in (23) and integrating by parts yields $z^*(0) = f(0) - u_0 - \alpha w_0$ a.e. Letting

$\lambda$ tend to infinity in (11), we observe that $z^*$ is the weak solution of the Stefan problem (SP). Since the weak solution of Problem (SP) is unique, the whole sequence converges, which completes the proof. □

Since $w^\lambda$ is uniformly bounded in $L^\infty(Q_T)$, $w^\lambda$ converges strongly in $L^p(Q_T)$ for all $p \geq 1$). Therefore, we deduce the following result from Lemma 3.2.

**Corollary 1.** *Suppose that* (H1) *holds. Let* $(u^\lambda, v^\lambda, w^\lambda)$ *be the weak solution of* $(\mathrm{P}^\lambda)$ *in which the function* $F$ *is defined as in* (2) *with* $\alpha \in \mathbb{N}$, $\beta = 0$ *or with* $\alpha = 1$, $\beta = -1$. *Then,*

$$\begin{aligned}
&u^\lambda \to f - Z^+ && \text{strongly in } L^p(Q_T)\ (\forall p \geq 1),\ \text{a.e. in } Q_T \\
& && \text{and weakly in } L^2(0,T;H^1(\Omega)), \\
&v^\lambda \to h + \tfrac{\beta}{\alpha} f - \tfrac{\beta}{\alpha} Z^+ && \text{strongly in } L^p(Q_T)\ (\forall p \geq 1),\ \text{a.e. in } Q_T \\
& && \text{and weakly in } L^2(0,T;H^1(\Omega)), \\
&w^\lambda \to \tfrac{1}{\alpha} Z^- && \text{strongly in } L^p(Q_T)\ (\forall p \geq 1) \text{ and a.e. in } Q_T
\end{aligned}$$

*as* $\lambda$ *tends to infinity. Here,* $h$ *and* $f$ *are the functions defined in* (3) *and* (H2), *respectively, and* $Z$ *is the weak solution of Problem* (SP).

## 5. **Rate of convergence.**

We obtain convergence rates with respect to $\lambda$ under some additional conditions.

**Lemma 5.1.** *In addition to the hypothesis* (H2), *we suppose that*

(H3) *There exists a positive constant* $\underline{g}$ *such that* $\underline{g} \leq g(x,t)$ *for a.e.* $(x,t) \in Q_T$.

*Then the following estimate holds for all* $u \in \mathbb{R}$ *and* $w \geq 0$ *and a.e.* $(x,t) \in Q_T$.

$$\left|(f(x,t) - u - \alpha w)^+ - (f(x,t) - u)\right| \leq \frac{1}{\underline{g}} \left| G\left(u, h(x,t) + \frac{\beta}{\alpha} u, w\right)\right|.$$

All the examples stated in Section 1 satisfy the condition (H3).

*Proof.* We denote the left-hand-side by $\mathcal{A}$ and the right-hand-side by $\mathcal{G}$. If $f - u > \alpha w > 0$, then it follows from the assumption that

$$\mathcal{A} = |-\alpha w| < |f - u| \leq \frac{1}{\underline{g}} |(u - f) g| = \frac{1}{\underline{g}} \mathcal{G}.$$

If $f - u > \alpha w = 0$, then

$$\mathcal{A} = 0 = \frac{1}{\underline{g}} \mathcal{G}.$$

If $f - u \leq \alpha w = 0$, then we have

$$\mathcal{A} = |u - f| = |(u - f)^+| \leq \frac{1}{\underline{g}} \left|(u - f)^+ g\right| = \frac{1}{\underline{g}} \mathcal{G}.$$

If $f - u \leq \alpha w$ and $w > 0$, then

$$\mathcal{A} = |u - f| \leq \frac{1}{\underline{g}} |(u - f) g| = \frac{1}{\underline{g}} \mathcal{G}.$$

This completes the proof of Lemma 5.1. □

**Theorem 5.2.** *Suppose that* (H1)–(H3) *are satisfied. Let* $(u^\lambda, v^\lambda, w^\lambda)$ *be the weak solution of* $(\mathrm{P})^\lambda$ *and let* $Z$ *be the weak solution of* (SP). *Set* $z^\lambda = f - u^\lambda - \alpha w^\lambda$. *If there is a positive constant* $M_w$ *independent of* $\lambda$ *such that*

$$\|w^\lambda\|_{L^\infty(Q_T)} \leq M_w, \tag{24}$$

*then there exists a positive constant $C$ independent of $\lambda$ such that*

$$\begin{aligned}
&\left\|u^{\lambda}-(f-Z^{+})\right\|_{L^2(Q_T)}+\left\|v^{\lambda}-\left(h+\frac{\beta}{\alpha}f-\frac{\beta}{\alpha}Z^{+}\right)\right\|_{L^2(Q_T)}\\
&+\left\|\int_0^t\left(u^{\lambda}-(f-Z^{+})\right)\right\|_{L^\infty(0,T;H^1(\Omega))}\\
&+\left\|\int_0^t\left(v^{\lambda}-\left(h+\frac{\beta}{\alpha}f-\frac{\beta}{\alpha}Z^{+}\right)\right)\right\|_{L^\infty(0,T;H^1(\Omega))}\\
&+\left\|z^{\lambda}-Z\right\|_{L^\infty(0,T;H^1(\Omega)^*)}\le C\lambda^{-1/2}. \qquad (25)
\end{aligned}$$

*Proof.* We define

$$e_z := z^{\lambda}-Z \text{ and } e_u := u^{\lambda}-f+Z^{+}.$$

We deduce from (11) and (9) that

$$\int_0^T\left\langle e_z,\frac{\partial\varphi}{\partial t}\right\rangle+\int_0^T\langle\nabla e_u,\nabla\varphi\rangle=0 \qquad (26)$$

for all $\varphi\in H^1(Q_T)$ with $\varphi(\cdot,T)=0$. Take

$$\varphi(x,t)=\begin{cases}\displaystyle\int_t^{t_0}e_u(s)ds & \text{for } 0\le t<t_0,\\ 0 & \text{for } t_0\le t\le T,\end{cases}$$

where $t_0$ is an arbitrary point in $(0,T)$. Then we get

$$-\int_0^{t_0}\langle e_z,e_u\rangle+\frac{1}{2}\left\|\nabla\int_0^{t_0}e_u\right\|^2_{L^2(\Omega)}=0.$$

We obtain the relation

$$\int_0^{t_0}\langle e_z,e_u\rangle=\int_0^{t_0}\left\langle z^{\lambda}-Z,u^{\lambda}-f+(z^{\lambda})^{+}\right\rangle+\int_0^{t_0}\left\langle z^{\lambda}-Z,-(z^{\lambda})^{+}+Z^{+}\right\rangle \qquad (27)$$

By the assumption (24) and the lemmas 3.1, 3.3 and 5.1, we obtain the estimate

$$\left|\int_0^{t_0}\left\langle z^{\lambda}-Z,u^{\lambda}-f+(z^{\lambda})^{+}\right\rangle\right|\le C\int_0^{t_0}\left\|u^{\lambda}-f+(z^{\lambda})^{+}\right\|_{L^1(\Omega)}\le C/\lambda.$$

Since the function $(\cdot)^{+}$ is a Lipschitz continuous and nondecreasing function with Lipschitz constant 1, the second term on the right-hand-side of (27) can be estimated from below as

$$\begin{aligned}
\int_0^{t_0}\left\langle z^{\lambda}-Z,(z^{\lambda})^{+}-Z^{+}\right\rangle&\ge\left\|(z^{\lambda})^{+}-Z^{+}\right\|^2_{L^2(Q_{t_0})}\\
&\ge\frac{1}{2}\|e_u\|^2_{L^2(Q_{t_0})}-\left\|(z^{\lambda})^{+}-(f-u^{\lambda})\right\|^2_{L^2(Q_{t_0})}\\
&\ge\frac{1}{2}\|e_u\|^2_{L^2(Q_{t_0})}-C/\lambda.
\end{aligned}$$

Collecting these inequalities yields

$$\|e_u\|^2_{L^2(Q_T)}+\left\|\int_0^t e_u\right\|^2_{L^\infty(0,T;H^1(\Omega))}\le C/\lambda. \qquad (28)$$

From the relation (3), we obtain the desired estimate for $v$.

Let $\zeta$ be a function belonging to $H^1(\Omega)$ and $t_0$ be an arbitrary point in $(0,T)$. The function $\chi_\delta = \chi_\delta(t)$ is defined as

$$\chi_\delta(t) = \begin{cases} 1 & t \in [0, t_0 - \delta], \\ (t_0 + \delta - t)/2\delta & t \in (t_0 - \delta, t_0 + \delta), \\ 0 & t \in [t_0 + \delta, T]. \end{cases}$$

The function $\chi_\delta$ converges in $L^2(0,T)$ to the characteristic function of $(0,t_0)$. Taking $\varphi(x,t) = \zeta(x)\chi_\delta(t)$ in (26), we obtain

$$-\frac{1}{2\delta}\int_{t_0-\delta}^{t_0+\delta} \langle e_z, \zeta\rangle + \int_0^T \chi_\delta \langle \nabla e_u, \nabla\zeta\rangle = 0.$$

Using the Lebesgue differentiation theorem and the Cauchy-Schwarz inequality, for a.e. $t_0 \in (0,T)$ and for all $\zeta \in H^1(\Omega)$, we have

$$|\langle e_z(t_0), \zeta\rangle| \le \left| \int_0^{t_0} \langle \nabla e_u, \nabla\zeta\rangle \right| \le \left\| \nabla \int_0^{t_0} e_u \right\|_{L^2(\Omega)} \|\zeta\|_{H^1(\Omega)} \le C\lambda^{-1/2} \|\zeta\|_{H^1(\Omega)}.$$

Thus we get

$$\|e_z\|_{L^\infty(0,T;H^1(\Omega)^*)} \le C\lambda^{-1/2},$$

which completes the proof. □

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