GLOBAL EXISTENCE AND ASYMPTOTIC BEHAVIOR OF MEASURE VALUED SOLUTIONS TO THE KINETIC KURAMOTO–DAIDO MODEL WITH INERTIA

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(Communicated by Corrado Lattanzio)

Abstract. We present the global existence and long-time behavior of measure-valued solutions to the kinetic Kuramoto–Daido model with inertia. For the global existence of measure-valued solutions, we employ a Neunzert's mean-field approach for the Vlasov equation to construct approximate solutions. The approximate solutions are empirical measures generated by the solution to the Kuramoto–Daido model with inertia, and we also provide an *a priori* local-in-time stability estimate for measure-valued solutions in terms of a bounded Lipschitz distance. For the asymptotic frequency synchronization, we adopt two frameworks depending on the relative strength of inertia and show that the diameter of the projected frequency support of the measure-valued solutions exponentially converge to zero.

1. **Introduction.** Synchronization of weakly coupled limit-cycle oscillators appears in many biological systems such as metabolic synchrony in yeast cell suspension, synchronous firing of a cardiac pacemaker, and the flashing of fireflies (see [4, 5, 16, 38] for details). Owing to emerging interest in complex networks in computer and social sciences, the synchronous dynamics of information flow through complex networks has attracted considerable interest among researchers from the nonlinear sciences (e.g., applied mathematics, control theory, and statistical physics). Thus far, several mathematical models have been proposed and used for simulating collective synchronization phenomena in complex networks [1, 26]. Among them, we focus on the Kuramoto-type model, which is a minimal prototype model for analytical treatment. The Kuramoto model (KM) [21, 22] is a system of first-order ordinary differential equations (ODEs) for the phase of weakly coupled

oscillators, and it can be formally derived from the coupled complex Ginzburg–Landau system for Landau–Stuart oscillators. This model is a type of analytically treatable minimal model, exhibiting a phase-transition-like phenomenon from disordered states to ordered states, when the strength of coupling is increased in the thermodynamic limit (or the mean-field limit). After Kuramoto's classical works, several Kuramoto-type models were proposed, including the replacement of sinusoidal phase coupling with a general coupling as in [12, 13] and the addition of phase-shift, inertia, and time-delay effects in the phase coupling and dynamics, respectively [2, 3, 16, 19, 20, 28, 31, 32]. For a detailed discussion on the variants of the KM, we refer the reader to [1, 30].

In this paper, we deal with a Kuramoto-type phase model incorporating a general periodic coupling and inertia effect in the original KM, and we call this model the Kuramoto–Daido model (KDM) with inertia. Consider an ensemble of N active rotors moving on a circle and let $z_i := e^{\sqrt{-1}\theta_i} \in \mathbb{S}$ be the locations of the ith rotors on this circle. We assume that the phase θ_i is governed by the KDM with inertia:

$$m\ddot{\theta}_i + \dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N}\Gamma(\theta_j - \theta_i), \quad t > 0, \quad i = 1, \cdots, N, \tag{1}$$

subject to suitable initial data:

$$(\theta_i, \dot{\theta}_i)(0) = (\theta_{i0}, \omega_{i0}), \tag{2}$$

where m, K, and N are the strength of inertia, coupling, and the system size, respectively, and Ω_i is the natural frequency of the ith oscillator, which is randomly extracted from the frequency distribution function $g = g(\Omega)$. The phase coupling function $\Gamma = \Gamma(\theta)$ satisfies the following properties:

$$\Gamma(\theta + 2\pi) = \Gamma(\theta), \quad |\Gamma(\theta) - \Gamma(\theta_*)| \leq \|\Gamma\|_{Lip}|\theta - \theta_*|,$$
$$\Gamma(0) = \Gamma(\pi) = 0, \quad \Gamma(\theta) > 0, \ \theta \in (0, \pi), \quad \Gamma(-\theta) = -\Gamma(\theta). \tag{3}$$

Note that the system (1) and (3) is translational invariant and that Kuramoto's coupling function $\sin\theta$ satisfies the structural condition (3). In the following, we briefly present previous mathematical results for the KDM with inertia and $\Gamma(\theta) = \sin\theta$.

The system (1) with sinusoidal coupling was first introduced by Ermentrout [16] for modeling slow relaxation in the synchronization process in certain biological systems (e.g., fireflies of the Pteroptyx malaccae). In the absence of inertia ($m = 0$), the system (1) corresponds to the Kuramoto–Daido model [12, 13] for a general coupling. The first-order harmonics of Γ correspond to the Kuramoto model with inertia and the system (1) is applied for modeling superconducting Josephson junction arrays [14, 33, 34, 35, 36, 37] and power networks [15]. Compared to the vast literature on the KM, there are few research papers on the KDM. The system (1) has several distinct dynamic features compared to the KM: For example, it is well known [1] that the KM exhibits a continuous second-order phase transition at the critical coupling strength K_{cr} for unimodal, symmetric, and long-range distribution functions g such as Lorentz and Gaussian distributions. In contrast, the KDM with *large inertia* shows a discontinuous first-order phase transition for the aforementioned distributions, and exhibits hysteresis [31, 32]. It is unknown whether the KDM with small inertia will show similar dynamic behaviors.

However, when the number of oscillators is sufficiently large, the kinetic version of the KDM is often used in the physics literature [2, 3] to study phase-transition

phenomena. Let $f = f(\theta, \omega, \Omega, t)$ be the one-oscillator distribution function in $[0, 2\pi) := \mathbb{R}/(2\pi\mathbb{Z})$ with frequency ω, and natural frequency Ω at time t. Then, the formal thermodynamic limit $(N \to \infty)$ using the Bogoliubov–Born–Green–Kirkwood–Yvon (BBGKY) hierarchy yields a Vlasov-type kinetic equation for f:

$$\partial_t f + \partial_\theta(\omega f) + \partial_\omega\big(\mathcal{A}[f]f\big) = 0, \quad (\theta, \omega, \Omega) \in [0, 2\pi) \times \mathbb{R} \times \mathbb{R}, \quad t > 0,$$

$$\mathcal{A}[f](\theta, \omega, \Omega, t) = \frac{1}{m}\Big[-\omega + \Omega$$

$$+ K \int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Gamma(\theta_* - \theta) f(\theta_*, \omega_*, \Omega_*, t) g(\Omega_*) d\theta_* d\omega_* d\Omega_* \Big].$$

$$(4)$$

As far as we know, the well-posedness issue and qualitative asymptotic behavior of the kinetic model (4) have not been addressed in the literature. In this paper, we study fundamental mathematical questions such as the existence of solutions and their asymptotic behavior via the finite-dimensional result for the KDM with inertia and a rigorous thermodynamic limit. Next, we briefly discuss our main results. First, we present a measure-theoretic formulation of the kinetic model (4) in Section 2.3. Because we are interested in the concentration of phases or frequencies, i.e., formation of a Dirac delta in phase and frequency, it is natural to include such singular measures in our concept of solutions. Therefore, measure-valued solutions to (4) are a natural class of solutions as far as asymptotics are concerned. Using the measure-theoretic formulation, we provide the existence of measure-valued solutions for an initial Radon measure with finite moments up to second order in Section 4. The approximate measure-valued solutions are empirical measures constructed from the corresponding finite-dimensional KDM with inertia and then the local-in-time stability result in Proposition 3 yields the convergence of the approximate measure-valued solutions. Second, for the asymptotic behavior of measure-valued solutions, we first establish the finite-dimensional result for the corresponding KDM with inertia, and then using the rigorous thermodynamic result in Section 4, we lift the finite-dimensional restriction to provide infinite-dimensional results.

The rest of the paper is outlined as follows: In Section 2, we briefly review the basic mathematical structure of the KDM and kinetic KDM with inertia, and present a measure-theoretic formulation of the kinetic KDM with inertia. In Section 3, we present a local-stability estimate of the KDM in terms of a bounded Lipschitz distance. In Section 4, we provide a global well-posedness of measure-valued solutions. In Section 5, we study the asymptotic synchronization property and contraction estimates for the measure-valued solutions. Finally, Section 6 is devoted to a summary of the main results and direction for future work. In Appendix A, we present the detailed proof of Theorem 5.2.

2. **Preliminaries.** In this section, we briefly review the basic properties of the kinetic KDM with inertia and present a measure-theoretic formulation of the kinetic KDM.

2.1. **The KDM with inertia.** Consider an ensemble of many weakly coupled limit-cycle oscillators under the effect of uniform inertia. In this case, the dynamics

of θ_i is governed by the following system of first-order ODEs:

$$\dot{\theta}_i = \omega_i, \quad i = 1, \cdots, N, \quad t > 0,$$

$$\dot{\omega}_i = \frac{1}{m}\left[-\omega_i + \Omega_i + \frac{K}{N}\sum_{j=1}^{N}\Gamma(\theta_j - \theta_i)\right], \tag{5}$$

$$\dot{\Omega}_i = 0.$$

Note that the system (5) is equivalent to the system (1), and here we added Ω_i as a part of the dynamic quantities, although it is invariant under the dynamics (5). We first observe that the system (5) is dissipative. This can be easily seen from the fact that the vector field generated by the first-order system (5),

$$\mathcal{F} = (\mathcal{F}_1, \cdots, \mathcal{F}_N), \quad \mathcal{F}_i := \left(\omega_i, \frac{1}{m}\left[-\omega_i + \Omega_i + \frac{K}{N}\sum_{j=1}^{N}\Gamma(\theta_j - \theta_i)\right], 0\right), \quad i = 1, \cdots, N,$$

has a negative divergence:

$$\mathrm{div}_{(\theta,\omega,\Omega)}\mathcal{F}(\theta,\omega,\Omega) = \sum_{i=1}^{N}\mathrm{div}_{(\theta_i,\omega_i,\Omega_i)}\mathcal{F}_i(\theta,\omega,\Omega) = -\frac{N}{m} < 0.$$

We also note that the equilibria to the system (5) correspond to equilibria to the KDM without inertia and vice versa, i.e.,

$$\omega_i = 0, \quad \Omega_i + \frac{K}{N}\sum_{j=1}^{N}\Gamma(\theta_j - \theta_i) = 0.$$

We now introduce average quantities and fluctuations around the averaged ones:

$$\theta_c := \frac{1}{N}\sum_{i=1}^{N}\theta_i, \quad \omega_c := \frac{1}{N}\sum_{i=1}^{N}\omega_i, \quad \Omega_c := \frac{1}{N}\sum_{i=1}^{N}\Omega_i,$$

$$\hat{\theta}_i := \theta_i - \theta_c, \quad \hat{\omega}_i := \omega_i - \omega_c, \quad \hat{\Omega}_i := \Omega_i - \Omega_c.$$

Note that the system can be completely decoupled as two independent systems for averages and fluctuations:

$$\dot{\theta}_c = \omega_c, \quad \dot{\omega}_c = -\frac{\omega_c}{m} + \frac{\Omega_c}{m}, \quad \dot{\Omega}_c = 0, \quad t > 0,$$

and

$$\dot{\hat{\theta}}_i = \hat{\omega}_i, \quad t > 0,$$

$$\dot{\hat{\omega}}_i = \frac{1}{m}\left[-\hat{\omega}_i + \hat{\Omega}_i + \frac{K}{N}\sum_{j=1}^{N}\Gamma(\hat{\theta}_j - \hat{\theta}_i)\right],$$

$$\dot{\hat{\Omega}}_i = 0.$$

Then it is easy to see that

$$\omega_c(t) = \Omega_c + (\omega_c(0) - \Omega_c)e^{-\frac{t}{m}}, \quad \Omega_c(t) = \Omega_c(0),$$

$$\theta_c(t) = \theta_c(0) + \Omega_c t + m(\omega_c(0) - \Omega_c)(1 - e^{-\frac{t}{m}}).$$

As far as long-time dynamics is concerned, for some well-prepared initial data with $\omega_c(0) = \Omega_c$, the dynamics of the averages is indistinguishable from that of the

KM. In this case, the inertia plays only the role of convergence rate toward the phase-locked states.

2.2. **The kinetic KDM with inertia.** In this part, we present several *a priori* estimates for the kinetic KDM with inertia. Consider the initial boundary value problem for the KDM:

$$\partial_t f + \partial_\theta(\omega f) + \partial_\omega\big(\mathcal{A}[f]f\big) = 0, \quad (\theta, \omega) \in [0, 2\pi) \times \mathbb{R}, \quad t > 0,$$

$$\mathcal{A}[f](\theta, \omega, \Omega, t) = \frac{1}{m}\Big[-\omega + \Omega$$
$$+ K \int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Gamma(\theta_* - \theta)f(\theta_*, \omega_*, \Omega_*, t)g(\Omega_*)d\theta_*d\omega_*d\Omega_* \Big],$$

$$\tag{6}$$

subject to initial and boundary conditions:

$$f(\theta, \omega, \Omega, 0) = f_0(\theta, \omega, \Omega) \quad \text{with} \int_0^{2\pi} \int_{-\infty}^{\infty} f_0(\theta, \omega, \Omega)d\omega d\theta = 1,$$
$$f(0, \omega, \Omega, t) = f(2\pi, \omega, \Omega, t), \quad \lim_{|\omega|\to\infty} |\omega|f(\theta, \omega, \Omega, t) = 0.$$

$$\tag{7}$$

Proposition 1. *Let* $f \in \mathcal{C}^\infty([0, 2\pi) \times \mathbb{R}^2 \times [0, T))$ *be a smooth solution to* (6) *and* (7). *Then we have*

$$\int_0^{2\pi} \int_{-\infty}^{\infty} f(\theta, \omega, \Omega, t)d\omega d\theta = \int_0^{2\pi} \int_{-\infty}^{\infty} f_0(\theta, \omega, \Omega)d\omega d\theta, \quad t > 0.$$

Proof. We integrate (6) with respect to $(\theta, \omega) \in [0, 2\pi] \times \mathbb{R}$ to get

$$\frac{d}{dt} \int_0^{2\pi} \int_{-\infty}^{\infty} f(\theta, \omega, \Omega, t)d\omega d\theta = 0.$$

By the normalization condition given in (7), we have

$$\int_0^{2\pi} \int_{-\infty}^{\infty} f(\theta, \omega, \Omega, t)d\omega d\theta = 1.$$

$$\square$$

For notational simplicity, we also introduce $\langle\langle \cdot \rangle\rangle$ to denote the integral over the phase space $[0, 2\pi) \times \mathbb{R}^2$ with respect to a measure $g(\Omega)d\theta d\omega d\Omega$:

$$\langle\langle h \rangle\rangle(t) := \int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\theta, \omega, \Omega, t)g(\Omega)d\theta d\omega d\Omega, \quad t > 0.$$

Note that since $\theta \in [0, 2\pi)$ is a 2π-periodic variable, $h(\theta, \omega, \Omega)$ is a 2π-periodic function with respect to θ on $[0, 2\pi) \times \mathbb{R} \times \mathbb{R}$.

Proposition 2. (Evolution of the moments) *Let* f *be a smooth solution to* (6) *and* (7). *Then for* $t > 0$, *we have*

(i) $\langle\langle f \rangle\rangle(t) = \langle\langle f_0 \rangle\rangle$.
(ii) $\langle\langle \omega f \rangle\rangle(t) = \langle\langle \Omega f_0 \rangle\rangle + (\langle\langle \omega f_0 \rangle\rangle - \langle\langle \Omega f_0 \rangle\rangle)\, e^{-\frac{t}{m}}$.
(iii) $\langle\langle \theta f \rangle\rangle(t) = \langle\langle \theta f_0 \rangle\rangle + \langle\langle \Omega f_0 \rangle\rangle t + m\left(\langle\langle \omega f_0 \rangle\rangle - \langle\langle \Omega f_0 \rangle\rangle\right)(1 - e^{-\frac{t}{m}})$.
(iv) $\langle\langle \Omega f \rangle\rangle(t) = \langle\langle \Omega f_0 \rangle\rangle$.

Proof. (i) We integrate (6) using the periodicity to find

$$\frac{d}{dt}\int_0^{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\theta,\omega,\Omega,t)g(\Omega)d\Omega d\omega d\theta = 0.$$

(ii) We multiply (6) by $\omega g(\Omega)$ to get

$$\partial_t(\omega f g) + \partial_\theta(\omega^2 f g) + \partial_\omega(\omega\mathcal{A}[f]fg) = \mathcal{A}[f]fg.$$

Note that

$$\int_0^{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathcal{A}[f]fg d\theta d\omega d\Omega$$

$$= \frac{1}{m}\Big[-\int_0^{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\omega f g d\theta d\omega d\Omega + \int_0^{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Omega f g d\theta d\omega d\Omega$$

$$+ \int_0^{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Gamma(\theta_* - \theta)f(\theta_*,\omega_*,\Omega,t)f(\theta,\omega,\Omega,t)g(\Omega_*)g(\Omega)d\theta d\theta_* d\omega d\omega_* d\Omega d\Omega_*$$

$$= -\frac{1}{m}\int_0^{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\omega f g d\theta d\omega d\Omega + \frac{1}{m}\int_0^{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Omega f_0 g d\theta d\omega d\Omega.$$

Hence, we have

$$\frac{d}{dt}\langle\langle\omega f\rangle\rangle = -\frac{1}{m}\langle\langle\omega f\rangle\rangle + \frac{1}{m}\langle\langle\Omega f_0\rangle\rangle.$$

This yields

$$\langle\langle\omega f\rangle\rangle(t) = \langle\langle\Omega f_0\rangle\rangle + \Big(\langle\langle\omega f_0\rangle\rangle - \langle\langle\Omega f_0\rangle\rangle\Big)e^{-\frac{t}{m}}.$$

(iii) We multiply (6) by $\theta g(\Omega)$ to get

$$\partial_t(\theta f g) + \partial_\theta\Big(\theta\omega f g\Big) + \partial_\omega(\theta\mathcal{A}[f]fg) = \omega f g.$$

We integrate this equation to find the desired result.
(iv) We first multiply (6) by $\Omega g(\Omega)$ to find

$$\partial_t(\Omega f g) + \partial_\theta(\Omega\omega f g) + \partial_\omega(\Omega\mathcal{A}[f]fg) = 0.$$

We integrate this equation over $[0,2\pi]\times\mathbb{R}^2$ to obtain

$$\frac{d}{dt}\int_0^{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Omega f(\theta,\omega,\Omega,t)g(\Omega)d\Omega d\omega d\theta = 0.$$

This yields the desired result.   □

2.3. **A measure theoretic formulation.** In this part, we present a measure-theoretic formulation of the kinetic KDM (6) with inertia. When the natural frequencies are distributed, i.e., nonidentical oscillators, the oscillators phases cannot collapse to a single phase asymptotically. In contrast, the frequencies of oscillators can be collapsed to a single frequency asymptotically. Hence, when the asymptotic complete-frequency synchronization occurs, the asymptotic limit of measure-valued solutions will have a Dirac measure concentrated on the average natural frequency as its component even for smooth initial data. Therefore, the suitable space for the asymptotic behavior of the solutions will be a measure space instead of the usual Sobolev space. In this manner, a measure-valued solution emerges as a natural concept for the solution to the kinetic KDM with inertia. For this, we adopt a standard Neunzert's framework from [25, 29]. Let $\mathcal{M}([0,2\pi]\times\mathbb{R}^2)$ be the set of nonnegative

Radon measures on $[0, 2\pi) \times \mathbb{R}^2$, and we use a standard duality relation for a Radon measure $\nu \in \mathcal{M}$ and its test function h:

$$\langle \nu, h \rangle = \int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\theta, \omega, \Omega) \nu(d\theta, d\omega, d\Omega), \quad h \in \mathcal{C}_0([0, 2\pi) \times \mathbb{R}^2).$$

We set $\mathcal{C}_w([0, T); \mathcal{M}([0, 2\pi) \times \mathbb{R}^2))$ a space of all weakly continuous time-dependent measures. Next, we present the definition of a measure-valued solution to (6) as follows.

Definition 2.1. For $T \in [0, \infty)$, let $\mu \in \mathcal{C}_w([0, T); \mathcal{M}([0, 2\pi) \times \mathbb{R}^2))$ be a measure-valued solution to (6) with initial Radon measure $\mu_0 \in \mathcal{M}([0, 2\pi) \times \mathbb{R}^2)$ if and only if μ satisfies the following conditions:

1. μ is weakly continuous; $\langle \mu_t, h \rangle$ is continuous as a function of t, $\forall h \in \mathcal{C}_0([0, 2\pi) \times \mathbb{R}^2)$.
2. μ_t satisfies the integral equation: $\forall h \in \mathcal{C}_0^1([0, 2\pi) \times \mathbb{R}^2 \times [0, T))$

$$\langle \mu_t, h(t) \rangle - \langle \mu_0, h(0) \rangle = \int_0^t \langle \mu_s, \partial_s h + \omega \partial_\theta h + \mathcal{A}(\theta, \omega, \Omega, \mu_s) \partial_\omega h \rangle ds,$$

$$\mathcal{A}(\theta, \omega, \Omega, \mu_s) := \frac{1}{m} \Big[-\omega + \Omega - K \int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Gamma(\theta - \theta_*) \mu_s(d\theta_*, d\omega_*, d\Omega_*) \Big].$$
(8)

Remark 1. 1. For a finite measure with compact support, we can use $g \in \mathcal{C}^1([0, 2\pi) \times \mathbb{R}^2)$ as a test function in (8).

2. Let $m_0(t)$ be the total mass of μ_t, i.e.,

$$m_0(t) := \int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu_t(d\theta, d\omega, d\Omega).$$

Then it follows from (8) that we have conservation of mass:

$$m_0(t) = m_0(0), \quad t \in [0, T).$$

3. **A local-in-time stability estimate.** In this section, we provide an *a priori* local-in-time stability estimate for the measure-valued solution to (6) and (7) using a bounded Lipschitz distance.

3.1. **Estimates on particle trajectories.** In this part, we first study several *a priori* estimates of particle trajectories generated from (6).

For $(\theta, \omega, \Omega, t) \in [0, 2\pi) \times \mathbb{R}^2 \times [0, T)$ and $\mu \in \mathcal{C}_w([0, T); \mathcal{M}([0, 2\pi) \times \mathbb{R}^2))$, we set

$$(\Theta_\mu(s), \Upsilon_\mu(s), \Phi_\mu(s)) := (\Theta_\mu(s; t, \theta, \omega, \Omega), \Upsilon_\mu(s; t, \theta, \omega, \Omega), \Phi_\mu(s; t, \theta, \omega, \Omega)),$$

to be the particle trajectory passing through (θ, ω, Ω) at time t, i.e.,

$$\begin{aligned}
\frac{d}{ds} \Theta_\mu(s) &= \Upsilon_\mu(s), \\
\frac{d}{ds} \Upsilon_\mu(s) &= -\frac{1}{m} \Upsilon_\mu(s) + \frac{\Phi_\mu(s)}{m} \\
&\quad - \frac{K}{m} \int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Gamma(\Theta_\mu(s) - \theta_*) \mu_s(d\theta_*, d\omega_*, d\Omega_*), \\
\frac{d}{ds} \Phi_\mu(s) &= 0,
\end{aligned}$$
(9)

subject to initial data

$$(\Theta_\mu(t), \Upsilon_\mu(t), \Phi_\mu(t)) = (\theta, \omega, \Omega).$$

We now introduce *a priori* conditions $(\mathcal{P})$ for measure-valued solutions to (6) and (7):

- $(\mathcal{P}1)$: For each $t \in [0, T]$, there exist nonnegative locally bounded functions $P(t)$, $Q(t)$, $R(t)$:

$$\text{supp}(\mu_t) \in B_{P(t)}(0) \times B_{Q(t)}(0) \times B_{R(t)}(0),$$

  where $\text{supp}(\mu)$ and $B_r(z)$ denote support of measure μ and ball of radius r around z, respectively, i.e., $\text{supp}(\mu) := \overline{\{\mathcal{B} \in [0, 2\pi) \times \mathbb{R}^2 : \mu(\mathcal{B}) > 0\}}$ and $B_r(z) := \{x : |x - z| < r\}$.
- $(\mathcal{P}2)$: The mass is uniformly bounded:

$$\int_0^{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty \mu_t(d\theta, d\omega, d\Omega) < \infty.$$

We set

$$F(\theta, \mu_s) := K \int_0^{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty \Gamma(\theta - \theta_*) \mu_s(d\theta_*, d\omega_*, d\Omega_*).$$

Lemma 3.1. *Let $\mu \in \mathcal{C}_w([0, T]; \mathcal{M}([0, 2\pi) \times \mathbb{R}^2))$ be a measure-valued function with the property $(\mathcal{P})$. Then, we have the following assertions:*

1. *The nonlinear term $F(\theta, \mu_t)$ satisfies*

$$|F(\theta, \mu_t)| \leq K\|\Gamma\|_{L^\infty} m_0(0), \quad |F(\theta_1, \mu_t) - F(\theta_2, \mu_t)| \leq K\|\Gamma\|_{Lip} m_0(0)|\theta_1 - \theta_2|.$$

2. *For any fixed $(\theta, \omega, \Omega, t) \in [0, 2\pi) \times \mathbb{R}^2 \times [0, T]$, there exists a unique global particle trajectory that is a $\mathcal{C}^1$ function of $s \in [0, T]$ and admits a unique inverse map in the form of*

$$\theta := \Theta_\mu(t; s, \bar\theta, \bar\omega), \quad \omega := \Upsilon_\mu(t; s, \bar\theta, \bar\omega),$$

   where

$$\bar\theta := \Theta_\mu(s; t, \theta, \omega), \quad \bar\omega := \Upsilon_\mu(s; t, \theta, \omega).$$

3. *Let $\widehat{P}(t)$, $\widehat{Q}(t)$, and $\widehat{R}(t)$ be the bounds for supports of the phase, velocity, and natural frequency variables, respectively. Then, we have*

$$\begin{aligned}
\widehat{P}(t) &\leq \widehat{P}(0) + \big(\widehat{Q}(0) + \widehat{R}(0) + K\|\Gamma\|_{L^\infty} m_0(0)\big)t, \\
\widehat{Q}(t) &\leq \widehat{Q}(0) + \widehat{R}(0) + K\|\Gamma\|_{L^\infty} m_0(0), \\
\widehat{R}(t) &= \widehat{R}(0).
\end{aligned}$$

Proof. (1) The first assertion directly follows from the boundedness of $\|\Gamma\|_{L^\infty}$. For the second assertion, we use Lipschitz continuity of Γ to get

$$\begin{aligned}
&|F(\theta_1, \mu_t) - F(\theta_2, \mu_t)| \\
&\leq K \int_0^{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty |\Gamma(\theta_1 - \theta_*) - \Gamma(\theta_2 - \theta_*)| \mu_t(d\theta_*, d\omega_*, d\Omega_*) \\
&\leq \|\Gamma\|_{Lip} K|\theta_1 - \theta_2| \int_0^{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty \mu_t(d\theta_*, d\omega_*, d\Omega_*) \\
&= K\|\Gamma\|_{Lip} m_0(0)|\theta_1 - \theta_2|.
\end{aligned}$$

(2) The proof is obtained from standard ordinary differential equation theory. For more details, we refer the reader to [18].

(3) Recall the characteristic equation (9)

$$\frac{d\omega(s)}{ds} = -\frac{\omega(s)}{m} + \frac{\Omega}{m} - \frac{F[\mu_s]}{m},$$

which leads to

$$\omega(t) = \omega_0 e^{-\frac{t}{m}} + \Omega(1 - e^{-\frac{1}{m}t}) - \frac{K}{m}\int_0^t e^{-\frac{t-s}{m}}F[\mu_s]ds.$$

We then use Lemma 3.1, (1) to see that

$$
\begin{aligned}
|\omega(t)| &\leq |\omega_0| + |\Omega| + \frac{K}{m}\int_0^t e^{-\frac{t-s}{m}}F[\mu_s]ds \\
&\leq |\omega_0| + |\Omega| + \frac{K\|\Gamma\|_{L^\infty}m_0(0)}{m}\int_0^t e^{-\frac{t-s}{m}}ds \\
&= |\omega_0| + |\Omega| + K\|\Gamma\|_{L^\infty}m_0(0)\left(1 - e^{-\frac{t}{m}}\right).
\end{aligned}
$$

Hence, we obtain the upper bound on the size of the velocity support of μ:

$$\widehat{Q}(t) \leq \widehat{Q}(0) + \widehat{R}(0) + K\|\Gamma\|_{L^\infty}m_0(0). \tag{10}$$

The bound on the support of θ follows from (10) and the following inequality:

$$
\begin{aligned}
|\theta(t)| &\leq |\theta(0)| + \int_0^t |\omega(s)|ds \\
&\leq \widehat{P}(0) + \int_0^t |\widehat{Q}(s)|ds \\
&\leq \widehat{P}(0) + \left(\widehat{Q}(0) + \widehat{R}(0) + K\|\Gamma\|_{L^\infty}m_0(0)\right)t.
\end{aligned}
$$

$\square$

Lemma 3.2. *Let $\mu \in \mathcal{C}_w([0,T); \mathcal{M}([0,2\pi) \times \mathbb{R}^2))$ be a measure-valued solution to (6) and (7) with the property $(\mathcal{P})$. Then, for any test function $h \in \mathcal{C}_0^1([0,2\pi) \times \mathbb{R}^2)$, we have*

$$\int_0^{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\theta,\omega,\Omega)\mu_t(d\theta,d\omega,d\Omega)$$
$$= \int_0^{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\Theta_\mu(t;s,\theta,\omega,\Omega),\Upsilon_\mu(t;s,\theta,\omega,\Omega),\Phi_\mu(t;s,\theta,\omega,\Omega))\mu_s(d\theta,d\omega,d\Omega).$$

Proof. Because μ is a measure-valued solution, it follows from the defining relation (8) that for any test function $\tilde{h} \in \mathcal{C}_0^1([0,2\pi) \times \mathbb{R}^2 \times [0,T))$, we have

$$\int_0^{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{h}(\theta,\omega,\Omega,t)\mu_t(d\theta,d\omega,d\Omega)$$
$$- \int_0^{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{h}(\theta_*,\omega_*,\Omega_*,s)\mu_s(d\theta_*,d\omega_*,d\Omega_*)$$
$$= \int_s^t\int_0^{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\partial_\tau\tilde{h} + \bar{\omega}\partial_{\bar\theta}\tilde{h} + \mathcal{A}(\bar\theta,\bar\omega,\bar\Omega,\tau)\partial_{\bar\omega}\tilde{h}\right)\mu_\tau(d\bar\theta,d\bar\omega,d\bar\Omega)d\tau. \tag{11}$$

We now choose a test function $\tilde{h}$ so that the right-hand side of (11) vanishes. For any $h \in \mathcal{C}_0^1([0,2\pi) \times \mathbb{R}^2)$ and fixed t, we set

$$\tilde{h}(\bar\theta,\bar\omega,\bar\Omega,\tau) := h(\Theta_\mu(t;\tau,\bar\theta,\bar\omega,\bar\Omega),\Upsilon_\mu(t;\tau,\bar\theta,\bar\omega,\bar\Omega),\Phi_\mu(t;\tau,\bar\theta,\bar\omega,\bar\Omega)).$$

By Lemma 3.1, (2), we have

$$\tilde{h}(\Theta_\mu(\tau;t,\theta,\omega,\Omega),\Upsilon_\mu(\tau;t,\theta,\omega,\Omega),\Phi_\mu(\tau;t,\theta,\omega,\Omega),\tau) = h(\theta,\omega,\Omega). \qquad (12)$$

Direct differentiation of (12) with respect to τ and Lemma 3.1 imply

$$\tilde{h} \in \mathcal{C}_0^1([0,2\pi) \times \mathbb{R}^2 \times [0,T)), \qquad \partial_\tau \tilde{h} + \bar{\omega}\partial_{\bar{\theta}}\tilde{h} + \mathcal{A}(\bar{\theta},\bar{\omega},\bar{\Omega},\tau)\partial_{\bar{\omega}}\tilde{h} = 0.$$

Hence, the relation (11) implies

$$\int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{h}(\theta,\omega,\Omega,t)\mu_t(d\theta,d\omega,d\Omega)$$
$$= \int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{h}(\theta,\omega,\Omega,s)\mu_s(d\theta,d\omega,d\Omega),$$

or

$$\int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\theta,\omega,\Omega)\mu_t(d\theta,d\omega,d\Omega)$$
$$= \int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\Theta_\mu(t;s,\theta,\omega,\Omega),\Upsilon_\mu(t;s,\theta,\omega,\Omega),\Phi_\mu(t;s,\theta,\omega,\Omega)\mu_s(d\theta,d\omega,d\Omega).$$

$$\square$$

3.2. **Local-in-time stability estimate.** In this part, we provide a local-in-time stability of measure-valued solutions to (6) and (7). We first introduce an admissible set Λ of test functions:

$$\Lambda := \Big\{ h : [0,2\pi) \times \mathbb{R}^2 \to \mathbb{R} \ : \ \|h\|_{L^\infty} \le 1, \quad \|h\|_{Lip} := \sup_{z_1 \ne z_2} \frac{|h(z_1) - h(z_2)|}{|z_1 - z_2|} \le 1 \Big\}.$$

Definition 3.3. Let μ,ν be two Radon measures. Then, the bounded Lipschitz distance $d(\mu,\nu)$ between μ and ν is given by

$$d(\mu,\nu) := \sup_{h \in \Lambda} \left| \int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h\,\mu(d\theta,d\omega,d\Omega) - \int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h\,\nu(d\theta,d\omega,d\Omega) \right|.$$

Remark 2. 1. $(\mathcal{M},d)$ is a complete metric space.

2. For any $h \in \mathcal{C}_0([0,2\pi) \times \mathbb{R}^2)$ with $\|h\|_{L^\infty} \le a$ and $\|h\|_{Lip} \le b$, we have

$$\left| \int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h\mu(d\theta,d\omega,d\Omega) - \int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h\nu(d\theta,d\omega,d\Omega) \right|$$
$$\le \max\{a,b\}d(\mu,\nu).$$

3. The bounded Lipschitz distance d is equivalent to the Wasserstein-1 distance (Kantorovich–Rubinstein distance) W_1:

$$W_1(\mu,\nu) := \inf_{\gamma \in \prod(\mu,\nu)} \int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |x - y|\gamma(dx,dy),$$

where $\prod(\mu,\nu)$ is the set of all product measures on $([0,2\pi) \times \mathbb{R}^2)^2$ such that their marginals are μ and ν.

Lemma 3.4. Let $\mu,\nu \in \mathcal{C}_w([0,T);\mathcal{M}([0,2\pi)\times\mathbb{R}^2))$ be two measure-valued solutions to (6) and (7) with the property $(\mathcal{P})$. Then for any $0 \le s \le T$, we have

$$|\Theta_\mu(s;t,\theta,\omega,\Omega) - \Theta_\nu(s;t,\theta,\omega,\Omega)| + |\Upsilon_\mu(s;t,\theta,\omega,\Omega) - \Upsilon_\nu(s;t,\theta,\omega,\Omega)|$$
$$\le \int_{\min\{s,t\}}^{\max\{s,t\}} \alpha(\tau;s)d(\mu_\tau,\nu_\tau)d\tau,$$

where α is a smooth function depending only on T, K, m_0, and m.

Proof. We set

$$
\begin{aligned}
x(s) &:= \Theta_\mu(s; t, \theta, \omega, \Omega) - \Theta_\nu(s; t, \theta, \omega, \Omega), \\
y(s) &:= \Upsilon_\mu(s; t, \theta, \omega, \Omega) - \Upsilon_\nu(s; t, \theta, \omega, \Omega).
\end{aligned}
$$

It follows from (9) that

$$
\frac{dx(\tau)}{d\tau} = y(\tau), \quad \frac{dy(\tau)}{d\tau} = -\frac{y(\tau)}{m} - \frac{1}{m}\big(F(\mu_\tau) - F(\nu_\tau)\big), \tag{13}
$$
$$
(x(t), y(t)) = (0, 0).
$$

We note that

$$
\begin{aligned}
&|F(\mu_\tau) - F(\nu_\tau)| \\
&\leq K \int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\Gamma(\Theta_\mu(\tau) - \theta_*) - \Gamma(\Theta_\nu(\tau) - \theta_*)| d\mu_\tau(d\theta_*, d\omega_*, d\Omega_*) \\
&+ K \Big| \int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Gamma(\Theta_\nu(\tau) - \theta_*)\big(d\mu_\tau(d\theta_*, d\omega_*, d\Omega_*) - d\nu_\tau(d\theta_*, d\omega_*, d\Omega_*)\big) \Big| \\
&\leq K\big(\|\Gamma\|_{L^\infty} m_0 |x(\tau)| + C_\Gamma d(\mu_\tau, \nu_\tau)\big),
\end{aligned}
$$

where C_Γ is a constant given by

$$
C_\Gamma := \max\{\|\Gamma\|_{L^\infty}, \|\Gamma\|_{Lip}\}.
$$

Therefore, we have from (13)

$$
\begin{aligned}
|x(s)| &\leq \int_t^s |y(\tau)| d\tau, \\
|y(s)| &\leq \Big(\|\Gamma\|_{L^\infty} m_0 \int_t^s e^{-\frac{s-\tau}{m}} |x(\tau)| d\tau + C_\Gamma \int_t^s e^{-\frac{s-\tau}{m}} d(\mu_\tau, \nu_\tau) d\tau\Big) \frac{K}{m} \\
&\leq \Big(\|\Gamma\|_{L^\infty} m_0 \int_t^s |x(\tau)| d\tau + C_\Gamma \int_t^s d(\mu_\tau, \nu_\tau) d\tau\Big) \frac{K}{m}.
\end{aligned}
$$

We add these inequalities to obtain

$$
|x(s)| + |y(s)| \leq \Big(1 + \frac{K\|\Gamma\|_{L^\infty}}{m} m_0\Big) \int_t^s (|x(\tau)| + |y(\tau)|) d\tau + \frac{K C_\Gamma}{m} \int_t^s d(\mu_\tau, \nu_\tau) d\tau.
$$

Then Gronwall lemma yields

$$
|x(s)| + |y(s)| \leq \frac{K C_\Gamma}{m} \int_t^s e^{-\big(1 + \frac{K\|\Gamma\|_{L^\infty}}{m} m_0\big)(\tau - s)} d(\mu_\tau, \nu_\tau) d\tau.
$$

$\square$

Proposition 3. (Local-in-time stability) *Let $\mu, \nu \in \mathcal{C}_w([0, T]; \mathcal{M}([0, 2\pi] \times \mathbb{R}^2))$ be measure-valued solutions to (6) and (7) with the property $(\mathcal{P})$. Then there exists a nonnegative function $C(T) = C(T, d, K, P, Q, m_0)$ satisfying*

$$
d(\mu_t, \nu_t) \leq C(T) d(\mu_0, \nu_0), \quad t \in [0, T).
$$

Proof. Let h be a test function in Λ, then we have from Lemmas 3.2 and 3.4

$$
\left| \int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h\,\mu_t(d\theta_*, d\omega_*, d\Omega) - \int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h\,\nu_t(d\theta, d\omega, d\Omega) \right|
$$

$$
\leq \int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| h(\Theta_\mu^0(t), \Upsilon_\mu^0(t), \Omega) - h(\Theta_\nu^0(t), \Upsilon_\nu^0(t), \Omega) \right| \mu_0
$$

$$
+ \left| \int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\Theta_\nu^0(t), \Upsilon_\nu^0(t), \Omega) \mu_0 - \int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\Theta_\nu^0(t), \Upsilon_\nu^0(t), \Omega) \nu_0 \right|
$$

$$
\leq \int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \Theta_\mu^0(t) - \Theta_\nu^0(t) \right| + \left| \Upsilon_\mu^0(t) - \Upsilon_\nu^0(t) \right| \mu_0 + d(\mu_0, \nu_0)
$$

$$
\leq \int_0^t \alpha(\tau; t) d(\mu_\tau, \nu_\tau) d\tau + d(\mu_0, \nu_0),
$$

where we employed the following notations for simplicity.

$$
\Theta_\mu^0(t) := \Theta_\mu(t, 0, \theta, \omega), \quad \Theta_\nu^0(t) := \Theta_\nu(t, 0, \theta, \omega),
$$

$$
\Upsilon_\mu^0(t) := \Upsilon_\mu(t, 0, \theta, \omega), \quad \Upsilon_\mu^0(t) := \Upsilon_\mu(t, 0, \theta, \omega).
$$

Therefore we have

$$
d(\mu_t, \nu_t) \leq \int_0^t \alpha(\tau; t) d(\mu_\tau, \nu_\tau) d\tau + d(\mu_0, \nu_0).
$$

The Gronwall lemma yields

$$
d(\mu_t, \nu_t) \leq d(\mu_0, \nu_0) e^{\int_0^t \alpha(\tau; t) d\tau}.
$$

$\square$

4. **A global existence of measure-valued solutions.** In this section, we provide a global existence of a measure-valued solution to (6) following the approach in [18, 25, 29].

4.1. **Construction of approximate solutions.** In this part, we present a construction of approximate solutions using the particle method [27].

Suppose that the initial Radon measure μ_0 has compact support in $[0, 2\pi) \times \mathbb{R}^2$, and it is included in a square $\mathcal{R}$, i.e.,

$$
\text{supp}(\mu_0) \in B_{P(0)}(0) \times B_{Q(0)}(0) \times B_{R(0)}(0) \subset \mathcal{R}.
$$

Then for a given positive integer n, we can divide the square $\mathcal{R}$ into n^3 subsquares $\mathcal{R}_i$, i.e.,

$$
\mathcal{R} = \bigcup_{i=1}^{n^3} \mathcal{R}_i.
$$

Let $z_i = (\theta_i, \omega_i, \Omega_i)$ be the center of $\mathcal{R}_i$. Then we construct the initial approximation μ_0^n as

$$
\mu_0^n := \sum_{i=1}^{n^3} c_i \delta(z - z_{i0}), \quad c_i := \int_{\mathcal{R}_i} \mu_0(d\theta, d\omega, d\Omega), \tag{14}
$$

and we define the approximate solution as

$$
\mu_t^n := \sum_{i=1}^{n^3} c_i \delta(z - z_i(t)),
$$

where $z_i(t) = (\theta_i(t), \omega_i(t), \Omega_i)$ is a solution of the Kuramoto–Daido model with inertia:

$$\frac{d\theta_i}{dt} = \omega_i,$$

$$\frac{d\omega_i}{dt} = \frac{1}{m}\Big[-\omega_i + \Omega_i + K\sum_{j=1}^{n^3} c_j \Gamma(\theta_j - \theta_i)\Big], \quad t > 0, \quad i = 1, \cdots, n^3,$$

$$\frac{d\Omega_i}{dt} = 0,$$

subject to initial data

$$(\theta_i, \omega_i, \Omega_i)(0) = (\theta_{i0}, \omega_{i0}, \Omega_{i0}). \tag{15}$$

Lemma 4.1. *Let $\mu \in \mathcal{C}_w([0,T]; \mathcal{M}([0, 2\pi) \times \mathbb{R}^2))$ be a given initial Radon measure on $[0, 2\pi) \times \mathbb{R}^2$ with compact support:*

$$supp(\mu_0) \subset B_{P(0)}(0) \times B_{Q(0)}(0) \times B_{R(0)}(0),$$

and let μ_0^n be the initial approximation given by (14). Then there exists a positive constant C such that

$$d(\mu_0^n, \mu_0) \leq \frac{C}{n}\|\mu_0\|,$$

where $\|\mu_0\| := \langle \mu_0, 1 \rangle$.

Proof. For $h \in \Lambda$, we have

$$\Big| \int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h \mu_0(d\theta, d\omega, d\Omega) - \int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h \mu_0^n(d\theta, d\omega, d\Omega) \Big|$$

$$\leq \sum_{i=1}^{n^3} \Big| \int_{\mathcal{R}_i} h(\theta, \omega, \Omega) \mu_0(d\theta, d\omega, d\Omega) - \int_{\mathcal{R}_i} h(\theta, \omega, \Omega) \mu_0^n(d\theta, d\omega, d\Omega) \Big|$$

$$\leq \sum_{i=1}^{n^3} \Big| \int_{\mathcal{R}_i} h(\theta, \omega, \Omega) \mu_0(d\theta, d\omega, d\Omega) - \int_{\mathcal{R}_i} h(\theta_i, \omega_j, \Omega_\ell) \mu_0(d\theta, d\omega, d\Omega) \Big|$$

$$\leq \sum_{i=1}^{n^3} \int_{\mathcal{R}_i} |h(\theta, \omega, \Omega) - h(\theta_i, \omega_j, \Omega_\ell)| \mu_0(d\theta, d\omega, d\Omega)$$

$$\leq \sum_{i=1}^{n^3} \int_{\mathcal{R}_i} |(\theta, \omega, \Omega) - (\theta_i, \omega_j, \Omega_\ell)| \mu_0(d\theta, d\omega, d\Omega)$$

$$\leq \frac{C}{n}\|\mu_0\|,$$

where C is a constant proportional to the diameter of the rectangle $\mathcal{R}$.   □

Lemma 4.2. *Let μ^n be the approximate measure-valued solution to (6) constructed by the procedure (14)–(15). Then we have*

$$\begin{aligned} P^n(t) &\leq P(0) + \Big(Q(0) + R(0) + K\|\Gamma\|_{L^\infty} m_0(0)\Big)t + \frac{C(1+t)}{n}, \\ Q^n(t) &\leq Q(0) + R(0) + K\|\Gamma\|_{L^\infty} m_0(0) + \frac{C}{n}, \\ R^n(t) &\leq R(0) + \frac{C}{n}. \end{aligned}$$

Proof. It follows from Remark 1 and Lemma 3.1 that we have

$$m_0^n(t) = m_0^n(0),$$

and

$$
\begin{array}{rcl}
P^n(t) & \leq & P^n(0) + \big(Q^n(0) + R^n(0) + K\|\Gamma\|_{L^\infty}m_0^n(0)\big)t, \\
Q^n(t) & \leq & Q^n(0) + R^n(0) + K\|\Gamma\|_{L^\infty}m_0^n(0), \\
R^n(t) & = & R^n(0).
\end{array}
$$

However, from the construction of the initial approximation μ_0^h, it is easy to prove that

$$m_0^n(0) \leq m_0 + \frac{C}{n},$$

and

$$P^n(0) \leq P(0) + \frac{C}{n}, \quad Q^n(0) \leq Q(0) + \frac{C}{n}, \quad R^n(0) \leq R(0) + \frac{C}{n}.$$

We then substitute these estimates into Lemma 3.1, (3) to obtain the desired estimates. □

4.2. **Convergence of approximate solutions.** In this part, we present the convergence of the approximate measure-valued solutions constructed in the previous subsection and establish the well-posedness of the global measure-valued solutions to the kinetic KDM.

Theorem 4.3. *Suppose that $\mu_0 \in \mathcal{M}([0,2\pi)\times\mathbb{R}^2))$ is a Radon measure with compact support satisfying*

$$supp(\mu_0) \in B_{P(0)}(0) \times B_{Q(0)}(0) \times B_{R(0)}(0),$$

and let μ_t^n be the approximate solution constructed by the procedure (14)–(15). Then there exists a unique measure-valued solution $\mu \in \mathcal{C}_w([0,T];\mathcal{M}([0,2\pi)\times\mathbb{R}^2))$ to (4) with initial data μ_0 such that μ_t is the weak-* limit of the approximate solutions, i.e.,

$$d(\mu_t, \mu_t^n) = 0 \quad as \quad n \to \infty.$$

Proof. We divide the estimates into several steps.
• **Step A.** (Select candidates for a measure-valued solution): We apply the local-in-stability results in Proposition 3 to μ^{n_1} and μ^{n_2}:

$$d(\mu_t^{n_1}, \mu_t^{n_2}) \leq Cd(\mu_0^{n_1}, \mu_0^{n_2}) \leq \frac{C}{\min\{n_1, n_2\}}\|\mu_0\|. \qquad (16)$$

This yields that the sequence of approximate solutions $\{\mu_t^n\}$ is a Cauchy sequence in the complete metric space $(\mathcal{M}([0,2\pi]\times\mathbb{R}^2), d(\cdot,\cdot))$. Therefore there exists a limit measure $\mu_t \in \mathcal{M}([0,2\pi]\times\mathbb{R}^2)$. However, since d convergence is equivalent to weak-* convergence, we know that μ_t is the weak-* limit of μ_t^n. The estimate (16) also implies

$$d(\mu_t, \mu_t^n) \leq \frac{C}{n}.$$

• **Step B.** (Use the weak-limit measure μ_t as the measure-valued solution to (4) in the sense of Definition 2.1)
Step B.1 (Check for weak Lipschitz continuity): We check (1) of Definition 2.1. We first observe from Lemma 3.1 that

$$
\begin{array}{rcl}
|(\omega, F[\mu_t^n])| & \leq & |\omega| + |F[\mu_t^n]| \\
& \leq & Q(0) + R(0) + K\|\Gamma\|_{L^\infty}m_0 + K\|\Gamma\|_{L^\infty}m_0 + \frac{C}{n}
\end{array}
$$

$$\leq \quad C(T, Q(0), R(0), K, \|\Gamma\|_{L^\infty}, m_0) < \infty.$$

This yields

$$|\Theta^n(t + \Delta t) - \Theta^n(t)| + |\Upsilon^n(t + \Delta t) - \Upsilon^n(t)| \leq C(T, Q(0), R(0), K, \|\Gamma\|_{L^\infty}, m_0)\Delta t.$$

Therefore, we have for $h \in \mathcal{C}_0^1([0, 2\pi) \times \mathbb{R}^2)$

$$\left|\langle \mu_{t+\Delta t}^n, h \rangle - \langle \mu_t^n, h \rangle\right| = \left|\int h(\Theta^n(t + \Delta t), \Upsilon^n(t + \Delta t)) - h(\Theta^n(t) - \Upsilon^n(t))\mu_t^n\right|$$

$$\leq \|h\|_{\mathcal{C}^1} m_0^n \big(|\Theta^n(t + \Delta t) - \Theta^n(t)| + |\Upsilon^n(t + \Delta t) - \Upsilon(t)|\big)$$

$$\leq C\Delta t.$$

Step B.2 (Check the defining condition (8) of Definition 2.1): We have

$$\langle \mu_t^n, h_t \rangle - \langle \mu_0^n, h_0 \rangle = \int_0^t \Big\langle \mu_s^n, \ \partial_t h + \omega \partial_\theta h - \frac{1}{m}(\omega - \Omega + F[\mu_t^n])\partial_\omega h \Big\rangle ds,$$

Since d convergence is equivalent to weak-* convergence, we have

$$\langle \mu_t^n, h_t \rangle - \langle \mu_0^n, h_0 \rangle \ \to \ \langle \mu_t, h_t \rangle - \langle \mu_0, h_0 \rangle \quad \text{as} \quad n \to \infty.$$

Therefore, we are done if we can show that, for any test function $g \in \mathcal{C}_0^1([0, 2\pi) \times \mathbb{R}^2 \times [0, T))$,

$$\int_0^t \Big\langle \mu_s^n, \partial_s h + \omega \partial_\theta h - \frac{1}{m}(\omega - \Omega + F[\mu_s^n]\partial_\omega h) \Big\rangle ds$$

$$\to \int_0^t \Big\langle \mu_s, \partial_s h + \omega \partial_\theta h - \frac{1}{m}(\omega - \Omega + F[\mu_s])\partial_\omega h \Big\rangle ds,$$

as $n \to \infty$. We will prove the following stronger estimate:

$$\left|\Big\langle \mu_t^n, \partial_s h + \omega \partial_\theta h - \frac{1}{m}(\omega - \Omega + F[\mu_s^n]\partial_\omega h) \Big\rangle\right.$$

$$\left. - \Big\langle \mu_t, \partial_s h + \omega \partial_\theta h - \frac{1}{m}(\omega - \Omega + F[\mu_s])\partial_\omega h \Big\rangle\right|$$

$$\leq \frac{C}{n}.$$

Note that

$$\left|\Big\langle \mu_t^n, \partial_s h + \omega \partial_\theta h - \frac{1}{m}(\omega - \Omega)\partial_\omega h \Big\rangle - \Big\langle \mu_t, \partial_s h + \omega \partial_\theta h - \frac{1}{m}(\omega - \Omega)\partial_\omega h \Big\rangle\right|$$

$$\leq \|\partial_s h + \omega \partial_\theta h - \frac{1}{m}(\omega - \Omega)\partial_\omega h\|_{\mathcal{C}^1} d(\mu_t^n, \mu_t) \leq \frac{C}{n},$$

and hence it is enough to show that

$$\left|\Big\langle \mu_s^n, \frac{K}{m} F[\mu_s^n]\partial_\omega h \Big\rangle - \Big\langle \mu_s, \frac{K}{m} F[\mu_s]\partial_\omega h \Big\rangle\right| \leq \frac{C}{n}.$$

To prove this claim, we first observe that

$$\left|\Big\langle \mu_s^n, \frac{1}{m} F[\mu_s^n]\partial_\omega h \Big\rangle - \Big\langle \mu_s, \frac{1}{m} F[\mu_s]\partial_\omega h \Big\rangle\right|$$

$$= \frac{1}{m}\langle \mu_s^n, \big(F[\mu_s^n] - F[\mu_s]\big)\partial_\omega h \rangle + \frac{1}{m}\langle \mu_s^n - \mu_s, F[\mu_s]\partial_\omega h \rangle$$

$$\equiv \mathcal{I}_1 + \mathcal{I}_2.$$

We now estimate $\mathcal{I}_1$ and $\mathcal{I}_2$ separately.

- (Estimate $\mathcal{I}_1$): We first recall Lemma 4.1 to see that

$$|F[\mu_s^n] - F[\mu_s]| = L\left|\int \Gamma(\theta - \theta_*)\mu_s^n - \int \Gamma(\theta - \theta_*)\mu_s\right| = Cd(\mu_s, \mu_s^n) \leq \frac{C}{n}.$$

This yields

$$\mathcal{I}_1 \leq \frac{C}{n}.$$

- (Estimate $\mathcal{I}_2$): To estimate the term $\mathcal{I}_2$, we use Lemma 3.1 to get

$$\|F[\mu_t]\partial_\omega h\|_{L^\infty} \leq K\|\Gamma\|_{L^\infty}\|\partial_\omega h\|_{L^\infty}m_0,$$
$$|F[\theta, \mu_t]\partial_\omega h(\theta, \omega) - F[\theta_*, \mu_t]\partial_\omega h(\theta_*, \omega_*)|$$
$$\leq |F[\theta, \mu_t] - F[\theta_*, \mu_t]|\|\partial_\omega h\|_{L^\infty} + \|F\|_{L^\infty}|\partial_\omega h(\theta, \omega) - \partial_\omega h(\theta_*, \omega_*)|$$
$$\leq (K\|\Gamma\|_{L^\infty}m_0\|\partial_\omega h\|_{L^\infty} + \|F\|_{L^\infty}\|h\|_{\mathcal{C}^2})|(\theta, \omega) - (\theta_*, \omega_*)|.$$

These two estimates lead to

$$\mathcal{I}_2 \leq (2K\|\Gamma\|_{L^\infty}m_0\|\partial_\omega h\|_{L^\infty} + \|F\|_{L^\infty}\|h\|_{\mathcal{C}^2})d(\mu_s^n, \mu_s) \leq \frac{C}{n}.$$

Step C. (Verify the uniqueness of the measure-valued solution): Let μ and μ' be the two measure-valued solutions in the sense of Definition 2.1 corresponding to the given initial Radon measure μ_0. Then it follows from Proposition 3 that

$$d(\mu_t, \mu_t') \leq C(T)d(\mu_0, \mu_0) = 0, \quad t \in (0, T).$$

Thus we have

$$d(\mu_t, \mu_t') = 0, \quad \text{i.e.,} \quad \mu_t = \mu_t', \qquad t \in (0, T).$$

Therefore, we have the uniqueness of measure-valued solution. □

Remark 3. 1. For the KM, similar results have been studied in [7, 18, 23, 24].
2. Note that the measure-valued solution μ has a bounded first moment for each time slice:

$$\int_0^{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty \mu_t(d\theta, d\omega, d\Omega) < \infty.$$

Moreover, μ has compact support for each time slice:

$$\text{supp}(\mu_t) \in B_{P(t)}(0) \times B_{Q(t)}(0) \times B_{R(t)}(0),$$

where $P(t)$, $Q(t)$, and $R(t)$ satisfy

$$\begin{aligned}
P(t) &\leq P(0) + (Q(0) + R(0) + K||\Gamma||_{L^\infty}m_0(0))t, \\
Q(t) &\leq Q(0) + R(0) + K||\Gamma||_{L^\infty}m_0(0), \\
R(t) &= R(0).
\end{aligned}$$

5. **Large-time behavior of the measure-valued solutions.** In this section, we present an asymptotic complete-frequency estimate for the measure-valued solutions whose existence is guaranteed by Theorem 4.3 in the previous section. For the desired synchronization estimates to the measure-valued solutions, we first establish the corresponding results at the oscillator level, and then using the rigorous mean-field limit, we obtain a synchronization estimate for the measure-valued solution. Without loss of generality, we assume that

$$\int_0^{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty \mu_0(d\theta, d\omega, d\Omega) = 1.$$

As a preliminary step for the complete synchronization, we consider the initial phase configuration consisting of a finite number of Dirac measures. For definiteness, we set

$$\mu_0^n := \sum_{i=1}^{n^3} c_i \delta(z - z_{i0}), \quad z_{i0} = (\theta_{i0}, \omega_{i0}, \Omega_{i0}), \tag{17}$$

where z_{i0} is defined as in Section 4.1. Then the unique measure-valued solution to (4) with the initial datum (17) is given by

$$\mu_t^n := \sum_{i=1}^{n^3} c_i \delta(z - z_i(t)),$$

where $z_i = (\theta_i, \omega_i, \Omega_i)$ is the unique solution of the Kuramoto–Daido model:

$$\begin{cases} \dfrac{d\theta_i}{dt} = \omega_i, \quad i = 1, \cdots, N, \ t > 0, \\ \dfrac{d\omega_i}{dt} = -\dfrac{1}{m}\omega_i + \dfrac{1}{m}\Omega_i + \dfrac{K}{m} \displaystyle\sum_{j=1}^{n^3} c_j \Gamma(\theta_j - \theta_i). \\ \dfrac{d\Omega_i}{dt} = 0, \\ \theta_i(0) = \theta_{i0}, \ \omega_i(0) = \omega_{i0}, \ \Omega_i(0) = \Omega_{i0} \qquad t > 0, \quad i = 1, \cdots, n^3. \end{cases} \tag{18}$$

In the following, we present asymptotic complete-frequency synchronization estimates and the contraction property of the system (18) with distributed natural frequencies. For the nonidentical Kuramoto oscillators, the phase-space support of μ_t does not collapse to a single point. However, we will show that the projected support of μ_t in frequency (ω) space will collapse to a single point as in the identical case.

Remark 4. If we consider the initial Radon measure that is absolutely continuous with respect to the Lebesgue measure $d\theta d\omega d\Omega$, i.e., $\mu_0 \ll d\theta d\omega d\Omega$, then we can choose the following approximation for μ_0 the following

$$\mu_0^n = \frac{1}{n^3} \sum_{i=1}^{n^3} \delta(z - z_{i0}),$$

using the similar argument in [23]. Later in Theorem 5.3 we will use this argument.

For convenience, we recall the following second-order differential inequality:

$$\begin{aligned} a\ddot{y} + b\dot{y} + cy + d &\leq 0, \quad t > 0, \\ y(0) = y_0, \quad \dot{y}(0) &= y_1, \end{aligned} \tag{19}$$

where $a > 0$, b, c, and d are constants.

Lemma 5.1. [10] *Let $y = y(t)$ be a nonnegative $\mathcal{C}^2$ function satisfying the differential inequality* (19). *Then we have following relations:*

(i) *If $b^2 - 4ac > 0$, then we have*
$$y(t) \leq \left(y_0 + \frac{d}{c}\right)e^{-\nu_1 t} + a\frac{e^{-\nu_2 t} - e^{-\nu_1 t}}{\sqrt{b^2 - 4ac}}\left(y_1 + \nu_1 y_0 + \frac{2d}{b - \sqrt{b^2 - 4ac}}\right) - \frac{d}{c},$$
(ii) *If $b^2 - 4ac \leq 0$, then we have*
$$y(t) \leq e^{-\frac{b}{2a}t}\left[y_0 + \frac{4ad}{b^2} + \left(\frac{b}{2a}y_0 + y_1 + \frac{2d}{b}\right)t\right] - \frac{4ad}{b^2},$$

where decay exponents ν_1 and ν_2 are given as

$$\nu_1 := \frac{b + \sqrt{b^2 - 4ac}}{2a}, \quad \nu_2 := \frac{b - \sqrt{b^2 - 4ac}}{2a}.$$

Before we present frameworks, we introduce some notation: For a given Radon measure μ_0, we set $\mathcal{P}(t)$, $\mathcal{Q}(t)$, and $\mathcal{R}(t)$ to be the orthogonal θ, ω, and Ω projections of $\text{supp}(\mu_t)$, respectively:

$$D_\theta(\mu_t) := \text{diam}(\mathcal{P}(t)), \quad D_\omega(\mu_t) := \text{diam}(\mathcal{Q}(t)),$$
$$D_\Omega(\mu_t) := \text{diam}(\mathcal{R}(t)), \quad C_k^\mu(m, \mu_0) := \max\{D_\theta(\mu_0), D_\theta(\mu_0) + kmD_\theta'(\mu_0)\},$$

for $k = 1, 2, \cdots$. Furthermore, we impose an extra assumption on $\Gamma \in \mathcal{C}^1$:

$$(\mathcal{P}3): \quad \Gamma' \text{ is a decreasing function on } (0, \theta^*), \qquad \Gamma_s := \sup_{\theta \in (0, \theta^*)} \Gamma(\theta), \qquad (20)$$

where $\theta^* := \inf\{\theta : \Gamma'(\theta) = 0\}$.

We next present two frameworks depending on small- and large-inertia regimes.

• **Framework A (Small-inertia regime)**: Parameters m, K and initial measure μ_0 satisfy

    1.

$$0 < \frac{D_\Omega(\mu_0)}{K} < \Gamma_s, \qquad mK < \frac{D^\infty}{4\Gamma(D^\infty)},$$

    where $D^\infty \in (0, \theta^*)$ is the root of $\Gamma(x) = \frac{D_\Omega(\mu_0)}{K}$.

    2.

$$0 < C_2^\mu(m, \mu_0) < D^\infty.$$

• **Framework B (Large-inertia regime)**: Parameters m, K and initial measure μ_0 satisfy

    1.

$$0 < 4mD_\Omega(\mu_0) < \theta^*, \qquad mK > \frac{\theta^*}{4\Gamma(\theta^*)}.$$

    2.

$$0 < C_2^\mu(m, \mu_0) < 4mD_\Omega(\mu_0).$$

Under these frameworks, we provide the complete-frequency synchronization to (6).

Theorem 5.2. (Complete-frequency synchronization) *Suppose that either Framework A or Framework B hold, and let $\mu_t \in \mathcal{M}([0, 2\pi) \times \mathbb{R}^2)$ be the measure-valued solution to* (6) *with μ_0. Then $\mathcal{Q}(t) = \mathbb{P}_\omega supp(\mu_t)$ shrinks to a single point at least exponentially fast:*

$$D_\omega(\mu_t) \le C \exp\left[-\left(\frac{1}{2m} - \eta\right)t\right], \quad t \ge 0,$$

where C is a positive constant depending only on m, Γ, K, $D_\Omega(\mu_0)$, $D_\theta(\mu_0)$, and $\dot{D}_\theta(\mu_0)$.

Proof. Although the proof is almost the same as in Theorem 5.1 [10], for the reader's convenience, we briefly sketch the proof below. For the detailed proof, see Appendix A.

Case A (Small-inertia regime). Suppose that Framework A holds, and we set

$$\bar{R}_1 := \frac{\Gamma(D^\infty)}{D^\infty}.$$

We first show that there exists a trapping region for $D_\theta(\mu_t^n)$. For this we use the following second-order differential inequality.

$$m\ddot{D}_\theta + \dot{D}_\theta + K\bar{R}_1 D_\theta - D_\Omega \leq 0, \quad \text{a.e. } t.$$

Then, from this inequality, we obtain

$$D_\theta(\mu_t^n) \leq D^\infty, \quad t \geq 0.$$

Next, we differentiate Equation (18) with respect to time t to get

$$m\ddot{\omega}_i + \dot{\omega}_i = K\sum_{j=1}^{n^3} c_j \Gamma'(\theta_j - \theta_i)(\omega_j - \omega_i).$$

By using the lower bound of Γ', we have

$$m\ddot{D}_\omega + \dot{D}_\omega + K\Gamma'(D^\infty)D_\omega \leq 0, \quad \text{a.e. } t. \tag{21}$$

We now apply Lemma 5.1 to (21) to obtain

$$D_\omega(\mu_t^n) \leq Ce^{-\gamma t},$$

where γ is a positive constant.

Case B (Large-inertia regime). Suppose that Framework B holds. In a manner similar to Case A, we have

$$m\ddot{D}_\theta + \dot{D}_\theta + K\bar{R}_2 D_\theta - D_\Omega \leq 0, \quad \text{a.e. } t,$$

where

$$\bar{R}_2 := \frac{\Gamma(4mD_\Omega)}{4mD_\Omega}.$$

Then we obtain the trapping region of $D_\theta(\mu_t^n)$ such that

$$D_\theta(\mu_t^n) \leq 4mD_\Omega, \quad t \geq 0,$$

and from this we have the complete-frequency synchronization:

$$D_\omega(\mu_t^n) \leq Ce^{-(\frac{1}{2m}-\eta)t},$$

where η is a positive constant. Hence by letting $n \to \infty$, we have the desired results. $\qquad\square$

Remark 5. 1. The synchronization problem for the Kuramoto phase model has been treated in [9, 11, 15, 17].

2. In contrast to [10], we cannot estimate the limit of the phase and frequency of the system (18), since the momentum of the system (18) is not conserved. However, from Theorem 5.2, we know that the supports of μ_t go to one point, as t goes to infinity. This implies that μ_t converges to the Dirac measure in the sense of the weak-* limit.

Finally, we present a contraction property of the kinetic Kuramoto–Daido model with finite inertia. In the absence of inertia, it is shown in [6] that the Kuramoto model has a contraction property in Wasserstein distance. The optimal mass transport approach for the contraction relies on the one-dimensional nature of the phase space. However, in our setting, our dynamic phase space is two-dimensional, i.e., $[0, 2\pi] \times \mathbb{R}$ in (θ, ω). Hence it seems that we cannot use the optimal mass transport technique directly as in [6].

For two measure-valued solutions μ and ν to (6), we introduce a functional $\mathcal{D}(\cdot, \cdot)$:

$$\mathcal{D}(\mu_t, \nu_t) := \text{diam}(\mathbb{P}_\theta \text{supp}(\mu_t - \nu_t)), \quad t \geq 0.$$

Theorem 5.3. *Suppose that initial Radon measures μ_0 and ν_0 satisfy the following:*

1. μ_0 and ν_0 have unit mass:

$$\int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu_0(d\theta, d\omega, d\Omega) = \int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \nu_0(d\theta, d\omega, d\Omega) = 1.$$

2. μ_0 and ν_0 are absolutely continuous.
3. μ_0 and ν_0 satisfy either Framework A or Framework B simultaneously.

Let μ_t and ν_t be the measure-valued solutions to (6) *with initial measures μ_0 and ν_0 such that the sum of the diameter of two trapping regions is less than θ^*. Then we have*

$$\mathcal{D}(\mu_t, \nu_t) \leq Ce^{-\beta t}, \quad t \geq 0,$$

where C, β are positive constants, and θ^* is a positive constant appearing in (20)*.*

Proof. From Remark 4, we can choose approximations for μ_0 and ν_0 as follows.

$$\mu_0^n := \frac{1}{n^3} \sum_{i=1}^{n^3} \delta(z - z_{i0}), \quad \nu_0^n := \frac{1}{n^3} \sum_{i=1}^{n^3} \delta(z - \tilde{z}_{i0}).$$

Suppose either Framework A or Framework B holds, and let θ^n and $\tilde{\theta}^n$ be the solution to the system (18) with $c_i = 1/n^3$. Then, thanks to Theorem 5.2, the two configurations $\theta^n(t)$ and $\tilde{\theta}^n(t)$ satisfy

$$D(\theta^n(t)) + D(\tilde{\theta}^n(t)) < \pi, \qquad t \geq 0.$$

We set

$$\alpha_i^n := \theta_i^n - \tilde{\theta}_i^n, \quad \alpha_M^n := \max_{1 \leq i \leq n^3} \alpha_i^n, \quad \alpha_m^n := \min_{1 \leq i \leq n^3} \alpha_i^n, \quad D(\alpha^n(t)) := \alpha_M^n - \alpha_m^n.$$

Since α_M^n is Lipschitz continuous, it is almost everywhere differentiable in time t. More precisely, from the similar arguments as in [10, Lemma 2.1] we know that collision times of the phases and frequencies are countable and isolated. This means that there exists $0 \leq t_0 < t_1 < \cdots$ such that

α_M^n, α_m^n are $\mathcal{C}^2$-differentiable in the time interval (t_{k-1}, t_k), $k = 1, 2, \cdots$.

For notational simplicity, from now on we suppress the n dependence in θ^n and α^n, i.e.,

$$\theta := \theta^n, \quad \alpha := \alpha^n.$$

By simple calculations and the mean-value theorem, we obtain

$$\begin{aligned}
m\frac{d^2\alpha_i}{dt^2} + \frac{d\alpha_i}{dt} &= \frac{K}{n^3} \sum_{j=1}^{n^3} \left(\Gamma(\theta_j - \theta_i) - \Gamma(\tilde{\theta}_j - \tilde{\theta}_i) \right) \\
&= \frac{K}{n^3} \sum_{j=1}^{n^3} \Gamma'(\theta_{ji})(\alpha_j - \alpha_i), \quad (t_{k-1}, t_k),
\end{aligned} \tag{22}$$

where θ_{ji} is a value between α_j and α_i.

• Step A (Derivation of Gronwall's inequality for $D(\alpha)$): It follows from (22) that

$$m\frac{d^2\alpha_M}{dt^2} + \frac{d\alpha_M}{dt} \leq \frac{K}{n^3}\Gamma'(D^*)\sum_{j=1}^{n^3}(\alpha_j - \alpha_M)$$

$$= -K\Gamma'(D^*)\Big(\alpha_M - \frac{1}{n^3}\sum_{j=1}^{n^3}\alpha_j\Big), \quad (t_{k-1}, t_k),$$

(23)

where D^* is the upper bound for the sum of the diameter of two trapping regions. Since D^* is less than θ^*, we have

$$\Gamma'(\theta_{ji}) \geq \Gamma'(D^*).$$

Similarly, we find

$$m\frac{d^2\alpha_m}{dt^2} + \frac{d\alpha_m}{dt} \geq -K\Gamma'(D^*)\Big(\alpha_m - \frac{1}{n^3}\sum_{j=1}^{n^3}\alpha_j\Big), \quad (t_{k-1}, t_k).$$

(24)

We combine the estimates (23) and (24) to find

$$m\frac{d^2D(\alpha)}{dt^2} + \frac{dD(\alpha)}{dt} + \bar{K}D(\alpha) \leq 0, \qquad \bar{K} := K\Gamma'(D^*).$$

(25)

• Step B (Decay estimates of $D(\alpha)$): We apply Lemma 5.1 to (25) to obtain

$$D(\alpha(t)) \leq \begin{cases} D(\alpha_0)e^{-\mu_1 t} + m\frac{e^{-\mu_2 t}-e^{-\mu_1 t}}{\sqrt{1-4m\bar{K}}}\Big(\dot{D}(\alpha_0) + \mu_1 D(\alpha_0)\Big), & 1 - 4m\bar{K} > 0, \\ e^{-\frac{t}{2m}}\Big[D(\alpha_0) + \Big(\frac{1}{2m}D(\alpha_0) + \dot{D}(\alpha_0)\Big)t\Big], & 1 - 4m\bar{K} \leq 0, \end{cases}$$

where

$$\mu_1 = \frac{1 + \sqrt{1-4m\bar{K}}}{2m}, \quad \mu_2 = \frac{1 - \sqrt{1-4m\bar{K}}}{2m}.$$

Hence, for any $\varepsilon \in \big(0, \frac{1}{2m}\big)$, we have

$$D(\mu_t^n, \nu_t^n) \approx Ce^{-\lambda(\varepsilon)t} \quad \text{for large time } t, \quad \lambda(\varepsilon) := \min\Big\{\mu_2, \frac{1}{2m} - \varepsilon\Big\}.$$

We finally have the desired result by letting $n \to \infty$. This completes the proof. □

6. **Conclusion.** The effect of inertia on the phase transition in the Kuramoto model with inertia has been extensively treated in the physics literature [2, 3, 8, 9, 16, 19, 31, 32]. As a result of large inertia, the convergence speed toward the phase-locked state is slower than that of the Kuramoto model without inertia, and the type of phase transition at the critical coupling strength can be dramatically changed. Moreover, hysteresis can also emerge by varying the coupling strength from zero to some large value or vice versa. Recently, the effect of inertia on the synchronization problem has been studied systematically using the Lyapunov functional approach in [10, 15]. The formal mean-field version of the Kuramoto–Daido phase model with inertia has been employed in the physics literature [2, 3]. However, there have been no systematic mathematical studies of the kinetic Kuramoto–Daido model with inertia involving the well-posedness issue and its asymptotic behavior. In this paper, we treated mathematical issues such as the well-posedness and asymptotic behaviors. More precisely, we provided a global well-posedness of measure-valued solutions to the kinetic Kuramoto–Daido model with finite inertia and their asymptotic behavior. For this, we first established the corresponding result to the

Kuramoto–Daido phase model with inertia at the level of phase, and then we lifted the phase result to the level of the kinetic version via the mean-field limit.

Appendix A. **Detailed proof of Theorem 5.2.** In this Appendix, we give a more detailed proof of Theorem 5.2. Since the proof of Case B is very similar to that of Case A, we only provide the proof of Case A (small-inertia regime).
• (Trapping region of $D_\theta(\mu_t^n)$) : Suppose that Framework A holds. Then $D_\theta(\mu_t^n) < D^\infty$.

Proof. We prove the theorem by contradiction. We set

$$\mathcal{T} := \{t \in [0, \infty) \ : \ D_\theta(\mu_t^n) < D^\infty\}, \quad T_* := \sup \mathcal{T}.$$

Note that since $0 \in \mathcal{T}$ and $D_\theta(\mu_t^n)$ is continuous, the set contains some small interval $[0, \varepsilon)$ for some small positive constant $0 < \varepsilon \ll 1$.

We claim

$$T_* = \infty.$$

Suppose not, i.e., $T_* < \infty$. Since $D_\theta(\mu_t^n)$ is continuous, we should have

$$\lim_{t \to T_*-} D_\theta(\mu_t^n) = D^\infty. \tag{26}$$

We next estimate the maximal and minimal fluctuations separately.
Step A (Maximal phase fluctuation): Since θ_M is Lipschitz continuous, it is almost everywhere differentiable in time t. More precisely, there exist at most countable number of times $0 := t_0 < t_1 < \cdots < t_\infty = T_*$ such that

$$\theta_M \text{ is differentiable in the time interval } (t_{k-1}, t_k), \ k = 1, 2, \cdots.$$

We now use

$$\begin{aligned}
\Gamma(x) &\leq \bar{R}_1 x, \quad x \in [-D_\theta^\infty, 0], \text{ where } \bar{R}_1 := \frac{\Gamma(D^\infty)}{D^\infty}, \\
-D^\infty &\leq -D_\theta(\mu_t^n) \leq \theta_i(t) - \theta_M(t) \leq 0, \quad \text{a.e. } t \in [0, T_*),
\end{aligned}$$

to derive a differential inequality:

$$\begin{aligned}
m\ddot{\theta}_M + \dot{\theta}_M &= \Omega_M + K \sum_{j=1}^{n^3} c_j \Gamma(\theta_j - \theta_M) \\
&\leq \Omega_M + K\bar{R}_1 \sum_{j=1}^{n^3} c_j(\theta_j - \theta_M) \\
&= \Omega_M - K\bar{R}_1 \theta_M + K\bar{R}_1 \sum_{j=1}^{n^3} c_j \theta_j, \quad t \in (t_{k-1}, t_k).
\end{aligned}$$

Step B (Minimal fluctuation): We use the same argument as in Case A to find

$$m\ddot{\theta}_m + \dot{\theta}_m \geq \Omega_m - K\bar{R}_1 \theta_m + K\bar{R}_1 \sum_{j=1}^{n^3} c_j \theta_j, \quad \text{a.e. } t.$$

We now combine Step A and Step B to obtain the following differential inequality:

$$m\ddot{D}_\theta + \dot{D}_\theta + K\bar{R}_1 D_\theta - D_\Omega \leq 0, \quad \text{a.e. } t.$$

Since $1 - 4mK\bar{R}_1 > 0$, we obtain

$$
\begin{aligned}
D_\theta(\mu_t^n) \quad \leq \quad & e^{-\bar{\mu}_1 t} D_\theta(0) \\
+ \quad & m\frac{e^{-\bar{\mu}_2 t} - e^{-\bar{\mu}_1 t}}{\sqrt{1 - 4mK\bar{R}_1}}\Big(\dot{D}_\theta(0) + \bar{\mu}_1 D_\theta(0) - \frac{2D_\Omega}{1 - \sqrt{1 - 4mK\bar{R}_1}}\Big) \\
+ \quad & \frac{D_\Omega}{K\bar{R}_1}\Big(1 - e^{-\bar{\mu}_1 t}\Big).
\end{aligned}
$$

By assumption, we have

$$\frac{D_\Omega}{K\bar{R}_1} = \frac{D^\infty D_\Omega}{K\Gamma(D^\infty)} = D^\infty.$$

This yields

$$
\begin{aligned}
D_\theta(\mu_t^n) \quad \leq \quad & D^\infty + (D_\theta(0) - D^\infty)e^{-\bar{\mu}_1 t} + m\frac{e^{-\bar{\mu}_2 t} - e^{-\bar{\mu}_1 t}}{\sqrt{1 - 4mK\bar{R}_1}} \\
\times \quad & \Big(\dot{D}_\theta(0) + \bar{\mu}_1 D_\theta(0) - \frac{2D_\Omega}{1 - \sqrt{1 - 4mK\bar{R}_1}}\Big), \quad t \in [0, T_*),
\end{aligned}
$$

where we used

$$\dot{D}_\theta(0) + \bar{\mu}_1 D_\theta(0) - \frac{2D_\Omega}{1 - \sqrt{1 - 4mK\bar{R}_1}} \leq 0$$

and

$$m\frac{e^{-\bar{\mu}_2 t} - e^{-\bar{\mu}_1 t}}{\sqrt{1 - 4mK\bar{R}_1}} \geq 0.$$

Hence we have

$$\lim_{t \to T_*-} D_\theta(\mu_t^n) < D^\infty.$$

This is a contradiction to (26). $\qquad\square$

• (Complete frequency synchronization): Suppose that Framework A holds. Then we have

$$D_\omega(\mu_t^n) \leq Ce^{-\gamma t},$$

where C is a positive constant depending only on m, Γ, K, $D_\Omega(\mu_0)$, $D_\theta(\mu_0)$, and $\dot{D}_\theta(\mu_0)$, and γ is given by

$$\gamma := \frac{1 - \sqrt{1 - 4mK\Gamma'(D^\infty)}}{2m}.$$

Proof. Step A (Maximal frequency fluctuation): Since ω_M is Lipschitz continuous, it is almost everywhere differentiable in time t. More precisely, there exist at most countable number of times $0 := t_0 < t_1 < \cdots < t_\infty \leq \infty$ such that

$$\omega_M \text{ is differentiable in the time interval } (t_{k-1}, t_k), \ k = 1, 2, \cdots.$$

For a given time zone (t_{k-1}, t_k), $k = 1, \cdots$, we choose an index i such that

$$\omega_i(t) = \omega_M(t), \quad t \in (t_{k-1}, t_k).$$

We use the above result,

$$|\theta_j(t) - \theta_i(t)| \leq D_\theta(\mu_t^n) \leq D^\infty < \theta^*,$$

to get

$$\Gamma'(\theta_j(t) - \theta_i(t)) = \Gamma'\big(|\theta_j(t) - \theta_i(t)|\big) \geq \Gamma'(D^\infty), \quad t \in (t_{k-1}, t_k).$$

We also obtain the following equation from (18):

$$m\ddot{\omega}_i + \dot{\omega}_i = K \sum_{j=1}^{n^3} c_j \Gamma'(\theta_j - \theta_i)(\omega_j - \omega_i), \quad t \in (t_{k-1}, t_k).$$

This yields

$$
\begin{aligned}
m\ddot{\omega}_M + \dot{\omega}_M &\leq K\Gamma'(D^\infty) \sum_{j=1}^{n^3} c_j(\omega_j - \omega_i) \\
&= -K\Gamma'(D^\infty)\big(\omega_M - \sum_{j=1}^{n^3} c_j\omega_j\big), \quad \text{a.e. } t.
\end{aligned}
$$

Step B (Minimal frequency fluctuation): In this case, we apply the same argument as Step A to find

$$m\ddot{\omega}_m + \dot{\omega}_m \geq -K\Gamma'(D^\infty)\big(\omega_m - \sum_{j=1}^{n^3} c_j\omega_j\big), \quad \text{a.e. } t.$$

We combine Step A and Step B to obtain the following differential inequality:

$$m\ddot{D}_\omega + \dot{D}_\omega + K\Gamma'(D^\infty)D_\omega \leq 0, \quad \text{a.e. } t. \tag{27}$$

By condition ($\mathcal{P}3$) and the assumption on Γ'

$$\frac{\Gamma(D^\infty)}{D^\infty} \geq \Gamma'(D^\infty).$$

The determinant of (27) satisfies

$$0 < 1 - 4mK\frac{\Gamma(D^\infty)}{D^\infty} \leq 1 - 4mK\Gamma'(D^\infty).$$

Hence we obtain the desired result. □

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