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SPECTRAL STIFF PROBLEMS IN DOMAINS SURROUNDED BY THIN STIFF AND HEAVY BANDS: LOCAL EFFECTS FOR EIGENFUNCTIONS

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Abstract. We consider the Neumann spectral problem for a second order differential operator, with piecewise constants coefficients, in a domain Ω_ε of $\mathbb{R}^2$. Here Ω_ε is $\Omega \cup \omega_\varepsilon \cup \Gamma$, where Ω is a fixed bounded domain with boundary Γ, ω_ε is a curvilinear band of variable width $O(\varepsilon)$, and $\Gamma = \overline{\Omega} \cap \overline{\omega}_\varepsilon$. The *density* and *stiffness* constants are of order $O(\varepsilon^{-m-1})$ and $O(\varepsilon^{-1})$ respectively in this band, while they are of order $O(1)$ in Ω; m is a positive parameter and $\varepsilon \in (0,1)$, $\varepsilon \to 0$. Considering the range of the low, middle and high frequencies, we provide asymptotics for the eigenvalues and the corresponding eigenfunctions. For $m > 2$, we highlight the middle frequencies for which the corresponding eigenfunctions may be localized asymptotically in small neighborhoods of certain points of the boundary.

1. Introduction and statement of the problem.

Neumann spectral stiff problems in domains surrounded by thin bands Ω_ε have been considered in [10] and [11], the thin band ω_ε being both stiff and heavy. The width of ω_ε is of order $O(\varepsilon)$ while the stiffness and density are of order $O(\varepsilon^{-t})$ and $O(\varepsilon^{-t-m})$ respectively, for t and $t+m$ two positive parameters, and for $\varepsilon \in (0,1)$, a parameter that converges towards zero: see Figure 1 and problem $(1)_1$, (2), $(1)_3$, (3) and $(1)_5$. An asymptotic study of the low frequencies, based on asymptotic expansions, is provided in [10] for different relations between t and $m+t$. Convergence for the low frequencies as $\varepsilon \to 0$ is also proved in [10] for $t = 1$, $m = 0$, and in [11] for $t > 1$ and $m = 0$,

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making it clear a different asymptotic behavior for the eigenvalues and eigenfunctions. We refer to [10] and [11] for a review of previous works in the literature on the subject which in fact are scarce; see also Remark 12 to compare with other vibrating systems with concentrated masses along curves.

In this paper, we consider problem (1), namely, $t = 1$ and $m > 0$ in (2) and (3). We address asymptotics for low, middle and high frequencies and characterize the asymptotic behavior for the corresponding eigenfunctions, which is very different depending on the range of frequencies under consideration, namely, frequencies of the order $O(\varepsilon^m)$, $O(\varepsilon^{m-2})$ or $O(1)$. It is self-evident that, for this problem, the frequencies of order $O(\varepsilon^{m-2})$ (the so-called *middle frequencies*) can only be considered when $m > 2$, and we prove that the behavior of the associated vibrations is also very different depending on the geometry of the band ω_ε, and more specifically on a function h (see Figure 1) which defines the boundary of Ω_ε. This being one of the main aims of the paper, we show that the points where h has a local maximum are points where certain middle frequency vibrations concentrate asymptotically their support.

Let Ω be a bounded domain of the plane $\mathbb{R}^2$ with a smooth boundary Γ and let (ν, τ) be the natural orthogonal curvilinear coordinates in a neighborhood of Γ: τ is the arc length and ν the distance along the normal vector to Γ; $\nu < 0$ inside Ω. Let ℓ denote the length of the curve Γ and $\varkappa(\tau)$ its curvature at the point τ. We assume that the domain Ω is surrounded by the thin band $\omega_\varepsilon = \{x : 0 < \nu < \varepsilon h(\tau)\}$ where $\varepsilon > 0$ is a small parameter and h is a strictly positive function of the τ variable, ℓ-periodic, $h \in C^\infty(\mathbb{S}_\ell)$ where $\mathbb{S}_\ell$ stands for the circumference of length ℓ. Let Ω_ε be the domain $\Omega_\varepsilon = \Omega \cup \omega_\varepsilon \cup \Gamma$ and $\Gamma_\varepsilon = \{x : \nu = \varepsilon h(\tau)\}$ the boundary of Ω_ε (see Figure 1).

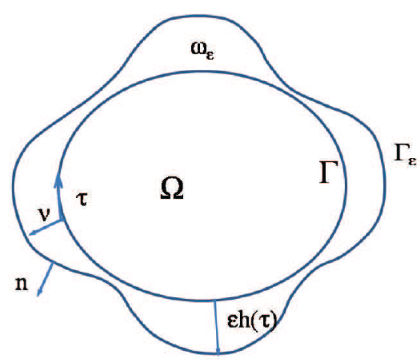

FIGURE 1. Possible geometry for Ω_ε

We consider the spectral Neumann problem in Ω_ε for a second order differential operator with piecewise constants coefficients:

$$\begin{cases} -A\Delta_x U^\varepsilon = \lambda^\varepsilon U^\varepsilon & \text{in } \Omega, \\ -a\varepsilon^{-1}\Delta_x u^\varepsilon = \lambda^\varepsilon \varepsilon^{-1-m} u^\varepsilon & \text{in } \omega_\varepsilon, \\ U^\varepsilon = u^\varepsilon & \text{on } \Gamma, \\ \varepsilon A \partial_\nu U^\varepsilon = a \partial_\nu u^\varepsilon & \text{on } \Gamma, \\ \partial_n u^\varepsilon = 0 & \text{on } \Gamma_\varepsilon. \end{cases} \tag{1}$$

Here, A and a are two positive constants while ∂_ν and ∂_n denote the derivatives along the outward normal vectors ν and n to the curves Γ and Γ_ε, respectively; m is a positive parameter. We study the asymptotic behavior, as $\varepsilon \to 0$, of the eigenvalues λ^ε of (1) and the corresponding eigenfunctions u_ε which we identify with pairs of functions $\{U^\varepsilon, u^\varepsilon\}$. In (1), U^ε stands for the restriction of u_ε to Ω and u^ε for the restriction of u_ε to ω_ε.

Problem (1) fits into the case where $t = 1$ in the set of problems considered in [10], depending on two parameters t and m, namely, the set of stiff problems with equations $(1)_1$, $(1)_3$, $(1)_5$, and

$$-a\varepsilon^{-t}\Delta_x u^\varepsilon = \lambda^\varepsilon \varepsilon^{-t-m} u^\varepsilon \quad \text{in} \quad \omega_\varepsilon, \tag{2}$$

$$A\partial_\nu U^\varepsilon = a\varepsilon^{-t}\partial_\nu u^\varepsilon \quad \text{on } \Gamma \tag{3}$$

for different values of t and m, provided that $t \geq 0$ and $t + m \geq 0$, and either $t > 0$ or $t + m > 0$; the eigenvalues always being in the range of the low frequencies. The parameters t and $t+m$ reflect the relative stiffness of the band and the dead-weight of the band respectively in mechanical problems. These problems are of interest, for instance, in the study of reinforcement problems for solid media and in vibrations for a two-phases system in fluid mechanics.

For strictly positive t, the band ω_ε is both stiffer and heavier simultaneously, and, it seems natural to have a different asymptotic behavior as $\varepsilon \to 0$ for the eigenpairs $\{\lambda^\varepsilon, u_\varepsilon\}$ of (1) depending on whether $m = 0$ or $m > 0$. We recall the results in [10] and [11] which are the closest in the literature to the problem under consideration. A characterization of the limiting problems for the eigenpairs of $(1)_1$, (2), $(1)_3$, (3) $(1)_5$, for the different values of t and m has been obtained in [10]. These limiting problems deal with the asymptotics for the low frequencies, which includes those of (1), but the characterization is obtained by means of asymptotic expansions.

In [10], we provide sharp bounds for convergence rates of the eigenpairs $\{\lambda^\varepsilon, u_\varepsilon\}$ in the case where $t = 1$ and $m = 0$ by using the so-called *inverse-direct reduction method* (cf. [18] and [19]). A different approach for the eigenpairs is provided in [11] for the case where $t > 1$ and $m = 0$ where, in addition to the convergence, a complete asymptotic expansion for the eigenpairs has been obtained, and a connection of this problem with Wentzell problems with small parameters has been shown. Also, both papers [10] and [11] deal with obtaining precise bounds for convergence rates for the low frequencies and the corresponding eigenfunctions in the cases mentioned above $m = 0$ and $t \geq 1$, but convergence results for the case where $t = 1$ and $m > 0$ were left as an open question to be considered. We refer to [10] and [11] for further references.

Here, we deal with the case where $t = 1$ and $m > 0$, namely with problem (1), and consider the low and high frequencies. We obtain the limiting problems associated with all these frequencies and provide information on the structure of the corresponding eigenfunctions.

As a matter of fact, there appear two limiting problems associated with (1) in a natural way. The first one is problem (6) which is associated with the low frequencies, namely the eigenvalues of order $O(\varepsilon^m)$, with the mass term appearing in the boundary condition on Γ. In this boundary condition also the second-order flexion terms of the stiffening band arise, while inside Ω the solutions are harmonic functions. The second problem is the Dirichlet problem (109). This is involved with the so-called *high frequencies*, namely, with the eigenvalues of (1) of order $O(1)$ which give rise to vibrations affecting the whole Ω and keeping the thin band at rest.

These two problems, (6) and (109), appear independently of the geometry of the band ω_ε, and we show that for $m > 2$ there are other limiting problems associated with the intermediate frequencies which strongly depend on this geometry: problem (35) when the function h is constant ($h \equiv h_0$) and problem (59) which depends on h at points τ_0 where the function h presents some kind of local extreme (a local maximum in this case).

Let us note that the order of magnitude of the low frequencies is provided by the estimate (5) for the k-th eigenvalue of problem (1). For fixed $k \in \mathbb{N} = \{1, 2, \cdots\}$, this estimate becomes optimal once that we prove Theorem 2.2. In fact, Theorem 2.2 shows the convergence of the positive re-scaled eigenvalues of (1), $\lambda_k^\varepsilon \varepsilon^{-m}$, towards the positive eigenvalues of (6) with conservation of the multiplicity, as $\varepsilon \to 0$. Also, a certain convergence for the corresponding eigenfunctions holds in the topology stated in Theorem 2.2 which implies the weak convergence in $H^1(\Omega)$. The results for the low frequencies are in Section 2.

We use a spectral convergence theorem for positive, symmetric and compact operators on parameter-dependent Hilbert spaces (cf. Lemma 2.1) to prove the above results of convergence, namely the results in Section 2, but it should be mentioned that they can be improved by using the technique of the inverse–direct reduction method applied in [10] to derive the convergence for the low frequencies of (1) in the case where $m = 0$. In this paper, we avoid using this method, which implies laborious computations, but we emphasize that the inverse-direct reduction method allows us to obtain precise bounds for convergence rates of the eigenpairs of (1). Specifying, under certain restrictions involving k and ε these bounds depend on ε and k in a explicit way and, as a consequence, the convergence with conservation of the multiplicity holds.

As regards higher order frequencies, the eigenvalues λ^ε of order $O(\varepsilon^\alpha)$ for $\alpha < m$, it is known (see Remark 15) that they accumulate on the whole positive axis. That is, for each $\lambda > 0$ there are sequences $\lambda^\varepsilon_{k(\varepsilon)} \varepsilon^{-\alpha} \to \lambda$ as $\varepsilon \to 0$, with the index $k(\varepsilon) \to \infty$ as $\varepsilon \to 0$ (see [9], [13], [15] and Chapter 7 of [22] for the technique). Nevertheless, information on the structure of the corresponding eigenfunctions has to be found. We first use asymptotic expansions to get this information for the values of $\alpha = m - 2$ and $\alpha = 0$, respectively (see Remark 16 for other values of α). When we deal with the high frequencies for $\alpha = m - 2$, the assumption of $m > 2$ must be understood, and these frequencies of order $O(\varepsilon^{m-2})$ are referred to as the middle frequencies since they are intermediate between the eigenvalues of order $O(\varepsilon^m)$ and of order $O(1)$. An important fact that we highlight in this paper is that the behavior of the eigenfunctions corresponding to the middle frequencies is very different depending on the geometry of the band, and more precisely, depending on whether the function h that describes this geometry is constant or not.

The asymptotic expansions for the middle frequencies are in Sections 3 in the case when h is constant, and in Section 4 when h is not constant, while the justifications of these results (cf. Remarks 6, 7 and 8 to compare) are in Section 5.1 and 5.2 respectively. We gather all the results for the high frequencies in Section 6.

When justifying the asymptotic expansions, Theorems 5.3 and 5.6 (Theorem 6.1, respectively) provide a certain sequence of positive numbers $\{\lambda_{0,j}\}_{j=1}^\infty$ such that there are sequences $\lambda^\varepsilon \varepsilon^{2-m} \to \lambda_{0,j}$ ($\lambda^\varepsilon \to \lambda_{0,j}$, respectively) as $\varepsilon \to 0$. These sequences of $\lambda_{0,j}$, limiting points of the middle frequencies (the high frequencies, respectively), are the eigenvalues of problem (35) or (59) depending on h ((109), respectively), and they can be referred to as *almost eigenfrequencies* since the re-scaled frequencies approaching $\lambda_{0,j}$ can give rise to some kinds of vibrations that cannot be detected with the low frequencies. Specifying, we can construct the so-called *quasimodes* of the problem which approach, in a certain topology, a linear combination of eigenfunctions corresponding to the re-scaled eigenvalues $\lambda^\varepsilon \varepsilon^{2-m}$ (λ^ε, respectively) in small intervals of the type $[\lambda^0 - d^\varepsilon, \lambda^0 + d^\varepsilon]$, for a certain sequence $d^\varepsilon \to 0$ which is also determined in the above mentioned theorems . These quasimodes allow us to construct *standing waves* which approach certain solutions of associated evolution problems for long times, and the time of approach can be determined in terms of ε and d^ε (see [21] in this connection). Also we show that determining the mentioned sequences of eigenvalues of (35) and (109) for middle and high frequencies respectively, along with a suitable normalization for the eigenfunctions, is somewhat optimal as stated by Theorems 5.4 and 6.2 respectively (cf. also Remarks 9 and 15, 14).

Finally, we point out that this is the first time in the literature of applied mathematics where we show localization effects for the eigenfunctions corresponding to the middle frequencies, in the case where the stiff and heavy band ω_ε has a variable width depending on τ. As a matter of fact, we construct approaches to eigenfunctions corresponding to certain eigenvalues of order $O(\varepsilon^{m-2})$ which concentrate asymptotically their support in $\varepsilon^{1/2}$-neighborhoods of points which are local maxima of h (see Remark 9 for more details). In addition, it should be noted that we do not exhaust the list of possible limiting problems associated with the middle frequencies (cf. Remarks 10 and 11 in this connection). We refer to [12], [17], [6] and [3] for different problems in thin domains where localization effects for the eigenfuncions arise: [12] and [17] deal with thin plate-like domains while [6] and [3] consider a thin rod structure in two and three dimensions respectively (cf. Remark 13 for more details). See [14] for references on other quite different localization effects at points for vibrating systems with concentrated masses.

We also emphasize that the technique here developed for asymptotic expansions, and to show the convergence, is original and it can be extended to highlight the localization phenomena for different types of local extremes and different values of the parameters t and m in (2)–(3) (Remarks 3, 4 and 10). Comparing with the case where h is constant, it should be noted that the change of variables (10) seems to be natural to the geometry of the problem (1), whose solutions present a boundary layer in ω_ε, but when the change (10) is applied to the case where h is not constant, it gives the limiting eigenvalue problem (31), (33), (34) which depends on τ. This shows a strong dependence of the asymptotic behavior of the eigenvalues of (1) on the geometry of the curve defining $\partial\Omega_\varepsilon \cap \partial\omega_\varepsilon$, and the change of variables (50) can reflect this dependence, providing an eigenvalue limiting problem constant in a small neighborhood of a particular point τ_0 (see also Remarks 5 and 11 to compare with the case where h is constant).

2. **Asymptotics for the low frequencies.**

The weak formulation of problem (1) reads: Find λ^ε and $\{U^\varepsilon, u^\varepsilon\} \in H^1(\Omega_\varepsilon)$, $\{U^\varepsilon, u^\varepsilon\} \neq 0$, satisfying

$$\begin{aligned} A\int_\Omega \nabla_x U^\varepsilon \cdot \nabla_x G\,dx + \frac{a}{\varepsilon}\int_{\omega_\varepsilon} \nabla_x u^\varepsilon \cdot \nabla_x g\,dx \\ = \lambda^\varepsilon \left(\int_\Omega U^\varepsilon\, G\,dx + \frac{1}{\varepsilon^{1+m}} \int_{\omega_\varepsilon} u^\varepsilon\, g\,dx \right) \quad \forall \{G,g\} \in H^1(\Omega_\varepsilon) \end{aligned} \tag{4}$$

Here, and in what follows, we identify a function g_ε in $L^2(\Omega_\varepsilon)$ ($H^1(\Omega_\varepsilon)$, respectively) with the pair of functions $\{G, g\}$ where G stands for the restriction of g_ε to Ω and g for the restriction of g_ε to ω_ε. In particular, the eigenelements formed by the eigenvalues λ^ε and the corresponding eigenfunctions u_ε read $(\lambda^\varepsilon, \{U^\varepsilon, u^\varepsilon\})$.

For each $\varepsilon > 0$, problem (4) is a standard spectral problem in the couple of spaces $H^1(\Omega_\varepsilon) \subset L^2(\Omega_\varepsilon)$, with a discrete spectrum. Let us consider

$$0 = \lambda_0^\varepsilon < \lambda_1^\varepsilon \le \lambda_2^\varepsilon \le \cdots \le \lambda_k^\varepsilon \le \cdots \xrightarrow{\mathrm{k}\to\infty} \infty$$

the sequence of eigenvalues repeated according to their multiplicities. Let us denote by $\{\{U_k^\varepsilon, u_k^\varepsilon\}\}_{k=0}^\infty$ the corresponding eigenfunctions which are subject to the orthonormalization condition

$$A\int_\Omega \nabla_x U_k^\varepsilon \cdot \nabla_x U_l^\varepsilon \, dx + \frac{a}{\varepsilon}\int_{\omega_\varepsilon} \nabla_x u_k^\varepsilon \cdot \nabla_x u_l^\varepsilon \, dx \\ + \varepsilon^m \int_\Omega U_k^\varepsilon \, U_l^\varepsilon \, dx + \frac{1}{\varepsilon}\int_{\omega_\varepsilon} u_k^\varepsilon \, u_l^\varepsilon \, dx = \delta_{k,l}$$

where $\delta_{k.l}$ denotes the Kronecker symbol.

Computations in [10] show the inequality

$$\lambda_k^\varepsilon \le C\varepsilon^m \mu_k, \quad k = 1, 2, \cdots \tag{5}$$

where μ_k denotes the $(k+1)$–th eigenvalue of problem (6) in the sequence $\{\mu_k\}_{k=0}^\infty$ and C is a constant independent of ε and k; $\varepsilon \in (0, 1)$. This estimate (5) indicates the order of magnitude of the so-called *low frequencies*; that is, for fixed k, $\lambda_k^\varepsilon = O(\varepsilon^m)$, and its asymptotic behavior as $\varepsilon \to 0$ and that of the corresponding eigenfunctions $\{U_k^\varepsilon, u_k^\varepsilon\}$ has been predicted by means of matched asymptotic expansions in [10] where the limiting spectral problem

$$\left\{ \begin{array}{ll} -A\Delta_x V = 0 & \text{in } \Omega, \\ A\partial_\nu V = \mu h V + a\partial_\tau \left(h \, \partial_\tau V\right) & \text{on } \Gamma. \end{array} \right. \tag{6}$$

has been outlined.

Let us introduce the functional space $\mathcal{H}^{1,1}(\Omega, \Gamma)$ as the completion of $C^\infty(\overline{\Omega})$ with respect to the norm

$$\|W\|_{\mathcal{H}^{1,1}(\Omega,\Gamma)} = \left(\|W\|^2_{H^1(\Omega)} + \|W\|^2_{H^1(\Gamma)} \right)^{1/2}. \tag{7}$$

The weak formulation of problem (6) reads: Find μ and $V \in \mathcal{H}^{1,1}(\Omega, \Gamma)$, $V \ne 0$, satisfying

$$A\int_\Omega \nabla_x V \cdot \nabla_x W \, dx + a\int_\Gamma h\partial_\tau V \partial_\tau W \, d\tau = \mu \int_\Gamma hVW \, d\tau \quad \forall W \in \mathcal{H}^{1,1}(\Omega, \Gamma). \tag{8}$$

Problem (8) has a non–negative discrete spectrum (see [10] for details). Let

$$0 = \mu_0 < \mu_1 \le \mu_2 \le \cdots \le \mu_k \le \cdots \xrightarrow{\mathrm{k}\to\infty} \infty$$

be the eigenvalues of (8) with the usual convention of repeated eigenvalues. We assume that the corresponding eigenfunctions $\{V_k\}_{k=0}^\infty$ are subject to the orthogonality condition

$$A\int_\Omega \nabla_x V_k \cdot \nabla_x V_l \, dx + a\int_\Gamma h\partial_\tau V_k \partial_\tau V_l \, d\tau + \int_\Gamma hV_k V_l \, d\tau = \delta_{k,l}. \tag{9}$$

It proves useful to introduce here some notations and bounds that we shall use throughout the paper. In this connection, in what follows $L^2_h(\Gamma)$ denotes the space $L^2(\Gamma)$ with the scalar product defined by

$$\int_\Gamma hFG\,d\tau \quad \forall F, G \in L^2(\Gamma)\,.$$

Also, in a neighborhood of Γ we introduce here the so-called *local coordinates*

$$(\zeta, \tau), \quad \zeta = \frac{\nu}{\varepsilon}, \tag{10}$$

ζ a *local variable*, which transforms the thin domain ω_ε into a band of length ℓ and width $O(1)$; namely, $\{(\nu, \tau) : \nu \in [0, \varepsilon h(\tau)), \tau \in \mathbb{S}_\ell\}$ into $\{(\zeta, \tau) : \zeta \in [0, h(\tau)), \tau \in \mathbb{S}_\ell\}$.

On the other hand, on account of the continuity of the function h and of the curvature $\varkappa$, for a certain sufficiently small $d > 0$, there exist constants c, C_1, C_2 and C_3 independent of ε such that

$$0 < c < K(\nu, \tau) < C_1 \quad \forall \nu \in [-d, d], \tau \in \mathbb{S}_\ell, \tag{11}$$

$$|1 - K(\nu, \tau)| \le C_2\varepsilon \quad \text{and} \quad |1 - K(\nu, \tau)^{-1}| \le C_3\varepsilon \quad \forall \nu \in [0, \varepsilon h(\tau)], \tau \in \mathbb{S}_\ell. \tag{12}$$

where $K(\nu, \tau) = 1 + \nu\varkappa(\tau)$ denotes the Jacobian of the transformation from (x_1, x_2) to (ν, τ). Formula (12) and that provided by the inequality

$$\left| Z(P) - \frac{1}{T}\int_0^T Z(t)\,dt \right| \le T^{1/2}\|Z'\|_{L^2(0,T)}, \quad \text{where } P = 0 \text{ or } P = T, \tag{13}$$

for any positive T and function $Z \in H^1(0, T)$, are used throughout the paper either in curvilinear or local coordinates in neighborhoods surrounding Γ (cf. [10] for instance).

For the sake of completeness, we first introduce a known result from the spectral perturbation theory, which allows us to prove the convergence for the low frequencies and the corresponding eigenfunctions. (see Chapter III of [20] for its proof).

Lemma 2.1. *Let H_ε and H_0 be separable Hilbert spaces with the scalar products $(\cdot,\cdot)_\varepsilon$ and $(\cdot,\cdot)_0$ respectively. Let $A^\varepsilon \in \mathcal{L}(H_\varepsilon)$ and $A^0 \in \mathcal{L}(H_0)$. Let $\mathcal{W}$ be a subspace of H_0 such that $Im\, A^0 = \{v \,|\, v = A^0 u \;:\; u \in H_0\} \subset \mathcal{W}$. We assume that the following properties are satisfied:*

a) There exists an operator $\mathcal{R}^\varepsilon \in \mathcal{L}(H_0, H_\varepsilon)$ and a constant $\mathrm{a} > 0$ such that, for any $f \in \mathcal{W}$, $\|\mathcal{R}^\varepsilon f\|_\varepsilon$ converge towards $\mathrm{a}\|f\|_0$ as $\varepsilon \to 0$.

b) A^ε and A^0 are positive, compact and self-adjoint operators on H^ε and H^0 respectively. Besides, the norms $\|A^\varepsilon\|_{\mathcal{L}(H_\varepsilon)}$ are bounded by a constant independent of ε.

c) For any $f \in \mathcal{W}$, $\|A^\varepsilon\mathcal{R}^\varepsilon f - \mathcal{R}^\varepsilon A^0 f\|_\varepsilon \to 0$ as $\varepsilon \to 0$.

d) The family of operators A^ε is uniformly compact, i.e., for any sequence f^ε in H_ε such that $\sup_\varepsilon \|f^\varepsilon\|_\varepsilon$ is bounded by a constant independent of ε, we can extract a subsequence $f^{\varepsilon'}$ verifying $\|A^{\varepsilon'} f^{\varepsilon'} - \mathcal{R}^{\varepsilon'} w^0\|_{\varepsilon'} \to 0$, as $\varepsilon' \to 0$, for certain $w^0 \in \mathcal{W}$.

Let $\{\mu_i^\varepsilon\}_{i=1}^\infty$ and $\{\mu_i^0\}_{i=1}^\infty$ be the sequences of the eigenvalues of A^ε and A^0, respectively, with the usual convention of repeated eigenvalues. Let $\{w_i^\varepsilon\}_{i=1}^\infty$ and ($\{w_i^0\}_{i=1}^\infty$, respectively) be the corresponding eigenfunctions in H_ε which are assumed to be orthonormal (H_0, respectively).

Then, for each fixed k there exists a constant C_k independent of ε and there is $\varepsilon_k > 0$ such that for $\varepsilon \leq \varepsilon_k$,

$$|\mu_k^\varepsilon - \mu_k^0| \leq C_k \sup \|A^\varepsilon \mathcal{R}^\varepsilon u - \mathcal{R}^\varepsilon A^0 u\|_\varepsilon$$

where the sup *is taken over all the functions u in the eigenspace associated with μ_k^0, u such that $\|u\|_0 = 1$. In addition, for any eigenvalue μ_k^0 of A^0 with multiplicity s ($\mu_k^0 = \mu_{k+1}^0 = \cdots = \mu_{k+s-1}^0$), and for any w eigenfunction corresponding to μ_k^0, with $\|w\|_0 = 1$, there exists w^ε, w^ε being a linear combination of eigenfunctions $\{w_j^\varepsilon\}_{j=k}^{j=k+s-1}$ of A^ε corresponding to $\{\mu_j^\varepsilon\}_{j=k}^{j=k+s-1}$, such that*

$$\|w^\varepsilon - \mathcal{R}^\varepsilon w\|_\varepsilon \leq C_k \|A^\varepsilon \mathcal{R}^\varepsilon w - \mathcal{R}^\varepsilon A^0 w\|_\varepsilon,$$

for a certain constant C_k independent of ε.

In order to apply Lemma 2.1 to derive the spectral convergence in our problem, let us introduce the norm $\|.\|_\varepsilon$ in $H^1(\Omega_\varepsilon)$ defined by the scalar product

$$\begin{aligned}(\{U,u\},\{V,v\})_\varepsilon = A\int_\Omega \nabla_x U\cdot\nabla_x V\,dx + \frac{a}{\varepsilon}\int_{\omega_\varepsilon}\nabla_x u\cdot\nabla_x v\,dx + \varepsilon^m\int_\Omega UV\,dx \\ + \frac{1}{\varepsilon}\int_{\omega_\varepsilon} uv\,dx \quad \forall\{U,u\},\{V,v\}\in H^1(\Omega_\varepsilon).\end{aligned} \tag{14}$$

Also, let us introduce the linear operator $\mathcal{R}^\varepsilon : \mathcal{H}^{1,1}(\Omega,\Gamma) \to H^1(\Omega_\varepsilon)$ defined by

$$\mathcal{R}^\varepsilon F = \{F, \tilde{F}\} \in H^1(\Omega_\varepsilon) \quad \text{where} \quad \tilde{F}(x) = F(0,\tau) \text{ for } x \in \omega_\varepsilon. \tag{15}$$

Here, we refer to $F(\nu,\tau)$ as the function $F(x)$ written in curvilinear coordinates, and, if no confusion arises, we do not distinguish between a point τ on the boundary Γ and its coordinate along Γ.

The main convergence results for the low frequencies of (1) with $m > 0$ is stated in the following theorem:

Theorem 2.2. *Let λ_k^ε be the eigenvalues of (1). Let m be positive. For each fixed $k \in \mathbb{N}$, the sequence $\lambda_k^\varepsilon/\varepsilon^m$ converges towards the eigenvalue μ_k of (8) as $\varepsilon \to 0$. Moreover, for any eigenvalue μ_k of (8) with multiplicity $\varkappa_k$ ($\mu_{k-1} < \mu_k = \mu_{k+1} = \cdots = \mu_{k+\varkappa_k-1} < \mu_{k+\varkappa_k}$), and for any eigenfunction V of (6) corresponding to μ_k, V verifying (9), there is a linear combination $\{\tilde{U}^\varepsilon, \tilde{u}^\varepsilon\}$ of eigenfunctions corresponding to the eigenvalues $\{\lambda_i^\varepsilon\}_{i=k}^{k+\varkappa_k-1}$, $\lambda_i^\varepsilon \varepsilon^{-m}$ converging towards μ_k, such that*

$$\|\{\tilde{U}^\varepsilon, \tilde{u}^\varepsilon\} - \mathcal{R}^\varepsilon V\|_\varepsilon \xrightarrow{\varepsilon\to 0} 0 \tag{16}$$

where $\|\cdot\|_\varepsilon$ denotes the norm associated with the scalar product (14) in $H^1(\Omega_\varepsilon)$ and $\mathcal{R}^\varepsilon$ is the operator defined by (15). In addition, for each sequence $\{U_k^\varepsilon, u_k^\varepsilon\}$ of eigenfunctions of (4), $\|\{U_k^\varepsilon, u_k^\varepsilon\}\|_\varepsilon = 1$, we can extract a subsequence (still denoted by ε) such that $U_k^\varepsilon \to V_k^*$ weakly in $H^1(\Omega)$, as $\varepsilon \to 0$, where V_k^* is an eigenfunction of (8) corresponding to μ_k and the set $\{V_k^*\}_{k=0}^\infty$ forms an orthonormal basis in the orthogonal complement of $\{V \in \mathcal{H}^{1,1}(\Omega,\Gamma) : V = 0 \text{ on } \Gamma\}$ in $\mathcal{H}^{1,1}(\Omega,\Gamma)$.

Proof. Let H_ε be the space $H^1(\Omega_\varepsilon)$ with the scalar product (14). Let A^ε be the positive, selfadjoint and compact operator defined on H_ε as follows: for any $\{F,f\} \in H_\varepsilon$, $A^\varepsilon\{F,f\} = \{U^\varepsilon, u^\varepsilon\}$ where $\{U^\varepsilon, u^\varepsilon\} \in H_\varepsilon$ is the unique solution of

$$(\{U^\varepsilon, u^\varepsilon\}, \{G,g\})_\varepsilon = \varepsilon^m \int_\Omega FG\,dx + \frac{1}{\varepsilon}\int_{\omega_\varepsilon} fg\,dx \quad \forall\{G,g\} \in H_\varepsilon. \tag{17}$$

Obviously, the eigenvalues of A^ε are $\{(1+\lambda_k^\varepsilon\varepsilon^{-m})^{-1}\}_{k=0}^\infty$ where $\{\lambda_k^\varepsilon\}_{k=0}^\infty$ are the eigenvalues of (4). Moreover, taking $\{G,g\}=\{U^\varepsilon,u^\varepsilon\}$ in (17), we have $\|A^\varepsilon\{F,f\}\|_\varepsilon^2 = \|\{U^\varepsilon,u^\varepsilon\}\|_\varepsilon^2 \leq \|\{F,f\}\|_\varepsilon\|\{U^\varepsilon,u^\varepsilon\}\|_\varepsilon$ and $\|A^\varepsilon\|_{\mathcal{L}(H_\varepsilon)}\leq 1$.

In a similar way, let H be the Hilbert space $\mathcal{H}^{1,1}(\Omega,\Gamma)$ with the scalar product

$$(U,V)_0 = A\int_\Omega \nabla_x U\cdot\nabla_x V\,dx + a\int_\Gamma h\partial_\tau U\partial_\tau V\,d\tau + \int_\Gamma hUV\,d\tau \quad \forall U,V\in\mathcal{H}^{1,1}(\Omega,\Gamma). \tag{18}$$

We consider the operator defined on H by $\mathsf{A}F=U$ where $U\in H$ is the unique solution of

$$(U,G)_0 = \int_\Gamma hFG\,d\tau \quad \forall G\in\mathcal{H}^{1,1}(\Omega,\Gamma).$$

A is a selfadjoint compact operator. The eigenvalues of A are $\{(1+\mu_k)^{-1}\}_{k=0}^\infty\bigcup\{0\}$ where $\{\mu_k\}_{k=0}^\infty$ are the eigenvalues of (8) with finite multiplicity while $\mu=0$ is an eigenvalue of infinite multiplicity; the eigenspace associated with $\mu=0$ is $\mathsf{W}=\{V\in\mathcal{H}^{1,1}(\Omega,\Gamma)\,:\,V=0 \text{ on } \Gamma\}$.

Let H_0 be the orthogonal complement of W in H. By definition of operator A, $Im(\mathsf{A})\subset H_0$, and we can consider the operator $A^0: H_0\to H_0$ as the restriction of A. It is clear that A^0 is now a positive, selfadjoint and compact operator whose eigenvalues are $\{(1+\mu_k)^{-1}\}_{k=0}^\infty$ where $\{\mu_k\}_{k=0}^\infty$ are the eigenvalues of (8).

Let $\mathcal{W}$ be the space H_0 and let $\mathcal{R}^\varepsilon$ be the linear, continuous operator $\mathcal{R}^\varepsilon: H_0\to H_\varepsilon$ defined by (15). In order to apply Lemma 2.1, we check the properties *a)–d)*.

First, let us note that property *b)* is satisfied by construction of operators A^ε and A^0. In addition, taking into account the definition of $\mathcal{R}^\varepsilon$ (15) and considering the integral in ω_ε of $\tilde F$ and $\nabla_x\tilde F$, the estimates (11) and (12) provide

$$\begin{gathered}\left|\frac{1}{\varepsilon}\|\tilde F\|^2_{L^2(\omega_\varepsilon)} - \|F\|^2_{L^2_h(\Gamma)}\right| \leq C\varepsilon\|F\|^2_{L^2_h(\Gamma)} \quad\text{and}\\ \left|\frac{1}{\varepsilon}\|\nabla_x\tilde F\|^2_{L^2(\omega_\varepsilon)} - \|\partial_\tau F\|^2_{L^2_h(\Gamma)}\right| \leq C\varepsilon\|\partial_\tau F\|^2_{L^2_h(\Gamma)}.\end{gathered} \tag{19}$$

Then, we have the convergence $\|\mathcal{R}^\varepsilon F\|_\varepsilon^2 = \|\{F,\tilde F\}\|_\varepsilon^2\to\|F\|_0^2$ as $\varepsilon\to 0$ for any $F\in H_0$, and property *a)* of Lemma 2.1 also holds.

Let us prove property *c)*. For each $\varepsilon>0$ and any fixed $F\in H_0$, we consider $\{U^\varepsilon,u^\varepsilon\}=A^\varepsilon\mathcal{R}^\varepsilon F$. Then, $\{U^\varepsilon,u^\varepsilon\}\in H_\varepsilon$ satisfies (17) for $f=\tilde F$. Taking $\{G,g\}=\{U^\varepsilon,u^\varepsilon\}$ in (17) for $f=\tilde F$ and using (19), we obtain that, for ε small enough,

$$\|\{U^\varepsilon,u^\varepsilon\}\|_\varepsilon\leq C \tag{20}$$

where C is a constant independent of ε. For each $\varepsilon>0$, we introduce in (20) the change of variables in ω_ε from Cartesian coordinates x_1,x_2 to local coordinates (10). Then, taking into account that bounds (11) also hold for $K_\varepsilon(\zeta,\tau)=1+\varepsilon\zeta\varkappa(\tau)$ $\forall\zeta\in[0,h(\tau)],\tau\in\mathbb{S}_\ell$ and sufficiently small ε, we can write

$$\|\nabla_x U^\varepsilon\|^2_{L^2(\Omega)}+\varepsilon^m\|U^\varepsilon\|^2_{L^2(\Omega)}+\|\mathsf{u}^\varepsilon\|^2_{L^2(\Pi)}+\|\partial_\tau\mathsf{u}^\varepsilon\|^2_{L^2(\Pi)}+\varepsilon^{-2}\|\partial_\zeta\mathsf{u}^\varepsilon\|^2_{L^2(\Pi)}\leq C \tag{21}$$

where $\mathsf{u}^\varepsilon(\zeta,\tau)$ denote the eigenfunctions $u^\varepsilon(x)$ in the local coordinates (ζ,τ) and Π is the domain $\{(\zeta,\tau)\,:\,\tau\in(0,\ell),\zeta\in(0,h(\tau))\}$. Now, since $U^\varepsilon=\mathsf{u}^\varepsilon$ on Γ, considering $\|U^\varepsilon\|_{H^1(\Omega)}\leq C[\|U^\varepsilon\|_{L^2(\Gamma)}+\|\nabla_x U^\varepsilon\|_{L^2(\Omega)}]$ and the trace inequality $\|\mathsf{u}^\varepsilon\|_{H^{1/2}(\Gamma)}\leq C\|\mathsf{u}^\varepsilon\|_{H^1(\Pi)}$, we conclude that U^ε is bounded in $H^1(\Omega)$. Therefore, we can extract a subsequence, still denoted by ε, such that $(U^\varepsilon,\mathsf{u}^\varepsilon)$ converges, as $\varepsilon\to 0$, towards some function (U^*,u^*) weakly in $H^1(\Omega)\times H^1(\Pi)$. Moreover, from

(21) it follows that $\|\partial_\zeta \mathsf{u}^\varepsilon\|^2_{L^2(\Pi)} \leq C\varepsilon^2$ and consequently $\partial_\zeta \mathsf{u}^* = 0$ in Π; thus, u^* does not depend on ζ and $\mathsf{u}^*(\zeta,\tau) = U^*(0,\tau)$ in Π.

In order to identify U^*, for any $V \in \mathcal{H}^{1,1}(\Omega,\Gamma)$ fixed, we consider the variational formulation of $\{U^\varepsilon, u^\varepsilon\} = A^\varepsilon \mathcal{R}^\varepsilon F$ with the test functions $\{G, g\} = \mathcal{R}^\varepsilon V = \{V, \tilde{V}\}$, and introduce (10):

$$A\int_\Omega \nabla_x U^\varepsilon \cdot \nabla_x V\,dx + a\int_0^\ell \int_0^{h(\tau)} \partial_\tau \mathsf{u}^\varepsilon \partial_\tau \tilde{V} K_\varepsilon^{-1}\,d\zeta d\tau + \varepsilon^m \int_\Omega U^\varepsilon V\,dx$$
$$+ \int_0^\ell \int_0^{h(\tau)} \mathsf{u}^\varepsilon \tilde{V} K_\varepsilon\,d\zeta d\tau = \varepsilon^m \int_\Omega FV\,dx + \int_0^\ell \int_0^{h(\tau)} \tilde{F}\tilde{V} K_\varepsilon\,d\zeta d\tau. \tag{22}$$

Taking into account that K_ε and K_ε^{-1} converge towards 1 in $L^\infty(\Pi)$ when $\varepsilon \to 0$, we pass to the limit in (22) and obtain

$$A\int_\Omega \nabla_x U^* \cdot \nabla_x V\,dx + a\int_\Gamma h\partial_\tau U^* \partial_\tau V\,d\zeta d\tau + \int_\Gamma hU^* V\,d\tau = \int_\Gamma hFV\,d\tau$$

since $\mathsf{u}^*(\zeta,\tau) = U^*(0,\tau)$ in Π; then, $U^* = \mathsf{A}F$. It is clear that $U^* \in H_0$ and $U^* = A^0 F$.

Finally, we prove $\|A^\varepsilon \mathcal{R}^\varepsilon F - \mathcal{R}^\varepsilon A^0 F\|_\varepsilon \to 0$ as $\varepsilon \to 0$. By virtue of the variational formulation of $\{U^\varepsilon, u^\varepsilon\}$, (15), the change to local variables and the fact that $\mathsf{u}^*(\zeta,\tau) = U^*(0,\tau)$ in Π, we can write

$$\begin{aligned}
&\|A^\varepsilon \mathcal{R}^\varepsilon F - \mathcal{R}^\varepsilon A^0 F\|^2_\varepsilon \\
=&\varepsilon^m \int_\Omega F(U^\varepsilon - U^*)\,dx + \int_0^\ell \int_0^{h(\tau)} \tilde{F}(\mathsf{u}^\varepsilon - \mathsf{u}^*) K_\varepsilon\,d\zeta d\tau \\
-&A\int_\Omega \nabla_x U^* \cdot \nabla_x (U^\varepsilon - U^*)\,dx - a\int_0^\ell \int_0^{h(\tau)} \partial_\tau \mathsf{u}^* \partial_\tau (\mathsf{u}^\varepsilon - \mathsf{u}^*) K_\varepsilon^{-1}\,d\zeta d\tau \\
-&\varepsilon^m \int_\Omega U^*(U^\varepsilon - U^*)\,dx - \int_0^\ell \int_0^{h(\tau)} \mathsf{u}^*(\mathsf{u}^\varepsilon - \mathsf{u}^*) K_\varepsilon\,d\zeta d\tau.
\end{aligned}$$

Now, we have that all the terms converge towards zero and property *c)* of Lemma 2.1 is satisfied.

In a similar way to property *c)*, we can prove that for any sequence $\{F^\varepsilon, f^\varepsilon\}$ in H_ε such that $\|\{F^\varepsilon, f^\varepsilon\}\|_\varepsilon$ is bounded by a constant independent of ε, we can extract a subsequence, still denoted by ε, verifying $\|A^\varepsilon\{F^\varepsilon, f^\varepsilon\} - \mathcal{R}^\varepsilon V_0\|_\varepsilon \to 0$ as $\varepsilon \to 0$, for a certain function $V_0 \in H_0$. Thus, A^ε is uniformly compact, and property *d)* of Lemma 2.1 holds.

Applying Lemma 2.1, we have that for each fixed k, $\lambda_k^\varepsilon \varepsilon^{-m}$ converge towards μ_k when $\varepsilon \to 0$. Moreover, for any eigenvalue μ_k of (8) with multiplicity $\varkappa_k$, and for any eigenfunction V of (8) corresponding to μ_k, V satisfying (9), there is a linear combination $\{\tilde{U}^\varepsilon, \tilde{u}^\varepsilon\}$ of eigenfunctions corresponding to $\{\lambda_i^\varepsilon\}_{i=k}^{k+\varkappa_k-1}$ such that (16) holds.

As regards the proof of the last statement in the theorem, we consider the sequence $\{U_k^\varepsilon, u_k^\varepsilon\}$ of eigenfunctions of (4), $\|\{U_k^\varepsilon, u_k^\varepsilon\}\|_\varepsilon = 1$. Taking into account the change to local coordinates (10) in ω_ε, Friedrichs' inequality in Ω and the trace theorem in Π, $(U_k^\varepsilon, \mathsf{u}_k^\varepsilon)$ is bounded in $H^1(\Omega) \times H^1(\Pi)$ and $\|\partial_\zeta \mathsf{u}_k^\varepsilon\|^2_{L^2(\Pi)} \leq C\varepsilon^2$, where by u_k^ε we denote the eigenfunctions u_k^ε written in the local coordinates. Then, using a classical argument of diagonalization we extract a subsequence (still denoted by ε) such that $(U_k^\varepsilon, \mathsf{u}_k^\varepsilon) \to (V_k^*, \mathsf{v}_k^*)$ weakly in $H^1(\Omega) \times H^1(\Pi)$, as $\varepsilon \to 0$. Besides,

$V_k^* \in \mathcal{H}^{1,1}(\Omega,\Gamma)$ and $\mathsf{v}_k^* = V_k^*(0,\tau)$ in Π. If we assume that this limit $V_k^* \neq 0$, since $\lambda_k^\varepsilon \varepsilon^{-m} \to \mu_k$ as $\varepsilon \to 0$ and $\mathsf{v}_k^* = V_k^*(0,\tau)$ in Π, we identify V_k^* with an eigenfunction of (8) corresponding to μ_k by taking limits in (4) for the test functions $\{G,g\} = \mathcal{R}^\varepsilon V$ with any fixed $V \in \mathcal{H}^{1,1}(\Omega,\Gamma)$, once we have performed the change to the local coordinates (ζ,τ) in ω_ε. Then, using again the convergence $\lambda_k^\varepsilon \varepsilon^{-m} \to \mu_k$ and (4) for $\{G,g\} = \{U_k^\varepsilon, u_k^\varepsilon\}$, we have $1 = \|\{U_k^\varepsilon, u_k^\varepsilon\}\|_\varepsilon^2 \to (\mu_k+1)[\|V_k^*\|^2_{L^2(\Omega)} + \|V_k^*\|^2_{L^2_h(\Gamma)}]$ and prove that $V_k^* \neq 0$.

The fact that the V_k^* are orthogonal in $\mathcal{H}^{1,1}(\Omega,\Gamma)$ for the scalar product (18) follows from the orthogonality condition for $\{U_k^\varepsilon, u_k^\varepsilon\}$. Then, we prove that the set $\{V_k^*\}_{k=0}^\infty$ forms a basis in the orthogonal complement of $\{V \in \mathcal{H}^{1,1}(\Omega,\Gamma) : V = 0 \text{ on } \Gamma\}$ in $\mathcal{H}^{1,1}(\Omega,\Gamma)$ for the scalar product (18) by contradiction (cf., for instance, Chapter II of [1] for the technique). Therefore, the theorem is proved. □

3. **The middle frequencies when h is a constant.** We study the asymptotic behavior of the eigenvalues of (1) of order $O(\varepsilon^{m-2})$ when $m > 2$, the so–called middle frequencies, and their corresponding eigenfunctions. Different limit behaviors appear for these frequencies depending on whether the function h defining the domain ω_ε is constant or not. In this section, we provide asymptotic expansions for the case where $h \equiv h_0$ with h_0 a positive constant while those for the case where h is not a constant are provided in Section 4. The justification for both asymptotic expansions is given in Section 5.

Since a boundary layer phenomenon appears in a neighborhood of Γ, it proves necessary to consider outer expansions for the eigenfunctions in Ω and inner expansions in a neighborhood of Γ in the local coordinates (ζ,τ) in (10). Thus, we write the Laplace operator in curvilinear coordinates,

$$\Delta_{\nu,\tau} = K(\nu,\tau)^{-1}\partial_\nu(K(\nu,\tau)\partial_\nu) + K(\nu,\tau)^{-1}\partial_\tau(K(\nu,\tau)^{-1}\partial_\tau), \tag{23}$$

and also the normal derivative at the boundary Γ_ε:

$$\partial_n = (1+\varepsilon^2 K(\nu,\tau)^{-2} h'(\tau)^2)^{-1/2}\left(\partial_\nu - \varepsilon h'(\tau) K(\nu,\tau)^{-2}\partial_\tau\right). \tag{24}$$

Then, we use the local variable (10) and we gather the coefficients at the different powers of ε. We write

$$\Delta_{\zeta,\tau} = \varepsilon^{-2}\,\partial_\zeta^2 + \varepsilon^{-1}\,\varkappa(\tau)\partial_\zeta - \varkappa(\tau)^2\zeta\partial_\zeta + \partial_\tau^2 + \cdots, \tag{25}$$

and

$$(1+\varepsilon^2 K(\nu,\tau)^{-2} h'(\tau)^2)^{1/2}\partial_n = \varepsilon^{-1}\partial_\zeta - \varepsilon h'(\tau)\partial_\tau + \cdots, \tag{26}$$

where here and in the sequel the dots denote further asymptotic terms of different powers of ε which in general are not used to derive our results.

We consider an asymptotic expansion for the eigenvalues λ^ε and for the corresponding eigenfunctions $\{U^\varepsilon, u^\varepsilon\}$ in Ω and ω_ε of the form:

$$\lambda^\varepsilon = \varepsilon^{m-2}(\lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \cdots), \tag{27}$$

$$U^\varepsilon(x) = V(x) + \varepsilon \mathsf{V}_1(x) + \varepsilon^2 \mathsf{V}_2(x) + \cdots, \qquad x \in \Omega, \tag{28}$$

$$u^\varepsilon(\zeta,\tau) = v_0(\zeta,\tau) + \varepsilon v_1(\zeta,\tau) + \varepsilon^2 v_2(\zeta,\tau) + \cdots, \qquad \zeta \in [0,h(\tau)), \tau \in \mathbb{S}_\ell, \tag{29}$$

respectively, where v_i are ℓ–periodic functions in τ. Besides, we assume that at least one of the functions V or v_0 in (28)–(29) are different from zero (see normalization in (75), convergence (77) and Remark 1).

By replacing expansions (27)–(29) in problem (1), after considering equations (25) and (26), we collect coefficients of the same powers of ε. In a first step, we see that the leading terms in (27)–(29) satisfy the following equations:

$$-A\Delta_x V = 0 \quad \text{in } \Omega, \tag{30}$$

$$-a\partial_\zeta^2 v_0 = \lambda_0 v_0, \quad \zeta \in (0, h(\tau)),\, \tau \in \mathbb{S}_\ell, \tag{31}$$

$$V = v_0 \quad \text{on } \Gamma, \tag{32}$$

$$a\partial_\zeta v_0(h(\tau), \tau) = 0, \quad \tau \in \mathbb{S}_\ell, \tag{33}$$

$$a\partial_\zeta v_0(0, \tau) = 0, \quad \tau \in \mathbb{S}_\ell. \tag{34}$$

From (31), (33) and (34), for $h \equiv h_0 > 0$, we deduce that λ_0 is an eigenvalue of

$$\begin{cases} -ay_0'' = \lambda_0 y_0 & \zeta \in (0, h_0), \\ y_0'(0) = y_0'(h_0) = 0 \end{cases} \tag{35}$$

and

$$v_0(\zeta, \tau) = y_0(\zeta) v(\tau), \quad \zeta \in (0, h_0),\, \tau \in \mathbb{S}_\ell, \tag{36}$$

where $y_0(\zeta)$ is an eigenfunction of (35) corresponding to λ_0 and $v(\tau)$, at this stage, is an arbitrary function of τ. Obviously, by assumption, $v(\tau)$ do not vanish in $\mathbb{S}_\ell$ (see (30), (32) and (36)). It is clear that the eigenvalues of (35) are given by

$$\lambda_{0,k} = \frac{ak^2\pi^2}{h_0^2} \qquad \text{for } k = 0, 1, 2 \dots \tag{37}$$

and the corresponding eigenfunctions can be chosen to be

$$y_{0,k}(\zeta) = \cos\left(\frac{k\pi}{h_0}\zeta\right) \qquad \text{for } k = 0, 1, 2 \dots, \tag{38}$$

while $v(\tau)$ in (36) has to been determined.

In a second step, we obtain the problem

$$-a\partial_\zeta^2 v_1 - a\varkappa\, \partial_\zeta v_0 = \lambda_0 v_1 + \lambda_1 v_0, \quad \zeta \in (0, h_0),\, \tau \in \mathbb{S}_\ell, \tag{39}$$

$$a\partial_\zeta v_1(h_0, \tau) = 0, \quad \tau \in \mathbb{S}_\ell, \tag{40}$$

$$a\partial_\zeta v_1(0, \tau) = 0, \quad \tau \in \mathbb{S}_\ell. \tag{41}$$

Since $v_0(\zeta, \tau) = y_0(\zeta) v(\tau)$ verifies (31), (33) and (34), the compatibility condition for the non–homogeneous Neumann problem (39)–(41) in the ζ–variable reads

$$-a\varkappa(\tau) v(\tau) \int_0^{h_0} y_0'(\zeta) y_0(\zeta)\, d\zeta = \lambda_1 v(\tau) \int_0^{h_0} y_0(\zeta)^2\, d\zeta, \quad \tau \in \mathbb{S}_\ell.$$

Moreover, by (38), $\int_0^{h_0} y_0' y_0\, d\zeta = \frac{1}{2}(y_0(h_0)^2 - y_0(0)^2) = 0$ and we have that $\lambda_1 = 0$.

Considering $\lambda_0 \neq 0$, (see Remark 1 otherwise), the functions v_1 satisfying (39)–(41) can be written in the form

$$v_1(\zeta, \tau) = \varkappa(\tau) v(\tau) y_1(\zeta), \quad \zeta \in (0, h_0),\, \tau \in \mathbb{S}_\ell,$$

where $y_1(\zeta)$ is a solution of

$$\begin{cases} -ay_1'' - \lambda_0 y_1 = ay_0' & \zeta \in (0, h_0), \\ y_1'(0) = y_1'(h_0) = 0. \end{cases}$$

In fact, for each fixed eigenpair (λ_0, y_0) of (35), we can choose the solution $y_1(\zeta)$ above to be the unique solution which satisfies $\int_0^{h_0} y_1(\zeta) y_0(\zeta)\, d\zeta = 0$, and then, for

$(\lambda_0, y_0) = (\lambda_{0,k}, y_{0,k})$ verifying (37) and (38) we have $v_1(\zeta,\tau) = v_{1,k}(\zeta,\tau)$ defined by

$$v_{1,k}(\zeta,\tau) = -\frac{1}{2}\varkappa(\tau)v(\tau)\left[\left(\zeta - \frac{h_0}{2}\right)\cos\left(\frac{k\pi}{h_0}\zeta\right) - \frac{h_0}{k\pi}\sin\left(\frac{k\pi}{h_0}\zeta\right)\right] \quad \text{for } k = 1, 2\dots \tag{42}$$

Following the process, in the next step, we have the problem for v_2:

$$\begin{aligned} &-a\partial_\zeta^2 v_2 - a\varkappa\,\partial_\zeta v_1 + a\varkappa^2\,\zeta\partial_\zeta v_0 - a\partial_\tau^2 v_0 \\ &= \lambda_0 v_2 + \lambda_1 v_1 + \lambda_2 v_0, \quad \zeta \in (0,h_0),\, \tau \in \mathbb{S}_\ell, \end{aligned} \tag{43}$$

$$a\partial_\zeta v_2(h_0,\tau) = 0, \quad \tau \in \mathbb{S}_\ell, \tag{44}$$

$$a\partial_\zeta v_2(0,\tau) = A\partial_\nu V(0,\tau), \quad \tau \in \mathbb{S}_\ell. \tag{45}$$

Since $\lambda_1 = 0$ and v_0 verifies (31), (33) and (34), the compatibility condition for the non–homogeneous Neumann problem (43)–(45) reads

$$\begin{aligned} A\partial_\nu V(0,\tau)y_0(0) - a\int_0^{h_0}(\varkappa(\tau)\,\partial_\zeta v_1(\zeta,\tau) - \varkappa(\tau)^2\,\zeta\partial_\zeta v_0(\zeta,\tau) + \partial_\tau^2 v_0(\zeta,\tau))y_0(\zeta)\,d\zeta \\ = \lambda_2\int_0^{h_0} v_0(\zeta,\tau)y_0(\zeta)\,d\zeta. \end{aligned} \tag{46}$$

Now, by virtue of (36), (38) and (42) we get

$$A\partial_\nu V(0,\tau) + \varkappa(\tau)^2 v(\tau)\frac{3ah_0}{8} - v''(\tau)\frac{ah_0}{2} = \lambda_2 v(\tau)\frac{h_0}{2}.$$

The last compatibility condition can be regarded as a boundary condition for the function V in equation (30). Thus, due to (32), (λ_2, V) in (27)–(28) is an eigenpair of the spectral problem

$$\begin{cases} -A\Delta_x V = 0 & \text{in } \Omega, \\ A\partial_\nu V = \dfrac{ah_0}{2}\partial_\tau^2 V - \dfrac{3ah_0}{8}\varkappa^2 V + \lambda_2\dfrac{h_0}{2}V & \text{on } \Gamma, \end{cases} \tag{47}$$

which does not depend on λ_0 (see Remark 5).

The weak formulation of (47) is: Find λ_2 and $V \in \mathcal{H}^{1,1}(\Omega,\Gamma)$, $V \neq 0$, such that

$$\begin{aligned} &A\int_\Omega \nabla_x V\cdot\nabla_x W\,dx + \frac{ah_0}{2}\int_\Gamma \partial_\tau V\partial_\tau W\,d\tau + \frac{3ah_0}{8}\int_\Gamma \varkappa^2 VW\,d\tau \\ &= \lambda_2\frac{h_0}{2}\int_\Gamma VW\,d\tau \quad \forall W \in \mathcal{H}^{1,1}(\Omega,\Gamma)\,. \end{aligned} \tag{48}$$

Since the left hand side of (48) defines a scalar product in $\mathcal{H}^{1,1}(\Omega,\Gamma)$ and the embedding of this space in $L^2(\Omega)$ (in $L^2(\Gamma)$, respectively) is compact, we can write an eigenvalue problem for a non–negative, symmetric and compact operator $\mathcal{A}$ on $\mathcal{H}^{1,1}(\Omega,\Gamma)$ defined by

$$(\mathcal{A}U, W) = \frac{h_0}{2}\int_\Gamma UW \quad \forall U, W \in \mathcal{H}^{1,1}(\Omega,\Gamma),$$

whose eigenvalues are 0, with the associated eigenspace $\{U \in \mathcal{H}^{1,1}(\Omega,\Gamma) : U = 0 \text{ on } \Gamma\}$, and $(\lambda_2)^{-1}$ with finite multiplicity and λ_2 an eigenvalue of (48). Therefore, (48) has a positive discrete spectrum which we denote by $\{\lambda_2^p\}_{p=1}^\infty$.

Hence, we have found the doubles sequences for the middle frequencies

$$\lambda^\varepsilon \sim \varepsilon^{m-2}\frac{ak^2\pi^2}{h_0^2} + \varepsilon^m\lambda_2^p, \qquad k,p = 1,2,\dots \tag{49}$$

for which the corresponding eigenfunctions $\{U^\varepsilon, u^\varepsilon\}$ have asymptotics in Ω_ε given by

$$U^\varepsilon(x) \sim V^p(x) \qquad x \in \Omega,$$

and

$$u^\varepsilon(\nu,\tau) \sim v^p_{0,k}(\frac{\nu}{\varepsilon},\tau) + \varepsilon v^p_{1,k}(\frac{\nu}{\varepsilon},\tau), \qquad x \in \omega_\varepsilon$$

where V^p is an eigenfunction of (47) corresponding to λ_2^p, $v^p_{0,k}(\zeta,\tau)=y_{0,k}(\zeta)V^p(0,\tau)$, $y_{0,k}$ is an eigenfunction of (35) corresponding to the eigenvalue $ak^2\pi^2h_0^{-2}$ (cf. (38)), and $v^p_{1,k}$ is given by (42) for $v(\tau) \equiv V^p(0,\tau)$.

Remark 1. We observe that for the first eigenvalue of (35), $\lambda_{0,0} = 0$, the functions $v_0(\zeta,\tau)$ and $v_1(\zeta,\tau)$ solutions of (31), (33), (34) and (39)–(41) respectively only depend on τ and, consequently, the compatibility condition (46) along with (30) and (32) lead us to problem (6) with $\mu = \lambda_2$. Thus, we are in the range of the low frequencies and the asymptotics in this section are in good agreement with the asymptotic expansions in [10] of (1) and with the convergence results in Section 2 of this paper.

In fact, for $\lambda_{0,0} \neq 0$, we have proved throughout this section that the assumption on $v_0 \neq 0$ or $V \neq 0$ in (29) and (28) implies that both functions are different from zero. Also, it should be noted that, in order to consider expansions (29) and (28), a certain normalization for the eigenfunctions $\{U^\varepsilon, u^\varepsilon\}$ corresponding to the eigenvalues (27) should be prescribed. In this connection, we refer to normalizations in $\mathcal{H}^\varepsilon$, and formulas (75), (80) and (81) which outline that the integrals on the τ-derivatives of v_0 in ω_ε compensate with the integrals on gradients of V in Ω.

Remark 2. It should be pointed out that formula (37), obtained from (31), (33) and (34), holds for $k > 1$ only if h is a constant function, namely, $h \equiv h_0$; otherwise, the asymptotic expansions have to be modified as we do in Section 4 in order to obtain the first term in (27) as an eigenvalue of a spectral problem independent of τ. As a matter of fact, the dependence on τ of the eigenvalue problem (31), (33) and (34) leads us to predict that in small neighborhoods of certain fixed points $\tau_0 \in \mathbb{S}_\ell$, asymptotically, we can consider an eigenvalue problem depending on τ_0; namely, depending on de geometry $\partial\Omega_\varepsilon$ defined by the function h at τ_0. This problem would keep τ as a parameter, in a similar way to (31), (33) and (34) for h constant. In Section 4, we show that this is possible assuming that τ_0 is a local maximum of h.

4. **Middle frequencies for a non-constant function h.** In this section we assume that h is not constant and that there is a point $\tau_0 \in \mathbb{S}_\ell$ where h has a local maximum and such that $h''(\tau_0) < 0$ (see Remark 10 for other possible cases of local maxima where $h''(\tau_0) = 0$).

In order to isolate a neighborhood of this point τ_0, it proves useful to introduce the *new local variables* defined by

$$\xi = \frac{\nu}{\varepsilon h(\tau)} \quad \text{and} \quad \eta = \frac{\tau - \tau_0}{\varepsilon^\gamma} \tag{50}$$

with γ a constant, $\gamma > 0$. For any $d > 0$, the change (50) transforms the narrow band $\{(\nu,\tau) : \nu \in [0, \varepsilon h(\tau)), |\tau - \tau_0| < d\}$ into the band $\{(\xi,\eta) : \xi \in [0,1), \eta \in (-d\varepsilon^{-\gamma}, d\varepsilon^{-\gamma})\}$ of width $O(1)$ and length $O(\varepsilon^{-\gamma})$, and it leads us to consider a local or limiting problem in $[0,1) \times \mathbb{R}$ independent of the geometry.

Taking into account the Taylor expansions of $h(\tau)$ and $\varkappa(\tau)$ in a neighborhood of τ_0, we introduce the new variables in the Laplacian, namely, in formula (23), and gather the different powers of ε. Since $h'(\tau_0) = 0$, we have

$$\begin{aligned}\Delta_{\xi,\eta} =&\varepsilon^{-2}\frac{1}{h(\tau_0)^2}\,\partial^2_\xi - \varepsilon^{2\gamma-2}\frac{h''(\tau_0)}{h(\tau_0)^3}\,\eta^2\,\partial^2_\xi + \varepsilon^{-1}\frac{\varkappa(\tau_0)}{h(\tau_0)}\,\partial_\xi + \varepsilon^{\gamma-1}\frac{\varkappa'(\tau_0)}{h(\tau_0)}\,\eta\,\partial_\xi \\ &+ \varepsilon^{-2\gamma}\partial^2_\eta + \cdots\end{aligned} \tag{51}$$

We proceed in a similar way for the normal derivative at the boundary Γ_ε, namely for (24), and we can write

$$(1+\varepsilon^2 K(\nu,\tau)^{-2}h'(\tau)^2)^{1/2}\partial_n = \varepsilon^{-1}\frac{1}{h(\tau_0)}\,\partial_\xi - \varepsilon^{2\gamma-1}\frac{h''(\tau_0)}{2h(\tau_0)^2}\,\eta^2\,\partial_\xi - \varepsilon h''(\tau_0)\,\eta\,\partial_\eta + \cdots \tag{52}$$

Following the idea in [12] and [17] for localized eigenfunctions, among the possible choices of γ we consider one that leads us to an eigenvalue problem in $L^2(\mathbb{R})$ for the Hermite differential operator in the "tangencial" variable η (cf. (67)). The idea is to obtain a limit problem which has solutions decaying exponentially at infinity in one of the variables (see Remark 13). Equalizing the exponents of ε in the second and fifth terms on the right hand side of (51) yields $\gamma = 1/2$ in (50) which also keeps the ansatz (27) valid for a local expansion of the eigenfunctions in the local variables (50) (see Remark 3 for other possible values of γ).

Consequently, for $\gamma = 1/2$ in (50), we consider the following asymptotic expansions for the eigenelements $(\lambda^\varepsilon, \{U^\varepsilon, u^\varepsilon\})$ of (1):

$$\lambda^\varepsilon = \varepsilon^{m-2}(\lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \cdots), \tag{53}$$

$$U^\varepsilon(x) = V^\varepsilon(x) + \varepsilon V_1^\varepsilon(x) + \varepsilon^2 V_2^\varepsilon(x) + \cdots, \qquad x \in \Omega, \tag{54}$$

$$u^\varepsilon(\xi,\eta) = v_0(\xi,\eta) + \varepsilon v_1(\xi,\eta) + \varepsilon^2 v_2(\xi,\eta) + \cdots, \qquad \xi \in [0,1), \eta \in \mathbb{R}. \tag{55}$$

Besides, we assume that V^ε in (54) or v_0 in (55) are different from zero. We observe that, as happens for (27), the first term λ_0 in (53) can be zero, but when dealing with $\lambda^\varepsilon = O(\varepsilon^{m-2})$ we must avoid this possibility (cf. Remark 1). Also, we note that we have assumed that the outer expansion (54) can be a non-regular expansion and we allow the terms arising in the expansion to be dependent on ε and x simultaneously and (55) is the expansion in the fast variables.

After considering equations (51) and (52), we replace expansions (53)–(55) in problem (1) and collect coefficients of the same powers of ε. In a first step, we have that the leading terms in (53)–(55) satisfy (30), (32),

$$-\frac{a}{h(\tau_0)^2}\,\partial^2_\xi v_0 = \lambda_0 v_0, \quad \xi \in (0,1), \eta \in \mathbb{R}, \tag{56}$$

$$\frac{a}{h(\tau_0)}\,\partial_\xi v_0(1,\eta) = 0, \quad \eta \in \mathbb{R}, \tag{57}$$

$$\frac{a}{h(\tau_0)}\,\partial_\xi v_0(0,\eta) = 0, \quad \eta \in \mathbb{R}. \tag{58}$$

Thus, λ_0 is an eigenvalue of

$$\begin{cases} -\dfrac{a}{h(\tau_0)^2}y_0'' = \lambda_0 y_0, \quad \xi \in (0,1), \\ y_0'(0) = y_0'(1) = 0, \end{cases} \tag{59}$$

and

$$v_0(\xi,\eta) = y_0(\xi)v(\eta), \quad \xi \in (0,1),\, \eta \in \mathbb{R}, \tag{60}$$

where $y_0(\xi)$ is an eigenfunction of (59) corresponding to λ_0 and $v(\eta)$ is an arbitrary function of η to be determined. The eigenvalues of (59) are given by

$$\lambda_{0,k} = \frac{ak^2\pi^2}{h(\tau_0)^2} \qquad \text{for } k = 0, 1, 2\dots \tag{61}$$

and the corresponding eigenfunctions can be chosen to be

$$y_{0,k}(\xi) = \cos(k\pi\xi) \qquad \text{for } k = 0, 1, 2\dots \tag{62}$$

In the second step, we have

$$\begin{aligned} &-\frac{a}{h(\tau_0)^2}\,\partial_\xi^2 v_1 + a\frac{h''(\tau_0)}{h(\tau_0)^3}\,\eta^2\,\partial_\xi^2 v_0 - a\frac{\varkappa(\tau_0)}{h(\tau_0)}\,\partial_\xi v_0 - a\partial_\eta^2 v_0 \\ &= \lambda_0 v_1 + \lambda_1 v_0, \quad \xi\in(0,1), \eta\in\mathbb{R}, \end{aligned} \tag{63}$$

$$\frac{a}{h(\tau_0)}\,\partial_\xi v_1(0,\eta) - \frac{ah''(\tau_0)}{2h(\tau_0)^2}\,\eta^2\,\partial_\xi v_0(0,\eta) = 0, \quad \eta\in\mathbb{R}, \tag{64}$$

$$\frac{a}{h(\tau_0)}\,\partial_\xi v_1(1,\eta) - \frac{ah''(\tau_0)}{2h(\tau_0)^2}\,\eta^2\,\partial_\xi v_0(1,\eta) = 0, \quad \eta\in\mathbb{R}. \tag{65}$$

Since $v_0(\xi,\eta) = y_0(\xi)v(\eta)$ verifies (56)–(58), we rewrite (64) and (65) as

$$\partial_\xi v_1(0,\eta) = \partial_\xi v_1(1,\eta) = 0, \quad \eta\in\mathbb{R}, \tag{66}$$

and, the compatibility condition for the non–homogeneous Neumann problem (63), (66) provides:

$$\begin{aligned} &\frac{ah''(\tau_0)}{h(\tau_0)^3}\eta^2 v(\eta)\int_0^1 y_0''(\xi)y_0(\xi)\,d\xi - \frac{a\varkappa(\tau_0)}{h(\tau_0)}v(\eta)\int_0^1 y_0'(\xi)y_0(\xi)\,d\xi \\ &\qquad - av''(\eta)\int_0^1 y_0(\xi)^2\,d\xi = \lambda_1 v(\eta)\int_0^1 y_0(\xi)^2\,d\xi, \quad \eta\in\mathbb{R}. \end{aligned}$$

Considering the explicit form of the solutions of (59), namely (62), we compute $\int_0^1 y_0' y_0\,d\xi = \frac{1}{2}(y_0(1)^2 - y_0(0)^2) = 0$ as well as the rest of the integrals in the compatibility condition above, and we get the equation for the eigenpair (λ_1, v):

$$-\lambda_0\frac{h''(\tau_0)}{h(\tau_0)}\eta^2 v(\eta) - av''(\eta) = \lambda_1 v(\eta), \quad \eta\in\mathbb{R}. \tag{67}$$

On account of the assumption $h''(\tau_0) < 0$, for each fixed $\lambda_0 > 0$, the changes

$$\widetilde{\eta} = \left(-\lambda_0\frac{h''(\tau_0)}{ah(\tau_0)}\right)^{1/4}\eta\,, \quad v(\eta) = exp(-\widetilde{\eta}^2/2)\widetilde{V}(\widetilde{\eta}), \quad \alpha = \frac{\lambda_1}{2a}\left(-\lambda_0\frac{h''(\tau_0)}{ah(\tau_0)}\right)^{-1/2} - \frac{1}{2}$$

in (67) lead us to the Hermite equation with parameter α, namely,

$$\widetilde{V}''(\widetilde{\eta}) - 2\widetilde{\eta}\widetilde{V}'(\widetilde{\eta}) + 2\alpha\widetilde{V}(\widetilde{\eta}) = 0\,,$$

which provides two linear independent solutions, and for any natural α, one of these solutions is a polynomial (cf. the Hermite polynomial of degree α). Therefore, by prescribing the condition that the solutions $exp(-\widetilde{\eta}^2/2)\widetilde{V}(\widetilde{\eta})$ belong to $L^2(\mathbb{R})$, (67) has a discrete spectrum (see, for example, Chapter IX of [5] for a proof) and we can compute the eigenvalues λ_1 and the corresponding eigenfunctions $v(\eta)$ as follows:

For each $\lambda_0 > 0$ eigenvalue of (59), we have the sequence of eigenvalues of (67)

$$\lambda_{1,p} = \left(-a\lambda_0\frac{h''(\tau_0)}{h(\tau_0)}\right)^{1/2}(2p-1) \quad \text{for } p = 1, 2\dots$$

while the corresponding eigenfunctions are

$$v^p(\eta) = C_p \exp\left(- \left(-\lambda_0 \frac{h''(\tau_0)}{a h(\tau_0)} \right)^{1/2} \frac{\eta^2}{2} \right) H_{p-1}\!\left(\left(-\lambda_0 \frac{h''(\tau_0)}{a h(\tau_0)} \right)^{1/4} \eta \right) \tag{68}$$

for $p = 1, 2 \dots$, where C_p are arbitrary constants and H_{p-1} are the Hermite polynomials of degree $p-1$.

Hence, we have identified the first two terms in the expansion (53) which shows a splitting of the middle frequencies into a double series

$$\lambda^\varepsilon = \varepsilon^{m-2} \frac{a k^2 \pi^2}{h(\tau_0)^2} + \varepsilon^{m-1} \left(-a \frac{h''(\tau_0)}{h(\tau_0)} \right)^{1/2} \frac{\sqrt{a} k \pi}{h(\tau_0)} (2p-1) \cdots \qquad k, p = 1, 2 \dots, \tag{69}$$

for which the first term in (55) is also determined by (60) with $v \equiv v^p$ given by (68).

Let us determine the second term in (55). By virtue of (60), (59) and (67), equation (63) becomes

$$-\frac{a}{h(\tau_0)^2} \partial_\xi^2 v_1 - a \frac{\varkappa(\tau_0)}{h(\tau_0)} \partial_\xi v_0 = \lambda_0 v_1, \quad \xi \in (0,1), \eta \in \mathbb{R},$$

and $v_1(\xi, \eta)$ can be obtained by separation of variables as

$$v_1(\xi, \eta) = v(\eta) y_1(\xi), \quad \xi \in (0,1),\ \eta \in \mathbb{R},$$

with $v(\eta)$ the function in (60) and $y_1(\xi)$ satisfying

$$\left\{ \begin{array}{l} -\dfrac{a}{h(\tau_0)^2} y_1'' - \lambda_0 y_1 = a \dfrac{\varkappa(\tau_0)}{h(\tau_0)} y_0', \quad \xi \in (0,1), \\ y_1'(0) = y_1'(1) = 0. \end{array} \right. \tag{70}$$

Indeed, by prescribing the condition $\int_0^1 v_1(\xi, \eta) y_0(\xi)\, d\xi = 0$, for each fixed eigenpair (λ_0, y_0) of (59), $(\lambda_0, y_0) \equiv (\lambda_{0,k}, y_{0,k})$ defined by (61) and (62), (70) determines uniquely $y_1(\xi)$ which can be explicitly computed. Hence, we have:

$$v_{1,k}(\xi, \eta) = -\frac{1}{2} \frac{\varkappa(\tau_0)}{h(\tau_0)} v(\eta) \left[\left(\xi - \frac{1}{2} \right) \cos(k\pi\xi) - \frac{1}{k\pi} \sin(k\pi\xi) \right] \qquad \text{for } k = 1, 2 \dots$$

From the reasoning above, it should be noted that searching for localized eigenfunctions we have determined the first two terms in (53) and (55), while the terms in the outer expansion (54) are yet to be computed in order that expansions (54) and (55) match up to a certain order.

To this end, we note that (55) is defined by the *fast* variable $\eta = (\tau - \tau_0)\varepsilon^{-1/2}$ and the function $v_0(\xi, \eta) + \varepsilon v_1(\xi, \eta)$ is somewhat localized in a neighborhood of $\eta = 0$, namely in $\{x \in \omega_\varepsilon : |\tau - \tau_0| < K\varepsilon^{1/2}, \nu \in (0, \varepsilon h(\tau))\}$ with K a positive constant, and it is exponentially small outside. Specifying further, $v(\eta)$ is exponentially small for τ satisfying $|\tau - \tau_0| = \varepsilon^p$ with any $p < 1/2$.

Hence, in order to get an approximation of $\{U^\varepsilon, u^\varepsilon\}$ in the whole Ω_ε, we introduce a cut–off function $\chi \in C^\infty(\mathbb{R})$ such that for a fixed d, $0 < d < \ell/2$,

$$\chi(s) = \left\{ \begin{array}{l} 1, \text{ as } |s| < d/2 \\ 0, \text{ as } |s| > d. \end{array} \right. \tag{71}$$

Then, we set

$$u^\varepsilon(\nu, \tau) \sim \chi(\tau - \tau_0)\, v\!\left(\frac{\tau - \tau_0}{\varepsilon^{1/2}} \right) \left[y_0\!\left(\frac{\nu}{\varepsilon h(\tau)} \right) + \varepsilon y_1\!\left(\frac{\nu}{\varepsilon h(\tau)} \right) \right] \quad \text{for } (\nu, \tau) \in \omega_\varepsilon, \tag{72}$$

where (λ_0, y_0) is an eigenpair of (59), (λ_1, v) is an eigenpair of (67) and y_1 is the solution of (70).

Now, the condition $(1)_3$, along with (53), (54) and (55), provides the first term in the outer expansion (54) to be the solution of the non-homogeneous Dirichlet problem

$$\begin{cases} -A\Delta_x V^\varepsilon = 0 & \text{in } \Omega, \\ V^\varepsilon(x) = \chi(\tau-\tau_0)\, v\Big(\dfrac{\tau-\tau_0}{\varepsilon^{1/2}}\Big)\,[y_0(0)+\varepsilon y_1(0)] & \text{on } \Gamma. \end{cases} \tag{73}$$

On account of the smoothness of the non-homogeneous data on Γ, for each fixed $\varepsilon > 0$, we have that the problem (73) has a unique solution $V^\varepsilon \in H^2(\Omega)$ and we can set

$$U^\varepsilon(x) \sim V^\varepsilon(x) \quad \text{for } x \in \Omega. \tag{74}$$

In addition, since the data is located at $supp(\chi)$ and the function v decays exponentially with the distance to τ_0, one may expect that V^ε, as well as its derivatives up to the order k, be of order $o(1)$ at a distance of the order $O(1)$ of τ_0 (also, at a distance $O(\varepsilon^{p_k})$ for a certain $p_k < 1/2$ depending on k).

Hence, formally, from (72) and (74), we have localized eigenfunctions corresponding to eigenvalues in the middle frequencies range in (69) which concentrate asymptotically their support in $C\varepsilon^{1/2}$–neighborhoods of τ_0. It remains to justify approximations (72) and (74) and to show the estimates above for V^ε; this is performed in Section 5.2 (see Theorem 5.6, Lemma 5.5, Remarks 8, 10 and 13).

Remark 3. It should be noted that, under the assumption $h''(\tau_0) < 0$, the choice of $\gamma = 1/2$ in (50) is crucial for obtaining the eigenvalue problem (67) and the results throughout this section. Indeed, the second and fifth terms on the right hand side of (51) allow us to obtain a countable set of eigenvalues of (67) and the exponential decay of their corresponding eigenfunctions. For any other value of $\gamma > 0$, the possible limiting problems do not have the above properties and the discrepancies between powers of ε can be larger when we replace (53) and (55) in $(1)_2$. See Remark 10 for the case where $h''(\tau_0) = 0$.

Remark 4. Also, we emphasize that the technique and the results throughout the section can be applied to the case where the parameter t dealing with the stiffness of the band ω_ε is greater than 1. Indeed, unlike the case where h is a constant function, we observe that here we only need to consider the first two terms in (53) and (55) to determine both $v(\eta)$ and $v_0(\xi, \eta)$. Thus, the leading terms of the asymptotic expansions (53), (54) and (55) of the eigenpairs of (1) coincide with those of problem $(1)_1$, $(1)_3$, $(1)_5$, (2) and (3) with $t > 1$. Consequently, the asymptotic expansions in this section hold for $t > 1$ up to the first order while higher order terms in (53), (54) and (55) can be different depending on t. Also, as regards the convergence, similar estimates to those obtained in Theorem 5.6 hold for $t > 1$.

Remark 5. We note that when the function h is constant (cf. Section 3), the resulting problem (47) for the leading term in (28) is independent of the positive eigenvalues and corresponding eigenfunctions of the limit problem (35). In fact, the resulting problem (47) provides the second term in the asymptotics (27), the first and second term in (29), and the first term in (28).

Instead, when h is not a constant and it has a local maximum at τ_0 (cf. Section 4), the resulting equation (67) in the Dirichlet problem for the leading term in (54) depends on the eigenvalue $\lambda_0 > 0$ of the limit problem (59). Also, the second term in the asymptotics (53) depends on this eigenvalue.

5. On the convergence for the middle frequencies.

In this section, we justify the asymptotic expansions in Sections 3 and 4. In particular, we provide estimates for convergence rates of the eigenvalues of (1) of order $O(\varepsilon^{m-2})$ and also of the corresponding eigenfunctions as stated in Theorems 5.3 and 5.6 depending on h. Throughout the section we assume that m is a parameter $m > 2$. The case where the function h is a constant is in Section 5.1 while the case where h is not constant but it has local maxima points is in Section 5.2.

We first introduce some notations and results of further use. For each $\varepsilon > 0$, let us denote by $\mathcal{H}^\varepsilon$ the space $H^1(\Omega_\varepsilon)$ with the scalar product

$$\begin{aligned}(\{U,u\},\{V,v\})_{\mathcal{H}^\varepsilon} =&\varepsilon^2 A\int_\Omega \nabla_x U\cdot\nabla_x V\,dx + \varepsilon a\int_{\omega_\varepsilon}\nabla_x u\cdot\nabla_x v\,dx\\ &+\varepsilon^m\int_\Omega UV\,dx+\frac{1}{\varepsilon}\int_{\omega_\varepsilon} uv\,dx \quad \forall\{U,u\},\{V,v\}\in H^1(\Omega_\varepsilon).\end{aligned} \tag{75}$$

Let $\mathcal{A}^\varepsilon$ be a positive, compact and symmetric operator on $\mathcal{H}^\varepsilon$ defined by

$$(\mathcal{A}^\varepsilon\{U,u\},\{G,g\})_{\mathcal{H}^\varepsilon} = \varepsilon^m\int_\Omega UG\,dx+\frac{1}{\varepsilon}\int_{\omega_\varepsilon} ug\,dx \qquad \forall\{U,u\},\{G,g\}\in H^1(\Omega_\varepsilon).$$

It is clear that the eigenvalues of $\mathcal{A}^\varepsilon$ are $\{(1+\lambda_k^\varepsilon\varepsilon^{2-m})^{-1}\}_{k=0}^\infty$ where $\{\lambda_k^\varepsilon\}_{k=0}^\infty$ are the eigenvalues of (1).

In order to prove convergence results, we use a classical result on "almost eigenvalues and eigenvectors" from the spectral perturbation theory, namely, Lemma 5.1 (see, for instance, Chapter III in [20] for a proof).

Lemma 5.1. *Let $A : H \longrightarrow H$ be a linear, self-adjoint, positive and compact operator on a separable Hilbert space H. Let $u \in H$, with $\|u\|_H = 1$ and λ, $r > 0$ such that $\|Au - \lambda u\|_H \le r$. Then, there exists an eigenvalue λ_i of the operator A satisfying the inequality $|\lambda - \lambda_i| \le r$. Moreover, for any $r^* > r$ there is $u^* \in H$, with $\|u^*\|_H = 1$, u^* belonging to the eigenspace associated with all the eigenvalues of the operator A lying on the segment $[\lambda - r^*, \lambda + r^*]$ and such that*

$$\|u-u^*\|_H \le \frac{2r}{r^*}.$$

For convenience, we also introduce here Lemma 5.2 which is related to the spectral convergence for large eigenvalues (see [4], for instance, for a proof.)

Lemma 5.2. *Let $\{\mathcal{T}^\varepsilon\}_{\varepsilon\in[0,1]}$ be a family of selfadjoint and compact operators on a Hilbert space H. For each ε, let $\{\mu_i^\varepsilon\}_{i=1}^\infty$ be the sequence of the eigenvalues of $\mathcal{T}^\varepsilon$ with the classical convention of repeated eigenvalues. Let us assume that the family $\mathcal{T}^\varepsilon$ satisfies the following property: for each $i\in\mathbb{N}$ the function $\mu^i(\varepsilon) = \mu_i^\varepsilon$ is continuous with respect to ε in $[0,1]$. Then, for each $\beta > 0$ and $\lambda > 0$ there exists a sequence $\varepsilon_j \to 0$ and a sequence of natural numbers $\{i(\varepsilon_j)\}_{j\in\mathbb{N}}$, $i(\varepsilon_j)\to\infty$, such that*

$$\left(\mu_{i(\varepsilon_j)}^{\varepsilon_j}\right)^{-1}\varepsilon_j^\beta = \lambda.$$

5.1. **Case where h is the constant h_0.** In this section, we justify the asymptotic expansions in Section 3 up to a certain degree which can be improved by constructing higher order terms in (27)–(29) (see Remark 6). When justifying (27), we provide estimates which establish the closeness of the eigenvalues of (1) and $\lambda_0+\varepsilon^2\lambda_2$, where (λ_0, λ_2) are pairs of eigenvalues of (35) and (47) (see (27) and (76)). When justifying asymptotics (28)–(29) for the eigenfunctions $\{U^\varepsilon, u^\varepsilon\}$, we deal with groups of eigenfunctions corresponding to eigenvalues λ^ε of (1) verifying (76) and the approaches hold in the topology for $H^1(\Omega_\varepsilon)$ given by the scalar product (75), in the way stated by the Theorem 5.3 (see (86)). We also provide a complementary result that allows us to assert that any approach of the re-scaled eigenvalues of (1), $\lambda^\varepsilon\varepsilon^{2-m}$, to other values different from the eigenvalues of (35) gives approaches through zero for the corresponding eigenfunctions in a certain topology stated in Theorem 5.4. Throughout the section we consider the change (10) and denote by ω the torus $(0,h_0)\times\mathbb{S}_\ell$. Let us note that choosing a suitable normalization for the eigenfunctions (cf. (75)) is essential in order to obtain the results throughout the section.

Theorem 5.3. *Let (λ_0, y_0) and (λ_2, V) be eigenpairs of (35) and (47) respectively, $\lambda_0 \neq 0$, such that $\|y_0\|^2_{L^2(0,h_0)} = \|V\|^{-2}_{L^2(\Gamma)} = h_0/2$. Let us consider $v_0(\zeta,\tau) = y_0(\zeta)V(0,\tau)$ for $(\zeta,\tau)\in\omega$. Then, if $m>2$, there are eigenvalues $\lambda^\varepsilon_{k(\varepsilon)}$ of problem (1) such that*

$$\left|\frac{\lambda^\varepsilon_{k(\varepsilon)}}{\varepsilon^{m-2}} - \lambda_0 - \varepsilon^2\lambda_2\right| \leq C(\varepsilon^3+\varepsilon^{m/2}) \tag{76}$$

where C is a constant independent of ε. Moreover, there is a linear combination of eigenfunctions $\{\tilde U^\varepsilon, \tilde u^\varepsilon\}\in H^1(\Omega_\varepsilon)$, $\{\tilde U^\varepsilon, \tilde u^\varepsilon\}$ corresponding to eigenvalues $\lambda^\varepsilon_{k(\varepsilon)}$ of (1) which satisfy $\lambda^\varepsilon_{k(\varepsilon)}\varepsilon^{2-m}\in[\lambda_0-K\varepsilon^\theta, \lambda_0+K\varepsilon^\theta]$ with $K>0$ and $0<\theta<\min(2,m/2-1)$, $\|\{\tilde U^\varepsilon,\tilde u^\varepsilon\}\|_{\mathcal{H}^\varepsilon}=1$, such that

$$\|\tilde U^\varepsilon - \alpha^\varepsilon V\|_{H^1(\Omega)} + \|\tilde{\mathsf{u}}^\varepsilon - \alpha^\varepsilon v_0\|_{H^1(\omega)} \leq C\varepsilon^{\min(2-\theta, m/2-1-\theta, 1)} \tag{77}$$

where $\tilde{\mathsf{u}}^\varepsilon(\zeta,\tau)\equiv\tilde u^\varepsilon(x)$ for $(\zeta,\tau)\in\omega$, and α^ε is a well determined constant (see (78) and (87), $\alpha^\varepsilon\to(\sqrt{1+\lambda_0}\,)^{-1}$, as $\varepsilon\to 0$. As a consequence, $\tilde U^\varepsilon$ ($\tilde{\mathsf{u}}^\varepsilon$, respectively) converge towards αV (αv_0, respectively) in $H^1(\Omega)$ ($H^1(\omega)$, respectively) as $\varepsilon\to 0$, the constant α being $\alpha=(\sqrt{1+\lambda_0}\,)^{-1}$.

Proof. Let $\lambda_0, \lambda_2, y_0, V$ and v_0 be as the theorem states. Let us consider $v_1, v_2\in H^1(\omega)$ satisfying periodic conditions on $\tau=0$ and $\tau=\ell$ and verifying problems (39)–(41) and (43)–(45) respectively for $\lambda_1=0$; v_1, v_2 are determined uniquely by prescribing the orthogonality conditions

$$\int_0^{h_0} v_i(\zeta,\tau)y_0(\zeta)\,d\zeta = 0 \quad \text{for } i=1,2.$$

Note that v_0, v_1, v_2 and V are smooth functions, in particular, $v_0, v_1, v_2\in H^2(\omega)$ and $V\in H^2(\Omega)$.

For sufficiently small ε, we consider the function $\{W^\varepsilon, w^\varepsilon\}$ defined by

$$\left\{\begin{array}{ll} W^\varepsilon(x) = V(x) + \varepsilon P v_1(x) + \varepsilon^2 P v_2(x) & \text{if } x\in\Omega, \\ w^\varepsilon(\nu,\tau) = v_0(\nu/\varepsilon,\tau) + \varepsilon v_1(\nu/\varepsilon,\tau) + \varepsilon^2 v_2(\nu/\varepsilon,\tau) & \text{if } 0\leq\nu\leq\varepsilon h_0, \tau\in\mathbb{S}_\ell, \end{array}\right. \tag{78}$$

where $P : H^2(\omega) \to H^2(\Omega)$ is the continuous operator such that Pv is a harmonic function and $(Pv)|_\Gamma = v(0,\tau)$ for any $v \in H^2(\omega)$. It is clear that $\{W^\varepsilon, w^\varepsilon\} \in H^1(\Omega_\varepsilon)$. In order to apply Lemma 5.1 we first prove the estimate

$$\begin{aligned} \Big|\Big(\mathcal{A}^\varepsilon\{\tilde{W}^\varepsilon, \tilde{w}^\varepsilon\} - \frac{1}{1+\lambda_0+\varepsilon^2\lambda_2}\{\tilde{W}^\varepsilon, \tilde{w}^\varepsilon\}, \{W, w\}\Big)_{\mathcal{H}^\varepsilon}\Big| \\ \leq C(\varepsilon^3 + \varepsilon^{m/2})\|\{W,w\}\|_{\mathcal{H}^\varepsilon} \quad \forall \{W,w\} \in \mathcal{H}^\varepsilon, \end{aligned} \tag{79}$$

where $\{\tilde{W}^\varepsilon, \tilde{w}^\varepsilon\} = \{W^\varepsilon, w^\varepsilon\}\|\{W^\varepsilon, w^\varepsilon\}\|_{\mathcal{H}^\varepsilon}^{-1}$ and C is a constant independent of ε.

Considering the definitions of $\mathcal{A}^\varepsilon$ and the scalar product $(\cdot,\cdot)_{\mathcal{H}^\varepsilon}$ in (75), we can write

$$\begin{aligned} &(1+\lambda_0+\varepsilon^2\lambda_2)\left(\mathcal{A}^\varepsilon\{W^\varepsilon, w^\varepsilon\} - \frac{1}{1+\lambda_0+\varepsilon^2\lambda_2}\{W^\varepsilon, w^\varepsilon\}, \{W,w\}\right)_{\mathcal{H}^\varepsilon} \\ =&(\lambda_0+\varepsilon^2\lambda_2)\left(\varepsilon^m\int_\Omega W^\varepsilon W\,dx + \frac{1}{\varepsilon}\int_{\omega_\varepsilon} w^\varepsilon w\,dx\right) \\ &-\varepsilon^2 A\int_\Omega \nabla_x W^\varepsilon\cdot\nabla_x W\,dx - \varepsilon a\int_{\omega_\varepsilon}\nabla_x w^\varepsilon\cdot\nabla_x w\,dx. \end{aligned}$$

For the integrals in ω_ε, we perform the change of variables (10), and we denote by $K_\varepsilon(\zeta,\tau) = 1 + \varepsilon\zeta\varkappa(\tau)$, namely $K(\nu,\tau)$ in the local coordinates (10). The integrals

$$\varepsilon\int_{\omega_\varepsilon}\nabla_x u\cdot\nabla_x v\,dx \quad \text{ and } \quad \frac{1}{\varepsilon}\int_{\omega_\varepsilon} uv\,dx \tag{80}$$

in (75) read

$$\int_\omega \frac{\partial u}{\partial\zeta}\frac{\partial v}{\partial\zeta}K_\varepsilon d\zeta d\tau + \varepsilon^2\int_\omega\frac{\partial u}{\partial\tau}\frac{\partial v}{\partial\tau}K_\varepsilon^{-1}\,d\zeta d\tau \quad \text{ and } \quad \int_\omega uv\,K_\varepsilon\,d\zeta d\tau \tag{81}$$

respectively, where now u and v are written in the new variables (ζ,τ). Thus, taking into account the definition (78) of $\{W^\varepsilon, w^\varepsilon\}$, (10) and the formulas above, we consider the decomposition

$$\begin{aligned} (1+\lambda_0+\varepsilon^2\lambda_2)&\Big(\mathcal{A}^\varepsilon\{W^\varepsilon, w^\varepsilon\} - \frac{1}{1+\lambda_0+\varepsilon^2\lambda_2}\{W^\varepsilon, w^\varepsilon\}, \{W,w\}\Big)_{\mathcal{H}^\varepsilon} \\ &= I_1+I_2+I_3+I_4+I_5+I_6 \end{aligned}$$

where

$$I_1 = \lambda_0\int_\omega v_0 w\,d\zeta d\tau - a\int_\omega \partial_\zeta v_0\partial_\zeta w\,d\zeta d\tau\,,$$

$$\begin{aligned} I_2 =\varepsilon\Big(&\lambda_0\int_\omega v_0 w\zeta\varkappa\,d\zeta d\tau + \lambda_0\int_\omega v_1 w\,d\zeta d\tau \\ &-a\int_\omega\partial_\zeta v_0\partial_\zeta w\zeta\varkappa\,d\zeta d\tau - a\int_\omega \partial_\zeta v_1\partial_\zeta w\,d\zeta d\tau\Big)\,, \end{aligned}$$

$$\begin{aligned} I_3 =\varepsilon^2\Big(&\lambda_0\int_\omega v_1 w\zeta\varkappa\,d\zeta d\tau + \lambda_0\int_\omega v_2 w\,d\zeta d\tau + \lambda_2\int_\omega v_0 w\,d\zeta d\tau - A\int_\Omega\nabla V\cdot\nabla W\,dx \\ &-a\int_\omega\partial_\tau v_0\partial_\tau w\,d\zeta d\tau - a\int_\omega\partial_\zeta v_1\partial_\zeta w\zeta\varkappa\,d\zeta d\tau - a\int_\omega\partial_\zeta v_2\partial_\zeta w\,d\zeta d\tau\Big), \end{aligned}$$

$$I_4 = \varepsilon^3 \left(\lambda_0 \int_\omega v_2 w \zeta \varkappa \, d\zeta d\tau + \lambda_2 \int_\omega v_0 w \zeta \varkappa \, d\zeta d\tau \right.$$
$$\left. + \lambda_2 \int_\omega (v_1 + \varepsilon v_2) w K_\varepsilon \, d\zeta d\tau - a \int_\omega \partial_\zeta v_2 \partial_\zeta w \zeta \varkappa \, d\zeta d\tau \right),$$
$$I_5 = -a\varepsilon^2 \int_\omega \partial_\tau v_0 \partial_\tau w (K_\varepsilon^{-1} - 1) \, d\zeta d\tau - a\varepsilon^3 \int_\omega \partial_\tau (v_1 + \varepsilon v_2) \partial_\tau w K_\varepsilon^{-1} \, d\zeta d\tau,$$
$$I_6 = (\lambda_0 + \varepsilon^2 \lambda_2) \varepsilon^m \int_\Omega W^\varepsilon W \, dx - A\varepsilon^3 \int_\Omega \nabla_x (P v_1 + \varepsilon P v_2) \cdot \nabla_x W \, dx \, .$$

Then, we prove the estimate (79) for each I_i above.

Indeed, the fact that v_0, v_1, v_2, V satisfy (31)–(34), (39)–(41), (43)–(45) and (47) respectively leads us to $I_1 = I_2 = I_3 = 0$. In addition, on account of (12), (75) and the change of variables (10), we have

$$|I_4| \leq C\varepsilon^3 \|\{W, w\}\|_{\mathcal{H}^\varepsilon} .$$

As regards I_5, integrating by parts in ω and taking into account the smoothness of v_0, v_1 and v_2 and the definition of K_ε and of the scalar product (75) yields

$$|I_5| \leq C\varepsilon^3 [\|\partial_\tau^2 v_0\|_{L^2(\omega)} \|w\|_{L^2(\omega)} + \|\partial_\tau v_0\|_{L^2(\omega)} \|w\|_{L^2(\omega)} + \|\partial_\tau^2 v_1\|_{L^2(\omega)} \|w\|_{L^2(\omega)}$$
$$+ \|\partial_\tau v_1\|_{L^2(\omega)} \|w\|_{L^2(\omega)} + \|\partial_\tau v_2\|_{L^2(\omega)} \|\partial_\tau w\|_{L^2(\omega)}] \leq C\varepsilon^3 \|\{W, w\}\|_{\mathcal{H}^\varepsilon} .$$

On the other hand, in order to obtain bounds for I_6, we first consider the trace inequality

$$\|w\|_{L^2(\Gamma)}^2 \leq C [\varepsilon^{-1} \|w\|_{L^2(\omega_\varepsilon)}^2 + \|w\|_{L^2(\omega_\varepsilon)} \|\partial_\nu w\|_{L^2(\omega_\varepsilon)}] \quad \forall w \in H^1(\omega_\varepsilon), \tag{82}$$

which is obtained by integrating over $\tau \in \Gamma$ the formula

$$|u(0,\tau)|^2 = -\int_0^{\varepsilon h(\tau)} \partial_\nu \left[\left(1 - \frac{\nu}{\varepsilon h(\tau)} \right) u(\nu,\tau)^2 \right] d\nu$$
$$\leq \int_0^{\varepsilon h(\tau)} \frac{1}{\varepsilon h(\tau)} |u(\nu,\tau)|^2 \, d\nu + 2 \int_0^{\varepsilon h(\tau)} |u(\nu,\tau) \partial_\nu u(\nu,\tau)| \, d\nu \quad \forall u \in C^1(\overline{\omega}_\varepsilon),$$

and allows us to obtain (see (75), (80) and (81))

$$\|w\|_{L^2(\Gamma)} \leq C \|\{W, w\}\|_{\mathcal{H}^\varepsilon} \quad \forall \{W, w\} \in H^1(\Omega_\varepsilon). \tag{83}$$

We also consider the trace inequality

$$\|\partial_\nu U\|_{L^2(\Gamma)} \leq C \|U\|_{H^2(\Omega)} \quad \forall U \in H^2(\Omega). \tag{84}$$

Then, integrating by parts in Ω and using the definition of the operator P, the fact that $W = w$ on Γ, the definition (75), formula (82) and the trace inequality $\|\partial_\nu (P v_1)\|_{L^2(\Gamma)} \leq C \|P v_1\|_{H^2(\Omega)}$ from (84), we obtain

$$|I_6| \leq C [\varepsilon^m \|W^\varepsilon\|_{L^2(\Omega)} \|W\|_{L^2(\Omega)} + \varepsilon^3 \|\partial_\nu (P v_1)\|_{L^2(\Gamma)} \|W\|_{L^2(\Gamma)}$$
$$+ \varepsilon^4 \|\nabla_x (P v_2)\|_{L^2(\Omega)} \|\nabla_x W\|_{L^2(\Omega)}] \leq C(\varepsilon^{m/2} + \varepsilon^3) \|\{W, w\}\|_{\mathcal{H}^\varepsilon} .$$

Finally, considering the local coordinates (10), we verify that $\|\{W^\varepsilon, w^\varepsilon\}\|_{\mathcal{H}^\varepsilon}^2 \to \|v_0\|_{L^2(\omega)}^2 + \|\partial_\zeta v_0\|_{L^2(\omega)}^2 = 1 + \lambda_0$ as $\varepsilon \to 0$, and we have proved that the estimate (79) holds for sufficiently small ε.

Now, we apply Lemma 5.1 for $H = \mathcal{H}^\varepsilon$, $A = \mathcal{A}^\varepsilon$, $\lambda = (1 + \lambda_0 + \varepsilon^2 \lambda_2)^{-1}$ and $u = \{\tilde{W}^\varepsilon, \tilde{w}^\varepsilon\}$ and $r = C(\varepsilon^3 + \varepsilon^{m/2})$ which provides, for sufficiently small ε, at least one eigenvalue $\lambda_{k(\varepsilon)}^\varepsilon$ of (1) verifying $|(1 + \lambda_{k(\varepsilon)}^\varepsilon \varepsilon^{2-m})^{-1} - (1 + \lambda_0 + \varepsilon^2 \lambda_2)^{-1}| \leq C(\varepsilon^3 + \varepsilon^{m/2})$, and consequently, we deduce (76). Moreover, if we take, for instance, $r^* = \varepsilon^\theta$ with

$0 < \theta < \min(2, m/2-1)$, Lemma 5.1 also provides a function $\{\tilde{U}^\varepsilon, \tilde{u}^\varepsilon\} \in \mathcal{H}^\varepsilon$, with $\|\{\tilde{U}^\varepsilon, \tilde{u}^\varepsilon\}\|_{\mathcal{H}^\varepsilon} = 1$, $\{\tilde{U}^\varepsilon, \tilde{u}^\varepsilon\}$ belonging to the eigenspace associated with all the eigenvalues $(1+\lambda^\varepsilon_{k(\varepsilon)}\varepsilon^{2-m})^{-1}$ of $\mathcal{A}^\varepsilon$ contained in

$$[(1+\lambda_0+\varepsilon^2\lambda_2)^{-1} - \varepsilon^\theta, (1+\lambda_0+\varepsilon^2\lambda_2)^{-1} + \varepsilon^\theta], \tag{85}$$

such that

$$\|\{\tilde{U}^\varepsilon, \tilde{u}^\varepsilon\} - \alpha^\varepsilon\{W^\varepsilon, w^\varepsilon\}\|_{\mathcal{H}^\varepsilon} \leq C(\varepsilon^{3-\theta} + \varepsilon^{m/2-\theta}) \tag{86}$$

is satisfied where

$$\alpha^\varepsilon = \|\{W^\varepsilon, w^\varepsilon\}\|^{-1}_{\mathcal{H}^\varepsilon}. \tag{87}$$

Now, from (78) and (75) and (10), we conclude that, for $m > 2$,

$$\|\tilde{\mathsf{u}}^\varepsilon - \alpha^\varepsilon v_0\|_{L^2(\omega)} + \|\partial_\zeta(\tilde{\mathsf{u}}^\varepsilon - \alpha^\varepsilon v_0)\|_{L^2(\omega)} \leq C\varepsilon^{\min(3-\theta, m/2-\theta)}$$

and

$$\|\partial_\tau(\tilde{\mathsf{u}}^\varepsilon - \alpha^\varepsilon v_0)\|_{L^2(\omega)} + \|\nabla_x(\tilde{U}^\varepsilon - \alpha^\varepsilon V)\|_{L^2(\Omega)} \leq C\varepsilon^{\min(2-\theta, m/2-1-\theta, 1)}.$$

Finally, since $\tilde{U}^\varepsilon|_\Gamma = \tilde{\mathsf{u}}^\varepsilon(0,\tau)$ and $V|_\Gamma = v_0(0,\tau)$, Friedrichs' inequality for $\tilde{U}^\varepsilon$ in Ω and the trace inequality for $\tilde{\mathsf{u}}^\varepsilon$ in ω lead us to assert estimate (77) and $\tilde{U}^\varepsilon$ ($\tilde{\mathsf{u}}^\varepsilon$, respectively) converge towards αV (αv_0, respectively) in $H^1(\Omega)$ ($H^1(\omega)$, respectively) as $\varepsilon \to 0$ being $\alpha = (\sqrt{1+\lambda_0})^{-1}$. Therefore, the theorem is proved. □

Remark 6. Let us note that (76) justifies the asymptotic expansions (27) up to order $O(\varepsilon^2)$ when $m > 4$. In the case where $2 < m \leq 4$, (76) and (86) also provide a justification for the first terms arising in (27)–(29) while it is necessary to construct explicitly the further terms in (28) to improve the estimate of $|I_6|$ and consequently the estimate in (76). We also note that in fact, the proof of Theorem 5.3 provides extra information on the approaches of the eigenpairs of (1) to the eigenpairs of (35) and (47) complementing that outlined in the statement of the theorem (see (85) and (86)).

Theorem 5.4. *Let λ_* be any positive real number which is not an eigenvalue of problem (31), (33) and (34) (problem (35) equivalently). Let m be $m > 2$, and let δ_ε denote any sequence of positive numbers converging towards zero as $\varepsilon \to 0$. Let us assume that there are re-scaled eigenvalues $\lambda^\varepsilon\varepsilon^{2-m}$ of problem (4) in the interval $[\lambda_*-\delta^\varepsilon, \lambda_*+\delta^\varepsilon]$. Let us consider $\{\tilde{U}^\varepsilon, \tilde{u}^\varepsilon\} \in H^1(\Omega_\varepsilon)$ any linear combination of eigenfunctions of (1) corresponding to the eigenvalues λ^ε such that $\lambda^\varepsilon\varepsilon^{2-m} \in [\lambda_*-\delta^\varepsilon, \lambda_*+\delta^\varepsilon]$, $\{\tilde{U}^\varepsilon, \tilde{u}^\varepsilon\}$ satisfying $\|\{\tilde{U}^\varepsilon, \tilde{u}^\varepsilon\}\|_{\mathcal{H}^\varepsilon} = 1$. Then, $\tilde{\mathsf{u}}^\varepsilon$ and $\partial_\zeta\tilde{\mathsf{u}}^\varepsilon$ converge towards zero in the weak topology of $L^2(\omega)$ as $\varepsilon \to 0$, being $\tilde{\mathsf{u}}^\varepsilon(\zeta,\tau) \equiv \tilde{u}^\varepsilon(x)$ for $(\zeta,\tau) \in \omega$.*

Proof. First, we consider the case where there is only one re-scaled eigenvalue $\lambda^\varepsilon\varepsilon^{2-m}$ in the interval of the statement. Let $\{U^\varepsilon, u^\varepsilon\}$ be the corresponding eigenfunction satisfying $\|\{U^\varepsilon, u^\varepsilon\}\|_{\mathcal{H}^\varepsilon} = 1$. Let us consider u^ε to be the function u^ε in the local coordinates (10), then, we prove that u^ε and $\partial_\zeta\mathsf{u}^\varepsilon$ converge towards zero in the weak topology of $L^2(\omega)$.

Considering the scalar product (75), and (80) and (81), the normalization in $\mathcal{H}^\varepsilon$ reads

$$\varepsilon^2 A\int_\Omega |\nabla_x U^\varepsilon|^2 dx + a\int_\omega |\partial_\zeta \mathsf{u}^\varepsilon|^2 K_\varepsilon\, d\zeta d\tau + \varepsilon^2 a\int_\omega |\partial_\tau \mathsf{u}^\varepsilon|^2 K_\varepsilon^{-1}\, d\zeta d\tau$$
$$+ \varepsilon^m\int_\Omega |U^\varepsilon|^2\, dx + \int_\omega |\mathsf{u}^\varepsilon|^2 K_\varepsilon\, d\zeta d\tau = 1,$$

where $K_\varepsilon(\zeta,\tau)=1+\varepsilon\zeta\varkappa(\tau)$. Taking into account (12), for sufficiently small ε, we have:

$$\varepsilon\|\nabla_x U^\varepsilon\|_{L^2(\Omega)}+\varepsilon^{m/2}\|U^\varepsilon\|_{L^2(\Omega)}+\|\mathsf{u}^\varepsilon\|_{L^2(\omega)}+\|\partial_\zeta \mathsf{u}^\varepsilon\|_{L^2(\omega)}+\varepsilon\|\partial_\tau \mathsf{u}^\varepsilon\|_{L^2(\omega)}\leq C. \tag{88}$$

Hence, we can extract a subsequence (still denote by ε) such that u^ε and $\partial_\zeta\mathsf{u}^\varepsilon$ converge, as $\varepsilon\to0$, weakly in $L^2(\omega)$, towards some functions u^* and ψ^* respectively. It is clear that $\psi^*=\partial_\zeta\mathsf{u}^*$ in $\mathcal{D}'(\omega)$. Assuming that the function u^* is not identically equal to zero, we identify it by limits in the weak formulation (4) for certain suitable test functions that we construct below.

Let v be any function $\mathsf{v}\in H^1(\omega)$ satisfying periodic conditions on $\tau=0$ and $\tau=\ell$. For sufficiently small ε, let us consider the function $\{V^\varepsilon,v^\varepsilon\}\in H^1(\Omega_\varepsilon)$ defined by

$$V^\varepsilon(x)=\begin{cases}\mathsf{v}(-\nu/\varepsilon,\tau)\phi(\nu/\varepsilon) & \text{if } (\nu,\tau)\in(-\varepsilon h_0,0)\times[0,\ell)\\ 0 & \text{outside}\end{cases} \tag{89}$$

and $v^\varepsilon(\nu,\tau)=\mathsf{v}(\nu/\varepsilon,\tau)$ if $(\nu,\tau)\in(0,\varepsilon h_0)\times[0,\ell)$, where $\phi\in C^\infty(\mathbb{R})$, $0\leq\phi\leq1$, $\phi(r)=0$ if $r\leq -h_0$ and $\phi(r)=1$ if $r\geq0$. Considering (10), (80) and (81) it can be verified that

$$\|V^\varepsilon\|_{L^2(\Omega)}\leq C\varepsilon^{1/2}\|\mathsf{v}\|_{L^2(\omega)} \quad\text{and}\quad \varepsilon^{1/2}\|\nabla_x V^\varepsilon\|_{L^2(\Omega)}\leq C\|\mathsf{v}\|_{H^1(\omega)}, \tag{90}$$

where C is a constant independent of ε and v.

Let us consider the variational formulation (4) multiplied by ε^2, with the test functions $\{G,g\}=\{V^\varepsilon,v^\varepsilon\}$ defined by (89), the change of variable (10), (80) and (81). We have:

$$\begin{aligned}\varepsilon^2A\int_\Omega\nabla_xU^\varepsilon\cdot\nabla_xV^\varepsilon\,dx+a\int_\omega\partial_\zeta\mathsf{u}^\varepsilon\partial_\zeta\mathsf{v}K_\varepsilon\,d\zeta d\tau+\varepsilon^2a\int_\omega\partial_\tau\mathsf{u}^\varepsilon\partial_\tau\mathsf{v}K_\varepsilon^{-1}\,d\zeta d\tau\\ =\frac{\lambda^\varepsilon}{\varepsilon^{m-2}}\left(\varepsilon^m\int_\Omega U^\varepsilon V^\varepsilon\,dx+\int_\omega\mathsf{u}^\varepsilon\mathsf{v}K_\varepsilon\,d\zeta d\tau\right).\end{aligned} \tag{91}$$

Taking into account (88), (90), (12), and the normalization of the eigenfunction in $\mathcal{H}^\varepsilon$ (see (75)), for $m>2$, we pass to the limit in (91) when $\varepsilon\to0$ and we get

$$a\int_\omega\partial_\zeta\mathsf{u}^*\partial_\zeta\mathsf{v}\,d\zeta d\tau=\lambda_*\int_\omega\mathsf{u}^*\mathsf{v}\,d\zeta d\tau$$

for all $\mathsf{v}\in H^1(\omega)$ satisfying periodic conditions on $\tau=0$ and $\tau=\ell$. Therefore, (λ_*,u^*) is an eigenpair of (31), (33) and (34), which contradicts the assumption on λ_* in the statement of the theorem. Therefore, $\mathsf{u}^*\equiv0$ in ω and the theorem holds.

Finally, we rewrite the above arguments with minor modifications in the general case where there are several re-scaled eigenvalues of (1), $\lambda^\varepsilon\varepsilon^{2-m}$, in the interval $[\lambda_*-\delta^\varepsilon,\lambda_*+\delta^\varepsilon]$. Indeed, let us denote by $\{\lambda^\varepsilon_{k(\varepsilon)+j}\}_{j=0}^J$ this set of eigenvalues, and $\{\{U^\varepsilon_{k(\varepsilon)+j},u^\varepsilon_{k(\varepsilon)+j}\}\}_{j=0}^J$ the set of the corresponding eigenfunctions; J being a certain natural that can depend on ε. Then, the $\{\tilde U^\varepsilon,\tilde u^\varepsilon\}$ in the statement of the theorem can be written as $\{\tilde U^\varepsilon,\tilde u^\varepsilon\}=\sum_{j=0}^J\alpha_j^\varepsilon\{U^\varepsilon_{k(\varepsilon)+j},u^\varepsilon_{k(\varepsilon)+j}\}$ for certain constants α_j^ε. Let us consider u^* and $\partial_\xi\mathsf{u}^*$ the weak limit in $L^2(\omega)$ of $\widetilde{\mathsf{u}}^\varepsilon$ and $\partial_\xi\widetilde{\mathsf{u}}^\varepsilon$ respectively. We rewrite the reasoning above for each eigenvalue and the corresponding eigenfunction of the set, and for $\{U^\varepsilon,u^\varepsilon\}\equiv\{U^\varepsilon_{k(\varepsilon)+j},u^\varepsilon_{k(\varepsilon)+j}\}$ and $\lambda^\varepsilon\equiv\lambda^\varepsilon_{k(\varepsilon)+j}$ we have equation (91) which we multiply by α_j^ε, for $j=0,1,\cdots,J$, and take the sum. Then, we take into account that $\max_{j=0,1,\cdots J}|\lambda^\varepsilon_{k(\varepsilon)+j}-\lambda_*|$ tends to zero as $\varepsilon \to 0$, and we obtain the equation above for $(\lambda_*, \mathsf{u}^*)$. Again the argument by contradiction leads us to assert that the result in the theorem holds. $\square$

5.2. **Case where h is not constant.** In this section, we justify the asymptotic expansions in Section 4. When justifying (53), we provide estimates which establish the closeness of the eigenvalues $\lambda^\varepsilon = O(\varepsilon^{m-2})$ of (1) and the sets $\lambda_0 + \varepsilon\lambda_1$ where λ_0 and λ_1 are eigenvalues of (59) and (67) respectively. We also give information on the structure of the eigenfunctions corresponding to λ^ε.

In particular, the results in this section show that in the case where the narrow band ω_ε is not constant there are "groups" of eigenfunctions of (1) corresponding to the middle frequencies which have supports localized asymptotically in small neighborhoods of points τ where the function h, which defines the geometry of the boundary $\partial\Omega_\varepsilon$, has a local maximum (see (99) and (94)).

Let us assume that the function h has a local maximum in τ_0 and $h''(\tau_0) < 0$. Let us recall the change of variable (50) from (x_1, x_2) to (ξ, η) for $\gamma = 1/2$, namely

$$\xi = \frac{\nu}{\varepsilon h(\tau)} \quad \text{and} \quad \eta = \frac{\tau - \tau_0}{\varepsilon^{1/2}} \tag{92}$$

Considering the change (92) in ω_ε, the scalar product (75) reads

$$\begin{aligned}
&(\{U,u\},\{V,v\})_{\mathcal{H}^\varepsilon}\\
=&\varepsilon^2 A\int_\Omega \nabla_x U \nabla_x V\,dx + \varepsilon^m \int_\Omega UV\,dx + \varepsilon^{1/2} a\int_{R_\varepsilon} \partial_\xi \mathrm{u}\,\partial_\xi \mathrm{v}\tilde{\mathrm{K}}_\varepsilon h_\varepsilon^{-1}\,d\xi d\eta\\
&+\varepsilon^{5/2} a\int_{R_\varepsilon} \partial_\xi \mathrm{u}\,\partial_\xi \mathrm{v}\xi^2\tilde{\mathrm{K}}_\varepsilon^{-1} h_\varepsilon^{-1}(h'_\varepsilon)^2\,d\xi d\eta + \varepsilon^{3/2} a\int_{R_\varepsilon} \partial_\eta \mathrm{u}\,\partial_\eta \mathrm{v}\tilde{\mathrm{K}}_\varepsilon^{-1} h_\varepsilon\,d\xi d\eta\\
&-\varepsilon^2 a\int_{R_\varepsilon}(\partial_\xi \mathrm{u}\,\partial_\eta \mathrm{v} + \partial_\eta \mathrm{u}\,\partial_\xi \mathrm{v})\xi\tilde{\mathrm{K}}_\varepsilon^{-1} h'_\varepsilon\,d\xi d\eta + \varepsilon^{1/2}\int_{R_\varepsilon} \mathrm{u}\,\mathrm{v}\tilde{\mathrm{K}}_\varepsilon h_\varepsilon\,d\xi d\eta,
\end{aligned} \tag{93}$$

where the functions $u, v \in H^1(\omega_\varepsilon)$, and the functions u, v, $\tilde{K}_\varepsilon$, h_ε and h'_ε are defined as: $\mathrm{u}(\xi,\eta) \equiv u(x)$, $\mathrm{v}(\xi,\eta) \equiv v(x)$, $\tilde{K}_\varepsilon(\xi,\eta) \equiv 1 + \varepsilon\xi h(\tau_0 + \varepsilon^{1/2}\eta)\varkappa(\tau_0 + \varepsilon^{1/2}\eta)$, $h_\varepsilon(\eta) \equiv h(\tau_0 + \varepsilon^{1/2}\eta)$ and $h'_\varepsilon(\eta) \equiv h'(\tau_0 + \varepsilon^{1/2}\eta)$ respectively for $(\xi,\eta) \in R_\varepsilon$. Here R_ε denotes the domain transformed of ω_ε with the change of variable (92).

The main result in this section is stated in Theorem 5.6. Previously, we provide a result which describes the behavior of the solution of problem (73) and which will be used for the proof.

Lemma 5.5. *Let $g \in C^\infty(\mathbb{R})$ be a function verifying*

$$|\nabla_s^k g(s)| \le C_k(1+s^2)^{-1-k/2} \quad \textit{for } s \in \mathbb{R} \textit{ and } k = 0,1,2\dots$$

For $\varepsilon > 0$, let V^ε be the solution of the problem

$$\begin{cases} -\Delta_x V^\varepsilon = 0 & \textit{in } \Omega\\ V^\varepsilon = \chi(\tau-\tau_0) g((\tau-\tau_0)/\varepsilon^\gamma) & \textit{on } \Gamma \end{cases}$$

where $\gamma > 0$, $\tau_0 \in \Gamma$, $\chi \in C^\infty(\mathbb{R})$ is a cut–off function such that $\chi(s) = 1$ as $|s| < d/2$ and $\chi(s) = 0$ as $|s| > d$ for sufficiently small $d > 0$. Then, the function V^ε satisfies for any $0 < \delta < 1$

$$|\nabla_x^k V^\varepsilon(x)| \le c_{k,\delta}\,\varepsilon^{\gamma(1-\delta)}(\varepsilon^{2\gamma} + r^2)^{(\delta-1-k)/2} \quad \forall x \in \Omega \textit{ and } k = 0,1,2\dots \tag{94}$$

r being dist(x,τ_0) and $c_{k,\delta}$ a constant independent of ε.

Proof. Let $G(s,t)$ be a function defined in the half-plane $\mathbb{R}^{2+} = \{(t,s)/s > 0\}$, such that $G(0,s) = g(s)$ and $|\nabla^k_{t,s} G(t,s)| \leq C_k(1+t^2+s^2)^{-1-k/2}$ for $(t,s) \in \mathbb{R}^{2+}$ and $k = 0,1,2\dots$ Then, the function V^ε can be written as $V^\varepsilon = V_1^\varepsilon + V_2^\varepsilon$ where $V_1^\varepsilon(x) = \chi(\nu)\chi(\tau-\tau_0)G(-\nu/\varepsilon^\gamma, (\tau-\tau_0)/\varepsilon^\gamma)$ and V_2^ε is the solution of

$$\begin{cases} -\Delta_x V_2^\varepsilon = F^\varepsilon & \text{in } \Omega \\ V_2^\varepsilon = 0 & \text{on } \Gamma \end{cases}$$

with $F^\varepsilon = \Delta_x V_1^\varepsilon$. By the definition of $G(t,s)$, it is easy to check that for sufficiently small $d > 0$,

$$|\nabla^k_x V_1^\varepsilon(x)| \leq C_k \varepsilon^{2\gamma}(\varepsilon^{2\gamma}+r^2)^{-1-k/2} \qquad \text{for } k = 0,1\dots \tag{95}$$

and consequently (94) holds for $V^\varepsilon = V_1^\varepsilon$.

In order to estimate $|\nabla^k_x V_2^\varepsilon(x)|$, we introduce the space $V^p_\beta(\Omega;\varepsilon^\gamma)$, defined by the Sobolev space $H^p(\Omega)$ equipped with the scalar product

$$(V,W)_{V^p_\beta(\Omega;\varepsilon^\gamma)} = \sum_{j=0}^{p} \int_\Omega (\varepsilon^\gamma + r)^{2(\beta-p+j)} \nabla^j_x V \cdot \nabla^j_x W \, dx \quad \text{for } \beta \in \mathbb{R}, \text{ and } p = 1,2\dots$$

Then, we use two results from the general theory of elliptic problems in domains with singular perturbed boundaries (see Chapter 4 of [7] and [16] for instance). The first result provides the estimate

$$\|V_2^\varepsilon\|_{V^{p+1}_\beta(\Omega;\varepsilon^\gamma)} \leq c_{\beta,p} \|F^\varepsilon\|_{V^{p-1}_\beta(\Omega;\varepsilon^\gamma)} \tag{96}$$

where the constant $c_{\beta,p}$ is independent of the parameter $\varepsilon \in (0,\varepsilon_0]$ if and only if $|\beta - p| < 1$. The second one, due to the embedding theorems of the Sobolev weighted spaces into the Hölder weighted spaces, gives us the relation

$$(\varepsilon^\gamma + r)^{\sigma+k} |\nabla^k_x V_2^\varepsilon(x)| \leq c_k \|V_2^\varepsilon\|_{V^{k+2}_{\sigma+k+1}(\Omega;\varepsilon^\gamma)}, \text{ for } k = 0,1,2\dots \tag{97}$$

Therefore, for each fixed k and $\delta \in (0,1)$, applying (97) and (96) with $\sigma = 1-\delta$, $p = k+1$ and $\beta = 2-\delta+k$, which satisfy $|\beta - p| < 1$, yields

$$(\varepsilon^{2\gamma}+r^2)^{(1-\delta+k)/2} |\nabla^k_x V_2^\varepsilon(x)| \leq c_k \|V_2^\varepsilon\|_{V^{k+2}_{2-\delta+k}(\Omega;\varepsilon^\gamma)} \leq c_{k,\delta} \|F^\varepsilon\|_{V^k_{2-\delta+k}(\Omega;\varepsilon^\gamma)}.$$

In addition, the definition of $V^p_\beta(\Omega;\varepsilon^\gamma)$, that of F^ε, and estimate (95) lead us to the inequalities:

$$\|F^\varepsilon\|^2_{V^k_{2-\delta+k}(\Omega;\varepsilon^\gamma)} \leq c_k \sum_{j=0}^{k} \int_\Omega (\varepsilon^\gamma + r)^{4-2\delta+2j} |\nabla^j_x F^\varepsilon|^2 \, dx$$

$$\leq c_k \sum_{j=0}^{k} \int_0^d (\varepsilon^\gamma + r)^{4-2\delta+2j} \varepsilon^{4\gamma} (\varepsilon^{2\gamma}+r^2)^{-2-j} r dr \leq c_k \varepsilon^{2\gamma(1-\delta)}.$$

Then, $(\varepsilon^{2\gamma}+r^2)^{(1-\delta+k)/2} |\nabla^k_x V_2^\varepsilon(x)| \leq c_{k,\delta} \varepsilon^{\gamma(1-\delta)}$ and (94) holds for $V^\varepsilon = V_2^\varepsilon$, which completes the proof. $\square$

Theorem 5.6. *Let $(\lambda_0, \lambda_1, y_0, v)$ be an eigenelement of (59) and (67), $\lambda_0 \neq 0$, such that $\|y_0\|^2_{L^2(0,1)} = \|v\|^{-2}_{L^2(\mathbb{R})} = 1/2$. Let us consider y_1 the solution of (70) orthogonal to y_0 in $L^2(0,1)$, and V^ε the solution of (73), where $\chi \in C^\infty(\mathbb{R})$ is defined by (71). Let m be $m > 2$. Then, for any $\delta \in (0,1/4)$, there are eigenvalues $\lambda^\varepsilon_{k(\varepsilon)}$ of problem (1) such that*

$$\left| \frac{\lambda^\varepsilon_{k(\varepsilon)}}{\varepsilon^{m-2}} - \lambda_0 - \varepsilon\lambda_1 \right| \leq C\varepsilon^{5/4-\delta} \tag{98}$$

where C is a constant independent of ε. Moreover, there is a linear combination of eigenfunctions $\{\tilde{U}^\varepsilon, \tilde{u}^\varepsilon\} \in H^1(\Omega_\varepsilon)$, $\{\tilde{U}^\varepsilon, \tilde{u}^\varepsilon\}$ corresponding to the eigenvalues $\lambda^\varepsilon_{k(\varepsilon)}$ of (1) which satisfy $\lambda^\varepsilon_{k(\varepsilon)}\varepsilon^{2-m} \in [\lambda_0 - K\varepsilon^\theta, \lambda_0 + K\varepsilon^\theta]$ with $K > 0$ and $0 < \theta < 1/2 - \delta$, $\|\{\tilde{U}^\varepsilon, \tilde{u}^\varepsilon\}\|_{\mathcal{H}^\varepsilon} = \varepsilon^{1/4}$, such that

$$\varepsilon^{1/2}\|\tilde{U}^\varepsilon - \beta^\varepsilon V^\varepsilon\|_{H^1(\Omega)} + \|\tilde{u}^\varepsilon - \beta^\varepsilon w^\varepsilon\|_{H^1(\omega_\varepsilon)} \le C\varepsilon^{1-\delta-\theta} \tag{99}$$

where w^ε is defined by $(100)_2$, β^ε is defined by (106) and $\beta^\varepsilon \to (\sqrt{(1+\lambda_0)h(\tau_0)})^{-1}$ as $\varepsilon \to 0$.

Proof. Let $(\lambda_0, \lambda_1, y_0, v)$, y_1 and V^ε be as the theorem states. For sufficiently small ε, we consider the function $\{W^\varepsilon, w^\varepsilon\}$ defined by

$$\begin{cases} W^\varepsilon(x) = V^\varepsilon(x) & \text{if } x \in \Omega, \\ w^\varepsilon(\nu,\tau) = \chi(\tau-\tau_0)\, v\!\left(\dfrac{\tau-\tau_0}{\varepsilon^{1/2}}\right) \left[y_0\!\left(\dfrac{\nu}{\varepsilon h(\tau)}\right) + \varepsilon y_1\!\left(\dfrac{\nu}{\varepsilon h(\tau)}\right)\right] & \text{if } (\nu,\tau) \in \omega_\varepsilon. \end{cases} \tag{100}$$

It is clear that $\{W^\varepsilon, w^\varepsilon\} \in H^1(\Omega_\varepsilon)$. In addition, considering Lemma 5.5 for $\delta \in (0, 1/2)$, we take integrals over Ω in (94) with $k = 0, 1, 2$, and we perform the change to polar coordinates around τ_0 in the integral on the right hand side of the inequality (94); then, we obtain the estimate

$$\varepsilon^{\delta/2}\|V^\varepsilon\|_{L^2(\Omega)} + \varepsilon^{1/2}\|\nabla_x V^\varepsilon\|_{L^2(\Omega)} + \varepsilon\|\nabla_x^2 V^\varepsilon\|_{L^2(\Omega)} \le C\varepsilon^{1/2} \tag{101}$$

with C a constant independent of ε.

In order to apply Lemma 5.1, we prove the estimate

$$\left|\left(\mathcal{A}^\varepsilon\{\tilde{W}^\varepsilon, \tilde{w}^\varepsilon\} - \frac{1}{1+\lambda_0+\varepsilon\lambda_1}\{\tilde{W}^\varepsilon, \tilde{w}^\varepsilon\}, \{W, w\}\right)_{\mathcal{H}^\varepsilon}\right| \le C\varepsilon^{5/4-\delta}\|\{W, w\}\|_{\mathcal{H}^\varepsilon} \tag{102}$$

for all $\{W, w\} \in \mathcal{H}^\varepsilon$, where $\{\tilde{W}^\varepsilon, \tilde{w}^\varepsilon\} = \{W^\varepsilon, w^\varepsilon\}\|\{W^\varepsilon, w^\varepsilon\}\|^{-1}_{\mathcal{H}^\varepsilon}$ and C is a constant independent of ε.

Taking into account the definition of the operator $\mathcal{A}^\varepsilon$, the scalar product $(\cdot,\cdot)_{\mathcal{H}^\varepsilon}$ and the function $\{W^\varepsilon, w^\varepsilon\}$ and introducing the change of variables (92) in the integrals in ω_ε, we can write

$$(1+\lambda_0+\varepsilon\lambda_1)\left(\mathcal{A}^\varepsilon\{W^\varepsilon, w^\varepsilon\} - \frac{1}{1+\lambda_0+\varepsilon\lambda_1}\{W^\varepsilon, w^\varepsilon\}, \{W, w\}\right)_{\mathcal{H}^\varepsilon} = J_1 + J_2 + J_3 + J_4$$

where

$$J_1 = \varepsilon^{1/2}\left(\lambda_0 \int_R \chi_\varepsilon v y_0 \mathrm{w}\tilde{\mathrm{K}}_\varepsilon h_\varepsilon \, d\xi d\eta - a\int_R \partial_\xi(\chi_\varepsilon v y_0)\partial_\xi \mathrm{w}\tilde{\mathrm{K}}_\varepsilon h_\varepsilon^{-1}\, d\xi d\eta\right),$$

$$\begin{aligned} J_2 = \varepsilon^{3/2}\Bigg(&\lambda_0 \int_R \chi_\varepsilon v y_1 \mathrm{w}\tilde{\mathrm{K}}_\varepsilon h_\varepsilon\, d\xi d\eta + \lambda_1 \int_R \chi_\varepsilon v y_0 \mathrm{w}\tilde{\mathrm{K}}_\varepsilon h_\varepsilon\, d\xi d\eta \\ &- a\int_R \partial_\xi(\chi_\varepsilon v y_1)\partial_\xi \mathrm{w}\tilde{\mathrm{K}}_\varepsilon h_\varepsilon^{-1}\, d\xi d\eta - a\int_R \partial_\eta(\chi_\varepsilon v y_0)\partial_\eta \mathrm{w}\tilde{\mathrm{K}}_\varepsilon^{-1} h_\varepsilon\, d\xi d\eta\Bigg), \end{aligned}$$

$$J_3 = (\lambda_0 + \varepsilon\lambda_1)\varepsilon^m \int_\Omega V^\varepsilon W\, dx - \varepsilon^2 A\int_\Omega \nabla_x V^\varepsilon \cdot \nabla_x W\, dx,$$

and

$$\begin{aligned}J_4 =&\varepsilon^{5/2}\left(\lambda_1\int_R \chi_\varepsilon v y_1 \mathrm{w}\tilde{\mathrm{K}}_\varepsilon h_\varepsilon\, d\xi d\eta - a\int_R \partial_\eta(\chi_\varepsilon v y_1)\partial_\eta \mathrm{w}\tilde{\mathrm{K}}_\varepsilon^{-1} h_\varepsilon\, d\xi d\eta\right.\\ &\left. - a\int_R \partial_\xi w^\varepsilon \partial_\xi \mathrm{w}\xi^2 (h_\varepsilon')^2 h_\varepsilon^{-1}\tilde{\mathrm{K}}_\varepsilon^{-1}\, d\xi d\eta\right)\\ &+\varepsilon^2 a\int_R (\partial_\xi w^\varepsilon \partial_\eta \mathrm{w} + \partial_\eta w^\varepsilon \partial_\xi \mathrm{w})\xi \tilde{\mathrm{K}}_\varepsilon^{-1} h_\varepsilon'\, d\xi d\eta;\end{aligned}$$

with w denoting the function $w\in H^1(\omega_\varepsilon)$ in the local variables (ξ,η), $R=(0,1)\times\mathbb{R}$ and $\chi_\varepsilon(\eta)=\chi(\varepsilon^{1/2}\eta)$. Therefore, in order to obtain estimate (102), it suffices to obtain the estimate for the terms J_1+J_2, J_3 and J_4.

The fact that (λ_0,y_0) is an eigenpair of (59) leads us to the relation

$$\begin{aligned}J_1 =&\varepsilon^{1/2}\left(\lambda_0\int_R \chi_\varepsilon v y_0 \mathrm{w}\left(\tilde{\mathrm{K}}_\varepsilon h_\varepsilon - h(\tau_0) - \varepsilon\frac{h''(\tau_0)}{2}\eta^2 - \varepsilon\xi h(\tau_0)^2\varkappa(\tau_0)\right) d\xi d\eta\right.\\ &- a\int_R \partial_\xi(\chi_\varepsilon v y_0)\partial_\xi \mathrm{w}\left(\tilde{\mathrm{K}}_\varepsilon h_\varepsilon^{-1} - \frac{1}{h(\tau_0)} + \varepsilon\frac{h''(\tau_0)}{2h(\tau_0)^2}\eta^2 - \varepsilon\xi\varkappa(\tau_0)\right) d\xi d\eta\\ &\left.+\varepsilon\lambda_0 h''(\tau_0)\int_R \chi_\varepsilon v y_0 \mathrm{w}\eta^2\, d\xi d\eta + \varepsilon a\varkappa(\tau_0)\int_R \chi_\varepsilon v y_0' \mathrm{w}\, d\xi d\eta\right).\end{aligned}$$

Besides, since (λ_1,v) is an eigenpair of (67) and y_1 is a solution of (70), we can write

$$\begin{aligned}J_2 = \varepsilon^{3/2}&\left(\lambda_0\int_R \chi_\varepsilon v y_1 \mathrm{w}\left(\tilde{\mathrm{K}}_\varepsilon h_\varepsilon - h(\tau_0)\right) d\xi d\eta + \lambda_1\int_R \chi_\varepsilon v y_0 \mathrm{w}\left(\tilde{\mathrm{K}}_\varepsilon h_\varepsilon - h(\tau_0)\right) d\xi d\eta\right.\\ &- a\int_R \partial_\xi(\chi_\varepsilon v y_1)\partial_\xi \mathrm{w}\left(\tilde{\mathrm{K}}_\varepsilon h_\varepsilon^{-1} - h(\tau_0)^{-1}\right) d\xi d\eta\\ &- a\int_R \partial_\eta(\chi_\varepsilon v y_0)\partial_\eta \mathrm{w}\left(\tilde{\mathrm{K}}_\varepsilon^{-1} h_\varepsilon - h(\tau_0)\right) d\xi d\eta - \lambda_0 h''(\tau_0)\int_R \chi_\varepsilon v y_0 \mathrm{w}\eta^2\, d\xi d\eta\\ &\left.- a\varkappa(\tau_0)\int_R \chi_\varepsilon v y_0' \mathrm{w}\, d\xi d\eta + a h(\tau_0)\int_R (\varepsilon\chi_\varepsilon'' v - 2\varepsilon^{1/2}\chi_\varepsilon' v')y_0 \mathrm{w}\, d\xi d\eta\right)\end{aligned}$$

and J_1+J_2 reads

$$\begin{aligned}&J_1+J_2\\ =&\varepsilon^{1/2}\left(\lambda_0\int_R \chi_\varepsilon v y_0 \mathrm{w}\left(\tilde{\mathrm{K}}_\varepsilon h_\varepsilon - h(\tau_0) - \varepsilon\frac{h''(\tau_0)}{2}\eta^2 - \varepsilon\xi h(\tau_0)^2\varkappa(\tau_0)\right) d\xi d\eta\right.\\ &\left.- a\int_R \partial_\xi(\chi_\varepsilon v y_0)\partial_\xi \mathrm{w}\left(\tilde{\mathrm{K}}_\varepsilon h_\varepsilon^{-1} - \frac{1}{h(\tau_0)} + \varepsilon\frac{h''(\tau_0)}{2h(\tau_0)^2}\eta^2 - \varepsilon\xi\varkappa(\tau_0)\right) d\xi d\eta\right)\\ &+\varepsilon^{3/2}\left(\lambda_0\int_R \chi_\varepsilon v y_1 \mathrm{w}\left(\tilde{\mathrm{K}}_\varepsilon h_\varepsilon - h(\tau_0)\right) d\xi d\eta + \lambda_1\int_R \chi_\varepsilon v y_0 \mathrm{w}\left(\tilde{\mathrm{K}}_\varepsilon h_\varepsilon - h(\tau_0)\right) d\xi d\eta\right.\\ &- a\int_R \partial_\xi(\chi_\varepsilon v y_1)\partial_\xi \mathrm{w}\left(\tilde{\mathrm{K}}_\varepsilon h_\varepsilon^{-1} - h(\tau_0)^{-1}\right) d\xi d\eta\\ &- a\int_R \partial_\eta(\chi_\varepsilon v y_0)\partial_\eta \mathrm{w}\left(\tilde{\mathrm{K}}_\varepsilon^{-1} h_\varepsilon - h(\tau_0)\right) d\xi d\eta\\ &\left.+ a h(\tau_0)\int_R (\varepsilon\chi_\varepsilon'' v - 2\varepsilon^{1/2}\chi_\varepsilon' v')y_0 \mathrm{w}\, d\xi d\eta\right),\end{aligned}$$

Now, for fixed ξ and ε, we consider the Taylor series at the point τ_0 of the functions $(\tilde{\mathrm{K}}_\varepsilon h_\varepsilon)(\xi,\tau) = (1+\varepsilon\xi h(\tau)\varkappa(\tau))h(\tau)$, $(\tilde{\mathrm{K}}_\varepsilon h_\varepsilon^{-1})(\xi,\tau) = (1+\varepsilon\xi h(\tau)\varkappa(\tau))h(\tau)^{-1}$

and $(\tilde{\mathrm{K}}_\varepsilon^{-1}h_\varepsilon)(\xi,\tau)=(1+\varepsilon\xi h(\tau)\varkappa(\tau))^{-1}h(\tau)$ for $\tau=\tau_0+\varepsilon^{1/2}\eta$. Then, taking into account the smoothness of h and $\varkappa$ in $\mathbb{S}_\ell$, that $h'(\tau_0)=0$ and $\|\eta^k v\|_{L^2(\mathbb{R})}$ with $k=2,4,6$ is bounded, and equation (93), we obtain

$$|J_1+J_2|\leq C_1\varepsilon^2(\|\mathrm{w}\|_{L^2(R)}+\|\partial_\xi\mathrm{w}\|_{L^2(R)}+\varepsilon^{1/2}\|\partial_\eta\mathrm{w}\|_{L^2(R)})\leq C_2\varepsilon^{7/4}\|\{W,w\}\|_{\mathcal{H}^\varepsilon}.$$

To estimate J_3, we take into account the definition of V^ε, the fact that $W=w$ on Γ, the trace inequalities (82) and (84) for V^ε, estimates (83) and (101) and equation (75). Then,

$$\begin{aligned}|J_3|&\leq(\lambda_0+\varepsilon\lambda_1)\varepsilon^m\|V^\varepsilon\|_{L^2(\Omega)}\|W\|_{L^2(\Omega)}+\varepsilon^2C_1\|\partial_\nu V^\varepsilon\|_{L^2(\Gamma)}\|W\|_{L^2(\Gamma)}\\&\leq C\varepsilon^{3/2-\delta}\|\{W,w\}\|_{\mathcal{H}^\varepsilon}.\end{aligned}$$

Finally, from (12) and (93), we check that the first three terms of J_4 are bounded by

$$\varepsilon^{5/2}C(\|\mathrm{w}\|_{L^2(R)}+\|\partial_\xi\mathrm{w}\|_{L^2(R)}+\|\partial_\eta\mathrm{w}\|_{L^2(R)})\leq C\varepsilon^{7/4}\|\{W,w\}\|_{\mathcal{H}^\varepsilon}.$$

To estimate the last term of J_4, we use (12), (93), the Taylor series of $h'(\tau)$ at τ_0, the fact $h'(\tau_0)=0$ and that $\|\eta v\|_{L^2(\mathbb{R})}$ is bounded. Thus,

$$|J_4|\leq C\varepsilon^{7/4}\|\{W,w\}\|_{\mathcal{H}^\varepsilon}.$$

As a result of the above estimates for J_1+J_2, J_3 and J_4, we have

$$\left|\left(\mathcal{A}^\varepsilon\{W^\varepsilon,w^\varepsilon\}-\frac{1}{1+\lambda_0+\varepsilon\lambda_1}\{W^\varepsilon,w^\varepsilon\},\{W,w\}\right)_{\mathcal{H}^\varepsilon}\right|\leq C\varepsilon^{3/2-\delta}\|\{W,w\}\|_{\mathcal{H}^\varepsilon}\quad\forall\{W,w\}\in\mathcal{H}^\varepsilon.$$

As regards the normalization of $\{\tilde{W}^\varepsilon,\tilde{w}^\varepsilon\}$ in $\mathcal{H}^\varepsilon$, we show

$$\begin{aligned}\varepsilon^{-1/2}\|\{W^\varepsilon,w^\varepsilon\}\|^2_{\mathcal{H}^\varepsilon}\quad\xrightarrow{\varepsilon\to0}\quad&ah(\tau_0)^{-1}\int_R|\partial_\xi(y_0v)|^2\,d\xi d\eta+h(\tau_0)\int_R|y_0v|^2\,d\xi d\eta\\&=(1+\lambda_0)h(\tau_0);\end{aligned}\tag{103}$$

which is obtained taking limits in (93), on account of (101), the normalization in the statement of the theorem for v and y_0 and the integral formulation of (59). Consequently, (102) holds due to the definition of $\{\tilde{W}^\varepsilon,\tilde{w}^\varepsilon\}$ and (103).

We apply Lemma 5.1 for $H=\mathcal{H}^\varepsilon$, $A=\mathcal{A}^\varepsilon$, $\lambda=(1+\lambda_0+\varepsilon\lambda_1)^{-1}$ and $u=\{\tilde{W}^\varepsilon,\tilde{w}^\varepsilon\}$ and $r=C\varepsilon^{5/4-\delta}$ which provides, for sufficiently small ε, at least one eigenvalue $\lambda^\varepsilon_{k(\varepsilon)}$ of (1) verifying $|(1+\lambda^\varepsilon_{k(\varepsilon)}\varepsilon^{2-m})^{-1}-(1+\lambda_0+\varepsilon\lambda_1)^{-1}|\leq C\varepsilon^{5/4-\delta}$, and consequently, we deduce (98). Moreover, if we take, for instance, $r^*=\varepsilon^\theta$ with $0<\theta<1/2-\delta$, Lemma 5.1 also provides a function $\{\hat{U}^\varepsilon,\hat{u}^\varepsilon\}\in\mathcal{H}^\varepsilon$, with $\|\{\hat{U}^\varepsilon,\hat{u}^\varepsilon\}\|_{\mathcal{H}^\varepsilon}=1$, $\{\hat{U}^\varepsilon,\hat{u}^\varepsilon\}$ belonging to the eigenspace associated with all the eigenvalues $(1+\lambda^\varepsilon_{k(\varepsilon)}\varepsilon^{2-m})^{-1}$ of operator $\mathcal{A}^\varepsilon$ contained in the interval

$$[(1+\lambda_0+\varepsilon\lambda_1)^{-1}-\varepsilon^\theta,(1+\lambda_0+\varepsilon\lambda_1)^{-1}+\varepsilon^\theta],\tag{104}$$

such that

$$\|\{\hat{U}^\varepsilon,\hat{u}^\varepsilon\}-\alpha^\varepsilon\{W^\varepsilon,w^\varepsilon\}\|_{\mathcal{H}^\varepsilon}\leq C\varepsilon^{5/4-\delta-\theta}\tag{105}$$

is satisfied where $\alpha^\varepsilon=\|\{W^\varepsilon,w^\varepsilon\}\|^{-1}_{\mathcal{H}^\varepsilon}$. Now, we set $\{\tilde{U}^\varepsilon,\tilde{u}^\varepsilon\}=\varepsilon^{1/4}\{\hat{U}^\varepsilon,\hat{u}^\varepsilon\}$ and $\beta^\varepsilon=\varepsilon^{1/4}\alpha^\varepsilon$, namely,

$$\beta^\varepsilon=\varepsilon^{1/4}\|\{W^\varepsilon,w^\varepsilon\}\|^{-1}_{\mathcal{H}^\varepsilon},\tag{106}$$

which converge towards $(\sqrt{(1+\lambda_0)h(\tau_0)})^{-1}$ as $\varepsilon\to0$ (see (103)).

Then, (75) (100) and (103) it follows

$$\varepsilon^{1/2}\|\nabla_x(\tilde{U}^\varepsilon-\beta^\varepsilon V^\varepsilon)\|_{L^2(\Omega)}+\varepsilon^{-1}\|\tilde{u}^\varepsilon-\beta^\varepsilon w^\varepsilon\|_{L^2(\omega_\varepsilon)}+\|\nabla_x(\tilde{u}^\varepsilon-\beta^\varepsilon w^\varepsilon)\|_{L^2(\omega_\varepsilon)} \leq C\varepsilon^{1-\delta-\theta},$$

and, since $(\tilde{U}^\varepsilon-\beta^\varepsilon V^\varepsilon)|_\Gamma=(\tilde{u}^\varepsilon-\beta^\varepsilon w^\varepsilon)_\Gamma$, using again Friedrichs' inequality and the trace inequality (82) and (83) with $w=\tilde{u}^\varepsilon-\beta^\varepsilon w^\varepsilon$, yields

$$\|\tilde{U}^\varepsilon-\beta^\varepsilon V^\varepsilon\|_{H^1(\Omega)}\leq C\varepsilon^{1/2-\delta-\theta}. \tag{107}$$

Thus, estimate (99) holds and the theorem is proved. □

Remark 7. Similar considerations to those in Remark 6 can be made in connection with the eigenfunctions corresponding to the eigenvalues in intervals or length $O(\varepsilon^\theta)$ in the statement of Theorem 5.6 (cf. (104) and (105)).

Remark 8. We note that Theorem 5.6 (see (99) and (107)) justifies, up to a certain order, asymptotic expansions (53), (54) and (55). In fact, under the assumptions of Theorem 5.6, for any fixed constant $K_1>0$ there exists $\varepsilon_1>0$ such that for $0<\varepsilon<\varepsilon_1$

$$\|\tilde{\mathrm{u}}^\varepsilon-\beta^\varepsilon(y_0+\varepsilon y_1)v\|_{L^2((0,1)\times(-K_1,K_1))}\leq C\varepsilon^{5/4-\delta-\theta}$$

and

$$\|\nabla_{\xi,\eta}(\tilde{\mathrm{u}}^\varepsilon-\beta^\varepsilon(y_0+\varepsilon y_1)v)\|_{L^2((0,1)\times(-K_1,K_1))}\leq C\varepsilon^{3/4-\delta-\theta}$$

where $\tilde{\mathrm{u}}^\varepsilon(\xi,\eta)=\tilde{u}^\varepsilon(\varepsilon\xi h(\tau_0+\varepsilon^{1/2}\eta),\tau_0+\varepsilon^{1/2}\eta)$ for $(\xi,\eta)\in(0,1)\times(-K_1,K_1)$. The estimates above are obtained from (105), (93) and the definition of the functions $\{\tilde{U}^\varepsilon,\tilde{u}^\varepsilon\}$ and $\{W^\varepsilon,w^\varepsilon\}$.

Remark 9. It should be pointed out that the "groups" of eigenfunctions $\{\widetilde{U}^\varepsilon,\widetilde{u}^\varepsilon\}$ corresponding to the frequencies λ^ε of (1) in the statement of the Theorem 5.6 are asymptotically localized in small neighborhoods of points of the thin band, namely, in neighborhoods of local maxima points of the function h which is related to the variable width and to the geometry of the band ω_ε. Indeed, estimates (99) and (107) along with Lemma 5.5 (cf. (94) for $r\geq O(\varepsilon^{p_k})$ for a certain $p_k<1/2$ depending on $k=0,1$) allow us to assert that the eigenfunctions $\{\widetilde{U}^\varepsilon,\widetilde{u}^\varepsilon\}$ are significant in $\varepsilon^{1/2}$–neighborhoods of τ_0 while they vanish asymptotically at a distance $O(\varepsilon^p)$ for a certain $p<1/2$.

Remark 10. It should be noted that similar results to those in Sections 4 and 5.2 can be obtained when h present a local maximum in τ_0 but $h''(\tau_0)=0$. If so, it is self-evident that we must introduce different variables and asymptotic expansions. As a matter of fact, if $h'(\tau_0)=h''(\tau_0)=\cdots=h^{(2n-1)}(\tau_0)=0$ and $h^{(2n)}(\tau_0)<0$ for certain $n>1$, the suitable variables to show the local effects for the eigenfunctions (cf. Remarks 8 and 9) are likely to be $\xi=\nu\varepsilon^{-1}h(\tau)^{-1}$ and $\eta=(\tau-\tau_0)\varepsilon^{-1/(n+1)}$.

Remark 11. Let us note that in contrast with the case where h constant, in this case, there can be different points where $h(\tau_0)$ has a local maximum, and even several different points with the same value for the second derivative $h''(\tau_0)$. Thus, without stronger restrictions for $h(\tau_0)$, the type of results in Theorem 5.4, which would complement those in Theorem 5.6, cannot be obtained.

Remark 12. It should be mentioned that asymptotics for low, middle and high frequencies have been considered in [8] for a problem different from (1); also, the results obtained are very different. In [8] a Dirichlet problem is considered in $\Omega_\varepsilon\equiv\Omega$

(that is, $\omega_\varepsilon \subset \Omega$ and $\partial\omega_\varepsilon \cap \partial\Omega = \emptyset$), the width of ω_ε is constant (that is, $\omega_\varepsilon \equiv \varepsilon\omega$) and the band ω_ε is only heavy (that is, $t = 0$). As a consequence, for the value of $m = 3$ considered in [8], different limiting problems arise and the localization phenomena has not been considered.

Remark 13. In contrast with Remark 12, localization effects for the eigenfunctions have been considered in previous papers always related to thin domains and homogeneous media. The oscillator harmonic equation (cf. (67)) appears in these problems as a consequence of an asymptotic analysis of the low or high frequencies depending on the geometry of the domain and of the boundary conditions. Let us mention [12], [17] and [6] in this connection; see [3] for further comments and references. In the present paper, we have another factor affecting the asymptotic behavior of the eigenfunctions which deal with the high contrast materials in both domains ω_ε and Ω. Consequently, we detect localized eigenfunctions corresponding to the middle frequencies. These eigenfunctions are localized asymptotically near points τ_0 which are local maxima of h. Specifying, they present an exponential decay in the tangential direction with the distance to τ_0 and a polynomial decay in the normal direction inside Ω while they behave as oscillating functions in the normal direction in the thin band ω_ε (see (72) and (74)). Lemma 5.5 is essential to prove the above asymptotic localization (cf. [2] for the value $\delta = 0$ in (94)). See [14] to compare with localization effects in models of vibrating systems with concentrated masses near the boundary.

6. **High frequencies.** In this section, we study the asymptotic behavior of the eigenvalues of (1) of order $O(1)$, the high frequencies. We show that the eigenvalues λ^ε asymptotically close to eigenvalues of the Dirichlet problem in Ω give rise to global vibrations in the way stated by Theorems 6.1 and 6.2: roughly speaking, *only the eigenfunctions corresponding to eigenvalues λ^ε asymptotically near an eigenvalue of the Dirichlet problem (109) can be asymptotically different from zero in $H^1(\Omega)$.*

Throughout the section we consider the case where $m > 0$. We first obtain the limiting problem associated with the eigenvalues λ^ε of (1) of order $O(1)$ by means of asymptotic expansions and then outline the proofs. It should be noted that convergence results hold for all $m > 0$, while some restrictions and extensions for the asymptotic expansions for certain values of m are in Remark 14.

For $m > 2$ (see Remark 14 for $m \in (0, 2]$), let us assume an asymptotic expansion for the eigenvalues λ^ε of the form

$$\lambda^\varepsilon = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \cdots \tag{108}$$

and an expansion for the corresponding eigenfunctions $\{U^\varepsilon, u^\varepsilon\}$ in Ω and ω_ε of the form (28) and (29) respectively. Then, by replacing (108), (28) and (29) in (1), on account of (25) and (26), we have that the leading terms in the asymptotic expansions satisfy the equations

$$-A\Delta_x V = \lambda_0 V \quad \text{in } \Omega,$$

$$0 = \lambda_0 v_0, \quad \zeta \in (0, h(\tau)), \tau \in \mathbb{S}_\ell,$$

and (32). Consequently, $\lambda_0 = 0$ or $v_0 \equiv 0$. Since we are dealing with the eigenvalues of order $O(1)$, we consider the case where $\lambda_0 \neq 0$, and consequently we have that

(λ_0, V) is an eigenpair of the Dirichlet problem

$$\left\{ \begin{array}{ll} -A\Delta_x V = \lambda_0 V & \text{in } \Omega, \\ V = 0 & \text{on } \Gamma. \end{array} \right. \tag{109}$$

As outlined for the asymptotics of the eigenfunctions corresponding to the low frequencies, an appropriate normalization for the eigenfunctions must be prescribed to obtain convergence for the high frequencies. Let us consider $\mathfrak{H}^\varepsilon$ the space $H^1(\Omega_\varepsilon)$ with the scalar product

$$\begin{aligned} (\{W,w\},\{G,g\})_{\mathfrak{H}^\varepsilon} =& A\int_\Omega \nabla_x W\cdot\nabla_x G\,dx + \frac{a}{\varepsilon}\int_{\omega_\varepsilon}\nabla_x w\cdot\nabla_x g\,dx \\ &+ \int_\Omega WG\,dx + \frac{1}{\varepsilon^{1+m}}\int_{\omega_\varepsilon} wg\,dx \quad \forall\{W,w\},\{G,g\}\in H^1(\Omega_\varepsilon). \end{aligned} \tag{110}$$

We follow the structure of Section 5, and more precisely that of Section 5.1. We use Lemma 5.1 to show the convergence of sequences of eigenvalues of (1) towards those of (109) and to obtain bounds for the convergence rates for the eigenvalues and eigenfunctions stated in Theorem 6.1 (cf. (111)). Theorem 6.2 shows that this result for the high frequencies is optimal, since, on account that any real λ_* is a limit point of sequences of eigenvalues $\lambda^\varepsilon = O(1)$ of (1) (cf. Lemma 5.2 and Remark 15), the normalization for the corresponding eigenfunctions (or linear combination of eigenfunctions) $\{U^\varepsilon, u^\varepsilon\}$ in $\mathfrak{H}^\varepsilon$ (see (110)), lead to possible limits being $(\lambda_*, 0)$ in $\mathbb{R}\times H^1(\Omega)$-weak in the case where λ_* is not an eigenvalue (109). For brevity, below we state the main results and outline the proofs which follow using the technique in Section 5.1.

Theorem 6.1. *Let (λ_0, V) be an eigenpair of the Dirichlet problem (109) such that $\|V\|_{L^2(\Omega)} = 1$. Then, for $m > 0$, there are eigenvalues $\lambda^\varepsilon_{k(\varepsilon)}$ of problem (1) such that*

$$|\lambda^\varepsilon_{k(\varepsilon)} - \lambda_0| \le C\varepsilon$$

where C is a constant independent of ε. In addition, there is a linear combination of eigenfunctions $\{\tilde U^\varepsilon, \tilde u^\varepsilon\} \in H^1(\Omega_\varepsilon)$, $\{\tilde U^\varepsilon, \tilde u^\varepsilon\}$ corresponding to the eigenvalues $\lambda^\varepsilon_{k(\varepsilon)}$ of (1) in the interval $[\lambda_0 - K\varepsilon^\theta, \lambda_0 + K\varepsilon^\theta]$ with $K > 0$ and $0 < \theta < \min(1, m/2)$, $\|\{\tilde U^\varepsilon, \tilde u^\varepsilon\}\|_{\mathfrak{H}^\varepsilon} = 1$, such that

$$\|\tilde U^\varepsilon - (\sqrt{1+\lambda_0}\,)^{-1} V\|_{H^1(\Omega)} \le C(\varepsilon^{1-\theta} + \varepsilon^{m/2-\theta}). \tag{111}$$

Proof. We apply Lemma 5.1 for $H = \mathfrak{H}^\varepsilon$ in (110), $A = \mathfrak{A}^\varepsilon$ the compact and symmetric operator on $\mathfrak{H}^\varepsilon$ defined by

$$(\mathfrak{A}^\varepsilon\{W,w\},\{G,g\})_{\mathfrak{A}^\varepsilon} = \int_\Omega WG\,dx + \frac{1}{\varepsilon^{1+m}}\int_{\omega_\varepsilon} wg\,dx \qquad \forall\{W,w\},\{G,g\}\in H^1(\Omega_\varepsilon);$$

$\lambda = (1+\lambda_0)^{-1}$ and $u = \{V, 0\}\|V\|^{-1}_{H^1(\Omega)} \in H^1(\Omega_\varepsilon)$ where (λ_0, V) is as the theorem states (see (14) and (17) to compare). Then, we rewrite the proof of Theorem 5.3 with the suitable simplifications, and the theorem holds. ☐

Theorem 6.2. *Let λ_* be any positive real number which is not an eigenvalue of the Dirichlet problem (109). Let m be $m > 0$, and let δ_ε denote any sequence of positive numbers converging towards zero as $\varepsilon \to 0$. Let us assume that there are eigenvalues λ^ε of problem (1) in the interval $[\lambda_* - \delta^\varepsilon, \lambda_* + \delta^\varepsilon]$. Let us consider $\{\tilde U^\varepsilon, \tilde u^\varepsilon\} \in H^1(\Omega_\varepsilon)$ any linear combination of eigenfunctions of (1) corresponding* *to the eigenvalues* $\lambda^\varepsilon_{k(\varepsilon)}$ *in the above interval,* $\{\tilde{U}^\varepsilon, \tilde{u}^\varepsilon\}$ *satisfying* $\|\{\tilde{U}^\varepsilon, \tilde{u}^\varepsilon\}\|_{\mathfrak{H}^\varepsilon} = 1$*. Then,* $\tilde{U}^\varepsilon$ *converge towards zero in the weak topology of* $H^1(\Omega)$ *as* $\varepsilon \to 0$*.*

Proof. We follow the technique in Theorem 5.4. First, we consider the case where there is only one eigenvalue $\lambda^\varepsilon = \lambda^\varepsilon_{k(\varepsilon)}$ in the interval of the statement. This amounts to taking $\delta_\varepsilon = |\lambda_* - \lambda^\varepsilon| \to 0$ as $\varepsilon \to 0$. Let $\{U^\varepsilon, u^\varepsilon\}$ be the corresponding eigenfunction of norm 1 in $\mathfrak{H}^\varepsilon$ (see 110). Thus, $\|\{U^\varepsilon, u^\varepsilon\}\|_{H^1(\Omega_\varepsilon)}$ is bounded by a constant independent of ε, and we can extract a subsequence (still denoted by ε) such that U^ε converges towards U^* in the weak topology of $H^1(\Omega)$. The normalization also gives us

$$\frac{1}{\varepsilon}\int_{\omega_\varepsilon} |u^\varepsilon|^2\, dx \le C\varepsilon^m\,,$$

and the strong convergence of U^ε in $L^2(\Gamma)$ and (13) imply that $U^* = 0$ on Γ. In order to identify U^*, we consider (4) for $G \in \mathcal{D}(\Omega)$ extended by zero to Ω_ε, we take limits as $\varepsilon \to 0$ and we obtain that (λ_*, U^*) satisfies

$$\int_\Omega \nabla_x U^*.\nabla_x G\, dx = \lambda_* \int_\Omega U^* G\, dx \quad \forall G \in H^1_0(\Omega)\,,$$

which is the weak formulation of (109). Consequently, if λ_* is not an eigenvalue of (109), then $U^* \equiv 0$.

Finally, we rewrite the above arguments with minor modifications in the general case where there are several eigenvalues of (1) in the interval $[\lambda_* - \delta^\varepsilon, \lambda_* + \delta^\varepsilon]$. Indeed, let us denote by $\{\lambda^\varepsilon_{k(\varepsilon)+j}\}_{j=0}^J$ the set of eigenvalues $[\lambda_* - \delta^\varepsilon, \lambda_* + \delta^\varepsilon]$, and by $\{\{U^\varepsilon_{k(\varepsilon)+j}, u^\varepsilon_{k(\varepsilon)+j}\}\}_{j=0}^J$ the set of the corresponding eigenfunctions; J being a certain natural that can depend on ε. Let us assume that $\{\tilde{U}^\varepsilon, \tilde{u}^\varepsilon\} = \sum_{j=0}^J \alpha_j^\varepsilon \{U^\varepsilon_{k(\varepsilon)+j}, u^\varepsilon_{k(\varepsilon)+j}\}$ for certain constants α^ε_j. We write the equation (4) for each eigenvalue and the corresponding eigenfunction of the set, and for $G \in H^1_0(\Omega)$, $g = 0$. Then, we take the sum after multiplying each equation by α^ε_j, j ranging from 0 to J. We take into account the convergence

$$\sum_{j=0}^J \alpha^\varepsilon_j(\lambda^\varepsilon_{k(\varepsilon)+j} - \lambda_*)\int_\Omega U^\varepsilon_{k(\varepsilon)+j} G\, dx \to 0 \quad \text{ as } \varepsilon \to 0,$$

and the result of the theorem holds. □

Remark 14. It should be noted that the technique of asymptotic expansions throughout this section also applies in the case where $m \in (0, 2)$ and we obtain the same limit problem (109). In this case we need using further terms of the asymptotic expansions of u^ε in ω_ε. As a matter of fact, for $m \neq 1$ the expansion (29) must be suitably modified by introducing other terms for different powers of ε, namely of the order $O(\varepsilon^p)$, with $p > 0$ a certain non-natural number depending on m.

In the case where $m = 2$ and h is constant, the asymptotic expansions (108) provide two possibilities for λ_0 that we state here without any proof. One is λ_0 to be eigenvalue of (109) and the other is λ_0 eigenvalue of (35). Now, it remains to identify the eigenfunction in (28) and (29) which involves another resulting problem in Ω with the spectral parameter λ_2 in (108) with $\lambda_1 = 0$.

In addition, comparing the asymptotic expansions for $m = 2$ with Theorem 6.2, we note that when h is constant the asymptotic expansion (27) coincides with

(108), and another normalization for the eigenfunctions, namely, a norm in $H^1(\Omega_\varepsilon)$ different from that in (110), will likely allows us to prove that the eigenvalues of (35) are other accumulation points of high frequencies which give rise to other kinds of vibrations (see norms (75) and (110) to compare). This case, as well as the case where h is not a constant with $m = 2$ remain as an open problem.

Remark 15. We emphasize that considering the results in Theorem 2.2, Lemma 5.2 shows that for fixed $\alpha < m$, any $\lambda > 0$ is an accumulation point of re-scaled sequences $\varepsilon^{-\alpha}\lambda^\varepsilon_{k(\varepsilon)}$ of eigenvalues $\lambda^\varepsilon_{k(\varepsilon)}$ of (1), for ε ranging in certain subsequences $\{\varepsilon_j\}_{j=1}^\infty$, $\varepsilon_j \to 0$ as $j \to \infty$. For the values of $m > 2$ and $\alpha = m-2$ ($m > 0$, $\alpha = 0$, respectively), Theorems 5.3 and 5.6 (Theorem 6.1 respectively) show that the results hold for certain well determined $\lambda > 0$ and a for a whole sequence $k(\varepsilon)$, with $\varepsilon \to 0$. Instead, obtaining this result for any $\lambda > 0$ in Theorem 5.4 and 6.2 could imply providing a very weak convergence of certain spectral families associated with (1) towards the spectral family of an operator with a continuous spectrum in $[0, \infty)$ and using Fourier transform. For brevity, we avoid this proof here and refer to [9], [13], [15] for instance, for the technique. In addition, we observe that Lemma 5.2 does not provide information for eigenfunctions of (1) corresponding to λ^ε, this being the aim of Sections 5 and 6 for the values of $\alpha = m-2$ and $\alpha = 0$ respectively (see Remark 16 for other values of α).

Remark 16. In Sections 3–6 we have addressed the asymptotic behavior of the eigenfrequencies corresponding to eigenvalues of order $O(\varepsilon^{m-2})$ and $O(1)$ for $m > 2$. On account of Remark 15, we examine the asymptotics for the eigenfunctions corresponding to eigenvalues $\lambda^\varepsilon = O(\varepsilon^\alpha)$ for $\alpha < m$, $\alpha \neq 0, m-2$. For the values of α, $\alpha < m-2$ or $m-2 < \alpha < m$, arguments similar to those in Section 3 lead us to state that if $\{\lambda^\varepsilon_{k(\varepsilon)}\}_\varepsilon$ is any sequence of eigenvalues of (1) such that $\lambda^\varepsilon_{k(\varepsilon)}\varepsilon^{-\alpha} \sim \lambda > 0$ as $\varepsilon \to 0$ and the corresponding eigenfunctions $\{U^\varepsilon_{k(\varepsilon)}, u^\varepsilon_{k(\varepsilon)}\}$ admit the expansions (28) and (29) in Ω and ω_ε respectively, then the functions V and v_0 in (28) and (29) are identically null. Consequently, assuming that the eigenfunctions of (1) admit the asymptotic expansions (28) and (29), the only converging sequences $\lambda^\varepsilon_{k(\varepsilon)}\varepsilon^{-\alpha}$ for which the corresponding eigenfunctions do not vanish asymptotically, in a certain topology, are likely to be those corresponding to eigenvalues $\lambda^\varepsilon_{k(\varepsilon)}$ of order $O(1)$, $O(\varepsilon^{m-2})$ and $O(\varepsilon^m)$. We also emphasize, that since the asymptotic expansions for the eigenfunctions rely on the normalization that we consider, other normalization for the eigenfunctions different from those in this paper might give rise to localizing other kinds of vibrations associated with $\lambda^\varepsilon = O(\varepsilon^\alpha)$ for $\alpha < m$ (see Remark 14 in this respect.)

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