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SYMMETRY AND POSITIVE DEFINITENESS OF THE TENSOR-VALUED SPRING CONSTANT DERIVED FROM P1-FEM FOR THE EQUATIONS OF LINEAR ELASTICITY

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Abstract. We study spring-block systems which are equivalent to the P1-finite element methods for the linear elliptic partial differential equation of second order and for the equations of linear elasticity. Each derived spring-block system is consistent with the original partial differential equation, since it is discretized by P1-FEM. Symmetry and positive definiteness of the scalar and tensor-valued spring constants are studied in two dimensions. Under the acuteness condition of the triangular mesh, positive definiteness of the scalar spring constant is obtained. In case of homogeneous linear elasticity, we show the symmetry of the tensor-valued spring constant in the two dimensional case. For isotropic elastic materials, we give a necessary and sufficient condition for the positive definiteness of the tensor-valued spring constant. Consequently, if Poisson's ratio of the elastic material is small enough, like concrete, we can construct a consistent spring-block system with positive definite tensor-valued spring constant.

1. **Introduction.** In computational analysis of deformation and stress field for several elastic materials such as metals and concrete, finite element analysis based on the equations of linear elasticity is widely used. On the other hand, spring-mass systems or spring-block systems are also used for several purposes, for example, a fracture model [8] etc.

The modelling of fracture phenomena is still one of the most interesting and challenging problems. Engineers have proposed many numerical methods in order to realize the fracture phenomena, e.g., the extended finite element method (X-FEM) [1], the discrete element method (DEM) [3, 10] and the particle discretization scheme (PDS-FEM) [4, 7]. Once a sort of spring-block type idea is introduced into a model of a fracture phenomenon, a spring constant is needed. It is, however, not easy to set the spring constant suitably. Indeed, in [7] it is pointed out that the so-called Poisson effect may not be properly expressed if the spring constant

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is not suitably given or the mathematical analysis of the model is not performed rigorously.

Recently, we have proposed a fracture model [8] and proved that the model is mathematically solvable if the spring-block system to be employed is *positively connected*, cf. [8] or Proposition 1 below. We note that the assumption, i.e., *positively connected*, includes symmetry and positive (non-negative) definiteness of the spring constant.

In this paper, symmetry and positive definiteness of scalar and tensor-valued spring constants of spring-block systems derived from the finite element method are studied.

There are springs and blocks in a spring-block system. A domain is divided into non-overlapping subblocks and there is a spring between each two adjacent subblocks. Each spring has a spring constant, which is assumed to be symmetric and positive (or non-negative) definite.

On the other hand, it is well known that the finite element method works well for many partial differential equations, e.g., [5, 2, 11]. We focus on the scalar elliptic partial differential equations of second order and the equations of linear elasticity in two dimensions. The finite element method with the piecewise linear element (P1-FEM) for each equation can be (formally) seen as a spring-block system and virtual scalar and tensor-valued spring constants are naturally derived from the discretization. In this paper we study symmetry and positive definiteness of the spring constants derived from P1-FEM.

In the case of the scalar elliptic equation, the spring constant is scalar, and it is positive (or non-negative) definite under the acuteness condition [9]. This fact is well known through the analysis of the discrete maximum principle, cf. [9]. We note that the obtained spring-block system is consistent with the elliptic equation since it is equivalent to P1-FEM.

In the case of the equations of linear elasticity with a given fourth-order stiffness tensor, symmetry and positive (or non-negative) definiteness of the derived spring constant are not trivial. Let $K_{ij}^{\mathrm{FE}} \in \mathbb{R}^{2\times 2}$ be the tensor-valued spring constant between two subblocks D_i and D_j $(i \neq j)$ derived from P1-FEM. The subblocks D_i and D_j are corresponding to two nodes P_i and P_j of a triangular mesh used in P1-FEM and the virtual spring is on $\overline{P_iP_j}$. Let the given fourth-order stiffness tensor $c = (c_{pqrs})$ satisfy the symmetry condition $c_{pqrs} = c_{rspq} = c_{qprs}$ $(p, q, r, s = 1, 2)$. The condition of c does not imply the symmetry $K_{ij}^{\mathrm{FE}} = (K_{ij}^{\mathrm{FE}})^T$ and just leads to $K_{ij}^{\mathrm{FE}} = (K_{ji}^{\mathrm{FE}})^T$, where the superscript T means transposition. We have computed K_{ij}^{FE} carefully and obtained symmetry of the tensor-valued spring constant K_{ij}^{FE} under some conditions including space homogeneity of c, which yields $K_{ij}^{\mathrm{FE}} = K_{ji}^{\mathrm{FE}}$. As for the positive definiteness of the spring constant, we present an equivalent condition under the acuteness condition for the triangular mesh and space homogeneity and isotropy of c. For the positive definiteness, Poisson's ratio should be sufficiently small depending on the regularity of the triangular mesh. In particular, if the mesh consists of equilateral triangles, K_{ij}^{FE} is positive definite if and only if Poisson's ratio ν is less than $1/4$. Materials with relatively large Poisson's ratio such as metals $(0.27 \leq \nu \leq 0.36)$ and rubber $(0.45 \leq \nu \leq 0.5)$ do not satisfy this condition, however, some materials with small Poisson's ratio such as concrete $(0.1 \leq \nu \leq 0.2)$ do, cf. [6] for the values of ν.

The paper is organized as follows. In Section 2, scalar and tensor-valued spring-block systems are introduced and known results are stated. In Section 3, a scalar spring constant is derived from P1-FEM for the scalar elliptic equation and positive definiteness is studied. In Section 4, a tensor-valued spring constant is derived from P1-FEM for the equations of linear elasticity. In Section 5, main results are presented, i.e., the symmetry and positive definiteness of the tensor-valued spring constant are well studied in two dimensions. In Section 6 conclusions are given.

In the rest of this section we prepare notations to be used in the paper. Let $n \in \mathbb{N}$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain. The n-dimensional volume of Ω is denoted by $|\Omega|$. We use the Lebesgue and Sobolev spaces $L^\infty(\Omega)$, $L^2(\Omega)$ and $H^1(\Omega)$ and define $\|\cdot\|_0 \equiv \|\cdot\|_{L^2(\Omega)}$. The space of continuous functions on $\bar{\Omega}$ is denoted by $C^0(\bar{\Omega})$. The dual pairing between a normed space X and the dual space X' is denoted by $\langle\cdot,\cdot\rangle$. For $i = 1, \cdots, n$, the partial derivative $\partial u/\partial x_i$ of a function u is simply denoted by $u_{,i}$. $\mathbb{R}^{n\times n}_{\mathrm{sym}}$ is a space of real symmetric matrices of size n. δ_{ij} is Kronecker's delta for $i, j = 1, \cdots, n$. We employ the same notations u, f, g, a, $\langle\cdot,\cdot\rangle$ and so on for scalar and tensor-valued spring constant models, since there is no confusion.

2. **Scalar and tensor-valued spring constant models.**

This section is devoted to state scalar and tensor-valued spring constant models.

Let $n \in \mathbb{N}$ be a number and $\Omega \subset \mathbb{R}^n$ be a bounded domain with a Lipschitz boundary $\Gamma \equiv \partial\Omega$. We divide Ω into N subblocks $\mathcal{D} = \{D_i\}_{i=1}^N$. We suppose that each block $D_i \subset \mathbb{R}^n$ is a nonempty connected open set and that the conditions,

$$\bar{\Omega} = \bigcup_{i=1}^N \bar{D}_i,\ D_i \cap D_j = \emptyset\ (i \neq j),$$

hold. If $n \geq 2$, we additionally suppose that D_i has a Lipschitz boundary for any $i \in \{1, \cdots, N\}$. In this paper, for simplicity, we call $\mathcal{D} = \{D_i\}_{i=1}^N$ a block division of Ω and assume the above conditions.

We introduce the following notation for adjacent blocks in a block division $\mathcal{D}$.

$$\begin{aligned}
D_{ij} &\equiv \bar{D}_i \cap \bar{D}_j\ (i,\ j = 1, \ldots, N,\ i \neq j),\\
d_{ij} &\equiv \mathcal{H}^{n-1}(D_{ij})\ (i,\ j = 1, \ldots, N,\ i \neq j),\\
\Lambda_i &\equiv \{j \in \{1, \cdots, N\} \setminus \{i\};\ d_{ij} > 0\}\ (i = 1, \cdots, N),\\
\Lambda &\equiv \{(i, j);\ 1 \leq i < j \leq N,\ d_{ij} > 0\},\\
\Sigma &\equiv \bigcup_{(i,j)\in\Lambda} D_{ij},
\end{aligned}$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure, cf. Figure 1.

We define function spaces of piecewise constant functions on D_i and D_{ij} as follows.

$$\chi_i(x) \equiv \begin{cases} 1 & (x \in D_i) \\ 0 & (x \in \Omega \setminus D_i) \end{cases} \quad (i = 1, \ldots, N),$$

$$\chi_{ij}(x) \equiv \begin{cases} 1 & (x \in D_{ij}) \\ 0 & (x \in \Sigma \setminus D_{ij}) \end{cases} \quad ((i, j) \in \Lambda),$$

$$V(\mathcal{D}) \equiv \Big\{v \in L^\infty(\Omega);\ v = \sum_{i=1}^N v_i\chi_i,\ v_i \in \mathbb{R},\ i = 1, \cdots, N\Big\},$$

$$W(\mathcal{D}) \equiv \Big\{\zeta \in L^\infty(\Sigma);\ \zeta = \sum_{(i,j)\in\Lambda} \zeta_{ij}\chi_{ij},\ \zeta_{ij} \in \mathbb{R}\Big\}.$$

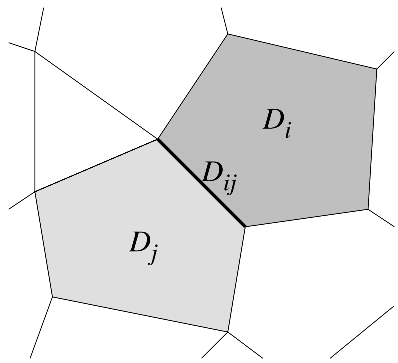

FIGURE 1. Sample blocks D_i and D_j and a common boundary D_{ij} of the two blocks.

In the following sections, we consider scalar or vector-valued displacement fields which belong to $V(\mathcal{D})$ and virtual springs between adjacent blocks. In order to set the Dirichlet boundary condition, we suppose

$$J = (J_0, J_1),\ J_0 \cup J_1 = \{1, \ldots, N\},\ J_0 \cap J_1 = \emptyset,\ J_0 \neq \emptyset,\ J_1 \neq \emptyset,$$

and that the balance of forces is considered at D_i for $i \in J_0$ and the displacement of D_i for $i \in J_1$ is a priori given. The displacement space $V(\mathcal{D})$ is a direct sum of the following subspaces,

$$V_l(\mathcal{D}) \equiv \Big\{ v \in V(\mathcal{D});\ v = \sum_{i \in J_l} v_i \chi_i,\ v_i \in \mathbb{R} \Big\} \qquad (l = 0,\, 1).$$

Now a tensor-valued spring constant model is constructed as follows. For a block division $\mathcal{D}$ of $\Omega \subset \mathbb{R}^n$ we consider a vector-valued displacement $u = \sum_{i=1}^N u_i \chi_i \in V(\mathcal{D})^n$, where $u_i \in \mathbb{R}^n$ is a vector and

$$V(\mathcal{D})^n \equiv \Big\{ v \in L^\infty(\Omega; \mathbb{R}^n);\ v = \sum_{i=1}^N v_i \chi_i,\ v_i \in \mathbb{R}^n,\ i = 1, \cdots, N \Big\}.$$

For $(i,j) \in \Lambda$ we consider a virtual spring between the adjacent blocks D_i and D_j with tensor-valued spring constant $K_{ij} \in \mathbb{R}^{n \times n}_{\mathrm{sym}}$. We suppose that the tensor-valued spring constant satisfies the condition,

$$K_{ij} = K_{ji} \geq O, \quad \forall (i,j) \in \Lambda, \tag{1}$$

where $K_{ij} \geq O$ means that K_{ij} is non-negative definite. If $K_{ij} \in \mathbb{R}^{n \times n}_{\mathrm{sym}}$ is positive definite, we denote it by $K_{ij} > O$. We additionally suppose that the vector-valued force acting on D_i from D_j is given as $K_{ij}(u_j - u_i) \in \mathbb{R}^n$. It is a sort of Hooke's law. Let K be a function defined by

$$K \equiv \sum_{(i,j) \in \Lambda} K_{ij} \chi_{ij} \in W(\mathcal{D})^{n \times n}.$$

Under the above situation, we call $(\mathcal{D}, K)$ *a tensor-valued spring-block system*, and call $(\mathcal{D}, K, J)$ *a tensor-valued spring-block system with Dirichlet boundary.*

We consider the following problem.

Problem 1. *Let $(\mathcal{D}, K, J)$ be a tensor-valued spring-block system with Dirichlet boundary in $\mathbb{R}^n$. For a given body force $f = \sum_{i \in J_0} f_i \chi_i \in V_0(\mathcal{D})^n$ with $F_i \equiv |D_i| f_i \in \mathbb{R}^n$ and a given displacement $g = \sum_{i \in J_1} g_i \chi_i \in V_1(\mathcal{D})^n$, find a displacement $u = \sum_{i=1}^N u_i \chi_i \in V(\mathcal{D})^n$ such that*

$$\sum_{j \in \Lambda_i} K_{ij}(u_j - u_i) + F_i = 0, \qquad \forall i \in J_0,$$
$$u_i = g_i, \qquad \forall i \in J_1.$$

We define a bilinear form $(\cdot,\cdot)_K$, a seminorm $|\cdot|_K$ and a constant $c_0 = c_0(\mathcal{D}, K, J)$ by

$$(u, v)_K \equiv \sum_{(i,j) \in \Lambda} \{K_{ij}(u_j - u_i)\} \cdot (v_j - v_i) \quad (u,\ v \in V(\mathcal{D})^n),$$
$$|v|_K \equiv \sqrt{(v, v)_K} \quad (v \in V(\mathcal{D})^n),$$
$$c_0 = c_0(\mathcal{D}, K, J) \equiv \inf_{v \in V_0(\mathcal{D})^n,\ \|v\|_0 \neq 0} \frac{|v|_K}{\|v\|_0} \geq 0.$$

Concerning the solvability of Problem 1, we introduce some non-degeneracy conditions of the spring constant K.

Definition 1. Let $(\mathcal{D}, K, J)$ be a tensor-valued spring-block system with Dirichlet boundary.

(i) $(\mathcal{D}, K, J)$ is called "positively connected" if the following condition is satisfied;

$$v \in V_0(\mathcal{D}) \text{ and } \sum_{K_{ij} > O} |v_j - v_i| = 0, \quad \text{iff} \quad v = 0 \in V(\mathcal{D}). \tag{2}$$

(ii) $(\mathcal{D}, K, J)$ is called "regular" if $c_0(\mathcal{D}, K, J) > 0$.

The condition (2) means that for any $i \in J_0$ block D_i is connected to a Dirichlet boundary block D_j $(j \in J_1)$ by a chain of positive definite springs. We also remark that, if $(\mathcal{D}, K, J)$ is regular, then the inequality

$$\|v\|_0 \leq c_0^{-1} |v|_K \quad (v \in V_0(\mathcal{D})^n)$$

holds. We give a proposition on the solvability of Problem 1, cf. [8].

Proposition 1 ([8])**.** *Let $(\mathcal{D}, K, J)$ be a tensor-valued spring-block system with Dirichlet boundary.*

(i) $(\mathcal{D}, K, J)$ is regular if it is positively connected.

(ii) Suppose $(\mathcal{D}, K, J)$ is regular. Then, there exists a unique solution $u \in V(\mathcal{D})^n$ of Problem 1.

Here we introduce a scalar spring constant model, which is constructed in a similar way to the tensor-valued spring constant model. We consider a scalar virtual spring between D_i and D_j and suppose that it has a spring constant $\kappa_{ij} \geq 0$ and that the force acting on D_i from D_j is given as $\kappa_{ij}(u_j - u_i) \in \mathbb{R}$. A corresponding condition to (1) is given as

$$\kappa_{ij} = \kappa_{ji} \geq 0, \quad \forall (i,j) \in \Lambda.$$

Let $\kappa \equiv \sum_{(i,j) \in \Lambda} \kappa_{ij} \chi_{ij} \in W(\mathcal{D})$. We call $(\mathcal{D}, \kappa)$ *a scalar spring-block system*, and call $(\mathcal{D}, \kappa, J)$ *a scalar spring-block system with Dirichlet boundary.*

The following problem is a scalar version of Problem 1. We note that a corresponding proposition of Proposition 1 holds, cf. [8].

Problem 2. *Let $(\mathcal{D}, \kappa, J)$ be a scalar spring-block system with Dirichlet boundary in $\mathbb{R}^n$. For a given body force $f = \sum_{i\in J_0} f_i\chi_i \in V_0(\mathcal{D})$ with $F_i \equiv f_i|D_i|$ and a given displacement $g = \sum_{i\in J_1} g_i\chi_i \in V_1(\mathcal{D})$, find a displacement $u = \sum_{i=1}^N u_i\chi_i \in V(\mathcal{D})$ such that*

$$\sum_{j\in\Lambda_i} \kappa_{ij}(u_j - u_i) + F_i = 0, \qquad \forall i \in J_0,$$
$$u_i = g_i, \qquad \forall i \in J_1.$$

3. **A scalar spring constant derived from P1-FEM for the elliptic equation.** In this section we derive a scalar spring constant from P1-FEM for the elliptic equation. The scalar spring-block system to be derived is an alternative expression of P1-FEM for the elliptic equation, which is well known to researchers of (theoretical) numerical analysis of FEM. This section is, however, set in order to easily understand a tensor-valued spring constant derived from P1-FEM for the equations of linear elasticity in Section 4 and main results, i.e., symmetry and positive definiteness of the tensor-valued spring constant in two dimensions in Section 5.

Let $B : \Omega \to \mathbb{R}_+(\equiv (0,\infty))$, $f : \Omega \to \mathbb{R}$ and $g : \Gamma \to \mathbb{R}$ be given functions. We assume $B \in L^\infty(\Omega)$, $f \in L^2(\Omega)$ and $g \in C^0(\Gamma) \cap H^{1/2}(\Gamma)$. The elliptic problem is to find $u : \Omega \to \mathbb{R}$ such that

$$-\nabla\cdot(B\nabla u) = f \quad \text{in } \Omega, \tag{3a}$$
$$u = g \quad \text{on } \Gamma. \tag{3b}$$

Let $X \equiv H^1(\Omega)$ and

$$V(g) \equiv \{v \in X;\ v = g \text{ on } \Gamma\}$$

for a given function $g : \Gamma \to \mathbb{R}$ be function spaces and set $V \equiv V(0)$. We define a bilinear form $a = a(\cdot,\cdot)$ on $X \times X$ and a linear functional $f \in X'$ by

$$a(u,v) \equiv \int_\Omega B(x)\nabla u(x)\cdot\nabla v(x)\ dx, \tag{4}$$
$$\langle f, v\rangle \equiv \int_\Omega f(x)v(x)\ dx,$$

respectively. Then, a weak formulation of problem (3) is to find $u \in V(g)$ such that

$$a(u,v) = \langle f, v\rangle, \quad \forall v \in V. \tag{5}$$

A discrete problem via P1-FEM for problem (5) is obtained as follows. Let $\mathcal{T}_h \equiv \{T\}$ be a triangulation of $\bar{\Omega}$ and

$$\Omega_h \equiv \operatorname{int} \bigcup_{T\in\mathcal{T}_h} T$$

be the approximate domain of Ω. For the sake of simplicity we assume $\Omega = \Omega_h$ in the rest of the paper. Let $P_1(T)$ be a polynomial space of linear functions on $T \in \mathcal{T}_h$, and X_h and $V_h(g)$ be finite element spaces defined by

$$X_h \equiv \{v_h \in C^0(\bar{\Omega});\ v_h|_T \in P_1(T),\ \forall T \in \mathcal{T}_h\},$$
$$V_h(g) \equiv \{v_h \in X_h;\ v_h(P) = g(P),\ \forall P:\ \text{node on } \Gamma\},$$

for a given function $g : \Gamma \to \mathbb{R}$, respectively, and set $V_h \equiv V_h(0)$. The discrete problem via P1-FEM for (5) is to find $u_h \in V_h(g)$ such that

$$a(u_h, v_h) = \langle f, v_h\rangle, \quad \forall v_h \in V_h. \tag{6}$$

Let $N^{\mathrm{FE}} \in \mathbb{N}$ be the total number of nodal points of $\mathcal{T}_h$ in $\bar{\Omega}$. We denote the i-th node by P_i ($\in \bar{\Omega}$). Let J_0^{FE} and J_1^{FE} be sets of indices with

$$J_0^{\mathrm{FE}} \cup J_1^{\mathrm{FE}} = \{1, \cdots, N^{\mathrm{FE}}\}, \quad P_i \in \Omega \ (\forall i \in J_0^{\mathrm{FE}}), \quad P_i \in \Gamma \ (\forall i \in J_1^{\mathrm{FE}}),$$

and set $J^{\mathrm{FE}} \equiv (J_0^{\mathrm{FE}}, J_1^{\mathrm{FE}})$. Let $\varphi_i \in X_h$ be the P1-basis function with respect to P_i, $\{\varphi_i : \bar{\Omega} \to \mathbb{R}\}_{i=1}^{N^{\mathrm{FE}}}$ be the set of P1-basis functions of X_h with $\varphi_i(P_j) = \delta_{ij}$, and Λ_i^{FE} and Λ^{FE} be a set of indices of adjacent nodes of P_i and a set of pairs of indices of adjacent nodes defined by

$$\Lambda_i^{\mathrm{FE}} \equiv \{j \in J_0^{\mathrm{FE}} \cup J_1^{\mathrm{FE}};\ j \neq i,\ \exists T \in \mathcal{T}_h \text{ s.t. } P_i \text{ and } P_j \in T\},$$
$$\Lambda^{\mathrm{FE}} \equiv \{(i, j);\ 1 \leq i < j \leq N^{\mathrm{FE}},\ \exists T \in \mathcal{T}_h \text{ s.t. } P_i \text{ and } P_j \in T\},$$

respectively. Since (6) is equivalent to

$$a(u_h, \varphi_i) = \langle f, \varphi_i \rangle, \quad i \in J_0^{\mathrm{FE}},$$

and $u_h \in V_h(g)$ is a function of the form

$$u_h = \sum_{j \in J_0^{\mathrm{FE}} \cup J_1^{\mathrm{FE}}} u_j \varphi_j \ \left(= \sum_{j \in J_0^{\mathrm{FE}}} u_j \varphi_j + \sum_{j \in J_1^{\mathrm{FE}}} g(P_j) \varphi_j \right)$$

for $\{u_j\}_{j \in J_0^{\mathrm{FE}} \cup J_1^{\mathrm{FE}}} = \{u_j\}_{j=1}^{N^{\mathrm{FE}}} \subset \mathbb{R}$, it holds that

$$\begin{aligned} a(u_h, \varphi_i) &= \sum_{j \in J_0^{\mathrm{FE}} \cup J_1^{\mathrm{FE}}} u_j\, a(\varphi_j, \varphi_i) = \sum_{j \in \Lambda_i^{\mathrm{FE}}} u_j\, a(\varphi_j, \varphi_i) + u_i\, a(\varphi_i, \varphi_i) \\ &= \sum_{j \in \Lambda_i^{\mathrm{FE}}} u_j\, a(\varphi_j, \varphi_i) + u_i\, a\Big(1 - \sum_{j \in \Lambda_i^{\mathrm{FE}}} \varphi_j, \varphi_i\Big) \quad \Big(\text{by } \sum_{j \in \Lambda_i^{\mathrm{FE}}} \varphi_j + \varphi_i = 1\Big) \\ &= \sum_{j \in \Lambda_i^{\mathrm{FE}}} u_j\, a(\varphi_j, \varphi_i) - \sum_{j \in \Lambda_i^{\mathrm{FE}}} u_i\, a(\varphi_j, \varphi_i) \\ &= \sum_{j \in \Lambda_i^{\mathrm{FE}}} a(\varphi_j, \varphi_i)(u_j - u_i) \end{aligned}$$

for $i \in J_0^{\mathrm{FE}}$. We can, therefore, set an equivalent problem to (6); find $\{u_j\}_{j=1}^{N^{\mathrm{FE}}} \subset \mathbb{R}$ such that

$$\sum_{j \in \Lambda_i} \kappa_{ij}^{\mathrm{FE}} (u_j - u_i) + F_i^{\mathrm{FE}} = 0, \qquad \forall i \in J_0^{\mathrm{FE}}, \tag{7a}$$

$$u_i = g_i^{\mathrm{FE}}, \qquad \forall i \in J_1^{\mathrm{FE}}, \tag{7b}$$

where notations $\kappa_{ij}^{\mathrm{FE}}$, F_i^{FE} and g_i^{FE} are defined by

$$\kappa_{ij}^{\mathrm{FE}} \equiv -a(\varphi_j, \varphi_i), \qquad F_i^{\mathrm{FE}} \equiv \langle f, \varphi_i \rangle, \qquad g_i^{\mathrm{FE}} \equiv g(P_i). \tag{8}$$

Let $\mathcal{D}^{\mathrm{FE}} = \{D_i^{\mathrm{FE}}\}_{i=1}^{N^{\mathrm{FE}}}$ be a block division of Ω defined by

$$D_i^{\mathrm{FE}} \equiv \big\{x \in \Omega;\ |x - P_i| < |x - P_j|,\ \forall j \in (J_0^{\mathrm{FE}} \cup J_1^{\mathrm{FE}}) \setminus \{i\}\big\}, \quad i = 1, \cdots, N^{\mathrm{FE}},$$

which is based on the Voronoi diagram [13, 12]. Problem (7) can be seen as a scalar spring-block system with Dirichlet boundary by setting

$$N = N^{\mathrm{FE}}, \quad \mathcal{D} = \mathcal{D}^{\mathrm{FE}}, \quad \kappa_{ij} = \kappa_{ij}^{\mathrm{FE}}, \quad J_l = J_l^{\mathrm{FE}}\ (l = 0, 1),$$
$$F_i = F_i^{\mathrm{FE}}, \quad g_i = g_i^{\mathrm{FE}}, \quad \Lambda_i = \Lambda_i^{\mathrm{FE}},$$

in Problem 2. Figure 2 shows a part of a sample block division $\mathcal{D} = \mathcal{D}^{\mathrm{FE}}$.

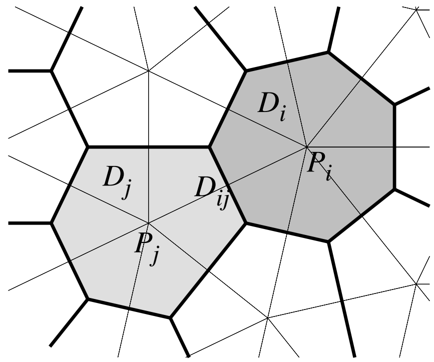

FIGURE 2. A part of a sample block division $\mathcal{D} = \mathcal{D}^{\rm FE}$. $D_i = D_i^{\rm FE}$ and $D_j = D_j^{\rm FE}$ are subblocks with respect to nodes P_i and P_j of a triangular mesh, respectively. Thin lines show the triangular mesh used for FEM.

After defining the acuteness condition [9] for $\mathcal{T}_h$, we present a proposition on symmetry and positivity of the scalar spring constant $\kappa_{ij}^{\rm FE}$.

Definition 2. Let $\mathcal{T}_h$ be a triangulation of $\Omega \subset \mathbb{R}^2$. We say $\mathcal{T}_h$ is "acute" if any interior angle of any element $T \in \mathcal{T}_h$ is less than $\pi/2$.

Proposition 2. *Let $\mathcal{T}_h$ be given. Suppose $(i,j) \in \Lambda^{\rm FE}$ with i or $j \in J_0^{\rm FE}$. Let $\kappa_{ij}^{\rm FE}$ be the scalar spring constant defined by* (8).
(i) Then, it holds that

$$\kappa_{ij}^{\rm FE} = \kappa_{ji}^{\rm FE}.$$

(ii) Suppose $n = 2$ and that $\mathcal{T}_h$ is acute. Then, $\kappa_{ij}^{\rm FE}$ is positive.

Proof. Let any $(i,j) \in \Lambda^{\rm FE}$ in the assumption be fixed. (i) is obvious by symmetry of $a = a(\cdot,\cdot)$, cf. (4). As for (ii) it holds that for any $T \in \mathcal{T}_h$

$$\nabla\varphi_j|_T \cdot \nabla\varphi_i|_T = \begin{cases} \big|\nabla\varphi_j|_T\big|\big|\nabla\varphi_i|_T\big| \cos\theta_T < 0 & (P_i \text{ and } P_j \in T) \\ 0 & (\text{otherwise}) \end{cases}$$

under the acuteness condition, where θ_T is the angle between the two vectors $\nabla\varphi_j|_T$ and $\nabla\varphi_i|_T$. It implies that

$$\kappa_{ij}^{\rm FE} = -a(\varphi_j, \varphi_i) = -\int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i \, dx = -\sum_{T \in \mathcal{T}_h} \int_T \nabla\varphi_j \cdot \nabla\varphi_i \, dx > 0,$$

which is the desired result. □

Remark 1. The scalar spring-block system (7) is consistent with the elliptic equation (3) since it is equivalent to P1-FEM.

4. **A tensor-valued spring constant derived from P1-FEM.** In this section we derive a tensor-valued spring constant from P1-FEM for the equations of linear elasticity.

Let $c_{pqrs} : \Omega \to \mathbb{R}$ $(p,q,r,s = 1,\cdots,n)$, $f : \Omega \to \mathbb{R}^n$ and $g : \Gamma \to \mathbb{R}^n$ be given functions. We assume $c_{pqrs} \in L^\infty(\Omega)$ $(p,q,r,s = 1,\cdots,n)$, $f \in L^2(\Omega)^n$, $g \in$

$C^0(\Gamma)^n \cap H^{1/2}(\Gamma)^n$ and that the fourth-order stiffness tensor c satisfies symmetry condition, i.e.,

$$c_{pqrs} = c_{rspq} = c_{qprs} \quad (p, q, r, s = 1, \cdots, n). \tag{9}$$

The elasticity problem is to find $u : \Omega \to \mathbb{R}^n$ such that

$$-\nabla\{\sigma(u)\} = f \quad \text{in } \Omega, \tag{10a}$$

$$u = g \quad \text{on } \Gamma. \tag{10b}$$

where $\sigma(u)$ and $\varepsilon(u)$ are the stress and strain tensors defined by

$$\sigma_{pq}(u) \equiv \sum_{r,s=1}^{n} c_{pqrs}\varepsilon_{rs}(u) \quad (p, q = 1, \cdots, n),$$

$$\varepsilon_{pq}(u) \equiv \frac{1}{2}(u_{p,q} + u_{q,p}) \quad (p, q = 1, \cdots, n).$$

Let $V^n(g)$ be the function space defined by

$$V^n(g) \equiv \{v \in X^n;\ v = g \text{ on } \Gamma\}$$

for a given function $g : \Gamma \to \mathbb{R}^n$ and set $V^n \equiv V^n(0)$. We define a bilinear form $a = a(\cdot, \cdot)$ on $X^n \times X^n$ and a linear functional $f \in (X^n)'$ by

$$a(u, v) \equiv \int_\Omega \sigma(u) : \varepsilon(v)\ dx,$$

$$\langle f, v\rangle \equiv \int_\Omega f(x) \cdot v(x)\ dx,$$

respectively. Then, a weak formulation of problem (10) is to find $u \in V(g)$ such that

$$a(u, v) = \langle f, v\rangle, \quad \forall v \in V^n. \tag{11}$$

A discrete problem for problem (11) via P1-FEM is obtained as follows. Let a triangulation $\mathcal{T}_h = \{T\}$ of $\bar{\Omega}$ be given and X_h^n and $V_h^n(g)$ be finite element spaces defined by

$$X_h^n \equiv \{v_h \in C^0(\bar{\Omega})^n;\ v_h|_T \in P_1(T)^n,\ \forall T \in \mathcal{T}_h\},$$

$$V_h^n(g) \equiv \{v_h \in X_h^n;\ v_h(P) = g(P),\ \forall P: \text{ node on } \Gamma\},$$

for a given function $g : \Gamma \to \mathbb{R}^n$, respectively, and set $V_h^n \equiv V_h^n(0)$. The discrete problem for (11) is to find $u_h \in V_h^n(g)$ such that

$$a(u_h, v_h) = \langle f, v_h\rangle, \quad \forall v_h \in V_h^n. \tag{12}$$

Let $e_k \equiv (\delta_{k1}, \cdots, \delta_{kn})^T \in \mathbb{R}^n$ $(k = 1, \cdots, n)$ be the orthogonal unit vectors. Since equation (12) is equivalent to

$$a(u_h, \varphi_i e_k) = \langle f, \varphi_i e_k\rangle, \quad \forall i \in J_0^{\mathrm{FE}},\ \forall k \in \{1, \cdots, n\},$$

and the solution $u_h \in V_h^n(g)$ is written as

$$u_h = \sum_{j \in J_0^{\mathrm{FE}} \cup J_1^{\mathrm{FE}}} u_j\varphi_j \ \Big(= \sum_{j \in J_0^{\mathrm{FE}}} u_j\varphi_j + \sum_{j \in J_1^{\mathrm{FE}}} g(P_j)\varphi_j\Big)$$

for $\{u_j\}_{j \in J_0^{\mathrm{FE}} \cup J_1^{\mathrm{FE}}} = \{u_j\}_{j=1}^{N^{\mathrm{FE}}} \subset \mathbb{R}^n$, it holds that

$$a(u_h, \varphi_i e_k) = \sum_{j \in J_0^{\mathrm{FE}} \cup J_1^{\mathrm{FE}}} a(u_j\varphi_j, \varphi_i e_k)$$

$$= \sum_{l=1}^{n} \Big\{ \sum_{j\in\Lambda_i^{\rm FE}} [u_j]_l \, a(\varphi_j e_l, \varphi_i e_k) + [u_i]_l \, a(\varphi_i e_l, \varphi_i e_k) \Big\}$$
$$= \sum_{l=1}^{n} \Big\{ \sum_{j\in\Lambda_i^{\rm FE}} [u_j]_l \, a(\varphi_j e_l, \varphi_i e_k) + [u_i]_l \, a\Big(\Big(1 - \sum_{j\in\Lambda_i^{\rm FE}} \varphi_j\Big) e_l, \varphi_i e_k \Big) \Big\}$$
$$= \sum_{l=1}^{n} \Big\{ \sum_{j\in\Lambda_i^{\rm FE}} [u_j]_l \, a(\varphi_j e_l, \varphi_i e_k) - \sum_{j\in\Lambda_i^{\rm FE}} [u_i]_l \, a(\varphi_j e_l, \varphi_i e_k) \Big\}$$
$$= \sum_{l=1}^{n} \sum_{j\in\Lambda_i^{\rm FE}} a(\varphi_j e_l, \varphi_i e_k)[u_j - u_i]_l$$

for $i \in J_0^{\rm FE}$. We can, therefore, set an equivalent problem to (12); find $\{u_j\}_{j=1}^{N^{\rm FE}} \subset \mathbb{R}^n$ such that

$$\sum_{j\in\Lambda_i^{\rm FE}} K_{ij}^{\rm FE}(u_j - u_i) + F_i^{\rm FE} = 0, \quad \forall i \in J_0^{\rm FE}, \tag{13a}$$

$$u_i = g_i^{\rm FE}, \quad \forall i \in J_1^{\rm FE}, \tag{13b}$$

where notations $K_{ij}^{\rm FE} \in \mathbb{R}^{n\times n}$, $F_i^{\rm FE} \in \mathbb{R}^n$ and $g_i^{\rm FE} \in \mathbb{R}^n$ are defined by

$$[K_{ij}^{\rm FE}]_{kl} \equiv -a(\varphi_j e_l, \varphi_i e_k), \qquad [F_i^{\rm FE}]_k \equiv \langle f, \varphi_i e_k \rangle, \qquad g_i^{\rm FE} \equiv g(P_i), \tag{14}$$

for $k, l = 1, \cdots, n$, respectively. Problem (13) can be (formally) seen as a tensor-valued spring-block system with Dirichlet boundary by setting

$$N = N^{\rm FE}, \quad \mathcal{D} = \mathcal{D}^{\rm FE}, \quad K_{ij} = K_{ij}^{\rm FE}, \quad J_l = J_l^{\rm FE} \ (l = 0, 1),$$
$$F_i = F_i^{\rm FE}, \quad g_i = g_i^{\rm FE}, \quad \Lambda_i = \Lambda_i^{\rm FE},$$

in Problem 1, while in general K_{ij} does not always have two properties, symmetry and positive definiteness, i.e., $K_{ij} = K_{ji}$ and $K_{ij} > O$, respectively. The two properties are studied in the next section.

5. Symmetry and positive definiteness of $K_{ij}^{\rm FE}$ in two dimensions.

In this section we study two properties, symmetry and positive definiteness, of the tensor-valued spring constant $K_{ij}^{\rm FE}$ under $n = 2$.

Throughout this section we assume $n = 2$ and often omit the superscript "FE" from $K_{ij}^{\rm FE}$ and $K_{ji}^{\rm FE}$ since there is no confusion. In the case of a scalar spring-block system with Dirichlet boundary via P1-FEM, the scalar spring constant $\kappa_{ij}^{\rm FE}$ defined in (8) is symmetric and positive positive definite if the triangulation satisfies the acuteness condition, cf. Proposition 2. Here we study symmetry and positive definiteness of tensor-valued spring constant K_{ij} defined in (14).

In order to state and prove the next theorem on symmetry and positive definiteness of K_{ij} we prepare notations to be used. Let $(i, j) \in \Lambda^{\rm FE}$ satisfying i or $j \in J_0^{\rm FE}$ be fixed and T^1 and $T^2 \in \mathcal{T}_h$ $(T^1 \neq T^2)$ be the two triangles including both P_i and P_j. Without loss of generality we consider the two triangles T^1 and T^2 as $T^1 = \triangle P_i P_j Q_1$ and $T^2 = \triangle P_i P_j Q_2$ and set $P_i \equiv (0, 0)^T$, $P_j \equiv (s, t)^T$, $Q_1 \equiv (a, b)^T$ and $Q_2 \equiv (c, d)^T$, cf. Figure 3. For a subscript $m = 1$ and 2 we prepare the following constants,

$$\theta_m \equiv \angle P_i Q_m P_j, \qquad \theta_{mi} \equiv \angle Q_m P_i P_j, \qquad \theta_{mj} \equiv \angle Q_m P_j P_i,$$
$$\theta_i \equiv \angle Q_1 P_i Q_2 = \theta_{1i} + \theta_{2i}, \qquad \theta_j \equiv \angle Q_1 P_j Q_2 = \theta_{1j} + \theta_{2j},$$

$$
\begin{aligned}
&k_{mi} \equiv \overline{Q_m P_i}, \qquad k_{mj} \equiv \overline{Q_m P_j},\\
&m_1 \equiv 2|T^1| = bs - at > 0, \quad m_2 \equiv 2|T^2| = ct - ds > 0,\\
&m_3 \equiv 2|\triangle P_i Q_2 Q_1| = k_{1i}k_{2i}\sin(\theta_{1i}+\theta_{2i}) = bc - ad > 0,\\
&m_4 \equiv 2|\triangle P_j Q_1 Q_2| = k_{1j}k_{2j}\sin(\theta_{1j}+\theta_{2j}) > 0,\\
&\bar{c}_{pqrs} \equiv \frac{1}{2}(c_{pqrs} + c_{psrq}),\\
&E \equiv E_1 + E_2, \quad E_1 \equiv \frac{(b-t)b}{2m_1}, \quad E_2 \equiv \frac{(d-t)d}{2m_2},\\
&F \equiv \frac{(s-a)b}{2m_1} + \frac{(s-c)d}{2m_2} = \frac{(t-b)a}{2m_1} + \frac{(t-d)c}{2m_2},\\
&G \equiv G_1 + G_2, \quad G_1 \equiv \frac{(a-s)a}{2m_1}, \quad G_2 \equiv \frac{(c-s)c}{2m_2}.
\end{aligned} \tag{15}
$$

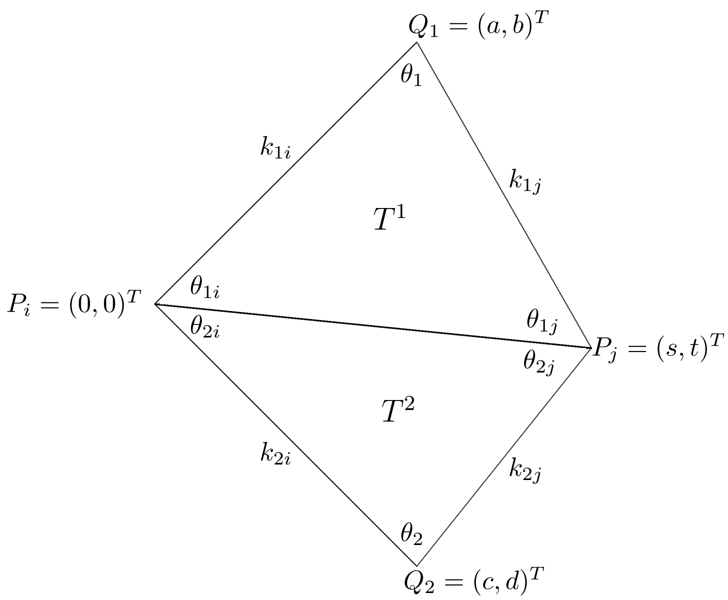

FIGURE 3. The two triangles including both P_i and P_j.

Theorem 1. *Suppose $n = 2$ and let $\mathcal{T}_h$ be given. Suppose $(i,j) \in \Lambda^{\mathrm{FE}}$ with i or $j \in J_0^{\mathrm{FE}}$ and that the stiffness tensor c with (9) is homogeneous, i.e., c does not depend on x. Let K_{ij} be the tensor-valued spring constant defined by (14).*

(i) Then, it holds that

$$K_{ij} = K_{ji} \in \mathbb{R}^{n\times n}_{\mathrm{sym}}.$$

(ii) Suppose that $\mathcal{T}_h$ is acute and that the stiffness tensor c satisfies the isotropy condition, i.e.,

$$c_{pqrs} = \lambda\delta_{pq}\delta_{rs} + \mu(\delta_{pr}\delta_{qs} + \delta_{ps}\delta_{qr}) \tag{16}$$

for positive constants λ and μ. Then, K_{ij} is positive definite if and only if

$$(0<)\alpha \equiv \frac{F^2-EG}{(E+G)^2} < 2 \quad \textit{and} \quad \frac{\lambda}{\mu} < f(\alpha) \equiv \frac{1-2\alpha+\sqrt{1+4\alpha}}{2\alpha}.$$

Remark 2. The definition of K_{ij} (14) with the assumption of c (9) implies that $K_{ij} = K_{ji}^T$, which is different from $K_{ij} = K_{ji}$.

Remark 3. For $\alpha > 0$ the function $f(\alpha)$ is strictly decreasing and satisfies $\lim_{\alpha \to +0} f(\alpha) = +\infty$ and $f(2) = 0$.

We prepare the following two lemmas which are employed in the proof of Theorem 1.

Lemma 1. *It holds that*

$$F^2 - EG = \frac{m_3 m_4}{4m_1 m_2} > 0.$$

Proof. From the two expressions of F in (15) we have

$$\begin{aligned} F^2 &= \left(\frac{(a-s)b}{2m_1} + \frac{(c-s)d}{2m_2}\right)\left(\frac{(b-t)a}{2m_1} + \frac{(d-t)c}{2m_2}\right) \\ &= \frac{(a-s)(b-t)ab}{4m_1^2} + \frac{(a-s)(d-t)bc+(b-t)(c-s)ad}{4m_1m_2} + \frac{(c-s)(d-t)cd}{4m_2^2} \\ &= E_1G_1 + \frac{(a-s)(d-t)bc+(b-t)(c-s)ad}{4m_1m_2} + E_2G_2. \end{aligned} \tag{17}$$

On the other hand, it holds that

$$\begin{aligned} EG &= E_1G_1 + E_2G_1 + E_1G_2 + E_2G_2 \\ &= E_1G_1 + \frac{(a-s)(d-t)ad+(b-t)(c-s)bc}{4m_1m_2} + E_2G_2. \end{aligned} \tag{18}$$

Subtracting (18) from (17), we have

$$\begin{aligned} F^2 - EG &= \frac{(a-s)(d-t)(bc-ad)+(b-t)(c-s)(ad-bc)}{4m_1m_2} \\ &= \frac{(bc-ad)(m_1+m_2-(bc-ad))}{4m_1m_2} = \frac{m_3m_4}{4m_1m_2} > 0, \end{aligned}$$

which completes the proof. □

Lemma 2. *(i) For $l = 1$ and 2 it holds that*

$$E_l + G_l = \frac{1}{2}\cot\theta_l.$$

(ii) We have

$$0 < \theta_l < \frac{\pi}{2} \quad \text{iff} \quad E_l + G_l > 0.$$

Proof. Since (ii) is easily obtained from (i), we omit the proof of (ii). We prove (i) with $l = 1$. From the law of cosines it holds that

$$\cos\theta_1 = \frac{\overline{P_iQ_1}^2 + \overline{P_jQ_1}^2 - \overline{P_iP_j}^2}{2\overline{P_iQ_1}\,\overline{P_jQ_1}} = \frac{a^2+b^2-(as+bt)}{\overline{P_iQ_1}\,\overline{P_jQ_1}},$$

$$\sin\theta_1 = \sqrt{1-\cos^2\theta_1} = \frac{\sqrt{(\overline{P_iQ_1}\,\overline{P_jQ_1})^2 - \{a^2+b^2-(as+bt)\}^2}}{\overline{P_iQ_1}\,\overline{P_jQ_1}}$$

$$= \frac{\sqrt{(a^2+b^2)(s^2+t^2)-(as+bt)^2}}{\overline{P_iQ_1}\ \overline{P_jQ_1}} = \frac{bs-at}{\overline{P_iQ_1}\ \overline{P_jQ_1}}.$$

We, therefore, have

$$\begin{aligned} E_1 + G_1 &= \frac{(b-t)b}{2m_1} + \frac{(a-s)a}{2m_1} = \frac{a^2+b^2-(bt+as)}{2m_1} \\ &= \frac{a^2+b^2-(bt+as)}{2(bs-at)} = \frac{\cos\theta_1}{2\sin\theta_1} = \frac{1}{2}\cot\theta_1, \end{aligned}$$

which completes the proof. The proof of (i) with $l=2$ is similar. □

Now we give the proof.

Proof of Theorem 1. We assume the situation mentioned just before the theorem, cf. Figure 3, and use the notations. First, we prove $K_{ij} \in \mathbb{R}^{2\times 2}_{\rm sym}$ of (i). For any fixed k and $l \in \{1,2\}$ it holds that

$$\begin{aligned} [K_{ij}]_{kl} &= -a(\varphi_j e_l, \varphi_i e_k) = -\int_\Omega \sigma(\varphi_j e_l) : \varepsilon(\varphi_i e_k)\ dx \\ &= -\sum_{p,q,r,s=1}^{2} \int_\Omega c_{pqrs}\varepsilon_{rs}(\varphi_j e_l)\varepsilon_{pq}(\varphi_i e_k)\ dx \\ &= -\frac{1}{4}\sum_{p,q,r,s=1}^{2} \int_\Omega c_{pqrs}\big([\varphi_j e_l]_{r,s} + [\varphi_j e_l]_{s,r}\big)\big([\varphi_i e_k]_{p,q} + [\varphi_i e_k]_{q,p}\big)\ dx \\ &= -\frac{1}{4}\sum_{p,q,r,s=1}^{2} \int_\Omega c_{pqrs}\big(\varphi_{j,s}\delta_{lr} + \varphi_{j,r}\delta_{ls}\big)\big(\varphi_{i,q}\delta_{kp} + \varphi_{i,p}\delta_{kq}\big)\ dx \\ &= -\frac{1}{4}\sum_{p,q,r,s=1}^{2} \int_\Omega c_{pqrs}\big(\varphi_{j,s}\varphi_{i,q}\delta_{lr}\delta_{kp} + \varphi_{j,s}\varphi_{i,p}\delta_{lr}\delta_{kq} \\ &\qquad + \varphi_{j,r}\varphi_{i,q}\delta_{ls}\delta_{kp} + \varphi_{j,r}\varphi_{i,p}\delta_{ls}\delta_{kq}\big)\ dx \\ &= -\frac{1}{4}\Big\{\sum_{q,s=1}^{2}\int_\Omega c_{kqls}\varphi_{j,s}\varphi_{i,q}\ dx + \sum_{p,s=1}^{2}\int_\Omega c_{pkls}\varphi_{j,s}\varphi_{i,p}\ dx \\ &\qquad + \sum_{q,r=1}^{2}\int_\Omega c_{kqrl}\varphi_{j,r}\varphi_{i,q}\ dx + \sum_{p,r=1}^{2}\int_\Omega c_{pkrl}\varphi_{j,r}\varphi_{i,p}\ dx\Big\} \\ &= -\frac{1}{4}\sum_{q,s=1}^{2}\int_\Omega \big(c_{kqls}\varphi_{j,s}\varphi_{i,q} + c_{qkls}\varphi_{j,s}\varphi_{i,q} + c_{kqsl}\varphi_{j,s}\varphi_{i,q} + c_{qksl}\varphi_{j,s}\varphi_{i,q}\big)\ dx \\ &= -\sum_{q,s=1}^{2}\int_\Omega c_{kqls}\varphi_{i,q}\varphi_{j,s}\ dx \quad (\text{by } (9)) \qquad (19) \\ &= -\sum_{q,s=1}^{2}\int_{T^1} c_{kqls}\varphi_{i,q}\varphi_{j,s}\ dx - \sum_{q,s=1}^{2}\int_{T^2} c_{kqls}\varphi_{i,q}\varphi_{j,s}\ dx \\ &\equiv [K_{ij}]^1_{kl} + [K_{ij}]^2_{kl}. \end{aligned}$$

Since functions φ_i, $\nabla\varphi_i$, φ_j and $\nabla\varphi_j$ in T^1 and T^2 are written as

$$\varphi_i|_{T^1}(x) = 1 + \frac{t-b}{m_1}x_1 + \frac{a-s}{m_1}x_2, \qquad \nabla\varphi_i|_{T^1}(x) = \Big(\frac{t-b}{m_1},\ \frac{a-s}{m_1}\Big)^T,$$
$$\varphi_j|_{T^1}(x) = \frac{b}{m_1}x_1 - \frac{a}{m_1}x_2, \qquad \nabla\varphi_j|_{T^1}(x) = \Big(\frac{b}{m_1},\ -\frac{a}{m_1}\Big)^T,$$
$$\varphi_i|_{T^2}(x) = 1 + \frac{d-t}{m_2}x_1 + \frac{s-c}{m_2}x_2, \qquad \nabla\varphi_i|_{T^2}(x) = \Big(\frac{d-t}{m_2},\ \frac{s-c}{m_2}\Big)^T,$$
$$\varphi_j|_{T^2}(x) = -\frac{d}{m_2}x_1 + \frac{c}{m_2}x_2, \qquad \nabla\varphi_j|_{T^2}(x) = \Big(-\frac{d}{m_2},\ \frac{c}{m_2}\Big)^T,$$

and c_{pqrs} is homogeneous, it holds that

$$\begin{aligned}
[K_{ij}]^1_{kl} &= -|T^1| \sum_{q,s=1}^{2} c_{kqls}\varphi_{i,q}\varphi_{j,s} \\
&= -\frac{|T^1|}{m_1^2}\{c_{k1l1}(t-b)b + c_{k1l2}(t-b)(-a) + c_{k2l1}(a-s)b + c_{k2l2}(a-s)(-a)\} \\
&= \frac{1}{2m_1}\{c_{k1l1}(b-t)b + c_{k1l2}(-m_1 + (s-a)b) + c_{k2l1}(s-a)b + c_{k2l2}(a-s)a\} \\
&= \frac{1}{2m_1}\{c_{k1l1}(b-t)b + 2\bar{c}_{k1l2}(s-a)b + c_{k2l2}(a-s)a\} - \frac{1}{2}c_{k1l2} \\
&\equiv [\widetilde{K_{ij}}]^1_{kl} - \frac{1}{2}c_{k1l2}, \\
[K_{ij}]^2_{kl} &= -|T^2| \sum_{q,s=1}^{2} c_{kqls}\varphi_{i,q}\varphi_{j,s} \\
&= -\frac{|T^2|}{m_2^2}\{c_{k1l1}(d-t)(-d) + c_{k1l2}(d-t)c + c_{k2l1}(s-c)(-d) + c_{k2l2}(s-c)c\} \\
&= \frac{1}{2m_2}\{c_{k1l1}(d-t)d + c_{k1l2}(m_2 + (s-c)d) + c_{k2l1}(s-c)d + c_{k2l2}(c-s)c\} \\
&= \frac{1}{2m_2}\{c_{k1l1}(d-t)d + 2\bar{c}_{k1l2}(s-c)d + c_{k2l2}(c-s)c\} + \frac{1}{2}c_{k1l2} \\
&\equiv [\widetilde{K_{ij}}]^2_{kl} + \frac{1}{2}c_{k1l2}.
\end{aligned}$$

We note that $[\widetilde{K_{ij}}]^m \in \mathbb{R}^{2\times 2}$ $(m = 1, 2)$ are symmetric from a relation, $\bar{c}^m_{kqls} = \bar{c}^m_{lqks}$. Then, the symmetric property of $[\widetilde{K_{ij}}]^m$ $(m = 1, 2)$ and the homogeneity of c_{pqrs} yield that

$$\begin{aligned}
[K_{ij}]_{kl} &= [K_{ij}]^1_{kl} + [K_{ij}]^2_{kl} = [\widetilde{K_{ij}}]^1_{kl} - \frac{1}{2}c_{k1l2} + [\widetilde{K_{ij}}]^2_{kl} + \frac{1}{2}c_{k1l2} \\
&= [\widetilde{K_{ij}}]^1_{kl} + [\widetilde{K_{ij}}]^2_{kl} = [\widetilde{K_{ij}}]^1_{lk} + [\widetilde{K_{ij}}]^2_{lk} = [K_{ij}]_{lk}.
\end{aligned}$$

Thus, we have $K_{ij} \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}$, which leads to the other property of (i), $K_{ij} = K_{ji}$ as

$$K_{ji} = K_{ij}^T = K_{ij},$$

where the first equality follows from (19) (i.e., $[K_{ij}]_{kl} = -\sum_{q,s=1}^{2} \int_\Omega c_{kqls}\varphi_{i,q}\varphi_{j,s}\, dx$).

Next, we prove (ii). It holds that

$$[K_{ij}]_{kl} = [\widetilde{K_{ij}}]^1_{kl} + [\widetilde{K_{ij}}]^2_{kl}$$

$$
\begin{aligned}
&= \frac{1}{2m_1}\big\{c_{k1l1}(b-t)b + 2\bar{c}_{k1l2}(s-a)b + c_{k2l2}(a-s)a\big\} \\
&\quad + \frac{1}{2m_2}\big\{c_{k1l1}(d-t)d + 2\bar{c}_{k1l2}(s-c)d + c_{k2l2}(c-s)c\big\} \\
&= c_{k1l1}E + 2\bar{c}_{k1l2}F + c_{k2l2}G. \qquad (20)
\end{aligned}
$$

From (16) and (20) the tensor-valued spring constant $K_{ij} \in \mathbb{R}^{2\times 2}_{\text{sym}}$ can be written as

$$K_{ij} = \begin{pmatrix} [K_{ij}]_{11} & [K_{ij}]_{12} \\ [K_{ij}]_{12} & [K_{ij}]_{22} \end{pmatrix}$$

for

$$[K_{ij}]_{11} \equiv (\lambda + 2\mu)E + \mu G, \quad [K_{ij}]_{22} \equiv \mu E + (\lambda + 2\mu)G, \quad [K_{ij}]_{12} \equiv (\lambda + \mu)F,$$

and it holds that K_{ij} is positive definite if and only if

$$\operatorname{tr} K_{ij} > 0 \quad \text{and} \quad \det K_{ij} > 0. \qquad (21)$$

From Lemma 2 the first condition of (21) holds. Letting

$$A \equiv \frac{\lambda + \mu}{\mu} = \frac{\lambda}{\mu} + 1 > 0,$$

we have

$$[K_{ij}]_{11} = \mu(EA + E + G), \qquad [K_{ij}]_{22} = \mu(GA + E + G), \qquad [K_{ij}]_{12} = \mu F A,$$

and

$$\det K_{ij} = -\mu^2 (E+G)^2 (\alpha A^2 - A - 1).$$

The above relation yields that the second condition of (21) is equivalent to

$$1 < A < \frac{1 + \sqrt{1 + 4\alpha}}{2\alpha} \quad \text{and} \quad 0 < \alpha < 2,$$

which leads to the desired result. □

Remark 4. When the stiffness tensor c is not homogeneous, we have in general $c_{k1l2}|_{T^1} \neq c_{k1l2}|_{T^2}$ which implies $[K_{ij}]_{kl} \neq [\widetilde{K_{ij}}]^1_{kl} + [\widetilde{K_{ij}}]^2_{kl}$. That is why K_{ij} is not always symmetric if c is not homogeneous.

From Theorem 1 the following two corollaries hold.

Corollary 1. *(i) α can be written as*

$$\alpha = \frac{\sin\theta_i \sin\theta_j \sin\theta_1 \sin\theta_2}{\sin^2(\theta_1 + \theta_2)}. \qquad (22)$$

(ii) Suppose the two triangles including P_i and P_j are equilateral in addition to the same assumptions of Theorem 1-(ii). Then, K_{ij} is positive definite if and only if

$$\lambda < \mu.$$

(iii) Under the same assumptions of (ii) and $\lambda = \mu$ the symmetric matrix K_{ij} has one positive and one zero eigenvalue.
(iv) Under the same assumptions of (ii) and $\mu < \lambda$ the symmetric matrix K_{ij} has one positive and one negative eigenvalue.

Proof. We firstly show (i). From Lemmas 1 and 2 the expression of α (22) is obtained by

$$\alpha = \frac{F^2 - EG}{(E+G)^2} = \frac{m_3 m_4}{4 m_1 m_2}\Big(\frac{\cot\theta_1 + \cot\theta_2}{2}\Big)^{-2}$$
$$= \frac{\sin\theta_i \sin\theta_j}{4\sin\theta_1 \sin\theta_2}\Big(\frac{\sin(\theta_1+\theta_2)}{2\sin\theta_1\sin\theta_2}\Big)^{-2} = \frac{\sin\theta_i\sin\theta_j\sin\theta_1\sin\theta_2}{\sin^2(\theta_1+\theta_2)}.$$

Next we prove (ii)–(iv). From (i) and the additional condition, i.e., the two triangles are equilateral, we have $\alpha = 3/4$, $f(\alpha) = f(3/4) = 1$ and

$$\det K_{ij} = -\frac{1}{4}\mu^2(E+G)^2(A-2)(3A+2)$$

with $A = \lambda/\mu + 1$, which implies $\det K_{ij} > 0$, $\det K_{ij} = 0$ and $\det K_{ij} < 0$ for the cases of (ii), (iii) and (iv), respectively, and yields the desired results of (ii)–(iv). □

Let θ_0 and θ_* be fixed positive constants defined by

$$\theta_0 \equiv \arctan(\sqrt{7}/3), \quad \theta_* \equiv \min\{\theta_{\min}, \pi - 2\theta_{\max}\},$$

where $\theta_{\min}$ and $\theta_{\max}$ are the minimum and the maximum interior angles of the triangulation, respectively.

Corollary 2. *In addition to the same assumptions of Theorem 1-(ii), suppose that*

$$\theta_0 < \theta < \frac{\pi - \theta_0}{2} \tag{23}$$

is satisfied for any interior angle θ of the triangulation and

$$\frac{\lambda}{\mu} < f\Big(\frac{1}{2(1-\cos\theta_*)}\Big). \tag{24}$$

Then, K_{ij} is positive definite for any $(i,j) \in \Lambda^{\mathrm{FE}}$ satisfying i or $j \in J_0^{\mathrm{FE}}$.

Proof. First we prove $\alpha < 2$. The definition of θ_* implies that

$$\sin\theta_1,\ \sin\theta_2 \in \Big[\sin\theta_*,\ \cos\frac{\theta_*}{2}\Big], \tag{25a}$$
$$\sin\theta_i,\ \sin\theta_j,\ \sin(\theta_1+\theta_2) \in [\sin\theta_*,\ 1]. \tag{25b}$$

Combining (25) with Corollary 1-(i), we have

$$\sin^4\theta_* \le \alpha \le \frac{1}{2(1-\cos\theta_*)}. \tag{26}$$

On the other hand, the inequality (23) is equivalent to $\theta_0 < \theta_*$, which yields another equivalent condition,

$$\frac{1}{2(1-\cos\theta_*)} < 2. \tag{27}$$

The inequality $\alpha < 2$, therefore, holds from (26) and (27).

Next we show $\lambda/\mu < f(\alpha)$. It is obtained as

$$\frac{\lambda}{\mu} < f\Big(\frac{1}{2(1-\cos\theta_*)}\Big) \le f(\alpha)$$

from (24), (26) and Remark 3. Thus, we have the desired result from Theorem 1-(ii). □

At the end of this section we mention about positive definiteness of the tensor-valued spring constant K_{ij}^{FE} for a specific material, i.e., concrete. Suppose $(i,j) \in \Lambda^{\mathrm{FE}}$ with i or $j \in J_0^{\mathrm{FE}}$. We consider a case that two triangles including nodes P_i and P_j are both equilateral. Isotropic homogeneous materials satisfy

$$\frac{\lambda}{\mu} = \frac{2\nu}{1-2\nu},$$

where ν is Poisson's ratio. Setting $\nu = 0.2$ as (representative) Poisson's ratio of concrete, we have $\lambda/\mu = 2/3 < 1$. The spring constant K_{ij}^{FE} derived from P1-FEM is, therefore, symmetric and positive definite.

6. **Conclusions.** We have studied symmetry and positive definiteness of scalar and tensor-valued spring constants derived from P1-FEM for the scalar elliptic equation and equations of linear elasticity in two dimensions. Each derived spring-block system with the spring constant is consistent in a sense that it is equivalent to P1-FEM. For the scalar case, it is always symmetric and positive definite under the acuteness condition. For the tensor case, it is symmetric if the fourth-order elastic stiffness tensor c satisfies condition (9) and is spatially homogeneous. It is also positive definite if isotropy of c and the acuteness condition are additionally satisfied. We found that materials with low Poisson's ratio can be approximated by the spring-block system which is consistent with the equations of linear elasticity. Concrete is a typical example of a low Poisson ratio material. To construct a mathematically sound spring constant having symmetry and positive definiteness for more general elastic tensor and/or dimension is a future work.

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