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HOMOGENIZATION OF A PORO-ELASTICITY MODEL COUPLED WITH DIFFUSIVE TRANSPORT AND A FIRST ORDER REACTION FOR CONCRETE

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Abstract. We study a two-scale homogenization problem describing the linearized poro-elastic behavior of a periodic two-component porous material exhibited to a slightly compressible viscous fluid flow and a first-order chemical reaction. One material component consists of disconnected parts embedded in the other component which is supposed to be connected. It is shown that a memory effect known from the purely mechanic problem gets inherited by the reaction component of the model.

1. **Introduction.** Building concrete is a mixture of cement, sand and other solid components such as gravel, reinforced by steel. Physico-chemical processes like carbonation and others pave the way to degradation of such structures. We look into a process which precedes the actual corrosion and takes a certain two-component structure of real concrete into account. A (not necessarily close) look reveals that some types of building concrete can be considered as a concrete-cement matrix with embedded gravel pieces (cf. Figure 1 for a real-life example).

Figure 1. A standard cylindrical concrete test specimen.

Matrix as well as gravel are porous, [6], and eligible of fluid transport accompanied by transport (dispersion (diffusion)) of diluted chemicals. In connection with the fluid transport concrete might exhibit poro-elastic behavior (cf. the discussion in [20], e.g.).

In this note we consider a generalization — a two-component porous solid undergoing slow viscous fluid flow and chemical transport and reaction. The material

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we have in mind is a concrete with many very small embedded (porous!) gravel pieces, cf. fig 1. Figures 2 show idealizations: fig. 2(a) depicts such a concrete piece from afar, fig. 2(b) indicates a magnification including the gravel pieces (small gray circles, summarized in Ω_2^ε, the whole setting idealized as a periodic structure) and fig. 2(c) shows a magnification of a single gravel piece and its cementitious neighborhood. An upscaled version of fig. 2(c) will serve as the "cell" Y in our homogenization setting below.

Let $\varepsilon_0 > 0$ stand for the "real", "typical" length of the small cube inside the larger cube in fig. 2(b). If ε_0 is "very small", then the (homogenization-) limit case $\varepsilon \to 0$ is a candidate for a good approximation of the case ε_0. Below we will make clear what is meant by "$\varepsilon \to 0$".

This note deals with two features — a (microscopic) double-poro-elasticity model $\left(M_{pe}^\varepsilon\right)$ based on Biot's system, complemented by Deresiewicz-Skalak interfacial (cf. [7]) and initial and boundary conditions (cf. (2a)-(2h) and [1]) and a diffusion-reaction equation based on the continuity equation. Implicitly we deal with a third scale by employing Darcy's law at the microscopic scale. The poro-elastic part has essentially been done by Ainouz, [1], and Showalter and Momken, [18]; the chemistry part seems to be new.[1] As a further simplification of the model, we assume the pores to be completely saturated.

In section 2 we introduce the formulations at the micro-level and obtain the "poro-elasticity problem" (M_{pe}^ε) and the combined "poro-elasticity problem with dispersion of a chemical", (M_{cpe}^ε). Section 3 deals with the existence of solutions of (M_{cpe}^ε) and ε-independent estimates. The fourth and final section is devoted to the two-scale convergence of the concentrations and the identification of the limit equations. As a particular feature, one might see the memory term (42b) in the concentration equations, which is inherited from the memory term in the limit equations for (M_{pe}^ε), cf. Ainouz [1] and (19b).

2. **Setting.** Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain with outward normal vector $\vec{\nu}$ and $Y = (0,1)^3$ be the open unit cell in $\mathbb{R}^3$. Let $Y_1, Y_2 \subset Y$ be two disjoint open sets, such that Y_1 is connected, such that $\Gamma := \overline{Y_1} \cap \overline{Y_2}$ is smooth and $\Gamma = \partial Y_2$, $\overline{Y_2} \subset Y$ and such that $Y = Y_1 \cup Y_2 \cup \Gamma$. Furthermore, let n denote the unit normal vector on Γ pointing into Y_1 and χ_1, χ_2 be the indicator functions of Y_1, Y_2 extended by Y-periodicity to the whole of $\mathbb{R}^3$, resp.

Let $\varepsilon > 0$ be a small parameter and define for $i \in \{1,2\}$ the sets

$$\Omega_i^\varepsilon = \{x \in \Omega : \chi_i^\varepsilon(x) = 1\} \quad \text{and} \quad \Gamma^\varepsilon = \overline{\Omega_1^\varepsilon} \cap \overline{\Omega_2^\varepsilon}, \tag{1}$$

where $\chi_i^\varepsilon(x) = \chi_i\left(\frac{x}{\varepsilon}\right)$ (see figure 2 (b), (c) for a two dimensional sketch). Note that $\Omega = \Omega_1^\varepsilon \cup \Omega_2^\varepsilon \cup \Gamma^\varepsilon$ by construction, that Ω_1^ε is connected and that Ω_2^ε is disconnected.

For $T > 0$ let $S := (0,T)$ denote a time interval; the space-time domains are $Q := S \times \Omega$ and $Q_i^\varepsilon := S \times \Omega_i^\varepsilon$, $i \in \{1,2\}$.

Both phases, i.e., Ω_1^ε and Ω_2^ε, are assumed to be occupied by a porous and slightly deformable material and to be saturated with a slightly compressible and viscous

[1] Actually, Ainouz, [1], assumes implicitely an additional interface condition (cf. our "note" following equation (6e)). This leads to a slight simplification of the problem. Therefore our relations (6c), (10a), (10b), (14c)-(14f), (15a) and (19a) have no or other counterparts in [1]. Despite this fact, we use the essential features of Ainouz' arguments.

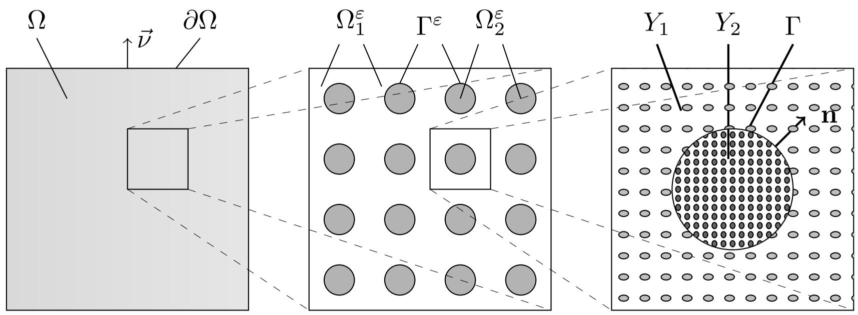

FIGURE 2. (a) Macroscale, (b) Mesoscale, (c) Microscale

fluid which moves through it. We denote by $u_i^\varepsilon = u_i^\varepsilon(t,x)$, $p_i^\varepsilon = p_i^\varepsilon(t,x)$ the *displacement field* and the *fluid pressure*, resp., $x \in \Omega_i^\varepsilon$, $t \in S$, $i \in \{1,2\}$.

Double poro-elasticity: The (linear) double poro-elasticity model (M_{pe}^ε) is formulated in the reference configuration and is based on Biot's system for consolidation processes in both phases. It is completed with *Deresiewicz-Skalak* interfacial, initial and boundary conditions (for details regarding the modeling, cf. [1, 5, 6, 18]):

$$-\operatorname{div}(\mathbb{A}_1^\varepsilon \mathrm{e}(u_1^\varepsilon)) + \alpha_1 \nabla p_1^\varepsilon = 0 \quad \text{in} \quad Q_1^\varepsilon, \tag{2a}$$

$$-\operatorname{div}(\mathbb{A}_2^\varepsilon \mathrm{e}(u_2^\varepsilon)) + \varepsilon\alpha_2 \nabla p_2^\varepsilon = 0 \quad \text{in} \quad Q_2^\varepsilon, \tag{2b}$$

$$\partial_t\left(c_1^\varepsilon p_1^\varepsilon + \alpha_1 \operatorname{div} u_1^\varepsilon\right) - \operatorname{div}(K_1^\varepsilon \nabla p_1^\varepsilon) = f_1^\varepsilon \quad \text{in} \quad Q_1^\varepsilon, \tag{2c}$$

$$\partial_t\left(c_2^\varepsilon p_2^\varepsilon + \varepsilon\alpha_2 \operatorname{div} u_2^\varepsilon\right) - \operatorname{div}(\varepsilon^2 K_2^\varepsilon \nabla p_2^\varepsilon) = f_2^\varepsilon \quad \text{in} \quad Q_2^\varepsilon, \tag{2d}$$

$$u_1^\varepsilon = u_2^\varepsilon, \quad \sigma_1^\varepsilon n^\varepsilon = \sigma_2^\varepsilon n^\varepsilon \quad \text{on} \quad S \times \Gamma^\varepsilon, \tag{2e}$$

$$K_1^\varepsilon \nabla p_1^\varepsilon \cdot n^\varepsilon = \varepsilon^2 K_2^\varepsilon \nabla p_2^\varepsilon \cdot n^\varepsilon = \varepsilon g^\varepsilon (p_1^\varepsilon - p_2^\varepsilon) \quad \text{on} \quad S \times \Gamma^\varepsilon, \tag{2f}$$

$$u_1^\varepsilon = 0, \quad p_1^\varepsilon = 0, \qquad \text{on} \quad S \times \partial\Omega, \tag{2g}$$

$$u_i^\varepsilon(0) = 0 \quad p_i^\varepsilon(0) = 0 \quad \text{in} \quad \Omega_i^\varepsilon. \tag{2h}$$

Here, f_i^ε are source densities,[2] α_i are the *Biot-Willis* parameters and $\sigma_i = \mathbb{A}_i^\varepsilon \mathrm{e}(u_i^\varepsilon) - \alpha_i p_i^\varepsilon \mathrm{I}_3$ the *effective stresses*, where $\mathbb{A}_i^\varepsilon$ are forth rank elasticity tensors, where $\mathrm{e}(u_i^\varepsilon) = \frac{1}{2}(\nabla u_i^\varepsilon + (\nabla u_i^\varepsilon)^t)$ is the linearized strain tensor, and where I_3 denotes the identity in $\mathbb{R}^{3\times 3}$. The coefficients c_i^ε and K_i^ε denote the porosity and the permeability, resp. Note that the K_i^ε and the c_i^ε are already averaged quantities and given by *Darcy's law* which is assumed to hold for both phases. The coefficient g^ε is the hydraulic permeability, which describes the quality of the hydraulic contact between the phases. The limit cases $g^\varepsilon = 0$ and $g^\varepsilon = \infty$, correspond to no transmission and perfect transmission, resp., (cf. [1, 18]).

Chemical transport: We include the dispersive/diffusive transport of a dissolved chemical substance. In Ω_i^ε its *concentration* is denoted by $b_i^\varepsilon = b_i^\varepsilon(t,x)$ ($x \in \Omega_i^\varepsilon$, $t \in S$). The transport is heavily influenced by the fluid pressure and the deformation but is assumed to have no reverse coupling to the poro-elasticity system.[3] We assume

[2] We exclude the case of volume distributed forces for the mechanical part since they can be eliminated by a simple translation (for details cf. [18]).

[3] Note: One could easily think of scenarios in which this is not the case, e.g., the chemical substance could have a directly affect the porosity or permeability.

that the advection can be neglected. In analogy to the poro-elasticity equations the linearized mass conservation then reads, cf. [8],

$$\partial_t(d_1^\varepsilon b_1^\varepsilon + \hat{\alpha}_1 \operatorname{div} u_1^\varepsilon + \hat{c}_1^\varepsilon p_1^\varepsilon) - \operatorname{div}(D_1^\varepsilon \nabla b_1^\varepsilon) = H_1(b_1^\varepsilon), \tag{2i}$$

$$\partial_t(d_2^\varepsilon b_2^\varepsilon + \varepsilon\hat{\alpha}_2 \operatorname{div} u_2^\varepsilon + \hat{c}_2^\varepsilon p_2^\varepsilon) - \operatorname{div}(\varepsilon^2 D_2^\varepsilon \nabla b_2^\varepsilon) = H_2(b_2^\varepsilon). \tag{2j}$$

Here, D_i^ε are the averaged dispersion/diffusion coefficients, d_i^ε are the porosities, $\hat{\alpha}_1$ and $\hat{c}_1^\varepsilon$ are essentially analogous to α_i and c_i^ε and h_i are concentration-dependent source densities.

The coupling terms, i.e., $(\hat{\alpha}_i \operatorname{div} u_i^\varepsilon)'$ and $(\hat{c}_i^\varepsilon p_i^\varepsilon)'$ $(i \in \{1,2\})$, account for the change of porosity and the change of pore pressure and arise via a linearization similar to that leading to Biot's system for consolidation.

Equations (2a)-(2h) are complemented by open-pore interface conditions, *Neumann* boundary conditions and non homogeneous initial values:

$$D_1^\varepsilon \nabla b_1^\varepsilon \cdot n^\varepsilon = \varepsilon^2 D_2^\varepsilon \nabla b_2^\varepsilon \cdot n^\varepsilon \quad \text{on} \quad S \times \Gamma^\varepsilon, \tag{2k}$$

$$D_1^\varepsilon \nabla b_1^\varepsilon \cdot n^\varepsilon = \varepsilon \beta^\varepsilon (b_1^\varepsilon - b_2^\varepsilon) \quad \text{on} \quad S \times \Gamma^\varepsilon, \tag{2l}$$

$$\nabla b_1^\varepsilon \cdot \vec{\nu} = 0 \quad \text{on} \quad S \times \partial\Omega, \tag{2m}$$

$$b_0 = \chi_1^\varepsilon b_1^\varepsilon(0) + \chi_2^\varepsilon b_2^\varepsilon(0) \quad \text{in} \quad \Omega. \tag{2n}$$

We summarize (2a)-(2n) as model (M_{cpe}^ε).

Note. Our choice of the ε-scalings of the permeabilities, diffusivities, Biot-Willis parameters and hydraulic permeabilities follows [1, 4, 15].

3. Well posedness of Model (M_{cpe}^ε).

We shall assume that there exist continuous and Y-periodic functions $\mathbb{A}_i : \mathbb{R}^3 \to \mathbb{R}^{3\times3\times3\times3}$, K_i, $D_i : \mathbb{R}^3 \to \mathbb{R}^{3\times3}$ and c_i, $\hat{c}_i$, d_i, g, $\beta : \mathbb{R}^3 \to \mathbb{R}$ and a $C > 0$, independent of ε, such that

$$\mathbb{A}_i^\varepsilon(x) = \mathbb{A}_i\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad \mathbb{A}_i(x)\Psi : \Psi \geq C(\Psi : \Psi), \tag{3a}$$

$$c_i^\varepsilon(x) = c_i\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad c_i(x) \geq C, \tag{3b}$$

$$K_i^\varepsilon(x) = K_i\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad K_i(x)\xi \cdot \xi \geq C\left|\xi\right|^2, \tag{3c}$$

$$\hat{c}_i^\varepsilon(x) = \hat{c}_i\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad \hat{c}_i(x) \geq C, \tag{3d}$$

$$d_i^\varepsilon(x) = d_i\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad d_i(x) \geq C, \tag{3e}$$

$$D_i^\varepsilon(x) = D_i\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad D_i(x)\xi \cdot \xi \geq C\left|\xi\right|^2, \tag{3f}$$

$$g^\varepsilon(x) = g\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad g_i(x) \geq C, \tag{3g}$$

$$\beta^\varepsilon(x) = \beta\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad \beta_i(x) \geq C \tag{3h}$$

for all $x \in \mathbb{R}^3$, for all symmetric matrices $\Psi \in \mathbb{R}^{3\times3}$ and for all vectors $\xi \in \mathbb{R}^3$, $i \in \{1,2\}$. Furthermore, let α_i and $\hat{\alpha_i}$ be positive constants.[4] In addition, we assume that we are given a *Caratheodory* function[5] h_1 and that there exists a function $a \in L^2(Q)$ such that

$$|h_1(t,x,u)| \leq C\left(|a(t,x)| + |u|\right) \quad \text{for all } (t,x,u) \in S \times \Omega \times \mathbb{R}. \tag{3i}$$

[4]Note: By these assumptions all coefficients have ε-independent L^∞ bounds.

[5]For details, we refer to [21, Chapter 26.3].

As a consequence, this *Caratheodory* function h_1 defines a continuous *Nemyckii* operator $H_1 : L^2(Q) \to L^2(Q)$ via $H_1(\cdot)(t,x) := h_1(t,x,\cdot)$. A second operator, $H_2 : L^2(Q) \to L^2(Q)$, is assumed to be given by the linear relation $H_2(b)(t,x) = h_2 + \tilde{h}_2(t,x)b(t,x)$ for some $h_2, \tilde{h}_2 \in L^2(Q)$ and for all $b \in L^2(Q)$.

We introduce the following spaces (cf. [1, 12]):

$$
\begin{aligned}
&H = L^2(\Omega), && \mathbf{H} = L^2(\Omega)^3, \qquad \mathbf{V} = H_0^1(\Omega)^3,\\
&\mathcal{A}_\varepsilon^1 = \{u \in H^1(\Omega_1^\varepsilon) \ : \ u = 0 \text{ on } \partial\Omega\}, && \mathcal{A}_\varepsilon^2 = H^1(\Omega_2^\varepsilon), \qquad \mathcal{A}_\varepsilon = \mathcal{A}_\varepsilon^1 \times \mathcal{A}_\varepsilon^2,\\
&\mathcal{B}_\varepsilon = H^1(\Omega_1^\varepsilon) \times H^1(\Omega_2^\varepsilon), && W = \left\{u \in L^2(S;\mathcal{B}_\varepsilon) \ : \ u' \in L^2(S;\mathcal{B}_\varepsilon')\right\}.
\end{aligned}
$$

The norms of the spaces $\mathcal{A}_\varepsilon$ and $\mathcal{B}_\varepsilon$ are defined for $p = (p_1, p_2) \in \mathcal{A}_\varepsilon$ and $b = (b_1, b_2) \in \mathcal{B}_\varepsilon$ by

$$\|p\|_{\mathcal{A}_\varepsilon}^2 := \|\nabla p_1\|_{L^2(\Omega_1^\varepsilon)}^2 + \varepsilon^2 \|\nabla p_2\|_{L^2(\Omega_2^\varepsilon)}^2 + \varepsilon \|p_1 - p_2\|_{L^2(\Gamma^\varepsilon)}^2, \tag{4}$$

$$\|b\|_{\mathcal{B}_\varepsilon}^2 := \|b_1\|_{H^1(\Omega_1^\varepsilon)}^2 + \|b_2\|_{L^2(\Omega_2^\varepsilon)}^2 + \varepsilon^2 \|\nabla b_2\|_{L^2(\Omega_2^\varepsilon)^3}^2 + \varepsilon \|b_1 - b_2\|_{L^2(\Gamma_\varepsilon)}^2. \tag{5}$$

A weak formulation of problem (M_{cpe}^ε) can be obtained as follows: For $f^\varepsilon := \chi_1^\varepsilon f_1^\varepsilon + \chi_2^\varepsilon f_2^\varepsilon \in L^2(Q)$ find $(u^\varepsilon, p^\varepsilon, b^\varepsilon) = (u^\varepsilon, (p_1^\varepsilon, p_2^\varepsilon), (b_1^\varepsilon, b_2^\varepsilon)) \in L^\infty(S;\mathbf{V}) \times L^2(S;\mathcal{A}_\varepsilon) \times L^2(S;\mathcal{B}_\varepsilon)$ such that[6]

$$(c_1^\varepsilon p_1^\varepsilon + \alpha_1 \operatorname{div} u^\varepsilon)' \in L^2(S;\mathcal{A}_\varepsilon^{1\prime}), \quad (d_1^\varepsilon b_1^\varepsilon + \tilde{c}_1^\varepsilon p_1^\varepsilon + \tilde{\alpha}_1 \operatorname{div} u^\varepsilon)' \in L^2(S;H^1(\Omega_1^\varepsilon)'), \tag{6a}$$

$$(c_2^\varepsilon p_2^\varepsilon + \alpha_2 \operatorname{div} u^\varepsilon)' \in L^2(S;\mathcal{A}_\varepsilon^{2\prime}), \quad (d_2^\varepsilon b_2^\varepsilon + \tilde{c}_2^\varepsilon p_2^\varepsilon + \varepsilon\tilde{\alpha}_2 \operatorname{div} u^\varepsilon)' \in L^2(S;H^1(\Omega_2^\varepsilon)'), \tag{6b}$$

such that $u^\varepsilon(0) = 0$, $p_i^\varepsilon(0) = 0$, $b_i^\varepsilon(0) = \chi_i^\varepsilon b_0$, $i \in \{1, 2\}$, and such that

$$\int_\Omega \mathbb{A}^\varepsilon \mathrm{e}(u^\varepsilon) : \mathrm{e}(v)\,\mathrm{d}x - \int_{\Omega_1^\varepsilon} \alpha_1 p_1^\varepsilon \operatorname{div} v\,\mathrm{d}x - \varepsilon \int_{\Omega_2^\varepsilon} \alpha_2 p_2^\varepsilon \operatorname{div} v\,\mathrm{d}x = 0, \tag{6c}$$

$$
\begin{aligned}
&\langle (c_1^\varepsilon p_1^\varepsilon + \alpha_1 \operatorname{div} u^\varepsilon)', q_1\rangle_{\mathcal{A}_\varepsilon^{1\prime}\mathcal{A}_\varepsilon^1} + \langle (c_2^\varepsilon p_2^\varepsilon + \varepsilon\alpha_2 \operatorname{div} u^\varepsilon)', q_2\rangle_{\mathcal{A}_\varepsilon^{2\prime}\mathcal{A}_\varepsilon^2}\\
&+ \int_{\Omega_1^\varepsilon} K_1^\varepsilon \nabla p_1^\varepsilon \cdot \nabla q_1\,\mathrm{d}x + \varepsilon^2 \int_{\Omega_2^\varepsilon} K_2^\varepsilon \nabla p_2^\varepsilon \cdot \nabla q_2\,\mathrm{d}x\\
&\qquad + \varepsilon \int_{\Gamma^\varepsilon} g^\varepsilon (p_1^\varepsilon - p_2^\varepsilon)(q_1 - q_2)\,\mathrm{d}s = \int_\Omega f^\varepsilon q\,\mathrm{d}x,
\end{aligned} \tag{6d}
$$

[7]

$$
\begin{aligned}
&\langle (d_1^\varepsilon b_1^\varepsilon + \tilde{c}_1^\varepsilon p_1^\varepsilon + \tilde{\alpha}_1 \operatorname{div} u^\varepsilon)', w_1\rangle_{H^1(\Omega_1^\varepsilon)' H^1(\Omega_1^\varepsilon)}\\
&\qquad + \langle (d_2^\varepsilon b_2^\varepsilon + \tilde{c}_2^\varepsilon p_2^\varepsilon + \varepsilon\tilde{\alpha}_2 \operatorname{div} u^\varepsilon)', w_2\rangle_{H^1(\Omega_2^\varepsilon)' H^1(\Omega_1^\varepsilon)}\\
&\qquad + \int_{\Omega_1^\varepsilon} D_1^\varepsilon \nabla b_1^\varepsilon \cdot \nabla w_1\,\mathrm{d}x + \varepsilon^2 \int_{\Omega_2^\varepsilon} D_2^\varepsilon \nabla b_2^\varepsilon \cdot \nabla w_2\,\mathrm{d}x\\
&+ \varepsilon \int_{\Gamma^\varepsilon} h^\varepsilon (b_1^\varepsilon - b_2^\varepsilon)(w_1 - w_2)\,\mathrm{d}s = \int_\Omega H_1(\chi_1^\varepsilon b_1^\varepsilon) w_1\,\mathrm{d}x + \int_\Omega H_2(\chi_2^\varepsilon b_2^\varepsilon) w_2\,\mathrm{d}x,
\end{aligned} \tag{6e}
$$

for all $(v, q, w) \in L^2(S;\mathbf{V}) \times L^2(S;\mathcal{A}_\varepsilon) \times L^2(S;\mathcal{B}_\varepsilon)$. Note that for such a weak solution we do not necessarily have *a priori* that $p^{\varepsilon\prime}$ or $b^{\varepsilon\prime}$ exists, though in our case they do exist in appropriate dual spaces, where they are also bounded with respect to the parameter ε. Also note that the weak formulation of the mechanical part (6) differs from the one in [1] since, although not explicitly stated, the author

[6]For a function $w \in L^2(Q_1^\varepsilon)$ we understand by $\chi_1^\varepsilon w$ the extension of f by zero into all of Q.

[7]Here, $\langle , \rangle_{\mathcal{A}_\varepsilon^{i\prime}\mathcal{A}_\varepsilon^i}$ stands for the dual pairing between $\mathcal{A}_\varepsilon^i$ and $\mathcal{A}_\varepsilon^{i\prime}$, $i \in \{1, 2\}$.

uses (slightly) different interface conditions (cf. (2e) here vs. $\mathbb{A}_1 \mathrm{e}(u_1) n^\varepsilon = \mathbb{A}_2 \mathrm{e}(u_2) n^\varepsilon$ on $S \times \Gamma^\varepsilon$ in [1]) for the (otherwise same!) problem. This leads to a different notion of "weak solution" and, as a consequence, to a slightly different homogenized system.

By the works of Ainouz [1, Theorem 2.2], Showalter and Momken [18, Theorem 1] and Showalter [17, Chapter 4, Theorem 3.3, Chapter 5, Theorem 2.1], there is the following existence and uniqueness result:

Theorem 3.1. *Let $f_\varepsilon \in L^2(Q)$ and let conditions (3a)-(3c) and (3g) be fulfilled. Then there exists a unique*

$$(u^\varepsilon, p^\varepsilon) \in C^1(\overline{S}; \boldsymbol{V}) \times C^1(\overline{S}; H)$$

such that $p(0) = \chi_1^\varepsilon p_1^\varepsilon(0) + \chi_2^\varepsilon p_2^\varepsilon(0) = 0$, $u(0) = \chi_1^\varepsilon u_1^\varepsilon(0) + \chi_2^\varepsilon u_2^\varepsilon(0) = 0$ and such that $(u^\varepsilon, p^\varepsilon)$ satisfies the system (M_{pe}^ε) in the weak sense (6c)-(6d).

If, in addition, $\sup_{\varepsilon>0} \|f^\varepsilon\|_{L^2(\Omega)} < \infty$ we have the following a-priori bounds:

$$\sup_{\varepsilon>0} \Big(\|u^\varepsilon\|_{L^\infty(S;\boldsymbol{V})} + \|u^{\varepsilon\prime}\|_{L^2(S;\boldsymbol{V})} + \|p^\varepsilon\|_{L^2(S;\mathcal{A}^\varepsilon)} \\ + \|p^\varepsilon\|_{L^\infty(S;H)} + \|p^{\varepsilon\prime}\|_{L^2(S;H)} \Big) < \infty. \quad (7)$$

Proof. For uniqueness and existence, cf. the references above. The estimates (7) follow by standard energy arguments. □

This implies the following result for problem (M_{cpe}^ε), cf. [8],

Theorem 3.2. *Let $(u^\varepsilon, p^\varepsilon) \in C^1(\overline{S}; \boldsymbol{V}) \times C^1(\overline{S}; H)$ be given and let conditions (3d), (3f), (3h),(3e) and (3i) be satisfied. Then there exists $b^\varepsilon \in W$ such that $b^\varepsilon(0) = b_0$ and such that $(u^\varepsilon, p^\varepsilon, b^\varepsilon)$ satisfies problem 6. Additionally we have that*

$$\sup_{\varepsilon>0} \Big(\|b^\varepsilon\|_{L^2(S;\mathcal{B})} + \|b^\varepsilon\|_{L^\infty(S;H)} + \left\|b^{\varepsilon\prime}\right\|_{L^2(S;\mathcal{B}_\varepsilon')} \Big) < \infty. \quad (8)$$

Proof. The existence can obtained by standard arguments for parabolic equations by virtue of Schauder's fixed-point theorem. The *a priori* estimates then follow by energy estimates. □

Remark 1. In this general setting uniqueness of the solution cannot be expected. Under stronger assumptions on the source terms, however, e.g., Lipschitz continuity of H_1, uniqueness can be established.

4. Homogenization.

In the following we will use the notion of two-scale convergence to derive a homogenized model. Our basic references for homogenization in general and two-scale convergence in particular are [2, 13, 19].

4.1. Review and extension of Ainouz' results.

Theorem 4.1. *Let $(u^\varepsilon, p^\varepsilon)$ be the sequence of solutions of Problem (P_{pe}^ε) given by Theorem 3.1. There exists a subsequence $(u^\varepsilon, p^\varepsilon)$, still denoted by ε, and there exist*

$$u \in L^\infty(S; \mathbf{V}), \quad \tilde{u} \in L^\infty(S; L^2(\Omega; H^1_\#(Y)/\mathbb{R})^3),$$
$$p_1 \in L^\infty(S; H^1_0(\Omega)), \quad \tilde{p}_1 \in L^2(Q; H^1_\#(Y)/\mathbb{R}), \quad p_2 \in L^\infty(S; L^2(\Omega; H^1_\#(Y)/\mathbb{R}))$$

such that

$$u^\varepsilon \xrightarrow{2} u, \qquad \operatorname{div} u^\varepsilon \xrightarrow{2} \operatorname{div} u + \operatorname{div}_y \tilde{u}, \tag{9a}$$

$$\chi_1^\varepsilon p_1^\varepsilon \xrightarrow{2} \chi_1 p_1, \qquad \chi_1^\varepsilon \nabla p_1^\varepsilon \xrightarrow{2} \chi_1(\nabla p_1 + \nabla_y \tilde{p}_1), \tag{9b}$$

$$\chi_2^\varepsilon p_2^\varepsilon \xrightarrow{2} \chi_2 p_2, \quad \varepsilon \chi_2^\varepsilon \nabla p_2^\varepsilon \xrightarrow{2} \chi_2 \nabla_y p_2. \tag{9c}$$

Moreover, we have that

$$\lim_{\varepsilon\to 0} \int_S \int_{\Gamma_\varepsilon} \varepsilon(p_1^\varepsilon - p_2^\varepsilon)\varphi^\varepsilon \, \mathrm{d}s \, \mathrm{d}t = \int_Q \int_\Gamma (p_1 - p_2)\varphi \, \mathrm{d}s \, \mathrm{d}x \, \mathrm{d}t, \tag{9d}$$

for any $\varphi \in C_0^\infty(Q; C_\#(Y))$ *with* $\varphi^\varepsilon(t,x) = \varphi(t,x,x/\varepsilon)$.

Proof. This result holds due to the a priori estimates established in Theorem 3.1, see also [1, Theorem 3.1]. □

Now, let $v \in C_0^\infty(\Omega)^3$, $\tilde{v} \in C_0^\infty(\Omega; C_\#^\infty(Y))^3$, $q_1 \in C_0^\infty(S \times \Omega)$ and $\tilde{q}_1$, $q_2 \in C_0^\infty(Q; C_\#^\infty(Y))$. Now, choosing $(v^\varepsilon, q_1^\varepsilon, q_2^\varepsilon)$ defined by $v^\varepsilon(x) = v(x) + \varepsilon v(x, x/\varepsilon)$, $q_1^\varepsilon(t,x) = q_1(t,x) + \tilde{q}_1(t,x,x/\varepsilon)$ and $q_2^\varepsilon(t,x) = q_2(t,x,x/\varepsilon)$ as a test function for the variational formulation of the poro-elasticity part, see equations (6c) and (6d), and letting $\varepsilon \to 0$, we obtain

$$\int_\Omega \int_Y \mathbb{A}(y)\big(\mathrm{e}(u) + \mathrm{e}_y(\tilde{u})\big) : \big(\mathrm{e}(v) + \mathrm{e}_y(\tilde{v})\big) \, \mathrm{d}y \, \mathrm{d}x - \int_{\Omega_1} \int_{Y_1} \alpha_1 p_1 \left(\operatorname{div} v + \operatorname{div}_y \tilde{v}\right) \mathrm{d}y \, \mathrm{d}x = 0, \quad t \in S, \tag{10a}$$

$$\begin{aligned} &- \int_Q \int_{Y_1} \left(c_1 p_1 + \alpha_1 \operatorname{div} u + \operatorname{div}_y \tilde{u}\right) \partial_t q_1 \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}t - \int_Q \int_{Y_2} c_2 p_2 \partial_t q_2 \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}t \\ &+ \int_Q \int_{Y_1} K_1(\nabla p_1 + \nabla_y \tilde{p}_1) \cdot (\nabla q_1 + \nabla_y \tilde{q}_1) \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}t + \int_Q \int_{Y_2} K_2 \nabla_y p_2 \cdot \nabla_y q_2 \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}t \\ &+ \int_Q \int_\Gamma g(p_1 - p_2)(q_1 - q_2) \, \mathrm{d}s \, \mathrm{d}x \, \mathrm{d}t = \int_Q \int_{Y_1} f q_1 \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}t + \int_Q \int_{Y_2} f q_2 \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}t. \end{aligned} \tag{10b}$$

Let $j,k \in \{1,2,3\}$ and define $d_{jk} = (y_j\delta_{1k}, y_j\delta_{2k}, y_j\delta_{3k})^T$ for $y \in Y$. We then denote by $w_{jk} \in \left(H_\#^1(Y)/\mathbb{R}\right)^3$ the unique solutions of the elasticity cell problems

$$-\operatorname{div}_y \left(\mathbb{A}_1 \mathrm{e}_y(w_{jk} + d_{jk})\right) = 0 \quad \text{in } Y_1, \tag{11a}$$

$$-\operatorname{div}_y \left(\mathbb{A}_2 \mathrm{e}_y(w_{jk} + d_{jk})\right) = 0 \quad \text{in } Y_2, \tag{11b}$$

$$\mathbb{A}_1 \mathrm{e}_y(w_{jk} + d_{jk})n = \mathbb{A}_2 \mathrm{e}_y(w_{jk} + d_{jk})n \quad \text{on } \Gamma, \tag{11c}$$

$$y \to w_{jk} \quad \text{Y-periodic.} \tag{11d}$$

In addition, let $\widetilde{w} \in \left(H_\#^1(Y)/\mathbb{R}\right)^3$ be the unique solution to the cell problem

$$-\operatorname{div}_y \left(\mathbb{A}_1 e_y(\widetilde{w})\right) = 0 \quad \text{in } Y_1, \tag{12a}$$

$$-\operatorname{div}_y \left(\mathbb{A}_2 e_y(\widetilde{w})\right) = 0 \quad \text{in } Y_2, \tag{12b}$$

$$\left(\mathbb{A}_1 e_y(\widetilde{w}) - \alpha_1\right) n = \mathbb{A}_2 e_y(\widetilde{w}) n \quad \text{on } \Gamma, \tag{12c}$$

$$y \to \widetilde{w} \quad \text{Y-periodic} \tag{12d}$$

and let $\pi_j \in H^1(Y_1)/\mathbb{R}$, $j \in \{1,2,3\}$, be the unique solutions of the following micro-pressure cell problems:

$$-\operatorname{div}_y(K_1(\nabla_y \pi_j + e_j)) = 0 \quad \text{in } Y_1, \tag{13a}$$
$$K_1(\nabla_y \pi_j + e_j)\cdot n = 0 \quad \text{on } \Gamma, \tag{13b}$$
$$y \to \pi_j \quad \text{Y-periodic}, \tag{13c}$$

where $\{e_1, e_2, e_3\}$ denotes the standard basis in $\mathbb{R}^3$.

Building on that, we introduce a constant forth rank tensor $\mathbb{A}^h \in \mathbb{R}^{3\times3\times3\times3}$ and some constant matrices K^h, Λ^h, $\alpha^h \in \mathbb{R}^{3\times3}$ with entries $(j,k,j_1,j_2,j_3,j_4 \in \{1,2,3\})$

$$(\mathbb{A}^h)_{j_1j_2j_3j_4} = \int_Y \mathbb{A}(y)\big(\mathrm{e}_y(w_{j_3j_4}(y) + d_{j_3j_4}(y))\big) : \big(\mathrm{e}_y(w_{j_1j_2}(y) + d_{j_1j_2}(y))\big)\,\mathrm{d}y, \tag{14a}$$

$$(K^h)_{jk} = \int_{Y_1} K_1(y)\,(\nabla_y \pi_j(y) + e_j)\cdot(\nabla_y \pi_k(y) + e_k)\,\mathrm{d}y, \tag{14b}$$

$$(\Lambda^h)_{jk} = \alpha_1 \left(\int_{Y_1} \operatorname{div}_y w_{jk}(y)\,\mathrm{d}y + |Y_1|\, e_j \cdot e_k\right)\,\mathrm{d}y, \tag{14c}$$

$$(\alpha^h)_{jk} = \alpha_1 \left(|Y_1|\, e_j\cdot e_k + \int_Y \mathbb{A}(y)\mathrm{e}_y(\widetilde{w})(y)e_j \cdot e_k\,\mathrm{d}y\right). \tag{14d}$$

In addition, we define an averaged source density

$$\tilde{f}\colon S\times\Omega \to \mathbb{R}, \quad \tilde{f}(t,x) := \int_{Y_1} f_1(t,x,y)\,\mathrm{d}y \tag{14e}$$

and some constant coefficients:

$$\tilde{c} := \int_{Y_1} \big(c_1(y) + \alpha_1 \operatorname{div}_y(\tilde{w}(y))\big)\,\mathrm{d}y, \tag{14f}$$

$$\tilde{g} := \int_\Gamma g(s)\,\mathrm{d}s. \tag{14g}$$

The two-scale homogenization limit of the poro-elasticity problem (M^ε_{cp}) then reads as (weak formulations of)

$$-\operatorname{div}(\mathbb{A}^h \mathrm{e}(u)) + \alpha^h \nabla p_1\,\mathrm{d}s = 0 \quad \text{in } Q, \tag{15a}$$

$$\partial_t(\tilde{c}p_1 + \Lambda^h \colon \mathrm{e}(u)) - \operatorname{div}(K^h\nabla p_1) + \tilde{g}p_1 - \int_\Gamma g p_2\,\mathrm{d}s = \tilde{f} \quad \text{in } Q, \tag{15b}$$

$$\partial_t(c_2p_2) - \operatorname{div}_y(K_2\nabla_y p_2) = f_2 \quad \text{in } Q\times Y_2, \tag{15c}$$

$$K_2\nabla_y p_2\cdot n = g(p_1 - p_2) \quad \text{on } Q\times\Gamma, \tag{15d}$$

$$u = 0,\ p_1 = 0 \quad \text{on } S\times\partial\Omega, \tag{15e}$$

$$y \to p_2 \quad \text{Y-periodic}, \tag{15f}$$

$$u(0) = 0,\ p_1(0) = 0 \quad \text{in } \Omega, \tag{15g}$$

$$p_2(0) = 0 \quad \text{in } \Omega\times Y_2. \tag{15h}$$

Moreover, in [1], Ainouz shows a memory-type result under the assumption that $f_2 \equiv 0$. In a similar way as for the system considered by Ainouz, the details of which are omitted, this can be established for our problem as well: By introducing $\zeta \in L^\infty(S; H^1_\#(Y_2))$ as the unique solution to the following *Robin*-type problem

$$\partial_t(c_2\zeta) - \operatorname{div}_y(K_2\nabla_y\zeta) = 0 \quad \text{in } S\times Y_2, \tag{16a}$$
$$K_2\nabla_y\zeta\cdot n = g(1-\zeta) \quad \text{on } S\times\Gamma, \tag{16b}$$
$$y\to\zeta \quad Y\text{-periodic}, \tag{16c}$$
$$\zeta(0) = 0 \quad \text{in } Y_2. \tag{16d}$$

we have for almost all $(t,x,y)\in S\times\Omega\times Y_2$ the identification[8]

$$p_2(t,x,y) = \int_0^t p_1(\tau,x)\partial_t\zeta(t-\tau,y)\mathrm{d}\tau. \tag{17}$$

If we then define an auxiliary function

$$\eta(t,\tau) = \int_\Gamma g\partial_t\zeta(t-\tau,y)\,\mathrm{d}s, \tag{18}$$

we eliminate p_2 and have

$$-\operatorname{div}(\mathbb{A}^h\mathrm{e}(u)) + \alpha^h\nabla p_1 = 0 \quad \text{in } Q, \tag{19a}$$
$$\partial_t(\tilde c p_1 + \Lambda^h : \mathrm{e}(u)) - \operatorname{div}(K^h\nabla p_1) + \tilde g p_1 - \int_0^t \eta(t,\tau)p_1(\tau,x)\mathrm{d}\tau = \tilde f \quad \text{in } Q, \tag{19b}$$
$$u = 0,\ p_1 = 0 \quad \text{on } S\times\partial\Omega, \tag{19c}$$
$$u(0) = 0,\ p_1(0) = 0 \quad \text{in } \Omega. \tag{19d}$$

Remark 2. Note that this System is different from the homogenized model derived in [1]: First, the homogenized coefficients α^h, Λ^h and $\tilde c$ given by (14c), (14d) and (14g) differ from these in [1], i.e., equations (2.21), (2.22) and (2.24). In addition, the effective momentum equation (19a) is independent of the micropressure $\zeta \in L^\infty(S; H^1_\#(Y_2))$, which is in opposition to the memory term emerging in [1]. Finally, the macroscopic pore pressure p_1 fulfills a homogeneous Dirichlet boundary condition as opposed to the Neumann condition in [1].

4.2. **The chemical part of the model.** With section 4.1 in mind we now attend to the homogenization of the chemical part

Theorem 4.2. *There exists a unique* $(b_1,\tilde b_1)\in L^2(S;\mathcal{B}_0)\times L^2(Q; H^1_\#(Y)/\mathbb{R})$ *and* $b_2\in L^\infty(S; L^2(\Omega; H^1_\#(Y)))$ *such that, up to a subsequence,*

$$\chi_1^\varepsilon b_1^\varepsilon \overset{2}{\rightharpoonup} \chi_1 b_1, \qquad \chi_1^\varepsilon\nabla b_1^\varepsilon \overset{2}{\rightharpoonup} \chi_1(\nabla b_1 + \nabla_y\tilde b_1), \tag{20}$$
$$\chi_2^\varepsilon b_2^\varepsilon \overset{2}{\rightharpoonup} \chi_2 b_2, \qquad \varepsilon\chi_2^\varepsilon\nabla b_2^\varepsilon \overset{2}{\rightharpoonup} \chi_2\nabla_y b_2. \tag{21}$$

Proof. The statement follows from estimates (8) (cf. [1, 2, 13, 14]). □

By choosing test functions $\varphi_1^\varepsilon(t,x) = \varphi_1(t,x) + \varepsilon\tilde\varphi_1(t,x,x/\varepsilon)$ and $\varphi_2^\varepsilon(t,x) = \varphi_2(t,x,x/\varepsilon)$ where $\varphi_1\in C^\infty(\overline{Q})$ and $\tilde\varphi_1,\varphi_2\in C^\infty\left(\overline{Q}; C^\infty_\#(Y)\right)$ such that $\varphi_1(T) =$

[8]This can be seen via Laplace transformation, cf. [1].

$\tilde{\varphi}_1(T) = \varphi_2(T) = 0$, we get after integrating by parts that

$$
\begin{aligned}
&- \int_{Q_1^\varepsilon} (d_1^\varepsilon b_1^\varepsilon + \hat{\alpha}_1 \operatorname{div} u^\varepsilon + \hat{c}_1^\varepsilon p_1^\varepsilon)\, \varphi_1^{\varepsilon\prime} \,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_1^\varepsilon} d_1^\varepsilon b_0 \varphi^\varepsilon(0) \,\mathrm{d}x \\
&\qquad - \int_{Q_2^\varepsilon} (d_2^\varepsilon b_2^\varepsilon + \varepsilon\hat{\alpha}_2 \operatorname{div} u^\varepsilon + \hat{c}_2^\varepsilon p_2^\varepsilon)\, \varphi_2^{\varepsilon\prime} \,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_1^\varepsilon} d_2^\varepsilon b_0 \varphi^\varepsilon(0) \,\mathrm{d}x \\
&\qquad + \int_{Q_1^\varepsilon} D_1^\varepsilon \nabla b_1^\varepsilon \cdot \nabla \varphi_1^\varepsilon \,\mathrm{d}x\,\mathrm{d}t + \varepsilon^2 \int_{Q_2^\varepsilon} D_2^\varepsilon \nabla b_2^\varepsilon \cdot \nabla \varphi_2^\varepsilon \,\mathrm{d}x\,\mathrm{d}t \\
&+ \varepsilon \int_{S\times\Gamma^\varepsilon} \beta^\varepsilon (b_1^\varepsilon - b_2^\varepsilon)\,(\varphi_1^\varepsilon - \varphi_2^\varepsilon) \,\mathrm{d}s\,\mathrm{d}t = \int_{Q_1^\varepsilon} H_1(\chi_1^\varepsilon b_1^\varepsilon)\varphi_1^\varepsilon \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_2^\varepsilon} H_2(\chi_2^\varepsilon b_2^\varepsilon)\varphi_2^\varepsilon \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{22}
$$

We examine the individual expressions and pass to the limit for $\varepsilon \to 0$: We have

$$
\begin{aligned}
&\int_{Q_1^\varepsilon} (d_1^\varepsilon b_1^\varepsilon + \widehat{\alpha}_1^\varepsilon \operatorname{div} u^\varepsilon + \widehat{c}_1^\varepsilon p_1^\varepsilon)\, \varphi_1^{\varepsilon\prime} \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad = \int_Q \Big[\chi_1^\varepsilon d_1^\varepsilon b_1^\varepsilon \varphi_1' + \chi_1^\varepsilon \widehat{\alpha}_1^\varepsilon \operatorname{div} u^\varepsilon \varphi_1' + \chi_1^\varepsilon \widehat{c}_1^\varepsilon p_1^\varepsilon \varphi_1' \\
&\qquad\qquad + \varepsilon\chi_1^\varepsilon d_1^\varepsilon b_1^\varepsilon \tilde{\varphi}_1^{\varepsilon\prime} + \varepsilon\chi_1^\varepsilon \widehat{\alpha}_1^\varepsilon \operatorname{div} u^\varepsilon \tilde{\varphi}_1^{\varepsilon\prime} + \varepsilon\chi_1^\varepsilon \widehat{c}_1^\varepsilon p_1^\varepsilon \tilde{\varphi}_1^{\varepsilon\prime}\Big] \,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
\tag{23}
$$

The coefficients d_1^ε, $\hat{c}_1^\varepsilon$ and $\hat{\alpha}_1$ are continuous. Furthermore, we know from [1, Theorem 3.1] and from Theorem 4.2, that

$$
\frac{\partial u^\varepsilon}{\partial x_j} \overset{2}{\rightharpoonup} \frac{\partial u}{\partial x_j} + \frac{\partial \tilde{u}}{\partial y_j}, \quad \chi_1^\varepsilon p_1^\varepsilon \overset{2}{\rightharpoonup} \chi_1 p_1 \quad \text{and} \quad \chi_1^\varepsilon b_1^\varepsilon \overset{2}{\rightharpoonup} \chi_1 b_1. \tag{24}
$$

Therefore we have, at least along a subsequence ε', still denoted by ε,[9]

$$
\begin{aligned}
&\int_Q \chi_1^\varepsilon d_1^\varepsilon b_1^\varepsilon(t,x)\varphi_1'(t,x) \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \to \int_Q \int_Y \chi_1(y) d_1(y) b_1(t,x)\varphi_1'(t,x) \,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
\tag{25a}
$$

$$
\begin{aligned}
&\int_Q \chi_1^\varepsilon \widehat{\alpha}_1^\varepsilon \operatorname{div} u^\varepsilon(t,x)\varphi_1'(t,x) \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \to \int_Q \int_Y \chi_1(y)\widehat{\alpha}_1(\operatorname{div} u(t,x) + \operatorname{div}_y \tilde{u}(t,x,y))\varphi_1'(t,x) \,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
\tag{25b}
$$

$$
\int_Q \chi_1^\varepsilon \widehat{c}_1^\varepsilon p_1^\varepsilon(t,x)\varphi_1'(t,x) \,\mathrm{d}x\,\mathrm{d}t \to \int_Q \int_Y \chi_1(y)\widehat{c}_1(y) p_1(t,x)\varphi_1'(t,x) \,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t. \tag{25c}
$$

The remaining terms of (23) are of first order with respect to ε and thus converge to 0 for $\varepsilon \to 0$, i.e.,

$$
\varepsilon \int_{Q_1} \chi_1^\varepsilon \Big[d_1^\varepsilon b_1^\varepsilon(t,x) + \widehat{\alpha}_1^\varepsilon \operatorname{div} u^\varepsilon(t,x) + \widetilde{c}_1^\varepsilon p_1^\varepsilon(t,x)\Big] \tilde{\varphi}_1^{\varepsilon\prime} \,\mathrm{d}x\,\mathrm{d}t \to 0. \tag{25d}
$$

[9]From here on all limits involving ε are to be understood in that sense.

We turn our attention to the second integral term in (22):

$$\int_{Q_2} (d_2^\varepsilon b_2^\varepsilon + \varepsilon\widehat{\alpha}_2^\varepsilon \operatorname{div} u^\varepsilon + \widehat{c}_2^\varepsilon p_2^\varepsilon)\, {\varphi_2^\varepsilon}' \,\mathrm{d}x\,\mathrm{d}t = \int_Q \left[\chi_2^\varepsilon d_2^\varepsilon b_2^\varepsilon {\varphi_2^\varepsilon}' + \chi_2^\varepsilon \varepsilon \widehat{\alpha}_2^\varepsilon \operatorname{div} u^\varepsilon {\varphi_2^\varepsilon}' + \chi_2^\varepsilon \widehat{c}_2^\varepsilon p_2^\varepsilon {\varphi_2^\varepsilon}'\right] \mathrm{d}x\,\mathrm{d}t. \tag{26}$$

The coefficients d_2^ε, $\hat{c}_2^\varepsilon$ and $\hat{\alpha}_2$ are continuous. We know from [1] and from Theorem 4.2 that

$$\frac{\partial u^\varepsilon}{\partial x_j} \xrightarrow{2} \frac{\partial u}{\partial x_j} + \frac{\partial \tilde{u}}{\partial y_j}, \quad \chi_2^\varepsilon p_2^\varepsilon \xrightarrow{2} \chi_2 p_2 \quad \text{and} \quad \chi_2^\varepsilon b_2^\varepsilon \xrightarrow{2} \chi_2 b_2. \tag{27}$$

This implies

$$\int_Q \chi_2^\varepsilon d_2^\varepsilon b_2^\varepsilon(t,x) {\varphi_2^\varepsilon}'(t,x)\,\mathrm{d}x\,\mathrm{d}t \to \int_Q \int_Y \chi_2(y) d_2(y) b_2(t,x,y) \varphi_2'(t,x,y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t, \tag{28a}$$

$$\int_Q \chi_2^\varepsilon \widehat{c}_2^\varepsilon p_2^\varepsilon(t,x) {\varphi_2^\varepsilon}'(t,x)\,\mathrm{d}x\,\mathrm{d}t \to \int_Q \int_Y \chi_2(y) \widehat{c}_2(y) p_2(t,x,y) \varphi_2'(t,x,y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \tag{28b}$$

and

$$\varepsilon \int_Q \chi_2^\varepsilon \widehat{\alpha}_2^\varepsilon \operatorname{div} u^\varepsilon(t,x) {\varphi_2^\varepsilon}'(t,x)\,\mathrm{d}x\,\mathrm{d}t \to 0. \tag{28c}$$

The limits of the initial value terms read as

$$\int_\Omega \chi_1^\varepsilon d_1^\varepsilon b_0 \left(\varphi_1(0,x) + \varepsilon\varphi_1(0,x,\frac{x}{\varepsilon})\right)\,\mathrm{d}x \to \int_\Omega \int_Y \chi_1(y) d_1(y) b_0 \varphi_1(0,x)\,\mathrm{d}x\,\mathrm{d}y, \tag{29a}$$

$$\int_\Omega \chi_2^\varepsilon d_2^\varepsilon b_0 \varphi_2^\varepsilon(0,x)\,\mathrm{d}x \to \int_\Omega \int_Y \chi_2(y) d_2(y) b_0 \varphi_2(0,x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{29b}$$

For the diffusion terms in Ω_1^ε and Ω_2^ε, resp., we have

$$\int_{Q_1^\varepsilon} D_1^\varepsilon \nabla b_1^\varepsilon \cdot (\nabla\varphi_1 + \varepsilon\nabla\tilde{\varphi}_1^\varepsilon + \nabla_y \tilde{\varphi}_1^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t = \int_Q \chi_1^\varepsilon D_1^\varepsilon \nabla b_1^\varepsilon \cdot \nabla\varphi_1\,\mathrm{d}x\,\mathrm{d}t + \varepsilon \int_Q \chi_1^\varepsilon D_1^\varepsilon \nabla b_1^\varepsilon \cdot \nabla\tilde{\varphi}_1^\varepsilon\,\mathrm{d}x\,\mathrm{d}t + \int_Q \chi_1^\varepsilon D_1^\varepsilon \nabla b_1^\varepsilon \cdot \nabla_y \tilde{\varphi}_1^\varepsilon\,\mathrm{d}x\,\mathrm{d}t, \tag{30a}$$

$$\varepsilon^2 \int_{Q_2^\varepsilon} D_2^\varepsilon \nabla b_2^\varepsilon \cdot \left(\nabla\varphi_2^\varepsilon + \frac{1}{\varepsilon}\nabla_y \varphi_2^\varepsilon\right)\,\mathrm{d}x\,\mathrm{d}t = \varepsilon^2 \int_Q \chi_2^\varepsilon D_2^\varepsilon \nabla b_2^\varepsilon \cdot \nabla\varphi_2^\varepsilon\,\mathrm{d}x\,\mathrm{d}t + \varepsilon \int_Q \chi_2^\varepsilon D_2^\varepsilon \nabla b_2^\varepsilon \cdot \nabla_y \varphi_2^\varepsilon\,\mathrm{d}x\,\mathrm{d}t. \tag{30b}$$

Since the D_i^ε are continuous, we obtain

$$\int_Q \chi_1^\varepsilon D_1^\varepsilon \nabla b_1^\varepsilon(t,x)\cdot\nabla\varphi_1(t,x)\,\mathrm{d}x\,\mathrm{d}t \\ \to \int_Q\int_Y \chi_1(y)D_1(y)\left(\nabla b_1(t,x)+\nabla_y\tilde{b}_1(t,x,y)\right)\cdot\nabla\varphi_1(t,x)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t, \tag{31a}$$

$$\int_Q \chi_1^\varepsilon D_1^\varepsilon \nabla b_1^\varepsilon(t,x)\cdot\nabla_y\tilde{\varphi}_1^\varepsilon(t,x)\,\mathrm{d}x\,\mathrm{d}t \\ \to \int_Q\int_Y \chi_1(y)D_1(y)\left(\nabla b_1(t,x)+\nabla_y\tilde{b}_1(t,x,y)\right)\cdot\nabla_y\tilde{\varphi}_1(t,x,y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t, \tag{31b}$$

$$\varepsilon\int_Q \chi_2^\varepsilon D_2^\varepsilon \nabla b_2^\varepsilon(t,x)\cdot\nabla_y\varphi_2^\varepsilon(t,x)\,\mathrm{d}x\,\mathrm{d}t \\ \to \int_Q\int_Y \chi_2(y)D_2(y)\nabla_y b_2(t,x,y)\cdot\nabla_y\varphi_2(t,x,y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \tag{31c}$$

and

$$\varepsilon\int_Q \chi_1^\varepsilon D_1^\varepsilon \nabla b_1^\varepsilon(t,x)\cdot\nabla\tilde{\varphi}_1^\varepsilon(t,x)\,\mathrm{d}x\,\mathrm{d}t \\ +\varepsilon^2\int_Q \chi_2^\varepsilon D_2^\varepsilon \nabla b_2^\varepsilon(t,x)\cdot\nabla\varphi_2^\varepsilon(t,x)\,\mathrm{d}x\,\mathrm{d}t \to 0. \tag{31d}$$

Next, we deal with the interface exchange terms:

$$\varepsilon\int_S\int_{\Gamma^\varepsilon}\beta^\varepsilon(b_1^\varepsilon-b_2^\varepsilon)\left(\varphi_1+\varepsilon\tilde{\varphi}_1^\varepsilon-\varphi_2^\varepsilon\right)\,\mathrm{d}s\,\mathrm{d}t \\ =\varepsilon\int_S\int_{\Gamma^\varepsilon}\beta^\varepsilon(b_1^\varepsilon-b_2^\varepsilon)\varphi_1\,\mathrm{d}s\,\mathrm{d}t+\varepsilon^2\int_{S\times\Gamma^\varepsilon}\beta^\varepsilon(b_1^\varepsilon-b_2^\varepsilon)\tilde{\varphi}_1^\varepsilon\,\mathrm{d}s\,\mathrm{d}t \\ -\varepsilon\int_S\int_{\Gamma^\varepsilon}\beta^\varepsilon(b_1^\varepsilon-b_2^\varepsilon)\varphi_2^\varepsilon\,\mathrm{d}s\,\mathrm{d}t, \tag{32}$$

where we find that (see for instance [3, Theorem 2.1])

$$\varepsilon\int_S\int_{\Gamma^\varepsilon}\beta^\varepsilon(b_1^\varepsilon(t,x)-b_2^\varepsilon(t,x))\varphi_1(t,x)\,\mathrm{d}s\,\mathrm{d}t \\ \to\int_Q\int_\Gamma\beta(y)(b_1(t,x)-b_2(t,x,y))\varphi_1(t,x)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t, \tag{33a}$$

$$\varepsilon\int_S\int_{\Gamma^\varepsilon}\beta^\varepsilon(b_1^\varepsilon(t,x)-b_2^\varepsilon(t,x))\varphi_2^\varepsilon(t,x)\,\mathrm{d}s\,\mathrm{d}t \\ \to\int_Q\int_\Gamma\beta(y)(b_1(t,x)-b_2(t,x,y))\varphi_2(t,x,y)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t \tag{33b}$$

and

$$\varepsilon^2\int_S\int_{\Gamma^\varepsilon}\beta^\varepsilon(b_1^\varepsilon(t,x)-b_2^\varepsilon(t,x))\tilde{\varphi}_1^\varepsilon(t,x)\,\mathrm{d}s\,\mathrm{d}t\to 0. \tag{33c}$$

For the right-hand side of equation (23) we have

$$\int_{Q_1^\varepsilon} H_1(\chi_1^\varepsilon b_1^\varepsilon)\varphi_1^\varepsilon \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_2^\varepsilon} H_2(\chi_2^\varepsilon b_2^\varepsilon)\varphi_2^\varepsilon \,\mathrm{d}x\,\mathrm{d}t = \int_Q \chi_1^\varepsilon H_1(\chi_1^\varepsilon b_1^\varepsilon)\varphi_1 \,\mathrm{d}x\,\mathrm{d}t + \varepsilon\int_Q \chi_1^\varepsilon H_1(\chi_1^\varepsilon b_1^\varepsilon)\tilde{\varphi}_1^\varepsilon \,\mathrm{d}x\,\mathrm{d}t + \int_Q \chi_2^\varepsilon H_2(\chi_2^\varepsilon b_2^\varepsilon)\varphi_2^\varepsilon \,\mathrm{d}x\,\mathrm{d}t. \tag{34}$$

With the extension theorem given in [10, Lemma 5], the boundedness of $\chi_1^\varepsilon b_1^\varepsilon$ in $L^2(S;H^1(\Omega))$ and of $\chi_1^\varepsilon b_1^{\varepsilon\prime} \in L^2(S;H^1(\Omega)')$ we have the strong convergence $\chi_1^\varepsilon b_1^\varepsilon \to b_1$ in $L^2(Q)$ for the same subsequence that two-scale converges by Theorem 4.2 (cf. also [11, Theorem 2.1]). Therefore there exist a subsequence of $\chi_1^\varepsilon b_1^\varepsilon$ and a function $r \in L^2(Q)$ such that $\chi_1^\varepsilon b_1^\varepsilon$ converges point-wise almost everywhere to b_1 and such that $|\chi_1^\varepsilon b_1^\varepsilon(t,x)| \le r(t,x)$ for almost all $(t,x)\in Q$. This implies the estimate

$$|(H_1(\chi_1^\varepsilon b_1^\varepsilon))(t,x)| = |h_1(t,x,\chi_1^\varepsilon b_1^\varepsilon(t,x))| \le C\left(|a_1(t,x)| + |b_1^\varepsilon(t,x)|\right) \tag{35}$$

$$\le C\left(|a_1(t,x)| + r(t,x)\right). \tag{36}$$

The continuity of H_1 implies $H_1(\chi_1^\varepsilon b_1^\varepsilon) \to H_1(b_1)$. As a consequence, *Lebesgue's* dominated convergence theorem yields

$$\int_Q \chi_1^\varepsilon (H_1(\chi_1^\varepsilon b_1^\varepsilon))(t,x)\varphi_1(t,x)\,\mathrm{d}x\,\mathrm{d}t \to \int_Q\int_Y \chi_1(y)(H_1(b_1))(t,x)\varphi_1(t,x)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y, \tag{37a}$$

$$\varepsilon\int_Q \chi_1^\varepsilon (H_1(\chi_1^\varepsilon b_1^\varepsilon))(t,x)\tilde{\varphi}_1^\varepsilon(t,x)\,\mathrm{d}x\,\mathrm{d}t \to 0. \tag{37b}$$

The second reaction term is actually more problematic: We do not expect the sequence b_2^ε to converge strongly since Ω_2^ε is disconnected and since the gradients of b_2^ε are not bounded independently of ε (cf. the definition (5) of the norm in B_ε). But since the reaction term in Ω_2^ε is assumed to be affine, we nonetheless are able to pass to the limit:

$$\int_Q \chi_2^\varepsilon H_2(\chi_2^\varepsilon b_2^\varepsilon)(t,x)\,\mathrm{d}x\,\mathrm{d}t = \int_Q \chi_2^\varepsilon(h_2 + \tilde{h}_2(t,x)b_2^\varepsilon(t,x))\varphi_1^\varepsilon(t,x)\,\mathrm{d}x\,\mathrm{d}t \to \int_Q\int_Y \chi_2(y)(h_2(t,x) + \tilde{h}_2(t,x)b_2(t,x,y))\varphi_2(t,x,y)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y. \tag{37c}$$

Let us summarize the above limits, that is equations (25), (28), (29), (31), (33) and (37):

$$-\int_Q\int_{Y_1}\Big[d_1 b_1 + \widehat{\alpha}(\operatorname{div} u + \operatorname{div}_y \tilde{u}) + \widehat{c}_1 p_1\Big]\varphi_1'\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y - \int_\Omega\int_{Y_1} d_1 b_0 \varphi_1(0)\,\mathrm{d}x\,\mathrm{d}y - \int_Q\int_{Y_2}\Big[d_2 b_2 + \widehat{c}_2 p_2\Big]\varphi_2'\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y - \int_\Omega\int_{Y_2} d_2 b_0\varphi_2(0)\,\mathrm{d}x\,\mathrm{d}y + \int_Q\int_{Y_1} D_1\Big(\nabla b_1 + \nabla_y \tilde{b}_1\Big)\cdot\Big(\nabla\varphi_1 + \nabla_y\tilde{\varphi}_1\Big)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y$$

$$+\int_Q\int_{Y_2} D_2\nabla_y b_2\cdot\nabla_y\varphi_2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y+\int_Q\int_\Gamma \beta(b_1-b_2)(\varphi_1-\varphi_2)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s$$
$$=\int_Q\int_{Y_1} H_1(b_1)\varphi_1\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y+\int_Q\int_{Y_2}(h_2+\tilde{h}_2b_2)\varphi_2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y, \quad (38)$$

which corresponds, as a weak formulation, to the following system of partial differential equations

$$-\operatorname{div}_y\left(D_1(\nabla b_1+\nabla_y\tilde{b}_1)\right)=0 \quad \text{in} \quad Q\times Y_1, \quad (39a)$$
$$\partial_t\left[d_2b_2+\hat{c}_2p_2\right]-\operatorname{div}_y\left(D_2\nabla_y b_2\right)=h_2+\tilde{h}_2b_2 \quad \text{in} \quad Q\times Y_2, \quad (39b)$$
$$\partial_t\left[\int_{Y_1}\left(d_1b_1+\hat{\alpha}(\operatorname{div}u+\operatorname{div}_y\tilde{u})+\hat{c}_1p_1\right)\mathrm{d}y\right]$$
$$-\operatorname{div}\left(\int_{Y_1}\left(D_1(\nabla b_1+\nabla_y\tilde{b}_1)\right)\mathrm{d}y\right) \quad (39c)$$
$$+\int_\Gamma\beta(b_1-b_2)\,\mathrm{d}s=\int_{Y_1}H_1(b_1) \quad \text{in} \quad Q,$$

completed by the boundary and initial conditions

$$D_1(\nabla b_1+\nabla_y\tilde{b}_1)\cdot n=0 \quad \text{in} \quad Q\times\Gamma, \quad (39d)$$
$$\int_{Y_1}D_1(\nabla b_1+\nabla_y\tilde{b}_1)\cdot\nu\,\mathrm{d}y=0 \quad \text{in} \quad S\times\partial\Omega\times Y_1, \quad (39e)$$
$$D_2\nabla_y b_2\cdot n=\beta(b_1-b_2) \quad \text{in} \quad Q\times\Gamma, \quad (39f)$$
$$y\to\tilde{b}_1,b_2 \quad \text{Y-periodic}, \quad (39g)$$
$$b_1(0)=|Y_1|\,b_0 \quad \text{in} \quad \Omega, \quad (39h)$$
$$b_2(0)=b_0 \quad \text{in} \quad \Omega\times Y_2. \quad (39i)$$

We continue by introducing a cell problem and some averaged quantities to arrive at a simplified form of problem (39a)-(39c). In this context, let $\tau_j\in H^1(Y_1)/\mathbb{R}$ be the unique solutions of the following cell problems

$$-\operatorname{div}_y(D_1(\nabla_y\tau_j+e_j))=0, \quad \text{in} \quad Y_1, \quad (40a)$$
$$D_1(\nabla_y\tau_j+e_j)\cdot n=0 \quad \text{on} \quad \Gamma, \quad (40b)$$
$$y\to\tau_j \quad \text{Y-periodic}. \quad (40c)$$

We then can write, up to an additive function $c\colon S\times\Omega\to\mathbb{R}$,

$$\tilde{b}_1(t,x,y)=\sum_{j=1}^3\nabla b_1(t,x)\cdot e_j\pi_j(y)+c(t,x).$$

Furthermore we introduce the homogenized tensors D^h, $\hat{\Lambda}$ and the averaged hydraulic permeability and source density:

$$(D^h)_{ij}=\int_{Y_1}D_1(y)\left(\nabla_y\tau_i+e_i\right)\left(\nabla_y\tau_j+e_j\right)\mathrm{d}y, \quad (41a)$$
$$(\hat{\Lambda}^h)_{jk}=\hat{\alpha}\left(\int_{Y_1}\operatorname{div}_y w_{jk}(y)\,\mathrm{d}y+e_j\cdot e_k\right)\mathrm{d}y, \quad (41b)$$

where w_{jk} is the solution of the cell problem (11a)-(11c), and some averaged quantities

$$\tilde{\beta} = \int_\Gamma \beta(y)\,\mathrm{d}s_y, \quad \tilde{d} = \int_{Y_1} d_1(y)\,\mathrm{d}y, \tag{41c}$$

$$\underline{\tilde{c}} = \int_{Y_1} (\hat{c}_1(y) + \hat{\alpha}_1(y)\,\mathrm{div}_y\,\tilde{w}(y))\,\mathrm{d}y. \tag{41d}$$

In addition, we define a scaled source density operator $\tilde{H}_1 : L^2(Q) \to L^2(Q)$ by $\tilde{H}_1(b_1) := |Y_1|\,H_1(b_1)$ and a microscale source density operator $\tilde{H}_2 : L^2(Q \times Y_2) \to L^2(Q \times Y_2)$ by $\tilde{H}_2(b) := h_2 + \tilde{h}_2 b$ for $b \in L^2(Q \times Y_2)$.[10]

The complete homogenized system of problem (M^ε_{cpe}) then reads as (15a)-(15h) supplemented by

$$\partial_t(\tilde{d}b_1 + \hat{\Lambda}^h : \mathrm{e}(u) + \underline{\tilde{c}}p_1) - \mathrm{div}(D^h\nabla b_1) + \tilde{\beta}b_1 - \int_\Gamma \beta b_2\,\mathrm{d}s_y = \tilde{H}_1(b_1) \quad \text{in } Q, \tag{42a}$$

$$\partial_t(d_2b_2 + \hat{c}_2p_2) - \mathrm{div}_y(D_2\nabla_y b_2) = \tilde{H}_2(b_2) \quad \text{in } Q \times Y_2, \tag{42b}$$

$$D_2\nabla_y b_2 \cdot n = \beta(b_1 - b_2) \quad \text{on } Q \times \Gamma, \tag{42c}$$

$$D^h\nabla b_1 \cdot \nu = 0 \quad \text{on } S \times \partial\Omega, \tag{42d}$$

$$y \to b_2 \quad \text{Y-periodic}, \tag{42e}$$

$$b_1(0) = |Y_1|\,b_0 \quad \text{in } \Omega, \tag{42f}$$

$$b_2(0) = b_0 \quad \text{in } \Omega \times Y_2. \tag{42g}$$

We define an auxiliary function by

$$\gamma(t,\tau,y) = \tilde{c}_2\partial_t\zeta(t-\tau,y), \tag{43}$$

eliminate p_2 and obtain

$$\partial_t(\tilde{d}b_1 + \hat{\Lambda}^h : \mathrm{e}(u) + \underline{\tilde{c}}p_1) - \mathrm{div}(D^h\nabla b_1) + \tilde{\beta}b_1 - \int_\Gamma \beta(s)b_2(s)\,\mathrm{d}s_y = \tilde{H}_1(b_1) \quad \text{in } Q, \tag{44a}$$

$$\partial_t\left(d_2b_2 + \int_0^t \gamma(\tau)p_1(\tau)\mathrm{d}\tau\right) - \mathrm{div}_y\,(D_2\nabla_y b_2) = \tilde{H}_2(b_2) \quad \text{in } Q \times Y_2, \tag{44b}$$

$$D_2\nabla_y b_2 \cdot n = \beta(b_1 - b_2) \quad \text{on } Q \times \Gamma, \tag{44c}$$

$$D^h\nabla b_1 \cdot \nu = 0 \quad \text{on } S \times \partial\Omega, \tag{44d}$$

$$y \to b_2 \quad \text{Y-periodic}, \tag{44e}$$

$$b_1(0) = |Y_1|\,b_0 \quad \text{in } \Omega, \tag{44f}$$

$$b_2(0) = b_0 \quad \text{in } \Omega \times Y_2. \tag{44g}$$

5. **Conclusion.** Using the two-scale convergence technique, we have derived the upscaled system (44) — a distributed microstructure system, cf. [16] — governing the effective dynamics (including a first-order reaction) of a chemical substance within a poro-elastic composite (consisting of a connected poro-elastic matrix and fully embedded micro-inclusions displaying very low diffusivities). The strong convergence of the concentrations b_1^ε has been established by using a compactness

[10] Recall that h_2 was introduced in Chapter 3.

criterion given by Meirmanov and Zimin [11]. Let us also point to the paper [9] in which a similar limit passage is discussed in the case where *both* Ω_1^ε and Ω_2^ε are connected (and the reaction terms in both domains are non-linear).

A very particular feature of system (44) is the memory term in equation (44b), which has been inherited from the memory term in equation (19b) of the macroscopic poro-elasticity System (19).

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