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A ONE DIMENSIONAL FREE BOUNDARY PROBLEM FOR ADSORPTION PHENOMENA

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Abstract. In this paper we deal with a one-dimensional free boundary problem, which is a mathematical model for an adsorption phenomena appearing in concrete carbonation process. This model was proposed in line of previous studies of three dimensional concrete carbonation process. The main result in this paper is concerned with the existence and uniqueness of a time-local solution to the free boundary problem. This result will be obtained by means of the abstract theory of nonlinear evolution equations and Banach's fixed point theorem, and especially, the maximum principle applied to our problem will play a very important role to obtain the uniform estimate to approximate solutions.

1. **Introduction.** In this paper, we study a free boundary problem proposed in our previous paper [8], as a mathematical model of an adsorption phenomena. In a study of a concrete carbonation process it is a crucial step how to describe the relationship between the humidity and the degree of saturation in an adsorption phenomena, since the graph of these two parameters draws a hysteresis loop, for instance in Maekawa-Ishida-Kishi [13] and Maekawa-Chaube-Kishi [12], Aiki-Kumazaki [1, 2, 3]. Now, a mathematical treatment of this kind of relationship is one of great interests in a filed of modeling. In view of this, we here propose a free boundary problem as a possible mathematical model to respond to the scientific interest.

First, we mention about the physical meaning of our free boundary problem. We consider a drying and wetting process in a porous medium. In this research we focus on one hole of the media and simplify the hole as a one-dimensional interval $[0, L]$. Here, the boundary point 0 and L denote the bottom and top of the hole, respectively. So, physically, it is supposed that the wall exists at $x = 0$, and the air comes from the region $\{x > L\}$ (see Figure 1). Also, the intervals $[0, s(t)]$ and

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$[s(t), L]$ indicate the water-drop (liquid) region and the air region in the hole (see Figure 2), respectively, and u is the relative humidity in the pore which is distributed in a space-time region, denoted by $Q_s(T)$:

$$Q_s(T) := \{(t, x) : 0 < t < T,\ s(t) < x < L)\}.$$

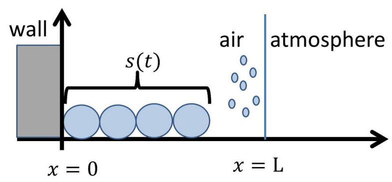

FIGURE 1.

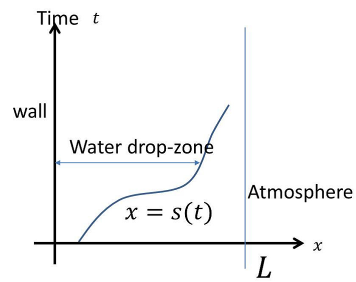

FIGURE 2.

On that basis, the curve s on $[0, T](0 < T < \infty)$ and function u on $Q_s(T)$ are supposed to fulfill the following system (1.1) $\sim$ (1.6), which is denoted by P:=P(s_0, u_0, g):

$$0 < s(t) < L \text{ for } 0 \le t \le T, \tag{1.1}$$

$$\rho_g u_t - \kappa u_{xx} = 0 \text{ in } Q_s(T), \tag{1.2}$$

$$u(t, L) = k(t) \text{ for } 0 < t < T, \tag{1.3}$$

$$s'(t) = \alpha(s(t), u(t, s(t))) \text{ for } 0 < t < T, \tag{1.4}$$

$$\kappa u_x(t, s(t)) = (\rho_a - \rho_g u(t, s(t)))s'(t) \text{ for } 0 < t < T, \tag{1.5}$$

$$s(0) = s_0, u(x, 0) = u_0(x) \text{ for } s_0 < x < L, \tag{1.6}$$

where ρ_a is a constant of the density of the aqueous-H_2O, ρ_g is a constant of the amount of saturated water vapor, κ is a diffusion constant of water in air, α is a Lipschitz continuous function on $\mathbb{R}^2$, k is a given boundary function on $[0, T]$, and s_0 and u_0 are the initial data.

Here, we note that (1.2) and (1.5) are derived from the mass conservation law for aqueous-H_2O and near the free boundary, respectively. Also, in this study, the relative humidity u is supposed to be measured in the outside region of the hole. This supposition is expressed by the Dirichlet boundary condition (1.3) of our problem P. Moreover, the free boundary condition (1.4) is based on the hypothesis mentioned in [8]. More precisely, we suppose that the growth rate $s'(t)$ of water-drop region is determined by means of:

- $u(t, s(t))$, i.e. the relative humidity of gaseous-H_2O at the free boundary,
- $s(t)$, i.e. the distance involved in the attracting force between the wall and the front of water-drop region.

Accordingly, the binary function α in (1.4) is to provide an input-output relation between $(s(t), u(t, s(t)))$ and $s'(t)$.

Our problem is somehow similar to one-phase Stefan problems in a one-dimensional domain. With regard to one-phase Stefan problem, there are a lot of mathematical results. However, we will find certain differences between our problem P and the Stefan type problems, in the conditions (1.4)-(1.5) on the free boundary.

In fact, our conditions (1.4)-(1.5) do not impose any value constraint for u, for instance $u(t, s(t)) = 0$ for $t \in [0, T]$, as in the classical Stefan problem. In addition, although Fasano and Primicerio dealt with a Stefan type problem under the following generalized condition:

$$\begin{cases} u(t, s(t)) = F(t, s(t)), \\ u_x(t, s(t)) = \lambda(t, s(t))s'(t) + \mu(t, s(t)), \end{cases} \text{for } t \in [0, T], \tag{1.7}$$

with given continuous functions F, λ and μ,

the above (1.7) will not be to cover our conditions (1.4)-(1.5). In fact, the condition (1.7) implies some kind of input-output relation between s and u, but meanwhile, the relation as in (1.4)-(1.5) must be more interactive. From this aspect, our problem P should be regarded as a separate type from the Stefan type.

On the other hand, the free boundary problems, including similar types of conditions to (1.4), were already proposed and studied by Muntean and Böhm [14]. Furthermore, this previous work has been extended by Aiki and Muntean [4, 5, 6, 7] to the mathematical studies of the existence and uniqueness (cf. [4]) and the large-time behavior (cf. [5, 6, 7]). However, the problems treated in [4, 5, 6, 7] are also different from ours in the point that the free boundary $s = s(t)$ must be monotone increasing in [4, 5, 6, 7], while it may not be so in our model. Hence, it should be noted that the problem P is complicated comparing to the those as in the previous works.

Now, the aim in this paper is to verify the solvability of the free boundary problem P. Based on this, we assume the smallness of the density of gaseous-H_2O, and prove the main theorem concerned with the existence and uniqueness of a time-local solution to P. The main theorem will be proved by means of the abstract theory for evolution equations governed by time-dependent subdifferentials and Banach's fixed-point theorem, and especially, the maximal principle for the humidity u will be a main tool to obtain the uniform estimates for approximate solutions.

2. **Main result.**

We begin with assumption for given data α, ρ_a, ρ_g and k.

(A1) $\alpha \in C^1(\mathbb{R}^2)$ and $\frac{\partial \alpha}{\partial s}$ are Lipschitz continuous with the (common) Lipschitz constant C_α and it holds that

$$\left|\frac{\partial \alpha}{\partial s}(s, u)\right| \leq C_\alpha \text{ and } \frac{\partial \alpha}{\partial u}(s, u) \geq \delta_0 \text{ for } (s, u) \in \mathbb{R}^2,$$

where δ_0 is a positive constant, and for any $s \in \mathbb{R}$

$$\alpha(s, u) \geq 0 \text{ if } u \geq 1 \text{ and } \alpha(s, u) \leq 0 \text{ if } u \leq 0.$$

For simplicity, we put

$$C_\alpha^0 = \sup\{|\alpha(s, u)| : 0 \leq s \leq L, 0 \leq u \leq 1\}.$$

(A2) ρ_a and ρ_g are positive constants such that:

$$\rho_a > \rho_g \text{ and } \rho_a \delta_0 - \rho_g(C_\alpha + C_\alpha^0) \geq 0.$$

Accordingly, it holds that

$$-\rho_g \alpha(s, u) + \rho_a \delta_0 - \rho_g \frac{\partial \alpha}{\partial u}(s, u) \geq 0 \text{ for } 0 \leq s \leq L, 0 \leq u \leq 1.$$

(A3) $k \in W^{1,2}(0, T)$ and $0 \leq k \leq k_* \leq 1$ on $[0, T]$, where k_* is a positive constant.

(A4) $s_0 \in (0, L)$ and $u_0 \in H^1(s_0, L)$ with $u_0(L) = k(0)$, $0 \leq u_0 \leq u_*$ on $[s_0, L]$, were u_* is a positive constant.

Definition 2.1. Let s and u be functions on $[0, T']$ and $Q_s(T')$, respectively, for $0 < T' \le T$. We call that a pair $\{s, u\}$ is a solution of P on $[0, T']$ if the conditions (S) and (1.2) $\sim$ (1.6) hold:

(S) $s \in W^{1,\infty}(0, T')$, $0 < s < L$ on $[0, T']$, $u \in L^\infty(Q_s(T'))$, $u_t, u_{xx} \in L^2(Q_s(T'))$, the function $t \in (0, T') \mapsto |u_x(\cdot, t)|_{L^2(s(t),L)} \in [0, \infty)$ is bounded.

In order to handle solutions of P easily, we consider the problem in a cylindrical domain obtained by change of variables. Let $\tilde{u}(t, y) := u(t, (1 - y)s(t) + Ly)$ for $(t, y) \in Q(T) := (0, T) \times (0, 1)$. Then it hold that

$$0 < s(t) < L \text{ for } 0 \le t \le T,$$

$$\rho_g \tilde{u}_t - \frac{\kappa}{(L-s)^2}\tilde{u}_{yy} = \frac{\rho_g(1-y)s'}{L-s}\tilde{u}_y \text{ in } Q(T), \tag{2.1}$$

$$\tilde{u}(t, 1) = k(t) \text{ for } 0 < t < T, \tag{2.2}$$

$$s'(t) = \alpha(s(t), \tilde{u}(t, 0)) \text{ for } 0 < t < T, \tag{2.3}$$

$$\frac{\kappa}{L - s(t)}\tilde{u}_y(t, 0) = (\rho_a - \rho_g \tilde{u}(t, 0))s'(t) \text{ for } 0 < t < T, \tag{2.4}$$

$$s(0) = s_0, \tag{2.5}$$

$$\tilde{u}(0, y) = \tilde{u}_0(y) \text{ for } 0 \le y \le 1, \tag{2.6}$$

where $\tilde{u}_0(y) = u_0((1 - y)s_0 + Ly)$ for $0 \le y \le 1$.

Then, obviously, we obtain the following set of conditions (S'1) and (S'2) which is equivalent to Definition 2.1:

(S'1) $s \in W^{1,\infty}(0, T')$, $0 < s < L$ on $[0, T']$, $\tilde{u} \in W^{1,2}(0, T'; L^2(0, 1)) \cap L^\infty(0, T'; H^1(0, 1)) \cap L^\infty(Q(T')) \cap L^2(0, T'; H^2(0, 1))$.

(S'2) (2.1) $\sim$ (2.6) hold.

Here, we introduce the following notations related to some function spaces: We put $H := L^2(0, 1)$, $V(T) = L^\infty(0, T; H) \cap L^2(0, T; H^1(0, 1))$, and $|z|_{V(T)} = |z|_{L^\infty(0,T;H)} + |z_y|_{L^2(0,T;H)}$ for $z \in V(T)$. As easily checked, $V(T)$ is a Banach space with the norm $|\cdot|_{V(T)}$.

Our main result of this paper is concerned with the existence and uniqueness of a time-local solution to P.

Theorem 2.2. *Let $T > 0$. If (A1) $\sim$ (A4) hold, $k_* \le 1$ and $u_* \le 1$ then there exists $T' \in (0, T]$ such that P has a unique solution $\{s, u\}$ on $[0, T']$ and $0 \le u(t) \le 1$ on $[s(t), L]$ for $t \in (0, T']$.*

Remark 1. Due to (A2), the value of the density ρ_g of aqueous-H_2O must be much smaller than that of the amount ρ_a of saturated water vapor. Meanwhile, at the temperature 30°C, the values of these constants are experimentally known that $\rho_a = 1.0 \times 10^6$ (g/m^3) and $\rho_g = 30.3$ (g/m^3), respectively. Moreover, we suppose that both C_α and C_α^0 are not too large, for instance, $C_\alpha + C_\alpha^0 \le 100$. Thus the condition (A2) can be said as a reasonable assumption from physical point of view. Also, u indicates the humidity so that the assumption $k_* \le 1$ and $u_* \le 1$ and the assertion $u \le 1$ should be quit natural.

3. **Auxiliary lemmas.**

In this section for given $s \in C([0, T])$ with $0 < s < L$ on $[0, T]$ we consider the following initial boundary value problem, denoted by

$\mathrm{AP}_1=\mathrm{AP}_1(\tilde{u}_0, s, f, k)$:

$$\rho_g \tilde{u}_t - \frac{\kappa}{(L-s)^2}\tilde{u}_{yy} = f \text{ in } Q(T), \tag{3.1}$$

$$\tilde{u}(t,1) = k(t) \text{ for } 0 < t < T, \tag{3.2}$$

$$\frac{\kappa}{L-s(t)}\tilde{u}_y(t,0) = (\rho_a - \rho_g\sigma(\tilde{u}(t,0)))\alpha(s(t), \sigma(\tilde{u}(t,0))) \text{ for } 0 < t < T, \tag{3.3}$$

$$\tilde{u}(0,y) = \tilde{u}_0(y) \text{ for } 0 \le y \le 1, \tag{3.4}$$

where f and $\tilde{u}_0$ are given functions on $Q(T)$ and $[0,1]$, respectively, and

$$\sigma(u) := \begin{cases} 1 & \text{if } u \ge 1, \\ u & \text{if } u < 1. \end{cases}$$

In order to solve this problem we introduce a family $\{\varphi^t\}_{t\in[0,T]}$ of time-dependent functionals $\varphi^t : H \to \mathbb{R} \cup \{\infty\}$ for $t \in [0,T]$, defined as follows: For $t \in [0,T]$,

$$\varphi^t(u) := \begin{cases} \dfrac{\kappa}{2(L-s(t))^2}\displaystyle\int_0^1 |u_y|^2 dy + b^t(u(1)) + \frac{1}{L-s(t)}\hat{b}_\sigma(s(t), u(0)) \\ \qquad\qquad\qquad\qquad\qquad \text{if } u \in H^1(0,1) \text{ and } u(1) = k(t), \\ +\infty \qquad\qquad \text{otherwise,} \end{cases}$$

where

$$b^t(r) = \begin{cases} 0 & \text{if } r = k(t), \\ +\infty & \text{otherwise,} \end{cases} \quad \text{for } r \in \mathbb{R},$$

$$\hat{b}_\sigma(s,r) = \int_0^r b_\sigma(s,\xi)d\xi \text{ and } b_\sigma(s,r) = (\rho_a - \rho_g\sigma(r))\alpha(s,\sigma(r)) \text{ for } (s,r) \in \mathbb{R}^2.$$

Clearly, for each $t \in [0,T]$ the effective domain $D(\varphi^t)$ of φ^t is given by $D(\varphi^t) = \{u \in H^1(0,1) : u(1) = k(t)\}$. Also, the function $\varphi^t(\cdot)$ is often denoted by $\varphi^t(s,\cdot)$ as appropriate.

The first lemma gives some useful inequalities concerned with $\hat{b}_\sigma$ and φ^t.

Lemma 3.1. *(1) If (A1) and (A2) hold, then*

$$|\tfrac{\partial}{\partial s} b_\sigma(s,u)| \le C'_\alpha(1+|u|) \text{ for } (s,u) \in \mathbb{R}^2, \tag{3.5}$$

$$|\tfrac{\partial}{\partial s} \hat{b}_\sigma(s,u)| \le C'_\alpha(1+u^2) \text{ for } (s,u) \in \mathbb{R}^2, \tag{3.6}$$

$$\hat{b}_\sigma(s,u) \ge -C'_\alpha|u| \text{ for } 0 \le s \le L, u \in \mathbb{R}, \tag{3.7}$$

where $C'_\alpha = \max\{(\rho_a+\rho_g)C_\alpha, \rho_a C^0_\alpha\}$.

(2) If (A1) $\sim$ *(A3) hold and* $s \in W^{1,2}(0,T)$ *with* $0 < s < L$ *on* $[0,T]$*, then there exist positive constants* C_0 *and* C_1 *depending only on* s, ρ_a, ρ_g, C_α, L, κ *and* k^* *such that*

$$\left.\begin{array}{l} |u(0)|^2 \le C_0\varphi^t(u) + C_1 \\ \dfrac{1}{L-s(t)}|\hat{b}_\sigma(s(t),u(0))| \le C_0\varphi^t(u) + C_1 \\ \dfrac{\kappa}{(L-s(t))^2}|u_y|_H^2 \le C_0\varphi^t(u) + C_1 \end{array}\right\} \text{ for } u \in D(\varphi^t) \text{ and } t \in [0,T].$$

Proof. (1) Let $s \in \mathbb{R}$. Since $|\sigma(u)| \le |u|$ for $u \in \mathbb{R}$, (3.5) immediately follows from (A1). By using (3.5) and (A1), it is easy to obtain (3.6).

If $u \le 0$, then (A1) implies that $\hat{b}_\sigma(s,u) \ge 0$. Otherwise,

$$\hat{b}_\sigma(s,u) \ge -C^0_\alpha \int_0^u |\rho_a - \rho_g\sigma(\xi)|d\xi.$$

Since $\rho_a - \rho_g\sigma(\xi) \geq \rho_a - \rho_g > 0$ for $\xi \in [0, u]$, we have $\hat{b}_\sigma(s, u) \geq -C^0_\alpha \rho_a u$. Thus (3.7) is true.

(2) Let $s \in W^{1,2}(0, T)$ with $0 < s < L$ on $[0, T]$, $t \in [0, T]$, $\delta_s > 0$ satisfying $s \leq L - \delta_s$ on $[0, T]$, and $u \in D(\varphi^t)$. Then by (3.7) it is easy to see that

$$\begin{aligned} |u(0)|^2 &= \left| \int_0^1 u_y dy + u(1) \right|^2 \\ &\leq 2(|u_y|_H^2 + |k(t)|^2) \\ &\leq \frac{4L^2}{\kappa}\left(\varphi^t(u) - \frac{1}{L - s(t)}\hat{b}_\sigma(s(t), u(0))\right) + 2k_*^2 \\ &\leq \frac{4L^2}{\kappa}\left(\varphi^t(u) + \frac{C'_\alpha}{\delta_s}|u(0)|\right) + 2k_*^2. \end{aligned}$$

Here, by applying the Young inequality we obtain

$$\frac{1}{2}|u(0)|^2 \leq \frac{4L^2}{\kappa}\varphi^t(u) + 8\left(\frac{L^2 C'_\alpha}{\kappa\delta_s}\right)^2 + 2k_*^2. \tag{3.8}$$

Thus we proved the first inequality of (2).

Next, we note the following facts: If $a \geq -b$ for $a, b \in \mathbb{R}$ with $b \geq 0$, then $|a| \leq a + 2b$. This together with (3.7) shows that

$$\begin{aligned} \frac{1}{L - s(t)}|\hat{b}_\sigma(s(t), u(0))| &\leq \frac{1}{L - s(t)}\left(\hat{b}_\sigma(s(t), u(0)) + 2C'_\alpha|u(0)|\right) \\ &\leq \varphi^t(u) + \frac{2C'_\alpha}{\delta}|u(0)|. \end{aligned} \tag{3.9}$$

(3.9) and (3.8) imply the second inequality of (2). Also, we can prove the third inequality of (2) in a similar way to the above inequalities. □

The following lemma guarantees the well-posedness of AP_1.

Lemma 3.2. *If (A1) $\sim$ (A3) hold, $s \in W^{1,2}(0, T)$ with $0 < s < L$ on $[0, T]$, $f \in L^2(0, T; H)$ and $\tilde{u}_0 \in H^1(0, 1)$ with $\tilde{u}_0(1) = k(0)$, then $AP_1(\tilde{u}_0, s, f, k)$ has a unique solution $\tilde{u} \in W^{1,2}(0, T; H) \cap L^\infty(0, T; H^1(0, 1))$ in the usual sense. Moreover, the function $t \to \varphi^t(\hat{u}(t))$ is absolutely continuous on $[0, T]$.*

Proof. First, for each $r \in [0, L]$, (A1) implies that $b_\sigma(r, \cdot)$ is monotone increasing on $\mathbb{R}$. Clearly, for each $t \in [0, T]$ φ^t is a proper l.s.c convex function, $\partial\varphi^t$ is single valued, $w^* = \partial\varphi^t(u)$ if and only if $w^* \in H$ and

$$\begin{cases} w^* = -\dfrac{\kappa}{(L - s(t))^2}u_{yy} \text{ on } (0, 1), \\ u(1) = k(t), \\ \dfrac{\kappa}{(L - s(t))}u_y(0) = b_\sigma(s(t), u(0)), \end{cases}$$

and there exists a positive constant C, depending on $\min\{L - s(t) \,|\, t \in [0, T]\} > 0$, such that for each $t_1, t_2 \in [0, T]$ with $t_1 \leq t_2$ and for any $u \in D(\varphi^{t_1})$ there exists $\hat{u} \in D(\varphi^{t_2})$ such that

$$|u - \hat{u}|_H \leq |k(t_1) - k(t_2)|(1 + |\varphi^{t_1}(u)|^{\frac{1}{2}})$$

and

$$|\varphi^{t_2}(\hat{u}) - \varphi^{t_1}(u)| \leq C(|s(t_1) - s(t_2)| + |k(t_1) - k(t_2)|)(1 + |\varphi^{t_1}(u)|).$$

In fact, by putting $\hat{u} = u + k(t_2) - k(t_1)$ and applying Lemma 3.1 we can show these two inequalities. Then, the theory of evolution equation governed by time-dependent subdifferentials, for instance [11, Theorems 1.1.2 and 1.5.1], implies the existence of $\tilde{u} \in W^{1,2}(0,T;H)$ such that $\varphi^{(\cdot)}(\tilde{u}(\cdot)) \in L^\infty(0,T)$, $t \to \varphi^t(\tilde{u}(t))$ is absolutely continuous on $[0,T]$ and

$$\rho_g \tilde{u}_t(t) + \partial\varphi^t(\tilde{u}(t)) = f(t) \text{ in } H \text{ for a.e. } t \in (0,T). \tag{3.10}$$

This $\tilde{u}$ is a unique solution of $\mathrm{AP}_1(\tilde{u}_0, s, f, k)$ on $[0,T]$. □

The next lemma is a direct result of Lemma 3.2.

Lemma 3.3. *If (A1) ∼ (A3) hold, $s \in W^{1,\infty}(0,T)$ with $0 < s < L$ on $[0,T]$, $f \in L^2(0,T;H^1(0,1))$ and $\tilde{u}_0 \in H^1(0,1)$ with $\tilde{u}_0(1) = k(0)$, then $AP_1(\tilde{u}_0, s, \frac{\rho_g(1-y)s'}{L-s} f_y, k)$ has a unique solution $\tilde{u} \in W^{1,2}(0,T;H) \cap L^\infty(0,T;H^1(0,1))$ on $[0,T]$.*

Next, we shall solve the initial boundary value problem $\mathrm{AP}_2(\tilde{u}_0, s, k) :=$ $\{(2.1), (2.2), (3.11), (2.6)\}$,

$$\frac{\kappa}{L - s(t)} \tilde{u}_y(t,0) = b_\sigma(s(t), \tilde{u}(t,0)) \text{ for } 0 < t < T. \tag{3.11}$$

Lemma 3.4. *If (A1) ∼ (A3) hold, $s \in W^{1,\infty}(0,T)$ with $0 < s < L$ on $[0,T]$ and $\tilde{u}_0 \in H^1(0,1)$ with $\tilde{u}_0(1) = k(0)$, then $AP_2(\tilde{u}_0, s, k)$ has a unique solution $\tilde{u} \in W^{1,2}(0,T;H) \cap L^\infty(0,T;H^1(0,1))$ on $[0,T]$.*

Proof. By Lemma 3.3 for any $f \in V(T)$ there exists a solution $\tilde{u} \in V(T)$ of $\mathrm{AP}_1(\tilde{u}_0, s, \frac{\rho_g(1-y)s'}{L-s} f_y, k)$ on $[0,T]$. Then we can define a solution operator $\Gamma_T : V(T) \to V(T)$, by putting $\Gamma_T(f) = \tilde{u}$. Besides, let us set $f_i \in V(T)$, $\tilde{u}_i = \Gamma_T(f_i)$ for $i = 1, 2$, and $\tilde{u} = \tilde{u}_1 - \tilde{u}_2$.

By multiplying the difference of the first equations of $\mathrm{AP}_1(\tilde{u}_0, s, \frac{\rho_g(1-y)s'}{L-s}(f_i)_y, k)$ for $i = 1, 2$ by $\tilde{u}$ and integrating it we observe that

$$\begin{aligned} & \frac{\rho_g}{2}\frac{d}{dt}|\tilde{u}(t)|_H^2 + \frac{\kappa}{(L-s(t))^2}\int_0^1 |\tilde{u}_y(t,y)|^2 dy \\ = \; & -\frac{1}{L-s(t)}(b_\sigma(s(t),\tilde{u}_1(t,0)) - b_\sigma(s(t),\tilde{u}_2(t,0)))\tilde{u}(t,0) \\ & + \frac{\rho_g s'(t)}{L-s(t)}\int_0^1 (1-y)(f_1(t,y) - f_2(t,y))_y \tilde{u}(t,y)dy \quad \text{ for a.e. } t \in (0,T). \end{aligned}$$

From the monotonicity of $b_\sigma(s, \sigma(\cdot))$ it follows that

$$\begin{aligned} & \frac{\rho_g}{2}\frac{d}{dt}|\tilde{u}(t)|_H^2 + \frac{\kappa}{(L-s(t))^2}|\tilde{u}_y(t)|_H^2 \\ \le \; & \frac{\rho_g |s'|_{L^\infty(0,T)}}{\delta_s}|(f_1 - f_2)_y(t)|_H |\tilde{u}(t)|_H \quad \text{ for a.e. } t \in (0,T), \end{aligned} \tag{3.12}$$

where δ_s is a positive constant such that $L - s \ge \delta_s$ on $[0,T]$.

Hence, by integrating both sides of the inequality on $[0, t_1]$, $0 < t_1 < T$, we have

$$\begin{aligned} & \frac{\rho_g}{2}|\tilde{u}(t_1)|_H^2 + \frac{\kappa}{L^2}\int_0^{t_1} |\tilde{u}_y(t)|_H^2 dt \\ \le \; & \frac{\rho_g |s'|_{L^\infty(0,T)}}{\delta} |\tilde{u}|_{L^\infty(0,t_1;H)} \int_0^{t_1} |(f_1 - f_2)_y(t)|_H dt \\ \le \; & \frac{\rho_g t_1^{\frac{1}{2}} |s'|_{L^\infty(0,T)}}{\delta} |\tilde{u}(t)|_{V(t_1)} |f_1 - f_2|_{V(t_1)} \quad \text{for } t \in [0, t_1]. \end{aligned}$$

Then by putting $\nu := \min\{\frac{\rho_g}{2}, \frac{\kappa}{L^2}\}$ we have

$$\nu |\tilde{u}|_{V(T_0)} \le \frac{\rho_g T_0^{\frac{1}{2}} |s'|_{L^\infty(0,T)}}{\delta} |f_1 - f_2|_{V(T_0)} \text{ for } 0 < T_0 \le T.$$

Hence, since we find a small constant $0 < T_0 \le T$ such that Γ_{T_0} is the contraction mapping, Banach's fixed point theorem implies that $\mathrm{AP}_2(\tilde{u}_0, s, k)$ has a unique solution $\tilde{u}$ on $[0, T_0]$. Clearly, the choice of T_0 is independent of initial values so that we have proved this lemma. □

In Lemma 3.5 we can relax the condition for s to have a solution of $\mathrm{AP}_2(\tilde{u}_0, s, k)$.

Lemma 3.5. *If (A1) ∼ (A3), $s \in W^{1,2}(0,T)$ with $0 < s < L$ on $[0,T]$, $\tilde{u}_0 \in H^1(0,1)$ with $\tilde{u}_0 \ge 0$ on $[0,1]$ and $\tilde{u}_0(1) = k(0)$, then $AP_2(\tilde{u}_0, s, k)$ has a unique solution on $[0,T]$.*

Proof. Choose a sequence $\{s_n\} \subset W^{1,\infty}(0,T)$ and $0 < \delta < L$ satisfying $L - s_n \ge \delta$ on $[0,T]$ for each n and $s_n \to s$ in $W^{1,2}(0,T)$ as $n \to \infty$, and put $\varphi_n^t(\cdot) = \varphi^t(s_n, \cdot)$. By Lemma 3.4 we can take a sequence $\{\tilde{u}_n\}$ of the solutions $\tilde{u}_n$ to $\mathrm{AP}_2(\tilde{u}_0, s_n, k)$ on $[0,T]$, for $n \in \mathbb{N}$. Then on account of Lemma 3.2 the function $t \to \varphi_n^t(\tilde{u}_n(t))$ is absolutely continuous on $[0,T]$ so that the function $t \to \frac{\kappa}{(L-s_n(t))^2}|\tilde{u}_{ny}(t)|_H^2$ is continuous on $[0,T]$. For a.e. $t \in (0,T)$, let $h > 0$ with $t - h > 0$. We multiply (2.1) by $\frac{\tilde{u}_n(t) - \tilde{u}_n(t-h)}{h} =: U_{nh}(t)$. Then by integration by parts we have

$$\begin{aligned} & \rho_g \int_0^1 (\tilde{u}_n)_t(t) U_{nh}(t) dy + \frac{\kappa}{(L - s_n(t))^2} \int_0^1 (\tilde{u}_n)_y(t) (U_{nh})_y(t) dy \\ & - \frac{\kappa}{(L - s_n(t))^2} (\tilde{u}_n)_y(t,1) U_{nh}(t,1) + \frac{\kappa}{(L - s_n(t))^2} (\tilde{u}_n)_y(t,0) U_{nh}(t,0) \\ \Big(=: & \sum_{i=1}^4 I_{ih}(t) \Big) \\ = \; & \int_0^1 \frac{(1-y)s_n'(t)}{L - s_n(t)} (\tilde{u}_n)_y(t) U_{nh}(t) dy \; (=: I_{5h}(t)) \quad \text{for a.e. } t \in (0,T). \end{aligned}$$

It is easy to see that

$$\lim_{h \downarrow 0} I_{1h}(t) = \rho_g |(\tilde{u}_n)_t(t)|_H^2, \; \lim_{h \downarrow 0} I_{3h}(t) = -\frac{\kappa}{(L - s_n(t))^2} (\tilde{u}_n)_y(t,1) k'(t),$$

and

$$\lim_{h \downarrow 0} I_{5h}(t) = \int_0^1 \frac{(1-y)s_n'(t)}{L - s_n(t)} (\tilde{u}_n)_y(t) (\tilde{u}_n)_t(t) dy$$

for a.e. $t \in (0,T)$. Also, we have

$$\begin{aligned} I_{2h}(t) &\geq \frac{\kappa}{2h(L-s_n(t))^2}(|(\tilde{u}_n)_y(t)|_H^2 - |(\tilde{u}_n)_y(t-h)|_H^2 \\ &= \frac{1}{2h}\left(\frac{\kappa}{(L-s_n(t))^2}|(\tilde{u}_n)_y(t)|_H^2 - \frac{\kappa}{(L-s_n(t-h))^2}|(\tilde{u}_n)_y(t-h)|_H^2\right) \\ &\quad -\frac{\kappa}{2h}\left(\frac{1}{(L-s_n(t))^2} - \frac{1}{(L-s_n(t-h))^2}\right)|(\tilde{u}_n)_y(t-h)|_H^2, \end{aligned}$$

and

$$\begin{aligned} & I_{4h}(t) \\ \geq\ & \frac{1}{h}\left(\frac{1}{L-s_n(t)}\hat{b}_\sigma(s_n(t),\tilde{u}_n(t,0)) - \frac{1}{L-s_n(t-h)}\hat{b}_\sigma(s_n(t-h),\tilde{u}_n(t-h,0))\right) \\ & -\frac{1}{h}\frac{1}{L-s_n(t)}\left(\hat{b}_\sigma(s_n(t),\tilde{u}_n(t-h,0)) - \hat{b}_\sigma(s_n(t-h),\tilde{u}_n(t-h,0))\right) \end{aligned}$$

for a.e. $t \in (0,T)$. Then we observe that

$$\begin{aligned} & I_{2h}(t) + I_{4h}(t) \\ \geq\ & \frac{1}{h}(\varphi_n^t(\tilde{u}_n(t)) - \varphi_n^{t-h}(\tilde{u}_n(t-h))) \\ & -\frac{\kappa}{2h}\left(\frac{1}{(L-s_n(t))^2} - \frac{1}{(L-s_n(t-h))^2}\right)|(\tilde{u}_n)_y(t-h)|_H^2 \\ & -\frac{1}{h}\frac{1}{L-s_n(t)}\left(\hat{b}_\sigma(s_n(t),\tilde{u}_n(t-h,0)) - \hat{b}_\sigma(s_n(t-h),\tilde{u}_n(t-h,0))\right) \end{aligned}$$

for a.e. $t \in (0,T)$. Here, Lemma 3.2 implies that

$$\begin{aligned} & \liminf_{h\downarrow 0}(I_{2h}(t) + I_{4h}(t)) \\ \geq\ & \frac{d}{dt}\varphi_n^t(\tilde{u}_n(t)) + \frac{\kappa s_n'(t)}{(L-s_n(t))^3}|\tilde{u}_{ny}(t)|_H^2 - \frac{s_n'(t)}{L-s_n(t)}\frac{\partial\hat{b}_\sigma}{\partial s}(s_n(t),\tilde{u}_n(t,0)) \end{aligned}$$

for a.e. $t \in (0,T)$. From the above calculations it follows that

$$\begin{aligned} & \rho_g|(\tilde{u}_n)_t(t)|_H^2 + \frac{d}{dt}\varphi_n^t(\tilde{u}_n(t)) \\ \leq\ & \frac{\kappa|s_n'(t)|}{(L-s_n(t))^3}|(\tilde{u}_n)_y(t)|_H^2 + \frac{|s_n'(t)|}{L-s_n(t)}\left|\frac{\partial\hat{b}_\sigma}{\partial s}(s_n(t),\tilde{u}_n(t,0))\right| \\ & +\frac{\kappa}{(L-s_n(t))^2}(\tilde{u}_n)_y(t,1)k'(t) + \frac{|s_n'(t)|}{L-s_n(t)}|(\tilde{u}_n)_y(t)|_H|(\tilde{u}_n)_t(t)|_H \\ =:\ & \sum_{i=1}^4 J_i(t) \quad \text{for a.e. } t \in (0,T). \end{aligned}$$

By Lemma 3.2 we have

$$J_1(t) \leq \frac{|s_n'(t)|}{\delta}(C_0\varphi_n^t(\tilde{u}_n(t)) + C_1),$$

$$\begin{aligned} J_2(t) &\leq \frac{C_\alpha'|s_n'(t)|}{L-s_n(t)}(1+|\tilde{u}_n(t,0)|^2) \\ &\leq \frac{C_\alpha'|s_n'(t)|}{\delta}(C_0\varphi_n^t(\tilde{u}_n(t)) + C_1 + 1), \end{aligned}$$

and

$$\begin{aligned} J_4(t) &\le \frac{|s_n'(t)|}{L-s_n(t)} |(\tilde{u}_n)_y(t)|_H |(\tilde{u}_n)_t(t)|_H \\ &\le \frac{\rho_g}{4} |(\tilde{u}_n)_t(t)|_H^2 + \frac{|s_n'(t)|^2}{\rho_g (L-s_n(t))^2} |(\tilde{u}_n)_y(t)|_H^2 \\ &\le \frac{\rho_g}{4} |(\tilde{u}_n)_t(t)|_H^2 + \frac{|s_n'(t)|^2}{\rho_g \kappa} (C_0 \varphi_n^t(\tilde{u}_n(t)) + C_1) \text{ for a.e. } t \in (0,T). \end{aligned}$$

Since $|(\tilde{u}_n)_y(t,1)| \le |(\tilde{u}_n)_{yy}(t)|_H + |(\tilde{u}_n)_y(t)|_H$, it holds that

$$|(\tilde{u}_n)_y(t,1)| \le \frac{\rho_g}{\kappa} |(\tilde{u}_n)_t(t)|_H + \frac{1}{\kappa} \left| \frac{s_n'(t)}{L-s_n(t)} (\tilde{u}_n)_y(t) \right|_H$$

so that

$$\begin{aligned} J_3(t) &\le \frac{\rho_g}{4} |(\tilde{u}_n)_t(t)|_H^2 + \frac{|s_n'(t)|^2}{\kappa} \frac{\kappa}{(L-s_n(t))^2} |(\tilde{u}_n)_y(t)|_H^2 \\ &\quad + \frac{\kappa}{2(L-s_n(t))^2} |(\tilde{u}_n)_y(t)|_H^2 + \left(\frac{\rho_g}{\delta^4} + \frac{1}{2\delta^4} + \frac{\kappa}{2\delta^4} \right) |k'(t)|^2 \\ &\le \frac{\rho_g}{4} |(\tilde{u}_n)_t(t)|_H^2 + \left(\frac{|s_n'(t)|^2}{\kappa} + 1 \right) (C_0 \varphi_n^t(\tilde{u}_n(t)) + C_1) \\ &\quad + \left(\frac{\rho_g}{\delta^4} + \frac{1}{2\delta^4} + \frac{\kappa}{2\delta^4} \right) |k'(t)|^2 \quad \text{for a.e. } t \in (0,T). \end{aligned}$$

Here, we applied Lemma 3.2, again.

Accordingly, there exists a positive constant C_5 independent of n such that

$$\begin{aligned} &\frac{\rho_g}{2} |(\tilde{u}_n)_t(t)|_H^2 + \frac{d}{dt} \left(\varphi_n^t(\tilde{u}_n(t)) + \frac{C_1}{C_0} \right) \\ &\le C_5 \{ (1 + |s_n'(t)|^2)(C_0 \varphi_n^t(\tilde{u}_n(t)) + C_1) + |k'(t)|^2 \} \text{ for a.e. } t \in (0,T). \end{aligned} \tag{3.13}$$

Hence, Gronwall's inequality guarantees that the sequence $\{u_n\}$ is bounded in $W^{1,2}(0,T;H)$ and $L^\infty(0,T;H^1(0,1))$. Then we can take a subsequence $\{n_j\}$ of $\{n\}$ and $\tilde{u} \in W^{1,2}(0,T;H) \cap L^\infty(0,T;H^1(0,1))$ such that $\tilde{u}_{n_j} \to \tilde{u}$ weakly in $W^{1,2}(0,T;H)$, weakly* in $L^\infty(0,T;H^1(0,1))$, and in $C(\overline{Q(T)})$ as $j \to \infty$. It is obvious that $\tilde{u}$ is a solution of $\mathrm{AP}_2(\tilde{u}_0, s, k)$ on $[0,T]$.

The uniqueness is easily obtained from (3.12), Schwartz's inequality and Gronwall's inequality. □

4. **The local existence in time.** The aim of this section is to prove Theorem 2.2 and we always assume (A1) $\sim$ (A4) throughout this section.

First, for $T > 0$ and $0 < s_0 < L' < L$ we put $S(T, s_0, L') := \{ s \in W^{1,2}(0,T) : 0 \le s \le L' \text{ on } [0,T],\ s(0) = s_0 \}$. Let $s \in S(T, s_0, L')$ and $\tilde{u}$ be a solution of $\mathrm{AP}_2(\tilde{u}_0, s, k)$ on $[0,T]$. Here, we define the operators $\Phi : S(T, s_0, L') \to V(T)$ and $\Lambda_T : S(T, s_0, L') \to W^{1,2}(0,T)$ by $\Phi s = \tilde{u}$ and $[\Lambda_T s](t) := \int_0^t \alpha(s(\tau), \tilde{u}(\tau, 0)) d\tau + s_0$ for $t \in [0,T]$ and $s \in S(T, s_0, L')$, respectively. Moreover, for any $M > 0$ we put $S_M(T) := S_M(T, s_0, L') := \{ s \in S(T, s_0, L') : |s|_{W^{1,2}(0,T)} \le M \}$.

Lemma 4.1. *For $M > 0$ and $T > 0$ there exists a positive constant $K_0(T, M)$ such that*

$$|\Phi s|_{W^{1,2}(0,T;H)} + |\Phi s|_{L^\infty(0,T;H^1(0,1))} \le K_0(T, M) \quad \textit{for any } s \in S_M(T).$$

This lemma is a direct consequence of (3.13).

Lemma 4.2. *For $0 < s_0 < L' < L$ and $M > 0$ there exists a positive constant $T_1 \leq T$ such that $\Lambda_{T_1} : S_M(T_1) \to S_M(T_1)$ and it becomes a contraction mapping on the closed subset $S_M(T_1)$ in $W^{1,2}(0, T_1)$.*

Proof. For $T > 0$ let $s \in S_M(T, s_0, L')$ and $\tilde{u} = \Phi s$. Lemma 4.1 guarantees the existence of a positive constant C_* such that

$$|\alpha(s(t), \tilde{u}(t,0))| \leq C_* \quad \text{for } 0 \leq t \leq T.$$

Accordingly, $|[\Lambda_T s](t) - s_0| \leq C_* t$ for $0 \leq t \leq T$. Then because of $0 < s_0 < L'$ we can take $T_0 \in (0, T]$ such that $\Lambda_{T_0} : S_M(T_0) \to S_M(T_0)$.

Next, for $s_i \in S_M(T)$ let $\tilde{u}_i = \Phi s_i$, $i = 1, 2$, $\tilde{u} := \tilde{u}_1 - \tilde{u}_2$, $s = s_1 - s_2$ and $\delta = L - L'$. Multiplying the both sides of the difference between the governing equations (2.1) for $\tilde{u}_i$, $i = 1, 2$, and integrating the result over $(0, 1)$, we have

$$\begin{aligned}
&\frac{\rho_g}{2}\frac{d}{dt}|\tilde{u}(t)|_H^2 - \int_0^1 \left(\frac{\kappa}{(L - s_1(t))^2}(\tilde{u}_1)_{yy}(t) - \frac{\kappa}{(L - s_2(t))^2}(\tilde{u}_2)_{yy}(t) \right) \tilde{u}(t) dy \\
= \;& \rho_g \int_0^1 \left(\frac{(1 - y)s_1'(t)}{L - s_1(t)}(\tilde{u}_1)_y(t) - \frac{(1 - y)s_2'(t)}{L - s_2(t)}(\tilde{u}_2)_y(t) \right) \tilde{u}(t) dy \qquad (4.1)
\end{aligned}$$

for a.e. $t \in (0, T)$. On the second term of the left hand side of (4.1), applying the integration by part yields that

$$\begin{aligned}
&(\text{ the second term of the left hand side}) \\
= \;& \int_0^1 \left(\frac{\kappa}{(L - s_1(t))^2}(\tilde{u}_1)_y(t) - \frac{\kappa}{(L - s_2(t))^2}(\tilde{u}_2)_y(t) \right) \tilde{u}_y(t) dy \\
& + \left(\frac{\kappa}{(L - s_1(t))^2}(\tilde{u}_1)_y(t,0) - \frac{\kappa}{(L - s_2(t))^2}(\tilde{u}_2)_y(t,0) \right) \tilde{u}(t,0) \\
=: \;& I_1(t) + I_2(t) \quad \text{for a.e. } t \in (0, T).
\end{aligned}$$

On account of Lemma 4.1 it clearly holds that

$$\begin{aligned}
I_1(t) \;=\;& \frac{\kappa}{(L - s_1(t))^2}|\tilde{u}_y(t)|_H^2 \\
& + \int_0^1 \left(\frac{\kappa}{(L - s_1(t))^2} - \frac{\kappa}{(L - s_2(t))^2} \right) (\tilde{u}_2)_y(t)\tilde{u}_y(t) dy (=: I_{1,2}(t)),
\end{aligned}$$

$$\begin{aligned}
|I_{1,2}(t)| \;\leq\;& \frac{2\kappa L|s(t)|}{\delta^2 (L - s_1(t))^2}|(\tilde{u}_2)_y(t)|_H |\tilde{u}_y(t)|_H \\
\leq\;& \int_0^1 \frac{\kappa}{8(L - s_1(t))^2}|\tilde{u}_y(t)|^2 dy + \frac{8\kappa L^2 |s(t)|^2}{\delta^4} K_0(T, M)^2,
\end{aligned}$$

$$\begin{aligned}
I_2(t) \;=\;& \left(\frac{1}{L - s_1(t)} - \frac{1}{L - s_2(t)} \right) b_\sigma(s_2(t), \tilde{u}_2(t,0))\tilde{u}(t,0) \\
& + \frac{1}{L - s_1(t)}(b_\sigma(s_1(t), \tilde{u}_1(t,0)) - b_\sigma(s_2(t), \tilde{u}_1(t,0))\tilde{u}(t,0) \\
& + \frac{1}{L - s_1(t)}(b_\sigma(s_2(t), \tilde{u}_1(t,0)) - b_\sigma(s_2(t), \tilde{u}_2(t,0)))\tilde{u}(t,0) \\
=: \;& I_{2,1}(t) + I_{2,2}(t) + I_{2,3}(t) \quad \text{for a.e. } t \in (0, T).
\end{aligned}$$

Because of the monotonicity of $b_\sigma(s_2(t),\cdot)$, we have $I_{2,3}(t) \ge 0$ for a.e. $t \in (0,T)$. Also, from (A1) and Lemma 3.1 it follows that

$$\begin{aligned}
&|I_{2,1}(t)| + |I_{2,2}(t)| \\
\le\ & \frac{C'_\alpha(1+|\tilde{u}_2(t,0)|)}{(L-s_1(t))(L-s_2(t))}|s(t)||\tilde{u}(t,0)| + \frac{C'_\alpha(1+|\tilde{u}_1(t,0)|)}{L-s_1(t)}|s(t)||\tilde{u}(t,0)| \\
\le\ & \frac{C'_\alpha(1+K_0(T,M))}{L-s_1(t)}\left(\frac{1}{\delta}+1\right)|s(t)||\tilde{u}_y(t)|_H \\
\le\ & \frac{\kappa}{4(L-s_1(t))^2}|\tilde{u}_y(t)|_H^2 + \frac{1}{\kappa}\left(\frac{1}{\delta}+1\right)^2\left(C'_\alpha(1+K_0(T,M))\right)^2|s(t)|^2
\end{aligned}$$

for a.e. $t \in (0,T)$.

Next, we consider the right hand side of (4.1) as follows:

$$\begin{aligned}
&(\text{ the right hand side}) \\
=\ & \rho_g\int_0^1 \frac{(1-y)s_1'(t)}{L-s_1(t)}\tilde{u}_y(t)\tilde{u}(t)dy + \rho_g\int_0^1 \frac{(1-y)s'(t)}{L-s_1(t)}(\tilde{u}_2)_y(t)\tilde{u}(t)dy \\
&+\rho_g\int_0^1\left(\frac{(1-y)}{L-s_1(t)} - \frac{(1-y)}{L-s_2(t)}\right)s_2'(t)(\tilde{u}_2)_y(t)\tilde{u}(t)dy \\
=:\ & I_3(t)+I_4(t)+I_5(t) \quad \text{for a.e. } t\in(0,T)
\end{aligned}$$

so that

$$I_3(t) \le \frac{\kappa}{8(L-s_2(t))^2}|\tilde{u}_y(t)|_H^2 + \frac{2\rho_g^2}{\kappa}|s_1'(t)|^2|\tilde{u}(t)|_H^2,$$

$$I_4(t) \le \frac{\rho_g}{2\delta}(K_0(T,M)^2|s'(t)|^2 + |\tilde{u}(t)|_H^2)$$

and

$$I_5(t) \le \frac{\rho_g}{2\delta^2}(K_0(T,M)^2|s(t)|^2 + |s_2'(t)|^2|\tilde{u}(t)|_H^2) \quad \text{for a.e. } t\in(0,T).$$

Then we see that there exists a positive constant B_2 such that for a.e. $t \in (0,T)$,

$$\frac{\rho_g}{2}\frac{d}{dt}|\tilde{u}(t)|_H^2 + \frac{\kappa}{2(L-s_1(t))^2}|\tilde{u}_y(t)|_H^2 \le B_2(F_1(t)+F_2(t)|\tilde{u}(t)|_H^2), \tag{4.2}$$

where $F_1(t) = |s(t)|^2 + |s'(t)|^2$ and $F_2(t) = |s_1'(t)|^2 + |s_2'(t)|^2 + 1$. By applying Gronwall's inequality to (4.2) we have

$$\begin{aligned}
\frac{\rho_g}{2}|\tilde{u}(t)|_H^2 + \frac{\kappa}{L^2}\int_0^t |\tilde{u}_y(\tau)|_H^2 d\tau &\le B_2\int_0^t F_1(\tau)d\tau \exp(B_2\int_0^t F_2(\tau)d\tau) \\
&\le B_3\int_0^t |s'(t)|^2 d\tau \text{ for } t\in[0,T], \qquad (4.3)
\end{aligned}$$

where $B_3 = B_2(1+T^2)\exp(B_2(T+2M^2))$. Moreover, since

$$\begin{aligned}
|\tilde{u}_1(t,0)-\tilde{u}_2(t,0)|^2 &\le \left|\int_0^1 \frac{\partial}{\partial y}(|\tilde{u}(t,y)|^2)dy\right| \\
&\le 2\int_0^1 |\tilde{u}_y(t,y)||\tilde{u}(t,y)|dy \\
&\le 2|\tilde{u}_y(t)|_H|\tilde{u}(t)|_H \quad \text{for a.e. } t\in(0,T),
\end{aligned}$$

we infer that for $0 < T_1 \le T_0$,

$$\begin{aligned} & |(\Lambda_{T_1} s_1)' - (\Lambda_{T_1} s_2)'|_{L^2(0,T_1)} \\ \le\ & |\alpha(s_1, \tilde{u}_1(\cdot,0)) - \alpha(s_2, \tilde{u}_2(\cdot,0))|_{L^2(0,T_1)} \\ \le\ & C_\alpha |s|_{L^2(0,T_1)} + C_\alpha |\tilde{u}_1(\cdot,0) - \tilde{u}_2(\cdot,0)|_{L^2(0,T_1)} \\ \le\ & C_\alpha T_1 |s'|_{L^2(0,T_1)} + \sqrt{2} C_\alpha \Big(\int_0^{T_1} |\tilde{u}_y|_H |\tilde{u}|_H dt \Big)^{1/2}. \end{aligned}$$

For any $\varepsilon > 0$ (4.3) leads to

$$\begin{aligned} & |(\Lambda_{T_1} s_1)' - (\Lambda_{T_1} s_2)'|_{L^2(0,T_1)} \\ \le\ & C_\alpha T_1 |s'|_{L^2(0,T_1)} + \varepsilon |s|_{W^{1,2}(0,T_1)} + \frac{B_4}{\varepsilon} \sqrt{T_1} |s|_{W^{1,2}(0,T_1)}, \end{aligned}$$

where B_4 is some positive constant. From this estimate it follows that

$$\begin{aligned} & |(\Lambda_{T_1} s_1) - (\Lambda_{T_1} s_2)|_{L^2(0,T_1)} \\ \le\ & T_1 ((C_\alpha T_1 + \varepsilon) |s|_{W^{1,2}(0,T_1)} + \frac{B_4}{\varepsilon} \sqrt{T_1} |s|_{W^{1,2}(0,T_1)}). \end{aligned}$$

Hence, by taking $\varepsilon > 0$ with $\varepsilon \le \frac{1}{2}$, it will be observed that for some small $T_1 \in (0, T_0]$, Λ_{T_1} will be a contraction on the closed subset $S_M(T_1)$ in $W^{1,2}(0,T_1)$. □

We note that Banach's fixed point theorem leads to the well-posedness of the system $\mathrm{AP}_3 = \mathrm{AP}_3(s_0, \tilde{u}_0, k) =: \{(2.1) \sim (2.3), (3.11), (2.5), (2.6)\}$ with (S'1). Namely, we have:

Proposition 4.3. *There exists a positive number $0 < T' \le T$ such that $AP_3(s_0, \tilde{u}_0, k)$ has a unique solution $\{s, u\}$ on $[0, T']$.*

Next, we shall prove the positivity and the boundedness of a solution to $\mathrm{AP}_3(s_0, \tilde{u}_0, k)$.

Lemma 4.4. *If $\{s, u\}$ is a solution of $AP_3(s_0, \tilde{u}_0, k)$ on $[0, T]$, $k_* \le 1$ and $u_* \le 1$, then $0 \le u \le 1$ on $Q_s(T)$.*

Proof. First, we easily obtain that (1.2) $\sim$ (1.5) hold. Then we multiply (1.2) by $[u-1]^+$ and observe that

$$\begin{aligned} & \frac{\rho_g}{2} \frac{d}{dt} \int_{s(t)}^L |[u(t) - 1]^+|^2 dx \\ =\ & \kappa \int_{s(t)}^L u_{xx}(t) [u(t) - 1]^+ dx - \frac{\rho_g}{2} s'(t) |[u(t, s(t)) - 1]^+|^2 \\ =\ & -\kappa |[u(t) - 1]_x^+|_H^2 - (\rho_a - \rho_g \sigma(u(t, s(t)))) s'(t) |[u(t, s(t)) - 1]^+|^2 \\ & - \frac{\rho_g}{2} s'(t) |[u(t, s(t)) - 1]^+|^2 \quad \text{for a.e. } t \in [0, T]. \end{aligned} \tag{4.4}$$

Here, it holds that $\frac{1}{2} \frac{d}{dt} \int_s^L |[u-1]^+|^2 dx \le 0$ a.e. on $[0, T]$. In fact, by (A2) $\rho_a - \rho_g \sigma(u(\cdot, s)) \ge 0$ and by (A1) it holds that $s' |[u(\cdot, s) - 1]^+|^2 \ge 0$ a.e. on $[0, T]$. Hence, we conclude that $u \le 1$ on $Q_s(T)$, namely, $\tilde{u} \le 1$ on $Q(T)$.

Similarly to (4.4), we have

$$\begin{aligned}
&\frac{\rho_g}{2}\frac{d}{dt}\int_s^L |[-u]^+|^2 dx \\
= \quad & -\kappa \int_s^L |[-u]_x^+|^2 dx + (\rho_a - \rho_g \sigma(u(\cdot, s)))s' |[-u(\cdot, s)]^+|^2 \\
& + \frac{\rho_g}{2} s' |[-u(\cdot, s)]^+|^2 \quad \text{a.e. on } (0, T).
\end{aligned}$$

From the above argument it follows that $u(t, x) \geq 0$ for a.e. $(t, x) \in Q_s(T)$. Thus we have proved this lemma. □

Proof of Theorem of 2.2. Proposition 4.3 guarantees the existence of a solution $\{s, \tilde{u}\}$ of $AP_3(s_0, \tilde{u}_0, k)$ on $[0, T']$ for some $0 < T' \leq T$. Here, by Lemma 4.4 we have $0 \leq \tilde{u}(t, 0) \leq 1$ for $0 \leq t \leq T'$. Therefore, $\{s, \tilde{u}\}$ must be a solution of P on $[0, T']$. Thus, we obtain our main result. □

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