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# PLASMONIC WAVES ALLOW PERFECT TRANSMISSION THROUGH SUB-WAVELENGTH METALLIC GRATINGS

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Abstract. We investigate the transmission properties of a metallic layer with narrow slits. Recent measurements and numerical calculations concerning the light transmission through metallic sub-wavelength structures suggest that an unexpectedly high transmission coefficient is possible. We analyze the time harmonic Maxwell's equations in the $H$-parallel case for a fixed incident wavelength. Denoting by $\eta > 0$ the typical size of the complex structure, effective equations describing the limit $\eta \to 0$ are derived. For metallic permittivities with negative real part, plasmonic waves can be excited on the surfaces of the channels. When these waves are in resonance with the height of the layer, the result can be perfect transmission through the layer.

## 1. Introduction.

The interest to construct small scale optical devices for technical applications has initiated much research in the fields of micro- and nano-optics. In structures of sub-wavelength size, the behavior of electromagnetic waves is often counterintuitive and its mathematical understanding requires to develop new analytical tools. One example is the behavior of metamaterials with a negative index [18].

In this contribution, we investigate another instance of the astonishing behavior of light in sub-wavelength structures — the high transmission of light through metallic layers with thin holes. As reported e.g. in [10], a metallic film with submicrometer cavities can display an highly unusual transmittivity. Since the openings are smaller than the wavelength of the incident photon, this high transmission is astonishing and contradicts classical aperture theory.

Many theoretical and numerical investigations of the effect are already available. The analysis given in [19] already establishes the connection of the effect to the excitation of surface plasmon polaritons. The photonic band structure of the

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surface plasmons is evaluated numerically, the contribution contains additionally two-dimensional calculations of typical electric fields in a neighborhood of the gratings. A semi-analytical calculation of transmission coefficients for lamellar grating is performed in [13], while the effect of surface plasmons on the upper and lower boundary of the layer is analyzed in [8]. Based on these investigations, the contribution [14] states that, in contrast to previously given explanations of the effect, the presence of surface plasmons has a negative effect on the transmission efficiency.

Further investigations focus on more specific topics. In [21], the effect of a finite conductivity is addressed. A relation between the high transmission effect and the negative index material obtained with a fishnet structure is made in [16]. An approach using homogenization theory is proposed in [11] where the authors emphasize the connection between the skin-depth of evanescent modes in the metal and the period of the gratings.

The aim of the contribution at hand is to provide, through a mathematical analysis of the scattering problem, a new rigorous approach to transmission properties of heterogeneous media, enlightening the role of plasmonic resonances. We show that high transmission effects can survive in a metallic grating even in an extreme sub-wavelength regime.

Our result is an effective scattering problem in which the metallic layer is replaced by an effective material with frequency dependent permittivity $\varepsilon_{\text{eff}}$ and permeability $\mu_{\text{eff}}$. The formulas for these effective parameters allow to evaluate the transmission coefficient $T = T(k, \theta)$ of the total structure in terms of the incident wave number $k$ and the incidence angle $\theta$. Formally, in the ideal case of a lossless metal with real and negative permittivity $\varepsilon_\eta$, we obtain that perfect transmission $|T| = 1$ occurs for every angle $\theta$ at an appropriate value of $k$. This value of $k$ is related to a resonance of the plasmonic wave with the height $h$ of the structure.

This article proceeds as follows. The problem is described in more detail in Subsection 1.1. In Subsection 1.2 we describe the geometry and the scattering problem in mathematical terms. The main result of this paper are effective equations for the structure, these are presented in Subsection 1.3. In that subsection, we also present the formula for the transmission properties of the effective structure. In Section 2 we derive rigorously the effective equations, using the analysis of the oscillatory behavior of solutions in the limit $\eta \to 0$. Section 3 contains the calculation of the transmission properties of the effective system.

The mathematical tools of this contribution are related to those of [2, 3, 5, 7, 15], where the Maxwell equations in other singular geometries have been investigated. Another application where the negative real part of the permittivity becomes relevant is cloaking by anomalous localized resonance, see [17] and the rigorous results in [6, 12].

1.1. **Problem description.** We assume that the metallic obstacle is invariant in one direction ($x_3$) and that the magnetic field is parallel to that direction *($H = (0, 0, u)$, magnetic transverse polarization)*. Accordingly, we can work with a two dimensional model, solving for $u = u(x_1, x_2)$. We investigate time-harmonic solutions with a fixed wave number $k$ and the corresponding wavelength $\lambda = \frac{2\pi}{k}$.

The obstacle is described by a metallic slab of finite length and finite height in $\mathbb{R}^2$, the slits (vacuum) are repeated periodically with a small period $\eta > 0$, compare Figure 1. The period $\eta$ will be infinitesimal with respect to the wavelength $\lambda$. The relative permittivity of the metal is denoted by $\varepsilon$. Since the permittivity of conductors has large absolute values, we allow it to depend on the small parameter $\eta$ and consider $\varepsilon = \varepsilon_\eta$. We obtain non-trivial effects due to plasmonic resonance for $|\varepsilon_\eta| \sim \eta^{-2}$, the scaling is identical to that of [3, 4]. When $\Sigma_\eta$ denotes the set of points occupied by the metal, we assume that the permittivity $\varepsilon_\eta$ is given by a number $\varepsilon_r \in \mathbb{C}$ as

$$\varepsilon_\eta(x) = \begin{cases} 1 & \text{for } x \notin \Sigma_\eta\,, \\ \dfrac{\varepsilon_r}{\eta^2} & \text{for } x \in \Sigma_\eta\,. \end{cases} \tag{1}$$

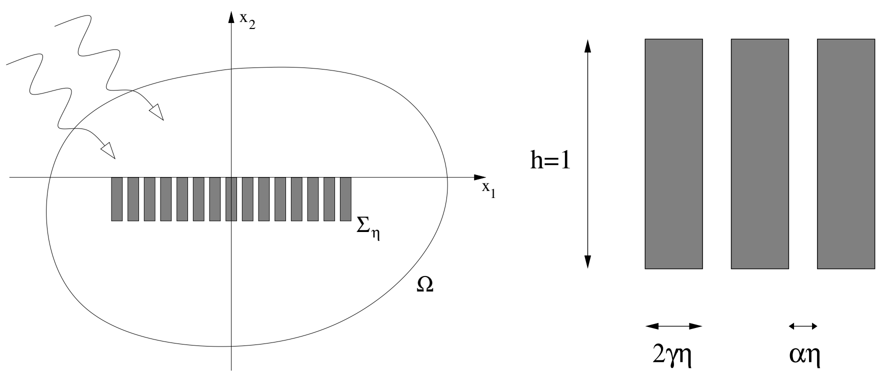


FIGURE 1. *Sketch of the non-dimensionalized scattering problem. Left: A metal layer with gratings is exposed to light at a fixed frequency. Because of the gratings, in two dimensions the metal occupies a number $N$ of disjoint rectangles. We study the case that $N \sim 1/\eta$ is large and that, at the same time, the permittivity $|\varepsilon_\eta| \sim 1/\eta^2$ is large in the metal. Right: Zoom with three of the small rectangles. The single metal component is thin and long (width $2\gamma\eta$ and height $h = 1$). The number $\alpha = 1 - 2\gamma \in (0, 1)$ is the aperture volume of the structure.*

Due to ohmic losses in the metal, the imaginary part of the permittivity is always positive in a physical system. Mathematically, we always assume $\Im\varepsilon_r \geq 0$ and $\varepsilon_r \neq 0$. We call a material with $\Im\varepsilon_r = 0$ a *lossless material.*

Our particular interest is the study of a lossless material with negative relative permittivity ($\Im\varepsilon_r = 0$ and $\Re\varepsilon_r < 0$). In this case, transverse evanescent modes are generated in the metal (due to the sign of $\varepsilon_r$, the Helmholtz equation (6) does not permit wave-like solutions, waves cannot penetrate the metal). These modes can penetrate only in a region that is determined by the skin-depth, in our case of order $\eta$ (solutions of (6) have the qualitative behavior $\bar{u}_\eta \sim \exp(-\bar{k}\sqrt{|\varepsilon_\eta|}\bar{x})$ in the metal, where $\bar{x}$ is the distance to the boundary; at distances $\bar{x} >> \eta$, the solution vanishes). We note that the scaling of the skin-depth coincides with the scaling of the metal thickness. The evanescent mode is related to a surface plasmon solution (in our case a solution that is non-vanishing in the slit, but which has exponential decay in the metal). For an appropriate wave-number $k$, the surface plasmon solution has a (vertical) wave-length that is in resonance with the height $h$ of the metallic layer. If this is the case, perfect transmission for the lossless material can occur.

Our formula for the effective transmission coefficient in (25) quantifies this effect for general permittivities.

**The $H$-parallel case in time-harmonic Maxwell equations.** The time-harmonic Maxwell equations read

$$\begin{aligned} \text{curl } E_\eta &= \quad i\omega\mu_0 H_\eta, && (2)\\ \text{curl } H_\eta &= -i\omega\varepsilon_\eta\varepsilon_0 E_\eta, && (3) \end{aligned}$$

with fixed positive real constants $\omega, \mu_0$ and $\varepsilon_0$ that denote the frequency of the incoming light and the permeability and permittivity of vacuum. The inclusion of a material in a region $\Sigma_\eta$ is described by a relative permittivity $\varepsilon_\eta$ which is different from 1.

We study a situation in which all quantities are $x_3$-independent and with a polarized magnetic field $H_\eta = (0, 0, \bar{u}_\eta)$; the overbar is introduced here to facilitate the non-dimensionalization later on. In this setting, the electric field has no third component, $E_\eta = (E_{x,\eta}, E_{y,\eta}, 0)$. The Maxwell equations then simplify to the two-dimensional system

$$\begin{aligned} \nabla^\perp \cdot (E_{x,\eta}, E_{y,\eta}) &= \quad i\omega\mu_0\bar{u}_\eta, && (4)\\ -\nabla^\perp\bar{u}_\eta &= -i\omega\varepsilon_\eta\varepsilon_0(E_{x,\eta}, E_{y,\eta}), && (5) \end{aligned}$$

where we used the two-dimensional orthogonal gradient, $\nabla^\perp u = (-\partial_2 u, \partial_1 u)$, and the two-dimensional curl, $\nabla^\perp \cdot (E_x, E_y) = -\partial_2 E_x + \partial_1 E_y$. The system can be described equivalently by a scalar Helmholtz equation. We multiply the second equation with the space dependent coefficient $\varepsilon_\eta^{-1} = \varepsilon_\eta^{-1}(x)$ and apply the operator $\nabla^\perp\cdot$ to the result. Since the permittivity is scalar, we can use the identity $\nabla^\perp \cdot (\varepsilon_\eta^{-1}\nabla^\perp\bar{u}_\eta) = \nabla\cdot(\varepsilon_\eta^{-1}\nabla\bar{u}_\eta)$. Setting $\bar{k}^2 = \omega^2\varepsilon_0\mu_0$ we obtain the Helmholtz equation

$$\nabla \cdot \left(\frac{1}{\varepsilon_\eta}\nabla\bar{u}_\eta\right) = -\bar{k}^2\bar{u}_\eta. \tag{6}$$

We will study the Helmholtz equation (6) in dimension-less quantities. We emphasize that the coefficient $a_\eta := \varepsilon_\eta^{-1}$ can have a negative real part and that it vanishes in the metal in the limit $\eta \to 0$.

**Non-dimensionalization.** We consider a physical situation with the following parameters (carrying dimensions): The periodicity of the gratings is $\bar{d}$, the height of the slab is $\bar{h}$, the aperture has the thickness $\bar{a} < \bar{d}$, the single metal piece has the thickness $\bar{d} - \bar{a}$, the wave-length of the incident wave is $\bar{\lambda}$ — all these quantities carry the unit of meters.

We choose the two length scales $\bar{d}$ and $\bar{h}$ to non-dimensionalize the problem and set set

$$\eta = \frac{\bar{d}}{\bar{h}}, \qquad \alpha = \frac{\bar{a}}{\bar{d}}, \qquad \gamma = \frac{1-\alpha}{2}, \qquad \lambda = \frac{\bar{\lambda}}{\bar{h}} \qquad k = \frac{2\pi}{\lambda}.$$

The physical spatial parameter $\bar{x} \in \bar{\Omega}$ is replaced by $x = \bar{x}/\bar{h}$ in the dimension-less domain $\Omega := \bar{\Omega}/\bar{h} \subset \mathbb{R}^2$.

From now on, we will work in this dimension-less setting. The aspect ratio $\eta = \bar{d}/\bar{h}$ of the periodic structure is the principal non-dimensional variable, we study here the limiting case of small $\eta > 0$. In this limit process, the relative aperture volume $\alpha$, the relative metal volume $2\gamma$, the height $h = 1$, and the non-dimensional wave-number $k$ are kept fixed. The relative permittivity $\varepsilon_\eta$ and its inverse $a_\eta$ are have been defined in a dimension-less form in the equations, they remain unchanged.

*Typical physical parameters.* To illustrate typical choices for the various parameters we refer to [8]. Figure 3 (b) of that work was obtained for periodicity length $\bar{d} = 3.5\mu m$, slit-width $\bar{a} = 0.5\mu m$, $\bar{h} = 3.0\mu m$, and wave-length $\bar{\lambda} = 7.5\mu m$. The corresponding quantities in the non-dimensional Helmholtz equation are

$$\eta = 7/6, \quad \alpha = 1/7, \quad \gamma = 3/7, \quad \lambda = 15/6, \quad k = 2\pi/\lambda \approx 2.51. \tag{7}$$

We use here the relative permittivity of silver as in [14], $\varepsilon_\eta = (0.12 + 3.7i)^2$. With the permittivity relation of (1), we choose $\varepsilon_r = \eta^2 \varepsilon_\eta = -\sigma^2$ with $\sigma = \eta(3.7 - 0.12i)$.

1.2. **Mathematical formulation.** Our interest is to study the Maxwell equations in a complex geometry and with high contrast permittivities. With the dimensionless number $\eta$ we indicate the small length scale that is present in the geometry, given by a set $\Sigma_\eta \subset \mathbb{R}^2$. At the same time, $\eta$ is used as an index to indicate large absolute values of the permittivity. We are led to the following problem.

We study the Helmholtz equation

$$\nabla \cdot (a_\eta \nabla u_\eta) = -k^2 u_\eta \tag{8}$$

on a domain $\Omega \subset \mathbb{R}^2$, where the coefficient $a_\eta$ is given as

$$a_\eta := (\varepsilon_\eta)^{-1} = \begin{cases} 1 & \text{in } \Omega \setminus \Sigma_\eta \\ \eta^2\, \varepsilon_r^{-1} & \text{in } \Sigma_\eta \end{cases} \tag{9}$$

The set $\Sigma_\eta \subset \Omega$ describes the complex geometry of the obstacle.

**Description of the complex geometry.** The two-dimensional metallic structure is contained in the closure of the open subset

$$R = (-l, l) \times (-h, 0) \subset \Omega\,.$$

We assume that the compact rectangle $\overline{R}$ contains $2N + 1$ small rectangles of width $2\gamma\eta$ and height $h$. The collection of the small rectangles is the domain $\Sigma_\eta$ that is occupied by metal,

$$\Sigma_\eta := \bigcup_{n=-N}^{N} (n\eta - \gamma\eta, n\eta + \gamma\eta) \times (-h, 0). \tag{10}$$

We always assume $0 < \gamma < 1/2$ such that the single rectangles do not intersect. The number $l = N\eta + \gamma\eta$ is the right end-point of the structure. In the following, we keep the number $l$ fixed. Sending the number $N$ of rectangles to infinity is then equivalent to sending $\eta = l/(N + \gamma)$ to zero. Due to non-dimensionalization, we are only interested in the height $h = 1$. The relative aperture volume is $\alpha = 1 - 2\gamma$. In $x_1$-direction, we denote a corresponding collection of intervals by $\Gamma_\eta := \eta\mathbb{Z} + \eta(-\gamma, \gamma) \subset \mathbb{R}$.

**Scattering problem.** We will analyze the effective behavior of solutions to (8) in two cases. In the first case we investigate an arbitrary bounded sequence of solutions on a bounded domain. In the second setting we investigate the scattering problem. This means that we study the Helmholtz equation (8) on the whole space $\mathbb{R}^2$. For a prescribed incident wave $u^i$, which solves the free space equation $\Delta u^i = -k^2 u^i$ on $\mathbb{R}^2$, we impose as a boundary condition that the scattered field $u^s_\eta = u_\eta - u^i$ satisfies the Sommerfeld condition

$$\partial_r u^s_\eta - ik u^s_\eta = o\left(r^{-1/2}\right) \tag{11}$$

for $r = |x| \to \infty$, uniformly in the angle variable.

1.3. **Main results.** The coefficients of the effective system are determined by a scalar, one-dimensional shape function $\Psi : \mathbb{R} \to \mathbb{C}$. This function has a graph similar to the one sketched in Figure 2, just that on every second interval, the function $\Psi$ is actually constant with value 1.

**The shape function $\Psi$.** The function $\Psi : \mathbb{R} \to \mathbb{C}$ is the continuous 1-periodic function that satisfies

$$\begin{aligned} \partial_z^2 \Psi(z) &= -k^2 \varepsilon_r \Psi(z) && \text{for } z \in (-\gamma, \gamma), \\ \Psi(z) &= 1 && \text{for } z \in [-1/2, 1/2] \setminus (-\gamma, \gamma). \end{aligned}$$

The function $\Psi$ and its average $\beta \in \mathbb{C}$ can be evaluated explicitly: With $\sigma \in \mathbb{C}$ satisfying $\sigma^2 = -\varepsilon_r$ we find

$$\Psi(z) = \begin{cases} \frac{\cosh(k\sigma z)}{\cosh(k\sigma\gamma)} & \text{for } |z| \le \gamma, \\ 1 & \text{for } \gamma < |z| \le 1/2, \end{cases} \qquad \beta := \int_{-1/2}^{1/2} \Psi(z)\, dz = \frac{2}{k\sigma} \frac{\sinh(k\sigma\gamma)}{\cosh(k\sigma\gamma)} + \alpha. \tag{12}$$

To fix a sign convention, we choose to consider always the root with $\Re\sigma \ge 0$.

Of special interest is the lossless case with negative $\varepsilon_r$, i.e. $\Im\varepsilon_r = 0$ and $\varepsilon_r < 0$. In this case, $\sigma$ and, hence, $\beta$ are real and positive numbers, $\Psi$ is a real and positive function.

**The effective coefficients.** With the help of the shape function $\Psi$ we have defined the average $\beta = \beta(k, \gamma, \varepsilon_r) \in \mathbb{C}$, which depends on the wave number $k$, the geometry parameter $\gamma$ (or $\alpha = 1 - 2\gamma$), and the permittivity parameter $\varepsilon_r$ through $\sigma = -i\sqrt{\varepsilon_r}$ (by our sign convention we have $\Im\varepsilon_r \ge 0$, $\Re\sigma \ge 0$, and $\Im\sigma \le 0$). The coefficient $\beta \in \mathbb{C}$ and the geometry parameter $\alpha \in \mathbb{R}$ provide the effective coefficients. We formulate the limit problem with the $x$-dependent effective coefficients $a_{\text{eff}} : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ and $\mu_{\text{eff}} : \mathbb{R}^2 \to \mathbb{C}$,

$$\begin{aligned} a_{\text{eff}}(x) &:= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mu_{\text{eff}}(x) := 1 \qquad \text{for } x \in \mathbb{R}^2 \setminus R, \\ a_{\text{eff}}(x) &:= \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix} \quad \text{and} \quad \mu_{\text{eff}}(x) := \beta \qquad \text{for } x \in R. \end{aligned} \tag{13}$$

It turns out that the effective permittivity tensor (formally $\varepsilon_{\text{eff}} = (a_{\text{eff}})^{-1}$) is infinite in the $x_1$-direction inside the scattering structure, that is

$$\varepsilon_{\text{eff}}(x) = \begin{pmatrix} +\infty & 0 \\ 0 & 1/\alpha \end{pmatrix} \qquad \text{for } x \in R.$$

As could be expected, the coefficient appearing in the $x_2$-direction is large if the aperture ratio $\alpha$ is small.

For a lossless metal of negative permittivity, $\beta$ and $\sigma$ are real and positive. Moreover, the derivative $\partial_\sigma \beta(\sigma)$ is negative and real for positive arguments $\sigma$. This implies that, for a slightly lossy material with small positive imaginary part of the permittivity, the imaginary part of the effective permeability in the homogenized medium satisfies $\Im\mu_{\text{eff}} = \Im\beta > 0$.

**Main results.** Let $\Omega \subset \mathbb{R}^2$ be an open set with $R \subset\subset \Omega$. We consider the geometry of the gratings given by $\Sigma_\eta \subset R \subset \Omega$ of (10). Let the inverse permittivity $a_\eta := \varepsilon_\eta^{-1}$ be given by (9) with $\varepsilon_r \neq 0$. We study solutions $u_\eta \in H^1_{\text{loc}}(\Omega)$ of (8),

$$\nabla \cdot (a_\eta \nabla u_\eta) = -k^2 u_\eta \quad \text{in } \Omega\,. \tag{14}$$

We do not specify the boundary conditions for this system and study an arbitrary sequence of solutions (but recall that the scattering problem of Theorem 1.2 has a unique solution $u_\eta$ for every $\eta$ in the case $\Im\varepsilon_r > 0$). In order to state our results, it is convenient to re-write this equation as a system,

$$\nabla \cdot j_\eta = -k^2 u_\eta\,, \tag{15}$$

$$j_\eta = a_\eta \nabla u_\eta\,. \tag{16}$$

Notice that here $j_\eta$ represents (up to a factor and a rotation) the horizontal electric field $E_\eta$. Recalling that the magnetic field is $H_\eta(x_1, x_2) = u_\eta(x_1, x_2)\, e_3$, we have simply chosen to write the system similar to its original form (4)–(5) of a Maxwell system.

**Theorem 1.1** (Homogenized system). *Let the geometry be given by $\Sigma_\eta$ of (10) on a domain $\Omega \subset \mathbb{R}^2$ and let the coefficient $a_\eta := \varepsilon_\eta^{-1}$ be as in (9). On $\varepsilon_r \in \mathbb{C}$ we assume that either $\Im\varepsilon_r > 0$ or $\varepsilon_r < 0 = \Im\varepsilon_r$, on $\beta$ of (12) we assume $\beta \neq 0$. Let $u_\eta$ be a sequence of solutions to (14) such that $u_\eta \rightharpoonup u$ in $L^2(\Omega)$ for $\eta \to 0$. We define $U \in L^2(\Omega)$ as the function*

$$U(x) := \begin{cases} u(x) & \text{for } x \in \Omega \setminus R\,,\\ \beta^{-1} u(x) & \text{for } x \in R\,. \end{cases} \tag{17}$$

*Then the distributional derivatives of $U$ satisfy $\partial_{x_1} U \in L^2(\Omega \setminus \overline{R})$ and $\partial_{x_2} U \in L^2(\Omega)$. The field $j_\eta = a_\eta \nabla u_\eta$ converges weakly in $L^2(\Omega)$ to the field $j : \Omega \to \mathbb{C}^2$ given by*

$$j \;=\; \begin{cases} (\partial_{x_1} U, \partial_{x_2} U) & \text{in } \Omega \setminus \overline{R}\,,\\ (0,\; \alpha\, \partial_{x_2} U) & \text{in } R\,. \end{cases} \tag{18}$$

*In particular, the limit functions satisfy on $\Omega$ the system*

$$\nabla \cdot j = -k^2\, u\,, \tag{19}$$

$$j = a_{\text{eff}}\, \nabla U\,. \tag{20}$$

Let us emphasize that, in the previous result, all derivatives of $U$ and $j$ are understood in the distributional sense. With this warning in mind, we may re-write the limit system in a condensed form as

$$\nabla \cdot (a_{\text{eff}} \nabla U) = -k^2 \mu_{\text{eff}}\, U \quad \text{in } \Omega\,.$$

By applying Theorem 1.1 with $\Omega$ being a ball of large radius, we are able to treat the scattering problem with an incoming wave generated at infinity. We obtain the strong convergence of the scattered field outside the obstacle and we identify the limit $U(x)$ as the solution of an effective diffraction problem. In the following we denote by $R^{\text{ext}}$ an exterior domain, the unbounded open set $R^{\text{ext}} := \mathbb{R}^2 \setminus \overline{R}$.

**Theorem 1.2** (Effective scattering problem). *Let the geometry of the gratings be given by $\Sigma_\eta$ of (10) and let the coefficient $a_\eta := \varepsilon_\eta^{-1}$ be as in (9) with the properties as in Theorem 1.1. Let $u^i$ be an incident wave, solving the free space equation $\Delta u^i = -k^2 u^i$ on $\mathbb{R}^2$. Let $u_\eta$ be the unique sequence of solutions to (14) such that $u_\eta^s = (u_\eta - u^i)$ satisfies the outgoing wave condition (11). We assume that the effective relative permeability coefficient $\beta$ of (12) satisfies $\Im\,\beta > 0$, and that the solution sequence satisfies, in the diffractive structure, the uniform bound*

$$\int_R |u_\eta|^2 \le C\,. \tag{21}$$

*Then, as $\eta \to 0$, there holds $u_\eta \to U$ strongly in $L^2_{\text{loc}}(R^{\text{ext}})$ with uniform convergence for all derivatives on any compact subset of $R^{\text{ext}}$. Here, the effective field $U : \mathbb{R}^2 \to \mathbb{C}$ is determined as the unique solution of the homogenized equation*

$$\nabla \cdot (a_{\text{eff}} \nabla U) = -k^2 \mu_{\text{eff}}\, U \quad \text{in } \mathbb{R}^2 \tag{22}$$

*with the outgoing wave condition (11) for the scattered field $(U - u^i)$. The effective parameters are given by (13).*

**Remark 1.** (Interface conditions and regularity issues). As pointed out before, the homogenized equation (22) has to be understood in the sense of distributions over the whole space $\mathbb{R}^2$. The exterior magnetic field $U(x)$ belongs to $H^1(B_r(0) \setminus \overline{R})$ for every large radius $r$, hence its trace $U^+$ on $\partial R$ from the outside is well-defined as an element of $H^{1/2}(\partial R)$. In contrast, no regularity holds a priori for $\partial_{x_1} U$ inside $R$. However, as $\partial_{x_2} U$ belongs to $L^2(B_r)$, the function $U(x_1, \cdot)$ is an element of $H^1_{\text{loc}}(\mathbb{R})$ for a.e. $x_1 \in (-l, l)$. This allows to define traces of $U$ on the horizontal boundary parts from the inside. Additionally, we have the information that the distributional divergence of the vector field $j = a_{\text{eff}} \nabla U$ is of class $L^2_{\text{loc}}(\mathbb{R}^2)$.

We decompose the boundary of $R$ into horizontal and vertical parts, $\partial R = \Gamma_{\text{hor}} \cup \bar{\Gamma}_{\text{ver}}$, with

$$\Gamma_{\text{hor}} := (-l, l) \times \{0, -h\}\,, \qquad \Gamma_{\text{ver}} := \{-l, l\} \times (-h, 0)\,.$$

Denoting with the superscript $+$ (respectively $-$) traces from outside (respectively inside), problem (22) can be re-written as:

$$\Delta U + k^2\, U = 0 \quad \text{in } R^{\text{ext}}, \qquad \alpha\, \partial^2_{x_2} U + \beta k^2 U = 0 \quad \text{in } R, \tag{23}$$

with the interface conditions

$$\begin{aligned} U^+ &= U^- && \text{on } \Gamma_{\text{hor}}\,, \\ \partial_{x_2} U^+ &= \alpha\, \partial_{x_2} U^- && \text{on } \Gamma_{\text{hor}}\,, \\ \partial_{x_1} U^+ &= 0 && \text{on } \Gamma_{\text{ver}}\,. \end{aligned} \tag{24}$$

It turns out that, together with the radiation condition (11) at infinity, this system (23)–(24) determines completely the effective scattered field $U$. We observe that the regularity of the trace $U^+$ on $\Gamma_{\text{hor}}$ implies, through the second equation of (23), some regularity of the solution in $R$. However, no continuity of the functions $U(\cdot, x_2)$ for $x_2 \in (-h, 0)$ can be expected at the points $x_1 = \pm l$.

**Remark 2.** (The case $\Re\varepsilon_r < 0$, plasmonic resonance). The above theorems allow to calculate the effective transmission properties of the structure. We perform this analysis in Section 3, where we obtain with (49) a formula for the transmission coefficient $T \in \mathbb{C}$. In terms of wave number $k$, height $h$, aperture ratio $\alpha$, incident angle $\theta$, and with $\tau := \sqrt{\beta/\alpha}$, we derive

$$T = \left( \cos(\tau k h) - \frac{i}{2} \left[ \frac{\alpha\tau}{\cos(\theta)} + \frac{\cos(\theta)}{\alpha\tau} \right] \sin(\tau k h) \right)^{-1}. \tag{25}$$

Perfect transmission $|T| = 1$ occurs for $\cos(\tau k h) = 1$. In particular, perfect transmission is possible when $\beta$ is real and $k$ is a resonant wave number, $kh\sqrt{\beta/\alpha} \in \pi\mathbb{N}$.

Let us again discuss the ideal case with $\Im\varepsilon_r = 0$ and $\Re\varepsilon_r < 0$ (and, hence, $\sigma > 0$ and $\beta > 0$). Relation (12) implies that $\beta$ has a limit for large values of $k$, namely $\alpha$. The same is true for large values of $\sigma$. The fact that $\beta$ depends in such a non-critical way on $k$ implies that also the transmission coefficient $T$ of (25) can be analyzed in more detail. Since $\beta$ (and therefore also $\tau$) stabilizes for large values of $k$, we find that $\cos(\tau kh) = 1$ occurs for an infinite discrete set of wave numbers $k$.

**Remark 3.** (The case $\Re\varepsilon_r > 0$). Let us compare the above discussion with the case of a lossless material with positive permittivity, $\varepsilon_r > 0$. In this case, $\sigma = -i\sqrt{\varepsilon_r} \in i\mathbb{R}$ is purely imaginary. The average $\beta$ of (12) is again real (as in the case $\varepsilon_r < 0$), its formula reduces to

$$\beta = \alpha + \frac{2}{k\sqrt{\varepsilon_r}} \frac{\sin(k\gamma\sqrt{\varepsilon_r})}{\cos(k\gamma\sqrt{\varepsilon_r})} \,.$$

We see that the dependence on the wave number $k$ is more critical than in the case $\varepsilon_r < 0$: negative values of $\beta$ can occur and even $\beta = \pm\infty$ is possible for resonant wave-numbers. In this situation, we do not find a resonance effect with the height $h$, but a resonance with the horizontal structure.

The role of the sign $\Re\varepsilon_r$ can be made even more apparent by expanding the function $\Psi$ in eigenfunctions corresponding to the cell-problem in $y$. In a similar way as in [4], the effective coefficient $\beta$ can be expressed in terms of the discrete set of resonance frequencies of the metallic inclusions (square-roots of eigenvalues) $\omega_n = (n + 1/2)\frac{\pi}{\gamma}$ for $n = 1, 2, \ldots$, which yields

$$\beta = \beta(k) = 1 + \sum_{n=1}^{\infty} \frac{4k^2\varepsilon_r}{(\omega_n^2 - k^2\varepsilon_r)\omega_n^2} \,.$$

In contrast to the "plasmonic" case, the dependence on $k$ becomes highly singular in case of small losses since the positive number $k^2\Re\varepsilon_r$ can be close to one of the numbers $\omega_n$. In the case $\varepsilon_r > 0$, we can expect to observe perfect transmission only for exceptional incident angles $\theta$.

**Remark 4.** (About the $L^2$-bound). Although assumption (21) seems physically reasonable, we have not been able to prove it as we did in [7]. The main difficulty is that we cannot exclude a priori strong variations of $u_\eta$ between successive slits (the geodesic distance between them is of order $h$). Technically, there is no uniformly bounded sequence of extension operators from $H^1(R \setminus \Sigma_\eta)$ to $H^1(R)$. This is in contrast to the situation where the metallic inclusion is disposed as compactly supported subsets of a standard periodicity cell $Y$.

## 2. **Derivation of the effective system.**

We will derive the effective equations with the tool of two-scale convergence as outlined in [1]. Inside the layer $R$, the function $u_\eta$ has oscillations in the horizontal direction ($x_1$-direction) the qualitative behavior is sketched in Figure 2. In the void, $u_\eta$ is approximately constant, $u_\eta \approx U$, in the metal, it has the shape of the function $\Psi$. This picture will be made precise in Lemma 2.2, where we show that the two-scale limit $u_0(x_1, x_2, y_1, y_2)$ of the sequence $u_\eta$ does not depend on the fast variable $y_2 = x_2/\eta$ and coincides with $U(x)\Psi(y_1)$.

### 2.1. **Improved bounds and two-scale limits.**

In this subsection, we consider a sequence of solutions to (14) as in Theorem 1.1, with the weak convergence $u_\eta \rightharpoonup u$ in $L^2(\Omega)$. We start by observing an improved bound for the solution sequence.

**Lemma 2.1** (Gradient estimate)**.** *Let $(u_\eta)_\eta$ be an $L^2(\Omega)$-bounded sequence of solutions to (14). Then, for every compactly contained subdomain $\Omega' \subset\subset \Omega$, there holds*

$$\int_{\Omega'} |a_\eta| \; |\nabla u_\eta|^2 \le C \,. \tag{26}$$

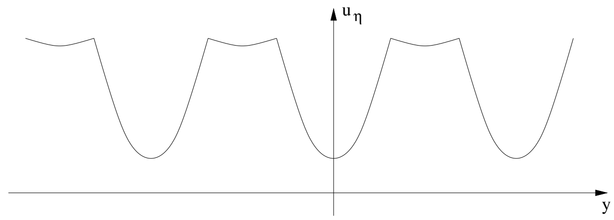


FIGURE 2. *Sketch of the magnetic field in a horizontal cross-section (for positive real $\sigma$). The field is almost constant in the slits, its value is approximately $U(x)$. The profile of $u_\eta$ is given by a* cosh-*function in $y_1 = x_1/\eta$ in the metal part.*

*The constant $C$ depends on $\Omega'$, but is independent of $\eta > 0$.*

*Proof.* We obtain (26) with a calculation as in the Cacciopoli inequality. Without loss of generality we assume $R \subset \Omega'$ and use a cut-off function $\Theta \in C_c^\infty(\Omega, [0, 1])$, which is identical to 1 on $\Omega'$. We multiply equation (14) with $\Theta^2(x)\bar{u}_\eta(x)$, where $\bar{u}_\eta(x)$ is the complex conjugate of the solution. Integrating over $\Omega$, we obtain

$$\int_\Omega a_\eta |\nabla u_\eta|^2 \Theta^2 = \int_\Omega k^2 |u_\eta|^2 \Theta^2 - \int_{\Omega\setminus\Omega'} a_\eta \nabla u_\eta \cdot \bar{u}_\eta \, 2\Theta\nabla\Theta \, .$$

We recall that the first integral on the right hand side is bounded by the $L^2(\Omega)$-boundedness assumption on $u_\eta$. In the second integral we estimate the integrand by $(\sqrt{|a_\eta|}\, |\nabla u_\eta|\, \Theta) \cdot (2\sqrt{|a_\eta|}\, |u_\eta|\, |\nabla\Theta|)$ and apply the Cauchy-Schwarz inequality. Regarding the second factor we observe that $\int_\Omega |a_\eta||u_\eta|^2|\nabla\Theta|^2$ is bounded by the boundedness of $|a_\eta|$. We can take the imaginary and the real part of the left hand side, apply the Young inequality and obtain (26). □

*The two-scale limit of $u_\eta$.* We will analyze the oscillatory behavior of $u_\eta$ with the tool of two-scale convergence. Although the relevant oscillations turn out to be only in the $x_1$-direction, we use in the following the periodicity cell $Y := (-1/2, +1/2)^2$. We note that the geometry is, inside $R$, not only $x_1$-periodic, but also $Y$-periodic. The metal part in the single periodicity cell is given by $\Sigma := (-\gamma, \gamma) \times (-1/2, +1/2) \subset Y$.

We recall that $u \in L^2(\Omega)$ is given as the weak limit of the sequence $u_\eta$. We introduce the following function $u_0(x, y) \equiv u_0(x_1, x_2, y_1, y_2) = u_0(x_1, x_2, y_1)$,

$$u_0(x, y) := \begin{cases} u(x) & \text{for } x \in \Omega \setminus R\,, \\ \beta^{-1} u(x)\, \Psi(y_1) & \text{for } x \in R\,, \end{cases} \tag{27}$$

where $\Psi$ is the continuous 1-periodic function defined in (12). Our definition of $u_0$ ensures that, for every point $x \in \Omega$, there holds $u(x) = \int_Y u_0(x, y)\, dy$. Our aim is to show that $u_\eta$ converges to $u_0$ in the sense of two-scale convergence.

We recall from [1] that a sequence $(f_\eta)_\eta$ in $L^2(\Omega)$ is said to converge weakly in two-scales to $f_0 \in L^2(\Omega \times Y)$ (denoted as $f_\eta \stackrel{2}{\rightharpoonup} f_0$) if there holds

$$\lim_{\eta\to 0} \int_\Omega f_\eta\, \varphi(x, x/\eta)\, dx \;=\; \int_\Omega \int_Y f_0(x, y)\varphi(x, y)\, dx\, dy \;, \tag{28}$$

for every smooth test function $\varphi$ on $\Omega\times Y$ such that $\varphi(x,\cdot)$ is $Y$-periodic. A classical compactness argument provides the existence of such a limit for subsequences, provided the initial sequence $(f_\eta)_\eta$ is bounded in $L^2(\Omega)$. In the following, we will use characteristic functions $\mathbf{1}_M$ for various Borel subsets $M\subset Y$, and use the following localization property:

$$f_\eta(x) \overset{2}{\rightharpoonup} f_0(x,y) \quad \Longrightarrow \quad f_\eta(x)\,\mathbf{1}_M(x/\eta) \overset{2}{\rightharpoonup} f_0(x,y)\,\mathbf{1}_M(y)\,. \tag{29}$$

**Lemma 2.2** (Two-scale limit of $u_\eta$)**.** *Let $u_\eta \rightharpoonup u$ in $L^2(\Omega)$ be a weakly convergent sequence of solutions to* (14)*. Then, with $u_0$ given in (27), there holds $u_\eta \overset{2}{\rightharpoonup} u_0$.*

*Outside $R$, we find the convergence $u_\eta \to u_0 = u$. More precisely, $u_\eta$ together with all derivatives converges uniformly on every compact subset $\Omega' \subset\subset \Omega\setminus\bar R$.*

*Proof. Two-scale convergence.* By our assumption on $u_\eta$ and estimate (26), the sequences $u_\eta$ and $\eta\nabla u_\eta$ are bounded in $L^2(\Omega)$. Therefore, possibly passing to a subsequence, we may assume that, for suitable $u_0$ and $\chi_0:\Omega\times Y\mapsto\mathbb{C}^2$, there holds

$$u_\eta \overset{2}{\rightharpoonup} u_0\,,\quad \eta\nabla u_\eta \overset{2}{\rightharpoonup} \chi_0$$

as $\eta\to 0$. Our goal is to show that $u_0$ agrees with the function defined in (27).

To that aim we use several test functions $\varphi$ of the form $\varphi(x,y)=\Theta(x)\,\psi(y)$ where $\Theta\in C_c^\infty(\Omega;[0,1])$ is a smooth cut-off function. By taking first $\psi:Y\mapsto\mathbb{C}^2$ smooth and periodic in $Y$, we obtain

$$\begin{aligned}\int_\Omega\int_Y \chi_0(x,y)\cdot\psi(y)\,\Theta(x)\,dx\,dy &= \lim_{\eta\to 0}\int_\Omega \eta\,\nabla u_\eta(x)\cdot\psi(x/\eta)\,\Theta(x)\,dx\\ = -\lim_{\eta\to0}\int_\Omega u_\eta(x)\,(\nabla\cdot\psi)(x/\eta)\,\Theta(x)\,dx &= -\int_\Omega\int_Y u_0(x,y)(\nabla\cdot\psi)(y)\,\Theta(x)\,dx\,dy\,,\end{aligned}$$

where in the second line we performed an integration by parts on $\Omega$. Since the localization function $\Theta$ was arbitrary, we find the identity $\nabla_y u_0(x,y)=\chi_0(x,y)$ in the distributional sense in $y\in Y$, for a.e. $x\in\Omega$. In particular, as a $Y$-periodic function, $u_0(x,\cdot)$ belongs to the Sobolev space $H^1_{\mathrm{loc}}(\mathbb{R}^2)$ and has a trace on $\partial\Sigma$. That implies that $u_0(x,\cdot)$ does not have a jump across $\partial\Sigma$.

Next we exploit that, on the set $\Omega\setminus\Sigma_\eta$, the coefficient is $a_\eta=1$. Taking into account the upper bound (26), we conclude that large gradients are excluded in this region. Formally, we argue as follows: $\eta\nabla u_\eta\mathbf{1}_{\Omega\setminus\Sigma_\eta}\to 0$ holds in $L^2(\Omega)$, hence, using the localization property (29), we infer that $\chi_0=\nabla_y u_0$ vanishes a.e. on $R\times(Y\setminus\Sigma)$ and on $(\Omega\setminus R)\times Y$. Therefore, by the periodicity condition, the function $u_0(x,\cdot)$ is constant on the strips $\{\gamma<|y_1|\le 1\}$ for $x\in R$, and it is constant everywhere for $x\notin\bar R$. We use this independence of $y$ to define a function $U\in L^2(\Omega;\mathbb{C})$,

$$u_0(x,y)=U(x)\qquad \text{for } (x,y)\in R\times(Y\setminus\Sigma)\cup(\Omega\setminus R)\times Y\,. \tag{30}$$

We emphasize that, at this stage of the proof, $u_0$ and $U$ are *defined* as the two scale limit of $u_\eta$ and by (30). We will show the characterizations (27) and (17) in the next steps.

*Characterization of the two-scale limit for $x\in R$.* We claim that, for a.e. $x\in R$, the function $w(y)=u_0(x,y)$, as an element $w\in H^1(\Sigma)$, solves the linear boundary value problem

$$\Delta w + k^2\,\varepsilon_r\,w=0\,,\quad w\,(\pm\gamma,\cdot)=U(x)\,,\quad w\left(\cdot,-\frac12\right)=w\left(\cdot,\frac12\right)\,, \tag{31}$$

where the differential equation holds in the distributional sense on $\Sigma = (-\gamma, \gamma) \times (-1/2, 1/2)$. In order to verify this fact, we use $\varphi$ of the form $\varphi(x, y) = \Theta(x)\,\psi(y)$, with $\Theta \in C_c^\infty(R; [0, 1])$ and $\psi \in C^\infty(Y; [0, 1])$ a periodic function on $\Sigma$, more precisely, with $\mathrm{supp}(\psi) \cap (Y \setminus \Sigma) = \emptyset$. Using $\varphi_\eta(x) = \varphi(x, x/\eta)$ as a test-function in equation (14) and inserting the coefficient $a_\eta = \varepsilon_r^{-1}\eta^2$, we obtain for $\eta \to 0$

$$k^2 \int_R \int_Y u_0(x, y)\psi(y)\, dy\, \Theta(x)\, dx \leftarrow k^2 \int_R u_\eta \varphi_\eta = \int_R a_\eta \nabla u_\eta \nabla \varphi_\eta$$
$$\to \varepsilon_r^{-1} \int_R \int_Y \nabla_y u_0(x, y) \cdot \nabla_y \psi(y)\, dy\, \Theta(x)\, dx.$$

Since $\Theta$ was arbitrary, we conclude (31).

It is easy to check that, with the function $\Psi$ of (12), the $y_2$-independent function $U(x)\,\Psi(y_1)$ is a solution of (31). But the solution is also unique: We note that for $U(x) = 0$, a solution $w$ can be trivially extended to a periodic function $w \in H^1(Y)$. Multiplication with $\bar{w}$ and an integration by parts yield

$$\int_\Sigma |\nabla w|^2 = k^2\, \varepsilon_r \int_\Sigma |w|^2 .$$

In both cases, $\Im(\varepsilon_r) > 0$ and $\varepsilon_r < 0$, we can conclude $w = 0$.

Summarizing, we find that the two-scale limit is $u_0(x, y) = U(x)\,\Psi(y_1)$ on $\Omega \times Y$. As a consequence, the weak limit $u$ satifies, for a.e. $x \in R$,

$$u(x) = \int_Y u_0(x, y)\, dy = U(x) \int_Y \Psi(y_1)\, dy = \beta\, U(x)\,,$$

and we obtain (17). The relation $u_0(x, y) = U(x)\,\Psi(y_1)$ then implies also (27).

*Strong convergence outside* $R$. We know already that $u_0(x, \cdot) = U(x) = u(x)$ holds for a.e. $x \in \Omega \setminus R$. Moreover, the strong convergence $u_\eta \to u$ in $L^2(\Omega \setminus R)$ holds, since (26) implies the uniform boundedness in $H^1(\Omega \setminus R)$. In view of the properties of hypoelliptic operators (see e.g. [20]), or using representation formulas (compare Theorem 2.2 in [9]), the uniform convergence on compact subsets of $u_\eta$ and of all its derivatives is a classical consequence of the fact that $u_\eta$ solves the Helmholtz equation $\Delta u_\eta + k^2 u_\eta = 0$ on the open set $\Omega \setminus \overline{R}$. □

2.2. **Limits of $j_\eta$ and proof of Theorem 1.1.** With the above lemma, the two-scale limit of $u_\eta$ is completely determined in $\Omega \times Y$ once we know the function $U(x)$. We are going now to characterize $U(x)$ as the solution of a non-isotropic equation, with no coupling in the $x_1$-direction, compare (13). To that aim, we write the equation as a system for the pair $(u_\eta, j_\eta)$, see (15)–(16). The next lemma collects properties of $j_\eta$, its weak limit $j$ and its two-scale limit $j_0$. It turns out that, in the scatterer $R$, the field $j$ must be pointing in the vertical direction, and that the two-scale limit $j_0(x, \cdot)$ vanishes in $\Sigma$.

**Proposition 2.3** (Limits of $j_\eta$)**.** *Let $u_\eta \rightharpoonup u$ be as in Theorem 1.1, let $U$ be given by* (17). *For $j_\eta = a_\eta \nabla u_\eta$ we assume $j_\eta \rightharpoonup j = (j_1, j_2)$ in $L^2(\Omega; \mathbb{C}^2)$. Then the limits are characterized as follows.*

*(i) The sequence $(j_\eta)_\eta$ converges in two scales to the field $j_0(x, y)$ given by*

$$j_0(x, y) := \begin{cases} j(x) & \text{for } x \in \Omega \setminus R\,, \\ \alpha^{-1}\, j_2(x)\, e_2\, \mathbf{1}_{\{|y_1| > \gamma\}}(y) & \text{for } x \in R\,. \end{cases} \tag{32}$$

*(ii) The distributional derivatives of $U$ satisfy $\partial_{x_1} U \in L^2(\Omega \setminus \overline{R})$ and $\partial_{x_2} U \in L^2(\Omega)$. Furthermore, the following relation holds almost everywhere:*

$$j(x) = \begin{cases} (\partial_{x_1} U(x), \partial_{x_2} U(x)) & \text{for } x \in \Omega \setminus R\,, \\ (0,\, \alpha\, \partial_{x_2} U(x)) & \text{for } x \in R\,. \end{cases} \tag{33}$$

*Proof.* The gradient estimate (26) implies the boundedness of $j_\eta$ in $L^2_{\rm loc}(\Omega)$. Possibly passing to a subsequence we may assume that $j_\eta$ two-scales converges to some field $j_0(x,y)$. The weak limit is then given by $j(x) = \int_Y j_0(x,y)\,dy$.

*The field outside $R$.* On the subset $\Omega \setminus \overline{R}$, the field $j_\eta$ agrees with $\nabla u_\eta$. As observed in Lemma 2.2, there holds the uniform convergence $\nabla u_\eta \to \nabla U$ on compact subsets of $\Omega \setminus \overline{R}$. This implies

$$j_0(x,\cdot) = j(x) = \nabla U(x) \qquad \text{for a.e. } x \in \Omega \setminus R\,. \tag{34}$$

*The field in the metal.* Since $|a_\eta| \le C\,\eta^2$ holds in $\Sigma_\eta$, the gradient estimate (26) implies $\int_{\Sigma_\eta} |j_\eta|^2 = \int_{\Sigma_\eta} |a_\eta|^2 |\nabla u_\eta|^2 \le C\,\eta^2$. It follows that $j_0$ vanishes in $R \times \Sigma$.

*Divergence of $j_0$.* The divergence of the fields $j_\eta$ is controlled by relation (15). Indeed, for an arbitrary smooth periodic test function $\psi : Y \to \mathbb{R}$ and arbitrary $\Theta \in C_c^\infty(\Omega)$, we find with an integration by parts

$$0 = \lim_{\eta\to 0} \int_\Omega \eta \nabla \cdot j_\eta\, \psi(x/\eta)\, \Theta(x)\, dx = -\int_\Omega \int_Y \nabla_y \psi(y) \cdot j_0(x,y)\, \Theta(x)\, dx\,.$$

Since $\Theta$ is arbitrary, we conclude that for a.e. $x \in \Omega$ there holds $\int_Y \nabla_y \psi(y) \cdot j_0(x,y)\,dy = 0$. This expresses that, in the sense of distributions, $j_0(x,\cdot)$ is divergence free, $\nabla_y \cdot j_0 = 0$.

*First component of $j$.* Let us now choose a particular periodic test function. We set $\psi(y) = \vartheta(y_1)$ where $\vartheta : \mathbb{R} \to \mathbb{R}$ is of period 1 and such that

$$\vartheta(y_1) = \begin{cases} -\alpha/2 - (y_1 + \gamma) & \text{for } -1/2 < y_1 \le -\gamma\,, \\ (\alpha/(2\gamma))\, y_1 & \text{for } -\gamma < y_1 \le \gamma\,, \\ \alpha/2 - (y_1 - \gamma) & \text{for } \gamma < y_1 \le 1/2\,. \end{cases}$$

This piecewise affine function is continuous with $\vartheta(\pm\gamma) = \pm\alpha/2$ and $\vartheta(1/2) = \alpha/2 - (1/2 - \gamma) = 0$. Since $j_0 = 0$ holds in $\Sigma$ and since $\vartheta' = -1$ holds for $|y_1| > \gamma$, we obtain

$$0 = \int_Y \nabla_y \psi(y) \cdot j_0(x,y)\, dy = -\int_Y e_1 \cdot j_0(x,y)\, dy = -j_1(x)\,.$$

This shows that the first component of $j$ vanishes for a.e. $x \in R$.

*A relation between $j_0$ and $U$.* In this step we have to exploit the two-scale convergence $j_\eta \overset{2}{\rightharpoonup} j_0$ with test functions in the class

$$\mathcal{A} := \left\{ \psi \in L^2_{\rm loc}(\mathbb{R}^2; \mathbb{C}^2) \mid \nabla \cdot \psi = 0,\ \psi \text{ is } Y\text{-periodic},\ \psi = 0 \text{ in } \Sigma \right\}.$$

We claim that, for $\psi \in \mathcal{A}$ and arbitrary $\Theta \in C_c^\infty(\Omega)$, there holds

$$\int_\Omega \int_Y j_0(x,y) \cdot \psi(y)\, dy\, \Theta(x)\, dx = -\int_\Omega U(x) \int_Y \psi(y)\, dy \cdot \nabla\Theta(x)\, dx\,. \tag{35}$$

We set once more $\varphi(x,y) = \Theta(x)\,\psi(y)$, with $\Theta \in C_c^\infty(R; [0,1])$, and use $\varphi_\eta(x) = \varphi(x, x/\eta)$ as a test-function. We use that $j_\eta = \nabla u_\eta$ holds in $\Omega \setminus \Sigma_\eta$, integrate by parts, and exploit $\nabla \cdot (\psi(x/\eta)\Theta(x)) = \psi(x/\eta) \cdot \nabla\Theta(x)$ to obtain

$$\begin{aligned}\int_\Omega \int_Y j_0(x,y) \cdot \psi(y)\, dy\, \Theta(x)\, dx &= \lim_{\eta\to 0} \int_\Omega j_\eta(x) \cdot \psi(x/\eta)\, \Theta(x)\, dx \\ &= \lim_{\eta\to 0} \int_\Omega \nabla u_\eta(x) \cdot \psi(x/\eta)\, \Theta(x)\, dx = -\lim_{\eta\to 0} \int_\Omega u_\eta(x)\, \psi(x/\eta) \cdot \nabla\Theta(x)\, dx \\ &= -\int_\Omega \int_Y u_0(x,y)\, \psi(y)\, dy \cdot \nabla\Theta(x)\, dx\,.\end{aligned}$$

This implies (35). Regarding the integral on the right hand side we note that $\psi = 0$ holds on $\Sigma$, such that, by (27), $u_0(x,y) = U(x)$ holds where $\psi$ is not vanishing.

*Remaining claims of (i).* We *define* a function $\tilde{j}_0$ by the right hand side of (32), i.e. by setting $\tilde{j}_0(x,\cdot) := \alpha^{-1} j_2(x)\, e_2\, \mathbf{1}_{\{|y_1|>\gamma\}}$. Item (i) is shown once we derive that, for a.e. $x \in R$, the two-scale limit $j_0(x,\cdot)$ coincides with $\tilde{j}_0(x,\cdot)$.

To show this fact, we consider the difference $\psi_0(\cdot) = \tilde{j}_0(x,\cdot) - j_0(x,\cdot)$. Since both functions $j_0$ and $\tilde{j}_0$ are divergence-free, we find $\nabla \cdot \psi_0 = 0$. Furthermore, since the $Y$-integral of both functions is $j_2(x) e_2$, we have $\int_Y \psi_0 = 0$. Additionally, $\psi_0 = 0$ holds in $\Sigma$.

These properties imply that $\psi_0$ and the complex conjugate $\bar{\psi}_0$ belong to the class $\mathcal{A}$. We can therefore use (35) with $\psi = \bar{\psi}_0$. The integral on the right hand side vanishes, since $\psi_0$ has a vanishing integral, and we obtain $\int_Y j_0(x,\cdot) \cdot \overline{\psi_0} = 0$. On the other hand, by the explicit formula for $\tilde{j}_0$, we can calculate $\int_Y \tilde{j}_0(x,\cdot)\overline{\psi_0} = \int_{Y\setminus\Sigma} \left\{\alpha^{-2}|j_2(x)|^2 - \alpha^{-1} j_2(x) e_2 \cdot \bar{j}_0(x,\cdot)\right\} = \alpha^{-1}|j_2(x)|^2 - \alpha^{-1}|j_2(x)|^2 = 0$. Taking the difference of the two equations, we find $\int_Y |\psi_0|^2 = 0$, hence $\psi_0 = 0$ a.e. in $Y$, and relation (32) is proved. The uniqueness of the limit implies the two-scale-convergence of the whole sequence $(j_\eta)_\eta$, which concludes the proof of assertion (i) of the Proposition.

*Remaining claims of (ii).* Outside $\bar{R}$, we have verified the claim already with (34). It remains to show that $\partial_{x_2} U$ belongs to $L^2(\Omega)$ and that $j_2 = \alpha\, \partial_{x_2} U$ holds in $R$. We exploit once more the relation (35), choosing now $\psi(y) := e_2\, \mathbf{1}_{\{|y_1|>\gamma\}}(y)$ as a test function; indeed $\psi \in \mathcal{A}$ is satisfied. By (32) and since $\int_Y \psi = \alpha\, e_2$, we infer that

$$\begin{aligned}-\alpha \int_\Omega U(x)\, e_2 \cdot \nabla\Theta(x)\, dx &= \int_\Omega \int_Y j_0(x,y) \cdot e_2 \mathbf{1}_{\{|y_1|>\gamma\}}(y)\, \Theta(x)\, dy\, dx \\ &= \int_{\Omega\setminus R} \alpha\, j_2(x)\, \Theta(x)\, dx + \int_R j_2(x)\, \Theta(x)\, dx\,,\end{aligned}$$

for every smooth $\Theta \in C_c^\infty(\Omega)$. It follows that the distribution $\partial_{x_2} U$ can be identified as an element of $L^2(\Omega)$. Furthermore, we find the characterization $j_2 = a(x)\, \partial_{x_2} U$, where $a(x) = 1$ for $x \in \Omega \setminus R$ and $a(x) = \alpha$ for $x \in R$. Taking into account (34) and that the first component of $j(x)$ vanishes, we have proved relation (33). □

*Conclusion in the proof of Theorem 1.1.* In the situation of Theorem 1.1, Lemmas 2.1 and 2.2 can be applied. From the former, and the fact that $|a_\eta|$ is uniformly bounded, we infer that the sequence $j_\eta = a_\eta \nabla u_\eta$ is bounded in $L^2(\Omega')$ for any open subset $\Omega'$ with $\bar{R} \subset \Omega' \subset\subset \Omega$. Passing to a subsequence if necessary, we obtain $j_\eta \rightharpoonup j$ weakly in $L^2(\Omega')$ for some limit $j$, such that also Proposition 2.3 can be applied (on the smaller domain $\Omega'$).

Proposition 2.3 provides with (33) the relation (18) between $U$ and $j$. Since the limit $j$ is characterized, the whole sequence $j_\eta$ has this limit. The limit equation

(19) is an immediate consequence of (15), taking the distributional limits. At first, since we were applying Proposition 2.3 on $\Omega'$, we obtain the relation only on $\Omega'$, but since this subdomain was arbitrary, the relations hold in the whole domain $\Omega$. □

2.3. **Proof of Theorem 1.2.** The proof is performed in three Steps.

**Uniqueness for the limit problem.** For fixed incident field $u^i$ we want to show that there exists at most one solution $U$ of the limit problem (22) of Theorem 1.2. To this end we consider the difference $u$ of two solutions, satisfying

$$\nabla \cdot (a_{\text{eff}} \nabla u) = -k^2 \mu_{\text{eff}} u \qquad \text{in } \mathbb{R}^2 \tag{36}$$

$$\partial_r u - iku = o\left(r^{-1/2}\right) \qquad \text{for } r \to \infty \tag{37}$$

We claim that every solution $u : \mathbb{R}^2 \to \mathbb{C}$ of (36)–(37) vanishes identically. We will show this result exploiting two facts: (i) outside $R$, the coefficients are $a_{\text{eff}} = \text{id}$ and $\mu_{\text{eff}} = 1$, hence we consider a standard Helmholtz equation. (ii) inside $R$, the coefficient matrix $a_{\text{eff}}$ is real and the coefficient $\mu_{\text{eff}}$ has a positive imaginary part. We follow standard arguments that are outlined e.g. in [9].

Denoting a sphere of radius $r$ by $S_r := \partial B_r(0)$, we deduce from (37) that

$$\lim_{r\to+\infty} \int_{S_r} \left[ |\partial_r u|^2 + k^2 |u|^2 + 2k\Im\left(u \partial_r \bar{u}\right) \right] = \lim_{r\to+\infty} \int_{S_r} |\partial_r u - iku|^2 = 0. \tag{38}$$

Here, the first equality is obtained simply by expanding the squared norm of the second integrand.

To study this integral further, we observe that, outside $R$, the divergence $\nabla \cdot (u \nabla \bar{u}) = -k^2|u|^2 + |\nabla u|^2$ is real. This implies that the surface integral $\int_{S_r} \Im\left(u\, \partial_r \bar{u}\right)$ is independent of the radius $r$, provided that $r$ is large enough to satisfy $R \subset B_r(0)$. In view of (38), we obtain for any such $r_0$ the inequality

$$\int_{S_{r_0}} \Im\left(u\, \partial_r \bar{u}\right) \leq 0. \tag{39}$$

After these preparations, the idea is now to multiply equation (36) for $u$ with $\bar{u}$, to integrate over the ball $B_{r_0}(0)$ and to integrate by parts. Since (36) holds only in the sense of distributions, we can not argue directly: due to possible jumps at the lateral sides of $R$, the function $\bar{u}$ is not necessarily of class $H^1(\mathbb{R}^2)$. Nevertheless, approximating $\bar{u}$ by smooth functions with large $x_1$-derivatives inside $R$, we find

$$\int_R \alpha |\partial_{x_2} u|^2 + \int_{B_{r_0}(0)\setminus R} |\nabla u|^2 - \int_{B_{r_0}(0)} k^2 \mu_{\text{eff}} |u|^2 = \int_{S_{r_0}} \partial_r u\, \bar{u}\,. \tag{40}$$

Regarding the outer boundary we note that, as a solution of $(\Delta + k^2)(u) = 0$ on $\mathbb{R}^2 \setminus \overline{R}$, the function $u$ is analytic on that domain. In particular, traces of $u$ and $\partial_r u$ are well defined and smooth on $S_{r_0}$.

We take the imaginary part in (40). Since $a_{\text{eff}}$ is real, $\mu_{\text{eff}}$ is real outside $R$, and $\mu_{\text{eff}} = \beta$ in $R$, we obtain (note that we performed a complex conjugation on the right hand side)

$$k^2\, \Im(\beta) \int_R |u|^2 = \int_{S_{r_0}} \Im\left(u\, \partial_r \bar{u}\right). \tag{41}$$

We combine the strict inequality $\Im(\beta) > 0$ with (39) to conclude that the expression in (41) vanishes; in particular, we find $u = 0$ on $R$. The fact that the boundary integral in (39) vanishes for every $r_0$, combined with (38), implies

$$\lim_{r\to+\infty}\int_{S_r}|u|^2 = 0\,. \tag{42}$$

A classical result, sometimes denoted as Rellich's first lemma, states that solutions $u$ of the Helmholtz equation on an exterior domain, satisfying property (42), vanish identically. We note that Rellich's first lemma is shown with an expansion of solutions in spherical harmonics, for a proof in three dimensions see Lemma 2.11 in [9].

Rellich's first lemma provides $u = 0$ in all of $\mathbb{R}^2$ and concludes the proof of the uniqueness property.

**Convergence to the limit problem assuming an $L^2_{\text{loc}}$-bound.** We analyze a sequence $u_\eta$ as in Theorem 1.2. We choose a radius $r_0 > 0$ such that $\bar{R} \subset B_{r_0}(0)$, and set $\Omega := B_{r_0}(0)$. In this step of the proof of Theorem 1.2, we assume that there holds

$$t_\eta := \left(\int_\Omega |u_\eta|^2\right)^{1/2} \le C\,, \tag{43}$$

uniformly in $\eta$. Based on the a priori estimate (43), we can construct a subsequence $\eta \to 0$, such that, for some limit function $u \in L^2(\Omega)$, there holds $u_\eta|_\Omega \rightharpoonup u$ weakly in $L^2(\Omega)$. To this subsequence we may apply Theorem 1.1. It follows that the function $U := u\,\mathbf{1}_{\Omega\setminus R} + \beta^{-1}\,u\,\mathbf{1}_R$ solves the limit system (19) in $\Omega$, relation (22) is shown.

It remains to verify the radiation condition (11) for $U - u^i$. We start by noting that Lemma 2.2 provides uniform convergence $u_\eta \to U$ and $\nabla u_\eta \to \nabla U$ on every compact subset of $\Omega \setminus \overline{R}$. In particular, let us choose $r < r_0$ such that $R \subset\subset B_r(0) \subset\subset \Omega$.

By the Sommerfeld radiation condition, the scattered field $u^s_\eta = u_\eta - u^i$ coincides on $\mathbb{R}^2 \setminus B_r(0)$ with its Helmholtz-representation through values and derivatives of $u_\eta - u^i$ on $\partial B_r(0)$ (see Theorem 2.4 of [9] in the three-dimensional case). With the same representation formula, using the values and derivatives of $U - u^i$ on $\partial B_r(0)$, we can extend $U$ to all of $\mathbb{R}^2$ to a solution of the Helmholtz equation $\Delta U + k^2 U = 0$ in all $R^{\text{ext}}$. By this construction, also $U - u^i$ satisfies the Sommerfeld radiation condition.

The convergence of values and derivatives of $u_\eta$ on $\partial B_r(0)$ to values and derivatives of $U$ imply the uniform convergence $u_\eta \to U$ (as well for derivatives) on all compact subset of $R^{\text{ext}}$, since both extensions are given by the same integral representation.

Our uniqueness result for the limit system implies the convergence $u_\eta \rightharpoonup u$ for the entire sequence $\eta \to 0$. This concludes the proof of Theorem 1.2 (under the boundedness assumption (43)).

**Boundedness of $t_\eta$.** In the previous step we have shown Theorem 1.2 under assumption (43) on $t_\eta$. We will now derive (43) with a contradiction argument. We assume that the solution sequence $u_\eta$ is such that $t_\eta \to \infty$ along a subsequence $\eta \to 0$. We then consider this subsequence, normalize $u_\eta$, and consider in the following

$$v_\eta := \frac{1}{t_\eta} u_\eta\,, \qquad \text{such that } \|v_\eta\|_{L^2(\Omega)} = 1\,. \tag{44}$$

By linearity, $v_\eta$ solves the original diffraction problem with the incident field $v_\eta^i := u^i/t_\eta \to 0$. Applying the previous step of the proof (which remains valid for sequences of incident fields), we deduce that $v_\eta$ converges weakly in $L^2_{\mathrm{loc}}(\mathbb{R}^2)$ to the function $v$, which is determined by: the function $V := v\,\mathbf{1}_{\Omega\setminus R} + \beta^{-1}\, v\,\mathbf{1}_R$ is the unique solution to (22) satisfying the outgoing wave condition (11) with vanishing incident field. As shown in the first step of the proof, we obtain $V = 0$ and therefore $v_\eta \rightharpoonup 0$ weakly in $L^2(\Omega)$.

Outside $R$, we can apply the gradient estimate (26) to the sequence $(v_\eta)_\eta$, and obtain that $v_\eta|_{\Omega\setminus R}$ remains in a bounded subset of $H^1(\Omega\setminus R)$. The Rellich compact embedding theorem implies $\lim_{\eta\to 0}\int_{\Omega\setminus R}|v_\eta|^2 = 0$.

Inside $R$, we exploit the boundedness assumption (21) on the sequence $(u_\eta)_\eta$. Because of $t_\eta \to \infty$, we find $\lim_{\eta\to 0}\int_R |v_\eta|^2 = 0$. Together with the convergence in the exterior, we find a contradiction to (44). This contradiction provides (43) and concludes the proof of Theorem 1.2. □

## 3. **Transmission properties of the effective layer.**

Theorems 1.1 and 1.2 provide the effective Helmholtz equation that describes the optical properties of the grated metallic structure. In this section we want to calculate the corresponding effective reflection and transmission properties of the structure.

In the following, we restrict ourselves to an effective structure that extends to infinity, i.e. $R = \mathbb{R}\times(-h, 0)$. Our aim is to study a planar front of waves that arrives from the upper part ($x_2 > 0$), hits the structure (that occupies the region $x_2 \in (-h, 0)$), and interacts with it. The incoming waves will be partially reflected and partially transmitted. With the incident angle $\theta \in (-\pi/2, \pi/2)$ we write the incoming wave in the form $e^{ik(\sin(\theta)x_1 - \cos(\theta)x_2)}$. We write $U$ for the effective field, which must solve the effective equation (22). We use an incident wave of unit amplitude, write $T \in \mathbb{C}$ for the complex amplitude (expressing amplitude and phase shift) of the transmitted wave, and $R \in \mathbb{C}$ for the complex amplitude of the reflected wave (we will not use $R$ for the rectangular slab structure in the following). Two amplitudes $A_1 \in \mathbb{C}$ and $A_2 \in \mathbb{C}$ are used to describe the solution in the structure. For a sketch of the reflection and transmission problem see Figure 3.

We make the solution ansatz

$$U(x_1, x_2) = \begin{cases} e^{ik(\sin(\theta)x_1 - \cos(\theta)x_2)} + Re^{ik(\sin(\theta)x_1 + \cos(\theta)x_2)} & \text{for } x_2 > 0\,,\\ \left(A_1\cos(\tau k x_2) + A_2\sin(\tau k x_2)\right) e^{ik(\sin(\theta)x_1)} & \text{for } 0 > x_2 > -h\,,\\ Te^{ik(\sin(\theta)x_1 - \cos(\theta)(x_2+h))} & \text{for } -h > x_2\,. \end{cases} \tag{45}$$

The parameter $\tau := \sqrt{\beta/\alpha}$ appears, since the effective equation (22) reads $\partial^2_{x_2}U = -k^2\tau^2 U$ in the structure. We observe that the ansatz (45) provides a solution $U$ of (22) in the three subdomains. If we can additionally satisfy the interface conditions, we have found a solution to the transmission problem. At this point we observe that the $x_1$ dependence is identical in all three subdomains ($e^{ik(\sin(\theta)x_1}$ as a factor). This fact implies that, if the interface conditions are satisfied at one point $x_1$, they are satisfied along the whole interface.

We will use the non-standard interface conditions at $x_2 = 0$ and at $x_2 = -h$ (two conditions at each interface) to determine the four complex constants $R$, $A_1$, $A_2$, and $T$. Our main interest is the real amplitude $|T|$ of the transmitted wave, since $|T| \approx 1$ relates to a high transmission of the effective structure.

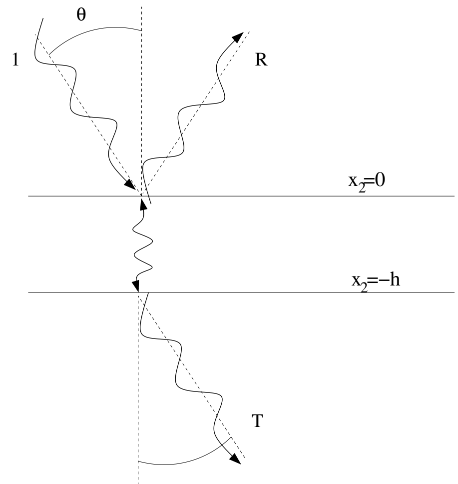


FIGURE 3. Illustration of the solution ansatz in the transmission problem. An incoming wave (from top) results in a reflected wave and a transmitted wave. The coupling across the slab occurs only in vertical direction.

**The transfer matrix $M$.** In the transfer matrix formalism one regards the slab $\mathbb{R} \times (-h, 0)$ as an abstract object that induces a relation between the solution characteristics on the upper boundary $x_2 = 0$ and the solution characteristics on the lower boundary $x_2 = -h$. More precisely, we will define and analyze a map $M : \mathbb{C}^2 \to \mathbb{C}^2$, that we write symbolically as

$$M : \begin{pmatrix} U(0) \\ \partial_{x_2} U(0) \end{pmatrix} \mapsto \begin{pmatrix} U(-h) \\ \partial_{x_2} U(-h) \end{pmatrix} . \tag{46}$$

Let us first give the precise definition of $M$: Let $v_1 \in \mathbb{C}$ and $v_2 \in \mathbb{C}$ be two complex numbers. On the interval $x_2 \in (-h, 0)$, we solve the ordinary differential equation $\partial_{x_2}^2 U = -k^2 \tau^2 U$. As boundary data (the initial data on $x_2 = 0$) we use $U(0) = v_1$ and $\partial_{x_2} U(0) = \alpha^{-1} v_2$. The unique solution $U$ provides two numbers, $U(-h) \in \mathbb{C}$ and $\partial_{x_2} U(-h) \in \mathbb{C}$. We can define the vector $M(v) := w$ by setting $w_1 := U(-h)$ and $w_2 := \alpha \partial_{x_2} U(-h)$. Since the differential equation is linear, the map $M$ is in fact linear and can therefore be expressed as a matrix, $M \in \mathbb{C}^{2\times 2}$.

We next want to explain the above definition of $M$. To this end, let $U$ be a smooth function in the domain $x_2 \geq 0$, and let $x_1 \in \mathbb{R}$ be an arbitrary point. We can evaluate (as traces from $x_2 > 0$) the two complex numbers $v_1 := U(x_1, 0+) \in \mathbb{C}$ and $v_2 := \partial_{x_2} U(x_1, 0+) \in \mathbb{C}$. Our aim is now to continue $U$ across $x_2 = 0$ in such a way, that it can be a solution of our effective system. The interface conditions at $x_2 = 0$ demand $U(x_1, 0-) = U(x_1, 0+)$ (continuity of the values) and $\partial_{x_2} U(x_1, 0-) = \alpha^{-1} \partial_{x_2} U(x_1, 0+)$ (continuity of the normal component of $a_{\text{eff}} \nabla U$). On the interval $x_2 \in (-h, 0)$ we want $U$ to solve the ordinary differential equation $\partial_{x_2}^2 U(x_1, \cdot) = -k^2 \tau^2 U(x_1, \cdot)$, just as in the definition of $M$. If we want to extend the solution across the lower boundary $x_2 = -h$, the interface conditions read $w_1 := U(x_1, -h-) = U(x_1, -h+)$ and $w_2 := \partial_{x_2} U(x_1, -h-) = \alpha \partial_{x_2} U(x_1, -h+)$. In this sense, the map $M$ is the map of (46): It maps value and normal derivative of a solution on $x_2 > 0$ to the corresponding value and normal derivative of a continuation of the solution on $x_2 < -h$.

**Calculation of $M$.** Our next step is to calculate the transfer matrix. The calculation is simplified by using only unit vectors as arguments: $(1,0)^T$ and $(0,1)^T$. The first column of $M$ is $M \cdot (1,0)^T$, the second column is $M \cdot (0,1)^T$.

*First column of $M$.* To calculate the first column of $M$, we have to investigate a solution $U : (-h, 0) \to \mathbb{C}$ with the properties $U|_{x_2=0+} = 1$ and $\partial_{x_2} U|_{x_2=0+} = 0$. We write the solution as $U(x_2) = a_1 \cos(\tau k x_2) + a_2 \sin(\tau k x_2)$. The transmission conditions imply

$$\begin{aligned} 1 &= a_1 \cos(\tau k\, 0) + a_2 \sin(\tau k\, 0) = a_1, \\ 0 &= \alpha \partial_{x_2} \left[a_1 \cos(\tau k x_2) + a_2 \sin(\tau k x_2)\right]\big|_{x_2=0} = \alpha \tau k a_2 \,. \end{aligned}$$

We find $a_1 = 1$ and $a_2 = 0$, and hence for the solution $U$ at $-h-0$ the value and the derivative

$$\begin{aligned} w_1 &= U|_{x_2=-h} = a_1 \cos(-\tau k h) + a_2 \sin(-\tau k h) = \cos(\tau k h), \\ w_2 &= \alpha \partial_{x_2} U|_{x_2=-h} = \alpha \tau k \sin(\tau k h). \end{aligned}$$

These two values provide the first column of $M$.

*Second column of $M$.* The calculation of the second column follows the same lines. The corresponding solution for $x_2 \in (-h, 0)$ reads $U(x_2) = (\alpha\tau k)^{-1} \sin(\tau k x_2)$.

Collecting the results, we find the following explicit expression for the transfer matrix $M$,

$$M = \begin{pmatrix} \cos(\tau k h) & -(\alpha \tau k)^{-1} \sin(\tau k h) \\ \alpha \tau k \sin(\tau k h) & \cos(\tau k h) \end{pmatrix}. \tag{47}$$

We recall that the parameter $\alpha = 1 - 2\gamma \in \mathbb{R}$ of the effective system stands for the relative slit width and that $\tau := \sqrt{\beta/\alpha}$ depends on the ratio of the effective parameters $\mu_{\mathrm{eff}}$ and $a_{\mathrm{eff}}$.

**The transmission coefficient.** The calculation of the transfer matrix was independent of the solution ansatz in the domain $x_2 > 0$. Our next aim is to calculate the transmission coefficient $T$, which will be obtained from the ansatz in (45) with the help of the transfer matrix $M$.

We study the ansatz (45) in the spirit of the transfer matrix formalism: at the line $x_2 = 0+$, the vector that comprises value and normal derivative is $(1 + R, ik\cos(\theta)(-1+R))e^{ik(\sin(\theta)x_1}$. The matrix $M$ maps these two data onto the corresponding values at $x_2 = -h-$, and, referring to (45), we want them to be $(T, -ik\cos(\theta)T)e^{ik(\sin(\theta)x_1}$. The dependence on $x_1$ is identical on both sides by our ansatz. It remains to solve, with the abbreviation $k_\theta := k\cos(\theta)$, the linear system

$$M \cdot \begin{pmatrix} 1 + R \\ ik_\theta(-1+R) \end{pmatrix} = T \begin{pmatrix} 1 \\ -ik_\theta \end{pmatrix}. \tag{48}$$

In this relation, the wave number $k$ of the incident field and the angle $\theta$ are known, hence we regard $k_\theta \in \mathbb{R}$ as given. Furthermore, the matrix $M$ is known from (47). We can use (48) to determine $R$ and $T$.

Since we are mainly interested in the number $T \in \mathbb{C}$, we will eliminate $R$. To this end, we introduce two special vectors $v \in \mathbb{C}^2$ and $w \in \mathbb{C}^2$ as

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} := M \cdot \begin{pmatrix} 1 \\ ik_\theta \end{pmatrix} = \begin{pmatrix} \cos(\tau kh) - i(\alpha\tau)^{-1}\cos(\theta)\sin(\tau kh) \\ \alpha\tau k \sin(\tau kh) + ik_\theta \cos(\tau kh) \end{pmatrix}$$

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} := v^\perp = \begin{pmatrix} -v_2 \\ v_1 \end{pmatrix} = \begin{pmatrix} -\alpha\tau k \sin(\tau kh) - ik_\theta \cos(\tau kh) \\ \cos(\tau kh) - i(\alpha\tau)^{-1}\cos(\theta)\sin(\tau kh) \end{pmatrix}.$$

Since on the left hand side of (48) appears the vector $Rv$, the scalar product of (48) with the vector $w$ eliminates $R$. We obtain

$$\begin{aligned} w \cdot M \begin{pmatrix} 1 \\ -ik_\theta \end{pmatrix} &= T\, w \cdot \begin{pmatrix} 1 \\ -ik_\theta \end{pmatrix} \\ &= T\left(-2ik_\theta \cos(\tau kh) - [\alpha\tau k + (\alpha\tau)^{-1} k_\theta \cos(\theta)]\sin(\tau kh)\right) \\ &= -ik_\theta T\, \left(2\cos(\tau kh) - i[\alpha\tau/\cos(\theta) + (\alpha\tau/\cos(\theta))^{-1}]\sin(\tau kh)\right). \end{aligned}$$

With the help of the expression (47) for $M$, we can evaluate the left hand side to

$$\begin{aligned} & w \cdot M \begin{pmatrix} 1 \\ -ik_\theta \end{pmatrix} \\ &= \begin{pmatrix} -\alpha\tau k \sin(\tau kh) - ik_\theta \cos(\tau kh) \\ \cos(\tau kh) - i(\alpha\tau)^{-1}\cos(\theta)\sin(\tau kh) \end{pmatrix} \cdot \begin{pmatrix} \cos(\tau kh) + i(\alpha\tau)^{-1}\cos(\theta)\sin(\tau kh) \\ \alpha\tau k \sin(\tau kh) - ik_\theta \cos(\tau kh) \end{pmatrix} \\ &= -2ik_\theta \cos^2(\tau kh) - 2ik_\theta \sin^2(\tau kh) = -2ik_\theta. \end{aligned}$$

Equating the two expressions provides the following expression for $T \in \mathbb{C}$,

$$T = \left(\cos(\tau kh) - \frac{i}{2}\left[\frac{\alpha\tau}{\cos(\theta)} + \frac{\cos(\theta)}{\alpha\tau}\right]\sin(\tau kh)\right)^{-1}. \tag{49}$$

Equation (49) determines the transmission coefficient $T = T(k, h, \tau, \theta)$ in dependence of the wave number $k$, the layer height $h$, the relative slit size $\alpha$, the effective material index $\tau := \sqrt{\beta/\alpha}$, and the angle $\theta$. We recall that $\beta$ is the average magnetic field across the metal part, when the magnetic field in the void part is 1, see (12) for the explicit expression. We emphasize that $T$ depends on $k$ also implicitly through $\beta = \beta(k)$. The graph of $|T|^2$ against the wave number $k$ can be evaluated from the explicit relations (12) and (49), see Figure 4.

Let us discuss once more the case of a material that permits perfect plasmon waves, i.e. of a lossless material with negative permittivity, $\varepsilon_r < 0$. For such a material, $\sigma$ and $\beta$ are positive real numbers by (12). In this case, the number in squared brackets of (49) is real and greater or equal to 2. Correspondingly, we find $|T| \le 1$. The value $|T| = 1$ is attained if and only if $\cos(\tau kh) = 1$. This corresponds to a resonance of the plasmon waves in the slab (solving $\partial_{x_2}^2 U = -k^2\tau^2 U$ for $x_2 \in (-h, 0)$) with the height $h$ of the slab.

We note that the effect can also be deduced from the transfer matrix $M$ of (47), since for $\cos(\tau kh) = 1$, we find the transfer matrix $M = \mathrm{id}$, corresponding to perfect transmission.

Figure 4 shows the transmission coefficient $|T|^2$ for physical parameter values. In dependence of the wave-number $k$, we observe pronounced peaks. Variations of the incident angle $\theta$ can lead to large variations, but we do not observe an oscillatory dependence. For normal incidence, the first local maximum $|T|^2 \approx 0.87$ is achieved for $k \approx 1.92$. The numerical experiments of [8] observed resonance at $k = 2.51$. We recall at this point that our theory investigates the thin-slit limit $\eta \to 0$, such that even a qualitative agreement is remarkable for the above experimental parameters.

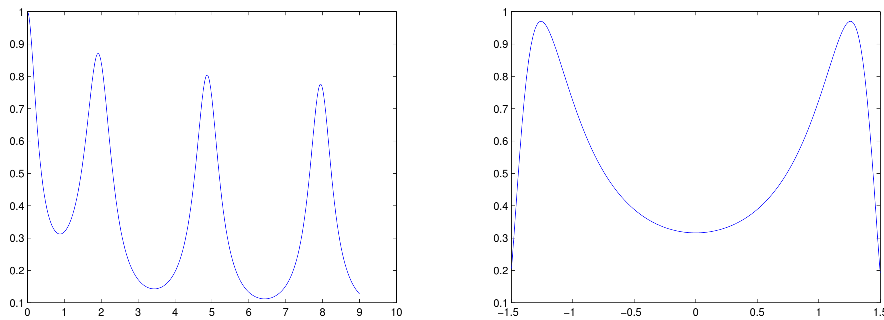

FIGURE 4. Numerical evaluation of the transmission coefficient $|T|^2$. Left: in dependence of the non-dimensional wave-number $k$ for normal incidence, $\theta = 0$. Right: in dependence of the angle $\theta$ for wave-number $k = 0.8$. In both figures, we used the non-dimensional geometrical quantities $\eta = 7/6$, $\alpha = 1/7$, and $\gamma = (1-\alpha)/2 = 3/7$ as mentioned in (7), the frequency independent relative permittivity $\varepsilon_\eta = (0.12 + 3.7i)^2$ is obtained by setting $\sigma = 4.32 - 0.14i$.

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