

## DYNAMIC TRANSITION AND PATTERN FORMATION FOR CHEMOTACTIC SYSTEMS

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**ABSTRACT.** The main objective of this article is to study the dynamic transition and pattern formation for chemotactic systems modeled by the Keller-Segel equations. We study chemotactic systems with either rich or moderated stimulant supplies. For the rich stimulant chemotactic system, we show that the chemotactic system always undergoes a Type-I or Type-II dynamic transition from the homogeneous state to steady state solutions. The type of transition is dictated by the sign of a non dimensional parameter  $b$ , which is derived by incorporating the nonlinear interactions of both stable and unstable modes. For the general Keller-Segel model where the stimulant is moderately supplied, the system can undergo a dynamic transition to either steady state patterns or spatiotemporal oscillations. From the pattern formation point of view, the formation and the mechanism of both the lamella and rectangular patterns are derived.

**1. Introduction.** Chemotaxis is an important phenomenon occurring in many biological individuals, and involves mobility and aggregation of the species in two aspects: one is random walk, and the other is the chemically directed movement. For example, in the slime mold *Dictyostelium discoideum*, the single-cell amoebae move towards regions of relatively high concentration of a chemical called cyclic-AAMP which is secreted by the amoebae themselves. Many experiments demonstrate that under proper conditions a bacterial colony can form a rather regular pattern, which is relatively stable in certain time scale. For example, a series of experimental results on the patterns formed by the bacteria *Escherichia coli* (*E. coli*) and *Salmonella typhimurium* (*S. Typhimurium*) were derived in [2, 3], where two types of experiments were conducted: one is in semi-solid medium, and the other

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is in liquid medium. Both showed that when the bacteria are exposed to intermediates of TCA cycle, they can form various regular patterns, typically as ringlike and sunflowerlike formations. In all these experiments, the bacteria are known to secrete aspartate, a potent chemoattractant; also see [10, 1].

In their pioneering work [5], E. F. Keller and L. A. Segel proposed a model in 1970, called the Keller-Segel equations, to describe the chemotactic behavior of the slime mold amoebae. In their equations, the growth rate of amoeba cells was ignored, i.e., the model can only depict the chemotaxis process in a small timescale, as exhibited in the liquid medium experiments with *E. Coli* and *S. Typhimurium* by [2, 3]. However, in the semi-solid medium experiments, the timescale of a pattern formation process is long enough to accommodate many generations of bacteria. Therefore, various revised models were presented by many authors, taking into consideration the effects of the stimulant (i.e. food source) and the growth rate of population; see among others [10] and the references therein. Also, there is a vast literature on the mathematical studies for the Keller-Segel model; see among others [13, 4, 12, 11].

The main objective of this article is to study the dynamic transition and pattern formation for chemotactic systems modeled by the Keller-Segel equations. The study is based on the dynamic transition theory developed recently by the authors. The theory studies dynamical transitions of dissipative systems in Nature. The key philosophy for the dynamic transition theory is to search for all transition states. The stability and the basin of attraction of the transition states provide naturally the mechanism of pattern formation associated with chemotactic systems.

One most important ingredient of this theory is the derivation of a general dynamic principle of dynamic transitions for dissipative systems., stating that all dynamic transitions of a dissipative system are classified into three categories: continuous, catastrophic, and random, which are also called respectively as Type-I, Type-II and Type-III.

The continuous transition amounts to saying that the control parameter crosses the threshold, the transition states stay in the close neighborhood of the basic state. In fact, continuous transition is essentially characterized by the attractor bifurcation theorem, which amounts to saying that when the system losses linear stability and the basic state is asymptotically stable at the critical parameter value  $\lambda_0$ , the system undergoes a continuous dynamic transition, which is described by the bifurcated attractor. There are many physical systems which can undergo a continuous transition. For example, consider the classical Bénard convection, as the Rayleigh number crosses the critical Rayleigh number, the system undergoes a continuous transition to an attractor, homeomorphic to an  $(m - 1)$ -dimensional sphere  $S^{m-1}$ , which consists of steady states and transients. Here  $m$  is the number of unstable modes of the linearized eigenvalue problem at the critical Rayleigh number, dictated by the spatial geometry, which also defines the pattern formation mechanism of the problem.

When the asymptotic stability of the basic state at the critical parameter is no longer valid, the system undergoes either catastrophic or random transitions, dictated by the nonlinear interactions as well. The dynamic transition theory gives a systematic approach to distinguish these transitions.

Intuitively speaking, catastrophic transition corresponds to the case where the system undergoes a more drastic change as the control parameter crosses the critical threshold. The random transition corresponds to the case where the neighborhood

(fluctuations) of the basic state can be divided into two regions such that fluctuations in one of the regions lead to continuous transitions and those in the other region lead to catastrophic transitions.

Two types of Keller-Segel models are addressed in this article. The first is the model for rich stimulant chemotactic systems (with rich nutrient supplies). In this case, the equations are a two-component system, describing the evolution of the population density of biological individuals and the chemoattractant concentration. We show that the chemotactic system always undergoes a Type-I or Type-II dynamic transition from the homogeneous state to steady state solutions. The type of transition is dictated by the sign of a nondimensional parameter  $b$ . This parameter is derived by carefully reducing the original model to the center manifold generated by the unstable modes, taking into consideration of the highly nonlinear interactions of both stable and unstable modes. Hence such a parameter cannot be derived from any linear theory, and provides a key characterization of the related phase transitions and pattern formations.

For example, in a non-growth system in a narrow domain the system undergoes a Type-I (continuous) transition if the spatial scale is smaller than a critical number. Otherwise the system undergoes a Type-II (catastrophic) transition, leading to a more complex pattern away from the basic homogeneous state.

The second is a more general Keller-Segel model where the stimulant is moderately supplied. This model is a three-component system describing the evolution of the population density of biological individuals, the chemoattractant concentration, and the stimulant concentration. In this case, the system undergoes a dynamic transition to either steady state patterns or spatiotemporal oscillations. In both transition scenarios, the transitions can be either a continuous or catastrophic dictated respectively by two nondimensional parameter  $b_0$  and  $b_1$ .

For simplicity, we consider in this article only the case where the first eigenvalue of the linearized problem around the homogeneous pattern is simple (real or complex), and we shall explore more general case elsewhere. In the case considered, for the continuous transition, when the linearized eigenvalue is simple, we show that both the lamella and rectangular pattern can form depending on the geometry of the spatial domain. Namely, for narrow domains, the lamella pattern forms, otherwise the rectangular pattern occurs. Of course, for catastrophic transitions, more complex patterns emerge far from the basic homogeneous state.

We end this section by mentioning that the main objective of this article is to determine the types of dynamic transitions, which are dictated by non-dimensional parameters, given in terms of key system parameters as well as the geometry of the spatial domain of the underlying system. These parameters are derived by taking into consideration of the highly nonlinear interactions of both stable and unstable modes. Further biological implications of the main theorems and the main results will be carried out elsewhere, and, as a motivation, biological implications on extreme cases are very briefly mentioned in Section 5.

The paper is arranged as follows. Section 2 introduces the Keller-Segel model. The rich stimulant case is addressed in Section 3, and the general three-component system is studied in Section 4. Section 5 explores some biological conclusions of the main theorems.

**2. Keller-Segel model.** The general form of the revised Keller-Segel model is given by

$$\begin{aligned}\frac{\partial u_1}{\partial t} &= k_1 \Delta u_1 - \chi \nabla(u_1 \nabla u_2) + \alpha_1 u_1 \left( \frac{\alpha_2 u_3}{\alpha_0 + u_3} - u_1^2 \right), \\ \frac{\partial u_2}{\partial t} &= k_2 \Delta u_2 + r_1 u_1 - r_2 u_2, \\ \frac{\partial u_3}{\partial t} &= k_3 \Delta u_3 - r_3 u_1 u_3 + q(x),\end{aligned}\tag{2.1}$$

where  $u_1$  is the population density of biological individuals,  $u_2$  is the chemoattractant concentration,  $u_3$  is the stimulant concentration,  $q(x)$  is the nutrient source, and  $\chi$  is a chemotactic response coefficient.

Equations (2.1) are supplemented with the Neumann condition:

$$\frac{\partial(u_1, u_2, u_3)}{\partial n} = 0 \text{ on } \partial\Omega.\tag{2.2}$$

For simplicity, we consider in this article the case where the spatial domain  $\Omega$  is a two-dimensional (2D) rectangle:

$$\Omega = (0, l_1) \times (0, l_2) \quad \text{for } l_1 \neq l_2.$$

It is convenient to introduce the nondimensional form of the model. For this purpose, let

$$\begin{aligned}t &= t'/r_2, \quad x = \sqrt{k_2/r_2} x', \\ u_1 &= \sqrt{\alpha_2} u'_1, \quad u_2 = k_2 u'_2/\chi, \quad u_3 = \alpha_0 u'_3,\end{aligned}\tag{2.3}$$

and we define the following non dimensional parameters:

$$\begin{aligned}\lambda &= r_1 \sqrt{\alpha_2} \chi / r_2 k_2, \quad \alpha = \alpha_1 \alpha_2 / r_2, \quad \mu = k_1 / k_2, \\ r &= k_3 / k_2, \quad \delta = r_3 \sqrt{\alpha_2} / r_2, \quad \delta_0 = q / r_2 \alpha_0.\end{aligned}\tag{2.4}$$

Then suppressing the primes, the non-dimensional form of the Keller-Segel model is given by:

$$\begin{aligned}\frac{\partial u_1}{\partial t} &= \mu \Delta u_1 - \nabla(u_1 \nabla u_2) + \alpha u_1 \left( \frac{u_3}{1 + u_3} - u_1^2 \right), \\ \frac{\partial u_2}{\partial t} &= \Delta u_2 - u_2 + \lambda u_1, \\ \frac{\partial u_3}{\partial t} &= r \Delta u_3 - \delta u_1 u_3 + \delta_0, \\ \frac{\partial u}{\partial n} \Big|_{\partial\Omega} &= 0, \\ u(0) &= u_0 \text{ in } \Omega.\end{aligned}\tag{2.5}$$

The non-dimensional of  $\Omega$  is written as

$$\Omega = (0, L_1) \times (0, L_2) \quad \text{with } L_1 \neq L_2.$$

Often times, the following form of the Keller-Segel equations is discussed in some literatures:

$$\begin{aligned}\frac{\partial u_1}{\partial t} &= \mu \Delta u_1 - \nabla(u_1 \nabla u_2) + \alpha u_1 \left( \frac{u_3}{1 + u_3} - u_1^2 \right), \\ \frac{\partial u_3}{\partial t} &= r \Delta u_3 - \delta u_1 u_3 + \delta_0, \\ -\Delta u_2 + u_2 &= \lambda u_1. \\ \frac{\partial u}{\partial n} \Big|_{\partial \Omega} &= 0, \\ u(0) &= u_0.\end{aligned}\tag{2.6}$$

The biological significance of (2.6) is that the diffusion and degradation of the chemoattractant secreted by the bacteria themselves are almost balanced by their production. The main advantage of (2.6) lies in its mathematical simplicity. Mathematically it is easy to see that all eigenvalues of the linear terms  $\Delta u_2 - u_2$  are all negative. Hence by the center manifold reduction technique, it is not hard to see that the solutions of (2.5) will converge to solutions of (2.6) as time goes to infinity. This is consistent that from the main results of this article, the main characteristics of the pattern formation associated with this reduced model are retained.

Of course, the reduced system may loss certain transient biological behavior, and it appears that it may provide a better description for the system where the chemoattractant is much faster the dynamics of the organisms.

### 3. Dynamic transitions for rich stimulant system.

**3.1. The model.** We know that as nutrient  $u_3$  is richly supplied, the Keller-Segel model (2.1) is reduced to a two-component system:

$$\begin{aligned}\frac{\partial u_1}{\partial t} &= \mu \Delta u_1 - \nabla(u_1 \nabla u_2) + \alpha u_1 (1 - u_1^2), \\ \frac{\partial u_2}{\partial t} &= \Delta u_2 - u_2 + \lambda u_1, \\ \frac{\partial(u_1, u_2)}{\partial n} \Big|_{\partial \Omega} &= 0, \\ u(0) &= u_0.\end{aligned}\tag{3.1}$$

It is easy to see that  $u^* = (1, \lambda)$  is a steady state of (3.1). Consider the deviation from  $u^*$ :

$$u = u^* + u'.$$

Suppressing the primes, the system (3.1) is then transformed into

$$\begin{aligned}\frac{\partial u_1}{\partial t} &= \mu \Delta u_1 - 2\alpha u_1 - \Delta u_2 - \nabla(u_1 \nabla u_2) - 3\alpha u_1^2 - \alpha u_1^3, \\ \frac{\partial u_2}{\partial t} &= \Delta u_2 - u_2 + \lambda u_1, \\ \frac{\partial(u_1, u_2)}{\partial n} \Big|_{\partial \Omega} &= 0, \\ u(0) &= u_0.\end{aligned}\tag{3.2}$$

**3.2. Dynamic transition and pattern formation for the diffusion and degradation balanced case.** We start with an important case where the diffusion and degradation of the chemoattractant secreted by the bacteria themselves are almost balanced by their production. In this case, the second equation in (3.2) is given by

$$0 = \Delta u_2 - u_2 + \lambda u_1.$$

With the Newman boundary condition for  $u_2$ , we have  $u_2 = \lambda[-\Delta + 1]^{-1}u_1$  and the functional form of the resulting equations are given by

$$\frac{\partial u_1}{\partial t} = \mathcal{L}_\lambda u_1 + G(u_1, \lambda), \quad (3.3)$$

where the operators  $\mathcal{L}_\lambda : H_1 \rightarrow H$  and  $G : H_1 \times \mathbb{R} \rightarrow \mathbb{R}$  are defined by

$$\begin{aligned} \mathcal{L}_\lambda u_1 &= \mu \Delta u_1 - 2\alpha u_1 - \lambda \Delta[-\Delta + I]^{-1}u_1, \\ G(u_1, \lambda) &= -\lambda \nabla(u_1 \nabla[-\Delta + I]^{-1}u_1) - 3\alpha u_1^2 - \alpha u_1^3. \end{aligned} \quad (3.4)$$

Here the two Hilbert spaces  $H$  and  $H_1$  are defined by

$$H = L^2(\Omega), \quad H_1 = \{u_1 \in H^2(\Omega) \mid \frac{\partial u_1}{\partial n} = 0 \text{ on } \Omega\}.$$

To study the dynamic transition of this problem, we need to consider the linearized eigenvalue problem of (3.3):

$$\mathcal{L}_\lambda e = \beta(\lambda)e. \quad (3.5)$$

Let  $\rho_k$  and  $e_k$  be the eigenvalues and eigenfunctions of  $-\Delta$  with the Neumann boundary condition given by

$$e_k = \cos \frac{k_1 \pi x_1}{L_1} \cos \frac{k_2 \pi x_2}{L_2}, \quad \rho_k = \pi^2 \left( \frac{k_1^2}{L_1^2} + \frac{k_2^2}{L_2^2} \right), \quad (3.6)$$

for any  $k = (k_1, k_2) \in \mathbb{N}_+^2$ . Here  $\mathbb{N}_+$  is the set of all nonnegative integers. In particular,  $e_0 = 1$  and  $\rho_0 = 0$ .

Obviously, the functions in (3.6) are also eigenvectors of (3.5), and the corresponding eigenvalues  $\beta_k$  are

$$\beta_k(\lambda) = -\mu \rho_k - 2\alpha + \frac{\lambda \rho_k}{1 + \rho_k}. \quad (3.7)$$

Define a critical parameter by

$$\lambda_c = \min_{\rho_k} \frac{(\rho_k + 1)(\mu \rho_k + 2\alpha)}{\rho_k}. \quad (3.8)$$

Let

$$\mathcal{S} = \{K = (K_1, K_2) \in \mathbb{N}_+^2 \text{ achieves the minimization in (3.8)}\}.$$

Then it follows from (3.7) and (3.8) that

$$\beta_K(\lambda) \begin{cases} < 0 & \text{if } \lambda < \lambda_c \\ = 0 & \text{if } \lambda = \lambda_c \\ > 0 & \text{if } \lambda > \lambda_c \end{cases} \quad \forall K = (K_1, K_2) \in \mathcal{S}, \quad (3.9)$$

$$\beta_k(\lambda_c) < 0 \quad \forall k \in \mathbb{Z}^2 \text{ with } k \notin \mathcal{S}. \quad (3.10)$$

Notice that for any  $K = (K_1, K_2) \in \mathcal{S}$ ,  $K \neq 0$ , and

$$\lambda_c = \frac{(\rho_K + 1)(\mu \rho_K + 2\alpha)}{\rho_K}. \quad (3.11)$$

We note that for properly choosing spatial geometry, we have

$$\rho_K = \pi^2 \left( \frac{K_1^2}{L_1^2} + \frac{K_2^2}{L_2^2} \right) = \sqrt{\frac{2\alpha}{\mu}} \quad \forall K = (K_1, K_2) \in \mathcal{S}, \quad (3.12)$$

$$\lambda_c = 2\alpha + \mu + 2\sqrt{2\alpha\mu}. \quad (3.13)$$

Conditions (3.9) and (3.10) give rise to a dynamic transition of (3.3) from  $(u, \lambda) = (0, \lambda_c)$ . For simplicity, we denote

$$K_1 \triangleq (K_1, 0), \quad K_2 \triangleq (0, K_2),$$

and  $K = (K_1, K_2) \in \mathcal{S}$ . Also, we introduce a parameter as

$$\begin{aligned} b = & -3\mu\rho_K + \left[ 12 - \frac{24 - 15\text{sign}(K_1 K_2)}{4 - 2\text{sign}(K_1 K_2)} \right] \alpha \\ & - \frac{(2\mu\rho_K + \alpha)(2\mu\rho_K^2 + 28\alpha\rho_K + 4\alpha - \mu\rho_K)}{[1 + \text{sign}(K_1 K_2)] \cdot [(\mu\rho_{2K} + 2\alpha)(1 + \rho_{2K}) - \lambda_c\rho_{2K}]} \\ & - \frac{2(2\mu\rho_K\rho_{K_1} + 4\alpha\rho_{K_1} - 3\alpha\rho_K)}{(1 + \text{sign}K_1)\rho_K^2[(\mu\rho_{2K_1} + 2\alpha)(1 + \rho_{2K_1}) - \lambda_c\rho_{2K_1}]} \\ & \times [(\mu\rho_K + 2\alpha)(2\rho_{K_1}^2 - 6\rho_{K_1}\rho_{K_2} - \rho_K) + 6\alpha\rho_K(4\rho_{K_1} + 1)] \\ & - \frac{2(2\mu\rho_K\rho_{K_2} + 4\alpha\rho_{K_2} - 3\alpha\rho_K)}{(1 + \text{sign}K_2)\rho_K^2[(\mu\rho_{2K_2} + 2\alpha)(1 + \rho_{2K_2}) - \lambda_c\rho_{2K_2}]} \\ & \times [(\mu\rho_K + 2\alpha)(2\rho_{K_2}^2 - 6\rho_{K_1}\rho_{K_2} - \rho_K) + 6\alpha\rho_K(4\rho_{K_2} + 1)]. \end{aligned} \quad (3.14)$$

The following is the main dynamic transition theorem, providing a precise criterion for the transition type and the pattern formation mechanism of the system.

**Theorem 3.1.** *Let  $b$  be the parameter defined by (3.14). Assume that the eigenvalue  $\beta_k$  satisfying (3.9) is simple. Then, for the system (3.3) we have the following assertions:*

- (1) *The system always undergoes a dynamic transition at  $(u, \lambda) = (0, \lambda_c)$ . Namely, the basic state  $u = 0$  is asymptotically stable for  $\lambda < \lambda_c$ , and is unstable for  $\lambda > \lambda_c$ .*
- (2) *For the case where  $b < 0$ , this transition is continuous (Type-I). In particular, the system bifurcates from  $(0, \lambda_c)$  to two steady state solutions on  $\lambda > \lambda_c$ , which can be expressed as*

$$u_1^\pm(x, \lambda) = \pm \frac{1}{2} \sqrt{\frac{\beta_K(\lambda)}{2|b|}} \cos \frac{K_1 \pi x_1}{L_1} \cos \frac{K_2 \pi x_2}{L_2} + o\left(\beta_K^{1/2}\right), \quad (3.15)$$

*and  $u_1^\pm(x, \lambda)$  are attractors.*

- (3) *For the case  $b > 0$ , this transition is jump (Type-II), and the system has two saddle-node bifurcation solutions at some  $\lambda^* (0 < \lambda^* < \lambda_c)$  such that there are two branches  $v_1^\lambda$  and  $v_2^\lambda$  of steady states bifurcated from  $(v^*, \lambda^*)$ , and there are two other branches  $v_3^\lambda$  and  $v_4^\lambda$  bifurcated from  $(u^*, \lambda^*)$ . In addition,  $v_1^\lambda$  and  $v_3^\lambda$  are saddles,  $v_2^\lambda$  and  $v_4^\lambda$  are attractors, with  $v_1^\lambda, v_3^\lambda \rightarrow 0$  as  $\lambda \rightarrow \lambda_c$ .*

Two remarks are now in order.

**Remark 3.1.** From the pattern formation point of view, for the Type-I transition, the patterns described by the transition solutions given in (3.15) are either lamella

or rectangular:

$$\begin{array}{ll} \text{lamella pattern} & \text{for } K_1 K_2 = 0, \\ \text{rectangular pattern} & \text{for } K_1 K_2 \neq 0. \end{array}$$

In the case where  $b > 0$ , the system undergoes a more drastic change. As  $\lambda^* < \lambda < \lambda_c$ , the homogeneous state, the new patterns  $v_2^\lambda$  and  $v_4^\lambda$  are metastable. For  $\lambda > \lambda_c$ , the system undergoes transitions to more complex patterns away from the basic homogeneous state form.

**Remark 3.2.** If we take the growth term  $f(u)$  as  $f = \alpha u_1(1 - u_1)$  instead of  $f = \alpha u_1(1 - u_1^2)$  in (3.1), (3.2) and (3.3), then Theorem 3.1 still holds true except the assertion on the existence of the two saddle-node bifurcation solutions, and the parameter should be replaced by

$$\begin{aligned} b = & -\mu\rho_K + \alpha - \frac{(2\mu\rho_K + \alpha)(2\mu\lambda_K^2 + 10\alpha\rho_K + \alpha - \mu\rho_K)}{2(1 + \text{sign}K_1K_2)[(\mu\rho_{2K} + \alpha)(1 + \rho_{2K}) - \lambda_c\rho_{2K}]} \\ & - \frac{(2\mu\rho_K\rho_{K_1} + 2\alpha\rho_{K_1} - \alpha\rho_K)}{(1 + \text{sign}K_1)\lambda_K^2[(\mu\rho_{2K_1} + \alpha)(1 + \rho_{2K_1}) - \lambda_c\rho_{2K_1}]} \\ & \times [(\mu\rho_K + \alpha)(2\lambda_{K_1}^2 - 6\rho_{K_1}\rho_{K_2} - \rho_K) + 2\alpha\rho_K(4\rho_{K_1} + 1)] \\ & - \frac{(2\mu\rho_K\rho_{K_2} + 2\alpha\rho_{K_2} - \alpha\rho_K)}{(1 + \text{sign}K_2)\lambda_K^2[(\mu\rho_{2K_2} + \alpha)(1 + \rho_{2K_2}) - \lambda_c\rho_{2K_2}]} \\ & \times [(\mu\rho_K + \alpha)(2\lambda_{K_1}^2 - 6\rho_{K_1}\rho_{K_2} - \rho_K) + 2\alpha\rho_K(4\rho_{K_2} + 1)]. \end{aligned}$$

**3.3. Pattern formation and dynamic transition for the general case.** Consider the general case (3.1). In this case, the unknown variable becomes  $u = (u_1, u_2)$ , and the basic function spaces are then defined by

$$H = L^2(\Omega, \mathbb{R}^2), \quad H_1 = \left\{ u \in H^2(\Omega, \mathbb{R}^2) \mid \frac{\partial u}{\partial n} = 0 \text{ on } \Omega \right\}.$$

Let  $L_\lambda : H_1 \rightarrow H$  and  $G : H_1 \rightarrow H$  be defined by

$$\begin{aligned} L_\lambda u &= \begin{pmatrix} \mu\Delta - 2\alpha & -\Delta \\ \lambda & \Delta - 1 \end{pmatrix} u, \\ G(u) &= \begin{pmatrix} -\nabla(u_1 \nabla u_2) - 3\alpha u_1^2 - \alpha u_1^3 \\ 0 \end{pmatrix}. \end{aligned} \quad (3.16)$$

The linearized eigenvalue problem of (3.2) is

$$L_\lambda \varphi = \beta \varphi, \quad (3.17)$$

where  $L_\lambda : H_1 \rightarrow H$  is defined by (3.16). Let  $B_k^\lambda$  be the matrices given by

$$B_k^\lambda = \begin{pmatrix} -(\mu\rho_k + 2\alpha) & \rho_k \\ \lambda & -(\rho_k + 1) \end{pmatrix}, \quad (3.18)$$

where  $\rho_k$  are the eigenvalues as in (3.6). It is easy to see that all eigenvectors  $\varphi_k$  and eigenvalues  $\beta_k$  of (3.17) can be expressed as follows

$$\varphi_k = \begin{pmatrix} \xi_{k1} e_k \\ \xi_{k2} e_k \end{pmatrix}, \quad (3.19)$$

$$B_k^\lambda \begin{pmatrix} \xi_{k1} \\ \xi_{k2} \end{pmatrix} = \beta_k \begin{pmatrix} \xi_{k1} \\ \xi_{k2} \end{pmatrix}, \quad (3.20)$$



where  $e_k$  are as in (3.6), and  $\beta_k$  are also the eigenvalues of  $B_k^\lambda$ . By (3.18),  $\beta_k$  can be expressed by

$$\beta_k^\pm(\lambda) = \frac{1}{2} \left[ -B \pm \sqrt{B^2 - 4((\rho_k + 1)(\mu\rho_k + 2\alpha) - \lambda\rho_k)} \right], \quad (3.21)$$

$$B = (\mu + 1)\rho_k + 2\alpha + 1.$$

Let  $\lambda_c$  be the parameter as defined by (3.8). It follows from (3.21) and (3.8) that

$$\beta_K^+(\lambda) \begin{cases} < 0 & \text{if } \lambda < \lambda_c, \\ = 0 & \text{if } \lambda = \lambda_c, \\ > 0 & \text{if } \lambda > \lambda_c, \end{cases} \quad (3.22)$$

$$\begin{cases} \operatorname{Re} \beta_k^-(\lambda_c) < 0 & \forall k \in \mathbb{Z}^2, \\ \operatorname{Re} \beta_k^+(\lambda_c) < 0 & \forall k \in \mathbb{Z}^2 \text{ with } \rho_k \neq \rho_K, \end{cases} \quad (3.23)$$

with  $K = (K_1, K_2)$  as in (3.11).

Then we have the following dynamic transition theorem.

**Theorem 3.2.** *Let  $b$  be the parameter defined by (3.14). Assume that the eigenvalue  $\beta_K^+$  satisfying (3.22) is simple. Then the assertions of Theorem 3.1 hold true for (3.2), with the expression (3.15) replaced by*

$$u_\lambda^\pm = \pm \sqrt{a\beta_K^+(\lambda)} \begin{pmatrix} \rho_K + 1 \\ \lambda_c \end{pmatrix} \cos \frac{K_1 \pi x_1}{L_1} \cos \frac{K_2 \pi x_2}{L_2} + o(|\beta_K^+|^{1/2}),$$

$$a = \frac{8(\mu\rho_K + \rho_K + 2\alpha + 1)}{(\rho_K + 1)^3 |b|}.$$

#### 3.4. Proof of main theorems.

*Proof of Theorem 3.1.* Assertion (1) follows directly from the general dynamic transition theorem in Chapter 2 of [7]. To prove Assertions (2) and (3), we need to reduce (3.3) to the center manifold near  $\lambda = \lambda_c$ . We note that although the underlying system is now quasilinear in this general case, the center manifold reduction holds true as well; see [6] for details.

To this end, let  $u = xe_k + \Phi$ , where  $\Phi(x)$  the center manifold function of (3.3). Since  $L_\lambda : H_1 \rightarrow H$  is symmetric, the reduced equation is given by

$$\frac{dx}{dt} = \beta_K(\lambda)x + \frac{1}{(e_K, e_K)} (G(xe_K + \Phi, \lambda), e_K), \quad (3.24)$$

where  $G : H_1 \rightarrow H$  is defined by (3.4), and

$$(e_K, e_K) = \int_{\Omega} e_K^2 dx = \frac{2 - \operatorname{sign}(K_1 K_2)}{4} |\Omega|. \quad (3.25)$$

It is known that the center manifold function satisfies that  $\Phi(x) = O(x^2)$ . A direct computation shows that

$$\begin{aligned} & \langle G(xe_K + \Phi, \lambda_c), e_K \rangle \\ &= -\alpha x^3 \int_{\Omega} e_K^4 dx - 6\alpha x \int_{\Omega} e_K^2 \Phi dx \\ & \quad + \lambda_c x \int_{\Omega} [e_K \nabla e_k \cdot \nabla (-\Delta + I)^{-1} \Phi + \Phi \nabla e_k \cdot \nabla (-\Delta + I)^{-1} e_K] dx + o(x^3). \end{aligned} \quad (3.26)$$

It is clear that

$$(-\Delta + I)^{-1} e_K = \frac{1}{\rho_K + 1} e_K, \quad \Delta e_K = -\rho_K e_K.$$

We infer from (3.26) that

$$\begin{aligned} (G(xe_K + \Phi, \lambda_c), e_K) &= -\alpha x^3 \int_{\Omega} e_K^4 dx - 6\alpha x \int_{\Omega} e_K^2 \Phi dx \\ &\quad + \lambda_c x \int_{\Omega} \left[ \frac{1}{\rho_K + 1} |\nabla e_K|^2 \Phi - |\nabla e_K|^2 (-\Delta_I)^{-1} \Phi \right. \\ &\quad \left. + \rho_K e_K^2 (-\Delta + I)^{-1} \Phi \right] dx + o(x^3). \end{aligned} \quad (3.27)$$

Using the approximation formula for center manifold functions given in (A.11) in [8],  $\Phi$  satisfies the equation

$$\begin{aligned} -L_{\lambda_c} \Phi &= G_2(xe_K, \lambda_c) + o(x^2) \\ &= x^2 \left[ \left( \frac{\rho_K \lambda_c}{\rho_K + 1} - 3\alpha \right) e_K^2 - \frac{\lambda_c}{\rho_K + 1} |\nabla e_K|^2 \right] + o(x^2). \end{aligned} \quad (3.28)$$

In view of (3.6), we find

$$\begin{aligned} e_K^2 &= \frac{1}{4} [e_0 + e_{2K_1} + e_{2K_2} + e_{2K}], \\ |\nabla e_K|^2 &= \frac{1}{4} [\rho_K e_0 + (\rho_{K_2} - \rho_{K_1}) e_{2K_1} + (\rho_{K_1} - \rho_{K_2}) e_{2K_2} - \rho_K e_{2K}]. \end{aligned} \quad (3.29)$$

Thus, (3.28) is written as

$$\begin{aligned} -L_{\lambda_c} \Phi &= \frac{x^2}{4} \left[ -3\alpha e_0 + \left( \frac{2\rho_{K_1} \lambda_c}{\rho_K + 1} - 3\alpha \right) e_{2K_1} \right. \\ &\quad \left. + \left( \frac{2\rho_{K_2} \lambda_c}{\rho_K + 1} - 3\alpha \right) e_{2K_2} + \left( \frac{2\rho_K \lambda_c}{\rho_K + 1} e_{2K} - 3\alpha \right) e_{2K} \right] + o(x^2). \end{aligned} \quad (3.30)$$

Denote by

$$\Phi = \Phi_0 e_0 + \Phi_{2K_1} e_{2K_1} + \Phi_{2K_2} e_{2K_2} + \Phi_{2K} e_{2K}. \quad (3.31)$$

Note that

$$-L_{\lambda_c} e_{2K} = \frac{1}{1 + \rho_{2K}} [(1 + \rho_{2K})(\mu \rho_K + 2\alpha) - \lambda_c \rho_{2K}] e_{2K}. \quad (3.32)$$

Then, by (3.11) and (3.30)-(3.32) we obtain

$$\begin{aligned} \Phi_0 &= -\frac{3}{8}, \\ \Phi_{2K_1} &= \frac{(1 + \rho_{2K_1})(2\mu \rho_K \rho_{K_1} + 4\alpha \rho_{K_1} - 3\alpha \rho_K)}{4\rho_K [(1 + \rho_{2K_1})(\mu \rho_{2K_1} + 2\alpha) - \rho_{2K_1} \lambda_c]}, \\ \Phi_{2K_2} &= \frac{(1 + \rho_{2K_2})(2\mu \rho_K \rho_{K_2} + 4\alpha \rho_{K_2} - 3\alpha \rho_K)}{4\rho_K [(1 + \rho_{2K_2})(\mu \rho_{2K_2} + 2\alpha) - \rho_{2K_2} \lambda_c]}, \\ \Phi_{2K} &= \frac{(1 + \rho_{2K})(2\mu \rho_K + \alpha)}{4[(1 + \rho_{2K})(\mu \rho_{2K} + 2\alpha) - \rho_{2K} \lambda_c]}. \end{aligned} \quad (3.33)$$

Inserting (3.31) and (3.6) into (3.27) we get

$$\begin{aligned}
 & \langle G(xe_K + \Phi, \lambda_c), e_K \rangle = -\alpha x^3 \int_{\Omega} e_K^4 dx \\
 & - \frac{6\alpha x(2 - \text{sign}(K_1 K_2))}{4} \int_{\Omega} [\Phi_0 e_0^2 + \Phi_{2K_1} e_{2K_1}^2 + \Phi_{2K_2} e_{2K_2}^2 + \Phi_{2K} e_{2K}^2] dx \\
 & + \frac{\lambda_c x(2 - \text{sign}(K_1 K_2))}{4(\rho_K + 1)} \int_{\Omega} [\rho_K \Phi_0 e_0^2 + (\rho_{K_2} - \rho_{K_1}) \Phi_{2K_1} e_{2K_1}^2 \\
 & + (\rho_{K_1} - \rho_{K_2}) \Phi_{2K_2} e_{2K_2}^2 - \rho_K \Phi_{2K} e_{2K}^2] dx \\
 & - \frac{\lambda_c x(2 - \text{sign}(K_1 K_2))}{4} \int_{\Omega} \left[ \rho_K \Phi_0 e_0^2 + \frac{\rho_{K_2} - \rho_{K_1}}{1 + \rho_{2K_1}} \Phi_{2K_1} e_{2K_1}^2 \right. \\
 & \left. + \frac{\rho_{K_1} - \rho_{K_2}}{1 + \rho_{2K_2}} \Phi_{2K_2} e_{2K_2}^2 - \frac{\rho_K}{1 + \rho_{2K}} \Phi_{2K} e_{2K}^2 \right] dx \\
 & + \frac{\lambda_c \rho_K x(2 - \text{sign}(K_1 K_2))}{4} \int_{\Omega} \left[ \Phi_0 e_0^2 + \frac{\Phi_{2K_1}}{1 + \rho_{2K_1}} e_{2K_1}^2 + \frac{\Phi_{2K_2}}{1 + \rho_{2K_2}} e_{2K_2}^2 \right. \\
 & \left. + \frac{\Phi_{2K}}{1 + \rho_{2K}} e_{2K}^2 \right] dx + o(x^3) \\
 & = -\alpha x^3 \int_{\Omega} e_K^4 dx + \frac{|\Omega| x(2 - \text{sign}(K_1 K_2))}{4} \\
 & \times \left[ (\mu \rho_K - 4\alpha) \Phi_0 + \frac{1}{1 + \text{sign} K_1} \left( \frac{\lambda_c(\rho_{K_2} - \rho_{K_1})}{1 + \rho_K} + \frac{2\lambda_c \rho_{K_1}}{1 + \rho_{2K_1}} - 6\alpha \right) \Phi_{2K_1} \right. \\
 & + \frac{1}{1 + \text{sign} K_2} \left( \frac{\lambda_c(\rho_{K_1} - \rho_{K_2})}{1 + \rho_K} + \frac{2\lambda_c \rho_{K_2}}{1 + \rho_{2K_1}} - 6\alpha \right) \Phi_{2K_2} \\
 & \left. + \frac{1}{2(1 + \text{sign}(K_1 K_2))} \left( -\frac{\lambda_c \rho_K}{1 + \rho_K} + \frac{2\lambda_c \rho_K}{1 + \rho_{2K}} - 6\alpha \right) \Phi_{2K} \right] dx + o(x^3).
 \end{aligned} \tag{3.34}$$

Also, we note that

$$\int_{\Omega} e_K^4 = \int_0^{L_1} e_{K_1}^4 dx_1 \int_0^{L_2} e_{K_2}^4 dx_2 = \frac{24 - 15\text{sign}(K_1 K_2)}{64}.$$

Then, putting (3.33) into (3.34) we get

$$\langle G(xe_K + \Phi, \lambda_c), e_K \rangle = \frac{(2 - \text{sign}(K_1 K_2))|\Omega|x^3}{32} b + o(x^3), \tag{3.35}$$

where  $b$  is the parameter given by (3.14).

By (3.24) and (3.35), we derive the reduced equation on the center manifold as follows:

$$\frac{dx}{dt} = \beta_K(\lambda)x + \frac{b}{8}x^3 + o(x^3). \tag{3.36}$$

Based on the dynamic transition theory developed in Chapter 2 in [7], we obtain Assertions (2) and (3), except that two saddle-node bifurcations occur at the same point  $\lambda = \lambda^*$ . To prove this conclusion, we note that if  $u^*(x)$  is a steady state solution of (3.3), then

$$v^*(x) = u^*(x + \pi) = u^*(x - \pi)$$

is also a steady state solution of (3.3). This is because the eigenvectors (3.6) form an orthogonal base of  $H_1$ . Hence, two saddle-node bifurcations on  $\lambda < \lambda_c$  imply

that they must occur at the same point  $\lambda = \lambda^*$ . Thus the proof of the theorem is complete.  $\square$

*Proof of Theorem 3.2.* Assertion (1) follows from (3.22) and (3.23). To prove Assertions (2) and (3), we need to get the reduced equation of (3.2) to the center manifold near  $\lambda = \lambda_c$ .

Let  $u = x \cdot \varphi_K + \Phi$ , where  $\varphi_K$  is the eigenvector of (3.17) corresponding to  $\beta_K$  at  $\lambda = \lambda_c$ , and  $\Phi(x)$  the center manifold function of (3.2). Then the reduced equation of (3.2) read

$$\frac{dx}{dt} = \beta_K^+(\lambda)x + \frac{1}{\langle \varphi_K, \varphi_K^* \rangle} \langle G(x \cdot \varphi_K + \Phi), \varphi_K^* \rangle, \quad (3.37)$$

Here  $\varphi_K^*$  is the conjugate eigenvector of  $\varphi_K$ .

By (3.19) and (3.20),  $\varphi_K$  is written as

$$\varphi_K = (\xi_1 e_k, \xi_2 e_K)^T, \quad (3.38)$$

with  $(\xi_1, \xi_2)$  satisfying

$$\begin{pmatrix} -(\mu\rho_K + 2\alpha) & \rho_K \\ \lambda_c & -(\rho_K + 1) \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0, \quad (3.39)$$

from which we get

$$(\xi_1, \xi_2) = (\rho_K + 1, \lambda_c). \quad (3.40)$$

Likewise,  $\varphi_K^*$  is

$$\varphi_K^* = (\xi_1^* e_k, \xi_2^* e_K)^T, \quad (3.41)$$

with  $(\xi_1^*, \xi_2^*)$  satisfying

$$\begin{pmatrix} -(\mu\rho_K + 2\alpha) & \lambda_c \\ \rho_K & -(\rho_K + 1) \end{pmatrix} \begin{pmatrix} \xi_1^* \\ \xi_2^* \end{pmatrix} = 0,$$

which yields

$$(\xi_1^*, \xi_2^*) = (\rho_{K+1}, \rho_K). \quad (3.42)$$

By (3.16), the nonlinear operator  $G$  is

$$\begin{aligned} G(u_1, u_2) &= G_2(u_1, u_2) + G_3(u_1, u_2), \\ G_2(u_1, u_2) &= -(\nabla u_1 \nabla u_2 + u_1 \Delta u_2 + 3\alpha u_1^2) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ G_3(u_1, u_2) &= -\alpha u^3 \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned}$$

It is known that the center manifold function

$$\Phi(x) = (\Phi_1(x), \Phi_2(x)) = O(x^2).$$

Then, in view of (3.38) and (3.40), by direct computation we derive that

$$\begin{aligned} & (G(x\xi_1 e_K + \Phi_1, x\xi_2 e_K + \Phi_2), \varphi_K^*) \\ &= (xG_2(\xi_1 e_K, \Phi_2) + xG_2(\Phi_1, \xi_2 e_K) + x^3 G_3(\xi_1 e_K, \xi_2 e_K), \varphi_K^*) + o(x^3) \\ &= x\xi_1^* \int_{\Omega} [\xi_2 \Phi_1 |\nabla e_K|^2 - \frac{1}{2} \xi_1 \Delta \Phi_2 e_K^2 - 6\alpha \xi_1 \Phi_1 e_K^2] dx \\ &\quad - \alpha \xi_1^* \xi_1^3 x^3 \int_{\Omega} e_K^4 dx + o(x^3). \end{aligned} \quad (3.43)$$

Using the approximation formula for center manifold functions given in (A.11) in [8],  $\Phi = (\Phi_1, \Phi_2)$  satisfies

$$\begin{aligned} -L_{\lambda_c} \Phi &= -x^2 G_2(\xi_1 e_k, \xi_2 e_K) + o(x^2) \\ &= -x^2 (\xi_1 \xi_2 |\nabla e_K|^2 + (3\alpha \xi_1^2 - \xi_1 \xi_2 \rho_K) e_K^2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + o(x^2). \end{aligned} \quad (3.44)$$

From (3.6) we see that

$$\begin{aligned} e_K^2 &= \frac{1}{4}(1 + e_{2K_1})(1 + e_{2K_2}) = \frac{1}{4}(e_0 + e_{2K_1} + e_{2K_2} + e_{2K}), \\ |\nabla e_K|^2 &= \frac{\rho_{K_1}}{4}(1 - e_{2K_1})(1 + e_{2K_2}) + \frac{\rho_{K_2}}{4}(1 + e_{2K_1})(1 - e_{2K_2}) \\ &= \frac{\rho_K}{4}e_0 + \frac{\rho_{K_2} - \rho_{K_1}}{4}e_{2K_1} + \frac{\rho_{K_1} - \rho_{K_2}}{4}e_{2K_2} - \frac{\rho_K}{4}e_{2K}. \end{aligned}$$

Thus, (3.44) is written as

$$\begin{aligned} L_{\lambda_c} \Phi &= -\frac{\xi_1 x^2}{4} (3\alpha \xi_1 e_0 + (3\alpha \xi_1 - 2\xi_2 \rho_{K_1}) e_{2K_1} \\ &\quad + (3\alpha \xi_1 - 2\xi_2 \rho_{K_2}) e_{2K_2} + (3\alpha \xi_1 - 2\xi_2 \rho_K) e_{2K}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + o(x^3). \end{aligned} \quad (3.45)$$

Let

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} \Phi_1^0 \\ \Phi_2^0 \end{pmatrix} e_0 + \begin{pmatrix} \Phi_1^{2K_1} \\ \Phi_2^{2K_1} \end{pmatrix} e_{2K_1} + \begin{pmatrix} \Phi_1^{2K_2} \\ \Phi_2^{2K_2} \end{pmatrix} e_{2K_2} + \begin{pmatrix} \Phi_1^{2K} \\ \Phi_2^{2K} \end{pmatrix} e_{2K} \quad (3.46)$$

It is clear that

$$L_{\lambda} \begin{pmatrix} \Phi_1^k \\ \Phi_2^k \end{pmatrix} e_k = B_k^{\lambda} \begin{pmatrix} \Phi_1^k \\ \Phi_2^k \end{pmatrix} e_k,$$

where  $B_k^{\lambda}$  is the matrix given by (3.18). Then by (3.45) and (3.46) we have

$$\begin{pmatrix} \Phi_1^{2k} \\ \Phi_2^{2k} \end{pmatrix} = -\frac{(3\alpha \xi_1^2 - 2\xi_1 \xi_2 \rho_k) x^2}{4} B_{2k}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

for  $k = K, K_1, K_2$ , and  $B_{2k} = B_{2k}^{\lambda_c}$ .

Direct computation shows that

$$\begin{pmatrix} \Phi_1^0 \\ \Phi_2^0 \end{pmatrix} = \frac{3\xi_1^2 x^2}{8} \begin{pmatrix} 1 \\ \lambda_c \end{pmatrix}, \quad (3.47)$$

$$\begin{pmatrix} \Phi_1^{2K_1} \\ \Phi_2^{2K_1} \end{pmatrix} = \frac{\xi_1(3\alpha \xi_1 - 2\xi_2 \rho_{K_1})}{4\det B_{2K_1}} \begin{pmatrix} 1 + \rho_{2K_1} \\ \lambda_c \end{pmatrix}, \quad (3.48)$$

$$\begin{pmatrix} \Phi_1^{2K_2} \\ \Phi_2^{2K_2} \end{pmatrix} = \frac{\xi_1(3\alpha \xi_1 - 2\xi_2 \rho_{K_2})}{4\det B_{2K_2}} \begin{pmatrix} 1 + \rho_{2K_2} \\ \lambda_c \end{pmatrix}, \quad (3.49)$$

$$\begin{pmatrix} \Phi_1^{2K} \\ \Phi_2^{2K} \end{pmatrix} = \frac{\xi_1(3\alpha \xi_1 - 2\xi_2 \rho_K)}{4\det B_{2K}} \begin{pmatrix} 1 + \rho_{2K} \\ \lambda_c \end{pmatrix}. \quad (3.50)$$

Inserting (3.47) into (3.43), by (3.40) and (3.42) we get

$$\begin{aligned}
\langle G(x\varphi_K + \Phi), \varphi_K^* \rangle &= \frac{(2 - \text{sign}K_1K_2)(\rho_K + 1)|\Omega|}{8} \\
&\times \left[ -\frac{8\alpha(\rho_K + 1)^3x^3 \int_{\Omega} e_K^4 dx}{(2 - \text{sign}K_1K_2)|\Omega|} + 2(\xi_2\rho_K - 6\alpha\xi_1)\Phi_1^0x \right. \\
&+ \frac{2}{1 + \text{sign}K_1}(\xi_2(\rho_{K_2} - \rho_{K_1}) - 6\alpha\xi_1)\Phi_1^{2K_1}x \\
&+ \frac{2}{1 + \text{sign}K_2}(\xi_2(\rho_{K_1} - \rho_{K_2}) - 6\alpha\xi_1)\Phi_2^{2K_2}x \\
&- \frac{2 - \text{sign}K_1K_2}{2}(\xi_2\rho_K + 6\alpha\xi_1)\Phi_1^{2K}x \\
&+ \frac{(\rho_K + 1)\rho_{2K_1}}{1 + \text{sign}K_1}\Phi_2^{2K_1}x + \frac{(\rho_K + 1)\rho_{K_2}}{1 + \text{sign}K_2}\Phi_2^{2K_2}x \\
&\left. + \frac{(\rho_K + 1)\rho_{2K}(2 - \text{sign}K_1K_2)}{4}\Phi_2^{2K}x \right] + o(x^3).
\end{aligned}$$

By definition, we have

$$\begin{aligned}
\rho_{K_1} + \rho_{K_2} &= \rho_K, \quad \rho_{2K} = 4\rho_K \quad \forall K = (K_1, K_2), \\
\langle \varphi, \varphi^* \rangle &= [(\rho_K + 1)^2 + \rho_K \lambda_c] \int_{\Omega} e_K^2 dx \\
&= \frac{2 - \text{sign}(K_1K_2)}{4}(\rho_K + 1)(\mu\rho_K + \rho_K + 2\alpha + 1)|\Omega|.
\end{aligned}$$

In view of (3.47)-(3.50), the reduced equation (3.37) is given by

$$\frac{dx}{dt} = \beta_K^+(\lambda)x + \frac{(\rho_K + 1)^3bx^3}{8(\mu\rho_K + \rho_K + 2\alpha + 1)} + o(x^3), \quad (3.51)$$

where  $b$  is the parameter as in (3.14). Then the theorem follows readily from (3.51). The proof is complete.  $\square$

#### 4. Transition of three-component systems.

**4.1. The model.** Hereafter  $\delta_0 \geq 0$  is always assumed to be a constant. Hence, (2.6) has a positive constant steady state  $u^*$  given by

$$(u_1^*, u_2^*, u_3^*) \quad \text{with} \quad u_1^* = \left( \frac{u_3^*}{1 + u_3^*} \right)^{1/2}, \quad u_2^* = \lambda u_1^*, \quad u_3^* u_1^* = \frac{\delta_0}{\delta}. \quad (4.1)$$

It is easy to see that  $u_3^*$  is the unique positive real root of the cubic equation

$$x^3 - \left( \frac{\delta_0}{\delta} \right)^2 x - \left( \frac{\delta_0}{\delta} \right)^2 = 0.$$

Consider the translation

$$(u_1, u_2, u_3) \rightarrow (u_1^* + u_1, u_2^* + u_2, u_1^* + u_1). \quad (4.2)$$

Then equations (2.6) are equivalent to

$$\begin{aligned}\frac{\partial u_1}{\partial t} &= \mu \Delta u_1 - 2\alpha u_1^{*2} u_1 - u_1^* \Delta u_2 + \frac{\alpha u_1^*}{(1+u_3^*)^2} u_3 + g(u), \\ \frac{\partial u_3}{\partial t} &= r \Delta u_3 - \delta u_1^* u_3 - \delta u_3^* u_1 - \delta u_1 u_3, \\ -\Delta u_2 + u_2 &= \lambda u_1, \\ \frac{\partial(u_1, u_2, u_3)}{\partial n} \Big|_{\partial\Omega} &= 0, \\ u(0) &= u_0,\end{aligned}\tag{4.3}$$

where  $u = (u_1, u_3)$ ,  $u_2 = \lambda[-\Delta + 1]^{-1}u_1$ , and

$$\begin{aligned}g(u) &= -\nabla(u_1 \nabla u_2) - 3\alpha u_1^* u_1^2 - \alpha u_1^3 + \frac{\alpha(u_1 + u_1^*)(u_3 + u_3^*)}{1 + u_3^* + u_3} \\ &\quad - \frac{\alpha u_1^* u_3^*}{1 + u_3^*} - \frac{\alpha u_1^* u_3}{(1 + u_3^*)^2} - \frac{\alpha u_3^* u_1}{1 + u_3^*}.\end{aligned}\tag{4.4}$$

The Taylor expansion of  $g$  at  $u = 0$  is expressed by

$$\begin{aligned}g(u) &= -\nabla(u_1 \nabla u_2) - 3\alpha u_1^* u_1^2 + \frac{\alpha u_1 u_3}{(1 + u_3^*)^2} - \frac{\alpha u_1^* u_3^2}{(1 + u_3^*)^3} \\ &\quad - \alpha u_1^3 - \frac{\alpha u_1 u_3^2}{(1 + u_3^*)^3} + \frac{\alpha u_1^* u_3^3}{(1 + u_3^*)^4} + o(3).\end{aligned}$$

Let

$$\begin{aligned}H &= L^2(\Omega, \mathbb{R}^2), \\ H_1 &= \{u \in H^2(\Omega, \mathbb{R}^2) \mid \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}.\end{aligned}$$

Define the operators  $L_\lambda : H_1 \rightarrow H$  and  $G_\lambda : H_1 \rightarrow H$  by

$$\begin{aligned}L_\lambda u &= \begin{pmatrix} \mu \Delta - 2\alpha u_1^{*2} - \lambda u_1^* \Delta[-\Delta + I]^{-1} & \frac{\alpha u_1^*}{(1+u_3^*)^2} \\ -\delta u_3^* & r \Delta - \delta u_1^* \end{pmatrix} \begin{pmatrix} u_1 \\ u_3 \end{pmatrix}, \\ G(u, \lambda) &= \begin{pmatrix} g(u) \\ -\delta u_1 u_3 \end{pmatrix},\end{aligned}\tag{4.5}$$

Then the problem (4.3) takes the following the abstract form:

$$\begin{aligned}\frac{du}{dt} &= L_\lambda u + G(u, \lambda), \\ u(0) &= u_0.\end{aligned}\tag{4.6}$$

It is known that the inverse mapping

$$[-\Delta + I]^{-1} : H \rightarrow H_1$$

is a bounded linear operator. Therefore we have

$$\begin{aligned}L_\lambda : H_1 \rightarrow H &\text{ is a sector operator, and} \\ G_\lambda : H_\theta \rightarrow H &\text{ is } C^\infty \text{ bounded operator for } \theta \geq \frac{1}{2}.\end{aligned}$$

We note that the transition of (4.3) from  $u = 0$  is equivalent to that of (2.6) from  $u = u^*$ .

Theorems 3.1 and 3.2 show that a two-component system undergoes only a dynamic transition to steady states. As we shall see, the transition for the three-component system (2.5) is quite different – it can undergo both steady state and spatiotemporal transitions.

**4.2. Linearized eigenvalue of (2.6).** The eigenvalue equations of (2.6) at the steady state  $(u_1^*, u_2^*, u_3^*)$  given by (4.1) in their abstract form are given by

$$L_\lambda \varphi = \beta \varphi, \quad (4.7)$$

where  $L_\lambda : H_1 \rightarrow H$  as defined in (4.5). The explicit form of (4.7) is given by

$$\begin{pmatrix} \mu\Delta - 2\alpha u_1^{*2} - \lambda u_1^* \Delta [-\Delta + I]^{-1} & \frac{\alpha u_1^*}{(1+u_3^*)^2} \\ -\delta u_3^* & r\Delta - \delta u_1^* \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_3 \end{pmatrix} = \beta \begin{pmatrix} \psi_1 \\ \psi_3 \end{pmatrix}.$$

As before, let  $\rho_k$  and  $e_k$  be the eigenvalue and eigenvector of  $-\Delta$  with Neumann boundary condition given by (3.6), and let

$$\psi_k = (\psi_1^k, \psi_3^k) = (\xi_{k1} e_k, \xi_{k3} e_k).$$

Then, it is easy to see that  $\psi_k$  is an eigenvector of (4.7) provided that  $(\xi_{k1}, \xi_{k3}) \in \mathbb{R}^2$  is an eigenvector of the matrix  $A_k^\lambda$ :

$$A_k^\lambda \begin{pmatrix} \xi_{k1} \\ \xi_{k3} \end{pmatrix} = \beta_k \begin{pmatrix} \xi_{k1} \\ \xi_{k3} \end{pmatrix},$$

with

$$A_k^\lambda = \begin{pmatrix} \frac{\lambda \rho_k u_1^*}{1+\rho_k} - \mu \rho_k - 2\alpha u_1^{*2} & \frac{\alpha u_1^*}{(1+u_3^*)^2} \\ -\delta u_3^* & -r \rho_k - \delta u_1^* \end{pmatrix}. \quad (4.8)$$

The eigenvalues  $\beta_k$  of  $A_k^\lambda$ , which are also eigenvalues of (4.7), are expressed by

$$\begin{aligned} \beta_k^\pm(\lambda) &= \frac{1}{2} \left[ a \pm \sqrt{a^2 - 4 \det A_k^\lambda} \right], \\ a &= \operatorname{tr} A_k^\lambda = \frac{\lambda \rho_k u_1^*}{1+\rho_k} - \mu \rho_k - 2\alpha u_1^{*2} - r \rho_k - \delta u_1^*. \end{aligned} \quad (4.9)$$

To derive the PES, we introduce two parameters as follows:

$$\Lambda_c = \min_{\rho_K} \frac{(\rho_K + 1)}{\rho_K u_1^*} [\mu \rho_K + 2\alpha u_1^{*2} + r \rho_K + \delta u_1^*], \quad (4.10)$$

$$\lambda_c = \min_{\rho_K} \frac{(\rho_K + 1)}{\rho_K u_1^*} \left[ \mu \rho_K + 2\alpha u_1^{*2} + \frac{\alpha \delta}{(1+u_3^*)^2 (r \rho_K + \delta u_1^*)} \right]. \quad (4.11)$$

Let  $K = (K_1, K_2)$  and  $K^* = (K_1^*, K_2^*)$  be the integer pairs such that  $\rho_K$  and  $\rho_{K^*}$  satisfy (4.10) and (4.11) respectively.

**Theorem 4.1.** *Let  $\Lambda_c$  and  $\lambda_c$  be the parameters defined by (4.10) and (4.11) respectively. Then we have the following assertions:*

- (1) As  $\Lambda_c < \lambda_c$ , the eigenvalues  $\beta_K^\pm(\lambda)$  of (4.9) are a pair of conjugate complex numbers near  $\lambda = \Lambda_c$ , and all eigenvalues of (4.9) satisfy

$$\operatorname{Re} \beta_K^\pm(\lambda) \begin{cases} < 0 & \lambda < \Lambda_c, \\ = 0 & \lambda = \Lambda_c, \\ > 0 & \lambda > \Lambda_c, \end{cases} \quad (4.12)$$

$$\operatorname{Re} \beta_k^\pm(\Lambda_c) < 0, \quad \forall k \in \mathbb{Z}^2 \text{ with } \rho_k \neq \rho_K \quad (4.13)$$



(2) As  $\lambda_c < \Lambda_c$ , the eigenvalue  $\beta_{K^*}^+(\lambda)$  is real near  $\lambda = \lambda_c$ , and all of (4.9) satisfy

$$\beta_{K^*}^+(\lambda) \begin{cases} < 0, & \lambda < \lambda_c, \\ = 0, & \lambda = \lambda_c, \\ > 0, & \lambda > \lambda_c, \end{cases} \quad (4.14)$$

$$\begin{cases} \operatorname{Re} \beta_k^+(\lambda_c) < 0, & \forall k \in \mathbb{Z}^2 \text{ with } \rho_k \neq \rho_{K^*}, \\ \operatorname{Re} \beta_k^-(\lambda_c) < 0, & \forall |k| \geq 0. \end{cases} \quad (4.15)$$

*Proof.* By (4.9) we can see that  $\beta_k^\pm(\lambda)$  are a pair of complex eigenvalues of (4.7) near some  $\lambda = \lambda^*$ , and satisfy

$$\operatorname{Re} \beta_k^\pm(\lambda) \begin{cases} < 0, & \lambda < \lambda^*, \\ = 0, & \lambda = \lambda^*, \\ > 0, & \lambda > \lambda^*, \end{cases}$$

if and only if

$$\operatorname{tr} A_k^{\lambda^*} = 0, \quad \det A_k^{\lambda^*} > 0.$$

Likewise,  $\beta_k^+(\lambda)$  is real near  $\lambda = \lambda^*$  and satisfies

$$\beta_k^+(\lambda) \begin{cases} < 0, & \lambda < \lambda^*, \\ = 0, & \lambda = \lambda^*, \\ > 0, & \lambda > \lambda^*, \end{cases}$$

if and only if

$$\operatorname{tr} A_k^{\lambda^*} < 0, \quad \det A_k^{\lambda^*} = 0$$

Due to the definition of  $\lambda_c$  and  $\Lambda_c$ , when  $\Lambda_c < \lambda_c$  we have

$$\begin{aligned} \operatorname{tr} A_{K^*}^{\Lambda_c} &= 0, \\ \operatorname{tr} A_k^{\Lambda_c} &< 0, \quad \forall k \in \mathbb{Z}^2 \text{ with } \rho_k \neq \rho_K, \\ \det A_k^{\Lambda_c} &> 0, \quad \forall |k| \geq 0, \end{aligned} \quad (4.16)$$

and when  $\lambda_c < \Lambda_c$ ,

$$\begin{aligned} \det A_{K^*}^{\lambda_c} &= 0, \\ \det A_k^{\lambda_c} &> 0, \quad \forall k \in \mathbb{Z}^2 \text{ with } \rho_k \neq \rho_K, \\ \operatorname{tr} A_k^{\lambda_c} &< 0, \quad \forall |k| \geq 0. \end{aligned} \quad (4.17)$$

It is known that the real parts of  $\beta_k^\pm$  are negative at  $\lambda$  if and only if

$$\det A_k^\lambda > 0, \quad \operatorname{tr} A_k^\lambda < 0.$$

Hence, Assertions (1) and (2) follow from (4.16) and (4.17) respectively. The theorem is proved.  $\square$

**4.3. Dynamic transition theorem for (2.6).** Based on Theorem 4.1, we immediately get the following transition theorem for (4.3).

**Theorem 4.2.** *Let  $\Lambda_c$  and  $\lambda_c$  be given by (4.10) and (4.11) respectively. Then the following assertions hold true for (4.3):*

- (1) *When  $\Lambda_c < \lambda_c$ , the system undergoes a dynamic transition to periodic solutions at  $(u, \lambda) = (0, \Lambda_c)$ . In particular, if the eigenvalues  $\beta_K^\pm$  satisfying (4.12) are complex simple, then there is a parameter  $b_0$  such that the dynamic transition is continuous (Type-I) as  $b_0 < 0$ , and is jump (Type-II) as  $b_0 > 0$  with a singularity separation of periodic solutions at some  $\lambda^* < \Lambda_c$ .*

- (2) When  $\lambda_c < \Lambda_c$ , the system undergoes a dynamic transition to steady states at  $(u, \lambda) = (0, \lambda_c)$ . If  $\beta_{K^*}^+(\lambda)$  satisfying (4.14) is simple, then there exists a parameter  $b_1$  such that the transition is continuous as  $b_1 < 0$ , and jumping as  $b_1 > 0$  with two saddle-node bifurcations at some  $\tilde{\lambda} < \lambda_c$  from  $(u^+, \tilde{\lambda})$  and  $(u^-, \tilde{\lambda})$ .

**Remark 4.1.** By applying the standard procedure used in the preceding sections, we can derive explicit formulas for the two parameters  $b_0$  and  $b_1$  in Theorem 4.2. However, due to their complexity, we omit the details. Instead in the following, we shall give a method to calculate  $b_0$ , and for  $b_1$  we refer the interested readers to the proof of Theorem 3.2.

**4.4. Computational procedure of  $b_0$ .** The procedure to compute the parameter  $b_0$  in Assertion (1) of Theorem 4.2 is divided into a few steps as follows.

**Step 1.** The reduced equations of (4.6) to center manifold at  $\lambda = \Lambda_c$  are expressed by

$$\begin{aligned} \frac{dx}{dt} &= -\rho y + \frac{1}{\langle \varphi, \varphi^* \rangle} \langle G(x\varphi + y\psi + \Phi, \Lambda_c), \varphi^* \rangle, \\ \frac{dy}{dt} &= \rho x + \frac{1}{\langle \psi, \psi^* \rangle} \langle G(x\varphi + y\psi + \Phi, \Lambda_c), \psi^* \rangle, \end{aligned} \quad (4.18)$$

where  $\varphi$  and  $\psi$  are the eigenvectors of  $L_\lambda$  at  $\lambda = \Lambda_c$ ,  $\varphi^*$  and  $\psi^*$  the conjugate eigenvectors, and  $L_\lambda, G_\lambda : H_1 \rightarrow H$  the operators defined by (4.5),  $\Phi$  is the center manifold function.

**Step 2.** Solving the eigenvectors  $\varphi, \psi$  and their conjugates  $\varphi^*, \psi^*$ . We know that  $\psi_i$  and  $\psi_i^*$  are

$$\begin{aligned} \varphi &= (\xi_1 e_K, \xi_2 e_K), & \psi &= (\eta_1 e_K, \eta_2 e_K), \\ \varphi^* &= (\xi_1^* e_K, \xi_2^* e_K), & \psi^* &= (\eta_1^* e_K, \eta_2^* e_K), \end{aligned} \quad (4.19)$$

and  $\xi_i, \xi_i^*$  satisfy

$$\begin{aligned} \begin{pmatrix} \frac{\Lambda_c \rho_K u_1^*}{1 + \rho_K} - \mu \rho_K - 2\alpha u_1^{*2} & \frac{\alpha u_1^*}{(1 + u_3^*)^2} \\ -\delta u_3^* & -r \rho_K - \delta u_1^* \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} &= \rho \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \\ \begin{pmatrix} \frac{\Lambda_c \rho_K u_1^*}{1 + \rho_K} - \mu \rho_K - 2\alpha u_1^{*2} & \frac{\alpha u_1^*}{(1 + u_3^*)^2} \\ -\delta u_3^* & -r \rho_K - \delta u_1^* \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} &= -\rho \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} \begin{pmatrix} \frac{\Lambda_c \rho_K u_1^*}{1 + \rho_K} - \mu \rho_K - 2\alpha u_1^{*2} & -\delta u_3^* \\ \frac{\alpha u_1^*}{(1 + u_3^*)^2} & -r \rho_K - \delta u_1^* \end{pmatrix} \begin{pmatrix} \xi_1^* \\ \xi_2^* \end{pmatrix} &= -\rho \begin{pmatrix} \eta_1^* \\ \eta_2^* \end{pmatrix} \\ \begin{pmatrix} \frac{\Lambda_c \rho_K u_1^*}{1 + \rho_K} - \mu \rho_K - 2\alpha u_1^{*2} & -\delta u_3^* \\ \frac{\alpha u_1^*}{(1 + u_3^*)^2} & -r \rho_K - \delta u_1^* \end{pmatrix} \begin{pmatrix} \eta_1^* \\ \eta_2^* \end{pmatrix} &= \rho \begin{pmatrix} \xi_1^* \\ \xi_2^* \end{pmatrix}, \end{aligned}$$

where  $\Lambda_c$  is as in (4.10),  $e_k$  as in (3.6), and

$$\begin{aligned} \frac{\Lambda_c \rho_K u_1^*}{1 + \rho_K} - \mu \rho_K - 2\alpha u_1^{*2} &= r \rho_K + \delta u_1^*, \\ \rho = \det A_K^{\Lambda_c} &= \frac{\alpha \delta_0}{(1 + u_3^*)^2} - (\gamma \rho_K + \delta u_1^*)^2. \end{aligned}$$

Here, we use that  $u_1^* u_3^* = \delta_0 / \delta$ . From these equations we obtain

$$\begin{aligned} \xi_1 &= -(r\rho_K + \delta u_1^*), & \xi_2 &= \delta u_3^*, \\ \eta_1 &= -\rho, & \eta_2 &= 0, \\ \xi_1^* &= 0, & \xi_2^* &= -\rho, \\ \eta_1^* &= \delta u_3^*, & \eta_2^* &= \gamma\rho_K + \delta u_1^*. \end{aligned} \quad (4.20)$$

Due to (4.19) and (4.20) we see that

$$\begin{aligned} \langle \varphi, \varphi^* \rangle &= \langle \psi, \psi^* \rangle = (\eta_1 \eta_1^* + \eta_2 \eta_2^*) \int_{\Omega} e_K^2 dx = -\delta \rho u_3^* \int_{\Omega} e_K^2 dx, \\ \langle \varphi, \psi^* \rangle &= \langle \psi, \varphi^* \rangle = 0. \end{aligned}$$

**Step 3.** We need to calculate

$$\langle G(x\varphi + y\psi + \Phi, \Lambda_c), \omega_j^* \rangle, \quad \text{with } \omega_1^* = \varphi^*, \omega_2^* = \psi^*.$$

By (4.5) we have  $G = G_2 + G_3$ , and

$$\begin{aligned} G_2(\omega, \lambda) &= \begin{pmatrix} -\lambda \nabla(\omega_1 \nabla(-\Delta + I)^{-1} \omega_1) - 3\alpha u_1^* \omega_1^2 + \frac{\alpha \omega_1 \omega_2}{(1+u_3^*)^2} - \frac{\alpha u_1^* \omega_2^2}{(1+u_3^*)^3} \\ -\delta \omega_1 \omega_2 \end{pmatrix}, \\ G_3(\omega, \lambda) &= \begin{pmatrix} -\alpha \omega_1^3 - \frac{\alpha \omega_1 \omega_2^2}{(1+u_3^*)^3} + \frac{\alpha u_1^* \omega_2^3}{(1+u_3^*)^4} \\ 0 \end{pmatrix}, \end{aligned}$$

for  $\omega = (\omega_1, \omega_2) \in H_1$ . By (4.20) we find

$$\langle G_3(x\varphi + y\psi + \Phi, \Lambda_c), \varphi^* \rangle = 0.$$

Noting that

$$\begin{aligned} \int_{\Omega} e_K e_J e_I dx &= 0, \quad \forall K, J, I \in \mathbb{Z}^2, \\ \Phi &= (\Phi_1, \Phi_2) = O(x^2), \end{aligned}$$

we have

$$\begin{aligned} &\langle G_2(x\varphi + y\psi + \Phi, \Lambda_c), \varphi^* \rangle \\ &= \int_{\Omega} [\xi_1^* e_K g_{21} + \xi_2^* e_K g_{22}] dx \\ &= (\text{by } \xi_1^* = 0) \\ &= \int_{\Omega} \xi_2^* e_K [-\delta(x\xi_1 e_K + y\eta_1 e_K + \Phi_1)(x\xi_2 + y\eta_2 + \Phi_2)] dx \\ &= -\delta \xi_2^* \left( \xi_2 x \int_{\Omega} \Phi_1 e_K^2 dx + \xi_1 x \int_{\Omega} \Phi_2 e_K^2 dx + \eta_1 y \int_{\Omega} \Phi_2 e_K^2 dx \right). \end{aligned}$$

Thus, we get

$$\begin{aligned} &\langle G(x\varphi + y\psi + \Phi, \Lambda_c), \varphi^* \rangle \\ &= -\delta \xi_2^* \left[ \xi_2 x \int_{\Omega} \Phi_1 e_K^2 dx + \xi_1 x \int_{\Omega} \Phi_2 e_K^2 dx + \eta_1 y \int_{\Omega} \Phi_2 e_K^2 dx \right] + o(3). \end{aligned} \quad (4.21)$$

In the same fashion, we derive

$$\begin{aligned}
& \langle G(x\varphi + y\psi + \Phi, \Lambda_c), \psi^* \rangle \\
&= \left( \frac{\alpha\xi_2\eta_1^*}{(1+u_3^*)^2} - 6\alpha u_1^*\xi_1\eta_1^* - \delta\xi_2\eta_2^* \right) x \int_{\Omega} \Phi_1 e_K^2 dx - 6\alpha u_1^*\eta_1\eta_1^* y \int_{\Omega} \Phi_1 e_K^2 dx \\
&\quad + \left( \frac{\alpha\eta_1\eta_1^*}{(1+u_3^*)^2} - \frac{2\alpha u_1^*\xi_2\eta_1^*}{(1+u_3^*)^3} - \delta\xi_1\eta_2^* \right) x \int_{\Omega} \Phi_2 e_K^2 dx \\
&\quad + \left( \frac{\alpha\eta_1\eta_1^*}{(1+u_3^*)^2} - \delta\eta_1\eta_3^* \right) y \int_{\Omega} \Phi_2 e_K^2 dx + \frac{\Lambda_c\xi_1\eta_1^*}{1+\rho_K} x \int_{\Omega} \Phi_1 |\nabla e_K|^2 dx \\
&\quad + \frac{\Lambda_c\eta_1\eta_1^*}{1+\rho_K} y \int_{\Omega} \Phi_1 |\nabla e_K|^2 dx - \frac{1}{2}\Lambda_c\xi_1\eta_1^* x \int_{\Omega} e_K^2 \Delta(-\Delta + I)^{-1} \Phi_1 dx \\
&\quad - \frac{1}{2}\Lambda_c\eta_1\eta_1^* y \int_{\Omega} e_K^2 \Delta(-\Delta + I)^{-1} \Phi_1 dx \\
&\quad - \left[ \eta_1^* \left( \frac{\alpha u_1^*}{(1+u_3^*)^4} \xi_2^3 - \frac{\alpha\xi_2^2\xi_1}{(1+u_3^*)^3} - \alpha\xi_1^3 \right) \int_{\Omega} e_K^4 dx \right] x^3 \\
&\quad - \left[ \eta_1^* \left( \frac{\alpha\xi_2^2\eta_1}{(1+u_3^*)^3} + 3\alpha\xi_1^2\eta_1 \right) \int_{\Omega} e_K^4 dx \right] x^2 y \\
&\quad - \left( 3\alpha\xi_1\eta_1^2\eta_1^* \int_{\Omega} e_K^4 dx \right) xy^2 - \left( \alpha\eta_1^3\eta_1^* \int_{\Omega} e_K^4 dx \right) y^3 + o(3).
\end{aligned} \tag{4.22}$$

**Step 4.** By the formula of center manifold function in the complex case in Theorem A.1 in [8], we have

$$\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} \Phi_1^1 \\ \Phi_2^1 \end{pmatrix} + \begin{pmatrix} \Phi_1^2 \\ \Phi_2^2 \end{pmatrix} + \begin{pmatrix} \Phi_1^3 \\ \Phi_2^3 \end{pmatrix} + o(3), \tag{4.23}$$

with

$$\begin{aligned}
& -L_{\Lambda_c} \begin{pmatrix} \Phi_1^1 \\ \Phi_2^1 \end{pmatrix} = x^2 G_{11} + xy(G_{12} + G_{21}) + y^2 G_{22}, \\
& -(L_{\Lambda_c}^2 + 4\rho^2)L_{\Lambda_c} \begin{pmatrix} \Phi_1^2 \\ \Phi_2^2 \end{pmatrix} = 2\rho^2 [(x^2 - y^2)(G_{22} - G_{11}) - 2xy(G_{12} + G_{21})], \\
& (L_{\Lambda_c}^2 + 4\rho^2) \begin{pmatrix} \Phi_1^3 \\ \Phi_2^3 \end{pmatrix} = \rho [(y^2 - x^2)(G_{12} + G_{21}) + 2xy(G_{11} - G_{22})].
\end{aligned} \tag{4.24}$$

Here  $G_{ij} = G_2(\Psi^i, \Psi^j, \lambda_c)$  with  $\Psi^1 = \varphi$  and  $\Psi^2 = \psi$ , and  $G_2$  is as defined in Step 3. Namely

$$G_{ij} = \begin{pmatrix} -\Lambda_c \nabla(\Psi_1^i \nabla(-\Delta + I)^{-1} \Psi_1^j) \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{\alpha\Psi_1^i \Psi_2^j}{(1+u_3^*)^2} - \frac{\alpha u_1^* \Psi_2^i \Psi_2^j}{(1+u_3^*)^2} - 3\alpha u_1^* \Psi_1^i \Psi_1^j \\ -\delta\Psi_1^i \Psi_2^j \end{pmatrix},$$

with  $\Psi_l^i = \Gamma_l^i e_K$ ,  $1 \leq i, l \leq 2$ , and

$$\Gamma_1^1 = \xi_1, \quad \Gamma_2^1 = \xi_2, \quad \Gamma_1^2 = \eta_1, \quad \Gamma_2^2 = \eta_2, \tag{4.25}$$

which are given by (4.20).

Direct calculation shows that

$$\begin{aligned} G_{ij} = & -\frac{\Lambda_c \Gamma_1^i \Gamma_1^j}{1 + \rho_K} \nabla(e_K \nabla e_K) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ & + e_K^2 \begin{pmatrix} \frac{\alpha \Gamma_1^i \Gamma_2^j}{(1+u_3^*)^2} - \frac{\alpha u_1^* \Gamma_2^i \Gamma_2^j}{(1+u_3^*)^3} - 3\alpha u_1^* \Gamma_1^i \Gamma_1^j \\ -\delta \Gamma_1^i \Gamma_2^j \end{pmatrix}. \end{aligned} \quad (4.26)$$

For simplicity, we only consider the case where  $K = (K_1, 0)$ . In this case, by (3.6) we can see that

$$e_K^2 = \frac{1}{2}(e_0 + e_{2K}), \quad \nabla(e_K \nabla e_K) = -\rho_K e_{2K}.$$

Then, by (4.26), we have

$$G_{ij} = \begin{pmatrix} h_{ij}^0 \\ g_{ij}^0 \end{pmatrix} e_0 + \begin{pmatrix} h_{ij}^{2K} \\ g_{ij}^{2K} \end{pmatrix} e_{2K}, \quad 1 \leq i, j \leq 2, \quad (4.27)$$

where

$$\begin{aligned} h_{ij}^0 &= \frac{1}{2} \left[ \frac{\alpha \Gamma_1^i \Gamma_2^j}{(1+u_3^*)^2} - \frac{\alpha u_1^* \Gamma_2^i \Gamma_2^j}{(1+u_3^*)^3} - 3\alpha u_1^* \Gamma_1^i \Gamma_1^j \right], \\ h_{ij}^{2K} &= \frac{\rho_K \lambda_c \Gamma_1^i \Gamma_1^j}{1 + \rho_K} + h_{ij}^0, \\ g_{ij}^0 &= g_{ij}^{2K} = -\frac{1}{2} \delta \Gamma_1^i \Gamma_2^j. \end{aligned} \quad (4.28)$$

Let

$$\begin{pmatrix} \Phi_1^K \\ \Phi_2^K \end{pmatrix} = \begin{pmatrix} \varphi_{k1}^0 \\ \varphi_{k2}^0 \end{pmatrix} e_0 + \begin{pmatrix} \varphi_{k1}^{2K} \\ \varphi_{k2}^{2K} \end{pmatrix} e_{2K}, \quad 1 \leq k \leq 3. \quad (4.29)$$

Then it follows from (4.24) and (4.27) that

$$\begin{aligned} \begin{pmatrix} \varphi_{11}^0 \\ \varphi_{12}^0 \end{pmatrix} &= B_0^{-1} \left[ x^2 \begin{pmatrix} h_{11}^0 \\ g_{11}^0 \end{pmatrix} + xy \begin{pmatrix} h_{12}^0 + h_{21}^0 \\ g_{12}^0 + g_{21}^0 \end{pmatrix} + y^2 \begin{pmatrix} h_{22}^0 \\ g_{22}^0 \end{pmatrix} \right], \\ \begin{pmatrix} \varphi_{11}^{2K} \\ \varphi_{12}^{2K} \end{pmatrix} &= B_{2K}^{-1} \left[ x \begin{pmatrix} h_{11}^{2K} \\ g_{11}^{2K} \end{pmatrix} + xy \begin{pmatrix} h_{12}^{2K} + h_{21}^{2K} \\ g_{12}^{2K} + g_{21}^{2K} \end{pmatrix} + y^2 \begin{pmatrix} h_{22}^{2K} \\ g_{22}^{2K} \end{pmatrix} \right], \\ \begin{pmatrix} \varphi_{21}^0 \\ \varphi_{22}^0 \end{pmatrix} &= \frac{2\rho^2}{B_0(B_0^2 + 4\rho^2 I)} \left[ (x^2 - y^2) \begin{pmatrix} h_{22}^0 - h_{11}^0 \\ g_{22}^0 - g_{11}^0 \end{pmatrix} - 2xy \begin{pmatrix} h_{12}^0 + h_{21}^0 \\ g_{12}^0 + g_{21}^0 \end{pmatrix} \right], \\ \begin{pmatrix} \varphi_{21}^{2K} \\ \varphi_{22}^{2K} \end{pmatrix} &= \frac{2\rho^2}{B_{2K}(B_{2K}^2 + 4\rho^2 I)} \left[ (x^2 - y^2) \begin{pmatrix} h_{22}^{2K} - h_{11}^{2K} \\ g_{22}^{2K} - g_{11}^{2K} \end{pmatrix} - 2xy \begin{pmatrix} h_{12}^{2K} + h_{21}^{2K} \\ g_{12}^{2K} + g_{21}^{2K} \end{pmatrix} \right], \\ \begin{pmatrix} \varphi_{31}^0 \\ \varphi_{32}^0 \end{pmatrix} &= \frac{\rho}{B_0^2 + 4\rho^2 I} \left[ (y^2 - x^2) \begin{pmatrix} h_{12}^0 + h_{21}^0 \\ g_{12}^0 + g_{21}^0 \end{pmatrix} + 2xy \begin{pmatrix} h_{11}^0 - h_{22}^0 \\ g_{11}^0 - g_{22}^0 \end{pmatrix} \right], \\ \begin{pmatrix} \varphi_{31}^{2K} \\ \varphi_{32}^{2K} \end{pmatrix} &= \frac{\rho}{B_{2K}^2 + 4\rho^2 I} \left[ (y^2 - x^2) \begin{pmatrix} h_{12}^{2K} + h_{21}^{2K} \\ g_{12}^{2K} + g_{21}^{2K} \end{pmatrix} + 2xy \begin{pmatrix} h_{11}^{2K} - h_{22}^{2K} \\ g_{11}^{2K} - g_{22}^{2K} \end{pmatrix} \right], \end{aligned} \quad (4.30)$$

where  $B_k = -A_k^{\lambda_c}$  with  $A_k^{\lambda_c}$  as defined by (4.8). By (4.23) and (4.29) we obtain an explicit expression of  $\Phi$  as follows:

$$\begin{aligned} \Phi_1 &= (\varphi_{11}^0 + \varphi_{21}^0 + \varphi_{31}^0) e_0 + (\varphi_{11}^{2K} + \varphi_{21}^{2K} + \varphi_{31}^{2K}) e_{2K} + o(2), \\ \Phi_2 &= (\varphi_{12}^0 + \varphi_{22}^0 + \varphi_{32}^0) e_0 + (\varphi_{12}^{2K} + \varphi_{22}^{2K} + \varphi_{32}^{2K}) e_{2K} + o(2). \end{aligned} \quad (4.31)$$

Here, by (4.30), (4.28), and (4.25),  $\varphi_{ij}^k$  are 2-order homogeneous functions of  $(x, y)$ , with the coefficients depending explicitly on the parameters defined in (2.4).

**Step 5.** Finally, inserting (4.31) into (4.21) and (4.22), we can write (4.18) in the following form

$$\begin{aligned}\frac{dx}{dt} &= -\rho y + a_{11}x^3 + a_{12}x^2y + a_{13}xy^2 + a_{14}y^3 + o(4), \\ \frac{dy}{dt} &= \rho x + a_{21}x^3 + a_{22}x^2y + a_{23}xy^2 + a_{24}y^3 + o(4).\end{aligned}$$

Then, by Theorem 2.4.5 in [7], the parameter  $b_0$  in Theorem 4.2 is obtained by

$$b_0 = 3a_{11} + 3a_{24} + a_{12} + a_{23},$$

where  $a_{11}, a_{24}, a_{12}, a_{23}$  can be explicitly expressed in the terms in (4.20)-(4.22).

**4.5. Transition for the system (2.5).** We are now in a position to discuss the transition of (2.5). With the translation (4.2), the system (2.5) is rewritten in the following form

$$\begin{aligned}\frac{\partial u_1}{\partial t} &= \mu \Delta u_1 - 2\alpha u_1^{*2} u_1 - u_1^* \Delta u_2 + \frac{\alpha u_1^* u + 3}{(1 + u_3^*)^2} + g(u), \\ \frac{\partial u_2}{\partial t} &= \Delta u_2 - u_2 + \lambda u_1, \\ \frac{\partial u_3}{\partial t} &= r \Delta u_3 - \delta u_1^* u_3 - \delta u_3^* u_1 - \delta u_1 u_3, \\ \frac{\partial u}{\partial n} \Big|_{\partial \Omega} &= 0, \\ u(0) &= u_0,\end{aligned}\tag{4.32}$$

where  $g(u)$  is as in (4.4). Here the notation  $u$  stands for three-component unknown:

$$u = (u_1, u_2, u_3).$$

Let

$$L_\lambda u = \begin{pmatrix} \mu \Delta - 2\alpha u_1^{*2} & -u_1^* \Delta & \frac{\alpha u_1^*}{(1+u_3^*)^2} \\ \lambda & \Delta - 1 & 0 \\ -\delta u_3^* & 0 & r \Delta - \delta u_1^* \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}.$$

Then, all eigenvalues  $\beta_k^j(\lambda)$  and eigenvectors  $\psi_k^j$  of  $L_\lambda$  satisfy

$$D_k^\lambda \begin{pmatrix} \xi_{k1}^j \\ \xi_{k2}^j \\ \xi_{k3}^j \end{pmatrix} = \beta_k^j(\lambda) \begin{pmatrix} \xi_{k1}^j \\ \xi_{k2}^j \\ \xi_{k3}^j \end{pmatrix}, \quad 1 \leq j \leq 3, \quad k \in \mathbb{Z}^2,$$

with

$$\psi_k^j = (\xi_{k1}^j e_k, \xi_{k2}^j e_k, \xi_{k3}^j e_k),$$

and  $e_k$  as in (3.6),  $D_k^\lambda$  is a  $3 \times 3$  matrix given by

$$D_k^\lambda = \begin{pmatrix} -(\mu \rho_k + 2\alpha u_1^{*2}) & u_1^* \rho_k & \frac{\alpha u_1^*}{(1+u_3^*)^2} \\ \lambda & -(\rho_k + 1) & 0 \\ -\delta u_3^* & 0 & -(r \rho_k + \delta u_1^*) \end{pmatrix}.$$

We introduce the following three parameters:

$$\begin{aligned} A_k^\lambda &= -\operatorname{tr} D_k^\lambda = \mu\rho_k + 2\alpha u_1^{*2} + \rho_k + 1 + r\rho_k + \delta u_1^*, \\ B_k^\lambda &= \det \begin{pmatrix} -(\mu\rho_k + 2\alpha u_1^{*2}) & u_1^*\rho_k \\ \lambda & -(\rho_k + 1) \end{pmatrix} \\ &\quad + \det \begin{pmatrix} -(\mu\rho_k + 2\alpha u_1^{*2}) & \frac{\alpha u_1^*}{(1+u_3^*)^2} \\ -\delta u_3^* & -(r\rho_k + \delta u_1^*) \end{pmatrix} + (\rho_k + 1)(r\rho_k + \delta u_1^*), \\ C_k^\lambda &= -\det D_k^\lambda = (\mu\rho_k + 2\alpha u_1^{*2})(\rho_k + 1)(r\rho_k + \delta u_1^*) \\ &\quad - u_1^*\rho_k\lambda(r\rho_k + \delta u_1^*) + \frac{\alpha u_1^*}{(1+u_3^*)^2}\delta u_3^*(\rho_k + 1). \end{aligned}$$

By the Routh-Hurwitz theorem, we know that all eigenvalues  $\beta_k^j$  of  $D_k^j$  have negative real parts if and only if

$$A_k^\lambda > 0, \quad A_k^\lambda B_k^\lambda - C_k^\lambda > 0, \quad C_k^\lambda > 0. \quad (4.33)$$

Let  $\Lambda^c$  and  $K = (K_1, K_2)$  satisfy

$$\begin{aligned} A_K^{\Lambda^c} &> 0, \quad A_K^{\Lambda^c} B_K^{\Lambda^c} - C_K^{\Lambda^c} = 0, \quad C_K^{\Lambda^c} > 0, \\ A_k^{\Lambda^c} &> 0, \quad A_k^{\Lambda^c} B_k^{\Lambda^c} - C_k^{\Lambda^c} > 0, \quad C_k^{\Lambda^c} > 0, \quad \forall k \text{ with } \rho_k \neq \rho_K. \end{aligned} \quad (4.34)$$

Then  $\Lambda_c$  satisfies that

$$\begin{aligned} \Lambda_c &= \inf_{\rho_k} \frac{1}{\rho_k u_1^*} [(\mu + r)\rho_k + 2\alpha u_1^{*2} + \delta u_1^*] \\ &\quad \times \left[ (r + 1)\rho_k + \delta u_1^* + 1 + \frac{\alpha\delta}{(\mu\rho_k + 2\alpha u_1^{*2} + \rho_k + 1)(1 + u_3^*)^2} \right], \end{aligned} \quad (4.35)$$

and  $\rho_K$  satisfies (4.35). In particular, under the condition (4.34), there is a pair of complex eigenvalues  $\beta_K^1(\lambda)$  and  $\beta_K^2(\lambda)$  of  $D_K^\lambda$ , such that

$$\operatorname{Re} \beta_K^{1,2}(\lambda) \begin{cases} < 0, & \lambda < \Lambda_c, \\ = 0, & \lambda = \Lambda_c, \\ > 0, & \lambda > \Lambda_c, \end{cases} \quad (4.36)$$

and the other eigenvalues  $\beta_k^j(\lambda)$  of  $L_\lambda$  satisfy

$$\begin{cases} \operatorname{Re} \beta_k^j(\Lambda_c) < 0, & \forall k \text{ with } \rho_k \neq \rho_K, \text{ and } 1 \leq j \leq 3, \\ \beta_K^3(\Lambda_c) < 0. \end{cases} \quad (4.37)$$

Let  $\lambda_c$  and  $K^* = (K_1^*, K_2^*)$  satisfy

$$\begin{aligned} A_{K^*}^{\lambda_c} &> 0, \quad A_{K^*}^{\lambda_c} B_{K^*}^{\lambda_c} - C_{K^*}^{\lambda_c} > 0, \quad C_{K^*}^{\lambda_c} = 0, \\ A_k^{\lambda_c} &> 0, \quad A_k^{\lambda_c} B_k^{\lambda_c} - C_k^{\lambda_c} > 0, \quad C_k^{\lambda_c} > 0, \quad \forall k \text{ with } \rho_k \neq \rho_{K^*}. \end{aligned} \quad (4.38)$$

Then  $\lambda_c$  is given by

$$\lambda_c = \inf_{\rho_k} \frac{(\rho_k + 1)}{\rho_k u_1^*} \left[ \mu\rho_k + 2\alpha u_1^{*2} + \frac{\alpha\delta}{(1 + u_3^*)^2(r\rho_k + \delta u_1^*)} \right], \quad (4.39)$$

and  $\lambda_c$  arrives its minimal at  $\rho_{K^*}$ . From the Routh-Hurwitz criterion (4.33), we deduce that with (4.38) there is a real eigenvalue  $\beta_{K^*}^1(\lambda)$  of  $D_{K^*}^{\lambda_c}$  satisfies

$$\beta_{K^*}(\lambda) \begin{cases} < 0, & \lambda < \lambda_c, \\ = 0, & \lambda = \lambda_c, \\ > 0, & \lambda > \lambda_c, \end{cases} \quad (4.40)$$

$$\begin{cases} \operatorname{Re} \beta_{K^*}^j(\lambda_c) < 0, & j = 2, 3, \\ \operatorname{Re} \beta_k^j(\lambda_c) < 0, & \forall k \in \mathbb{Z}^2 \text{ with } \rho_k \neq \rho_{K^*} \text{ and } 1 \leq j \leq 3. \end{cases} \quad (4.41)$$

It is clear that (4.36) and (4.37) hold true as  $\Lambda_c < \lambda_c$ , and (4.39)-(4.40) hold true as  $\lambda_c < \Lambda_c$ . Hence, we have the following transition theorem for (4.32).

**Theorem 4.3.** *Let  $\Lambda_c$  and  $\lambda_c$  be given by (4.35) and (4.39) respectively. Then, Assertions (1) and (2) of Theorem 4.2 hold true for the system (4.32).*

**5. Biological conclusions.** The main objective of this article is to determine the types of dynamic transitions, which are dictated by non-dimensional parameters, given in terms of key system parameters as well as the geometry of the spatial domain of the underlying system. In this section, we give some biological implications on extreme cases, and further biological implications of the main theorems and the main results will be carried out elsewhere.

**5.1. Biological significance of transition theorems.** Pattern formation is one of the characteristics for bacteria chemotaxis, and is fully characterized by the dynamic transitions. Theorems 3.1–4.3 tell us that the nondimensional parameter  $\lambda$ , given by

$$\lambda = \frac{\sqrt{\alpha_2} r_1 \chi}{r_2 k_2}, \quad (5.1)$$

plays a crucial role to determine the dynamic transition and pattern formation. Actually, the key factor in (5.1) is the product of the chemotactic coefficient  $\chi$  and the production rate  $r_1 : \chi r_1$ , which depends on the type of bacteria. When  $\lambda$  is less than some critical value  $\lambda_c$ , the uniform distribution of biological individuals is a stable state. When  $\lambda$  exceeds  $\lambda_c$ , the bacteria cells aggregate to form more complex and stable patterns.

As seen in (3.11), (4.10), (4.11) and (4.35), under different biological conditions, the critical parameter  $\lambda_c$  takes different forms and values. But, a general formula for  $\lambda_c$  is of the following type:

$$\lambda_c = a_0 + \inf_{\rho_k} \left( a_1 \rho_k + \frac{a_2}{\rho_k} + \frac{a_3}{b_1 \rho_k + b_0} + \frac{a_4}{\rho_k (b_1 \rho_k + b_0)} \right), \quad (5.2)$$

where  $\rho_k$  are taken as the eigenvalues of  $-\Delta$  with the Neumann boundary condition. When  $\Omega$  is a rectangular region,  $\rho_k$  are given by (3.6), and the coefficients  $a_j$  ( $1 \leq j \leq 4$ ),  $b_0, b_1 \geq 0$  depend on the parameters in (2.4), with

$$a_0, a_1, a_2, b_0, b_1 > 0, \quad a_3, a_4 \geq 0.$$

In particular, for the system with rich nutrient supplies, (5.2) becomes

$$\lambda_c = a_0 + \inf_{\rho_k} \left[ a_1 \rho_k + \frac{a_2}{\rho_k} \right].$$



The eigenvalues  $\rho_k$ , depending on the geometry of  $\Omega$ , satisfy

$$\begin{aligned} 0 &= \rho_0 < \rho_1 \leq \dots \leq \rho_k \leq \dots, \\ \rho_k &\rightarrow \infty \quad \text{if } k \rightarrow \infty, \\ \rho_1 &\propto \frac{1}{L^2}, \end{aligned} \tag{5.3}$$

where  $L$  is the length scale of  $\Omega$ .

We infer from (5.2) and (5.3) that

$$\lambda_c \rightarrow \infty \quad \text{as } |\Omega| \rightarrow 0 \quad (L \rightarrow 0).$$

It implies that when the container  $\Omega$  is small, the homogenous state is state and there is no pattern formation of bacteria under any biological conditions.

**5.2. Spatiotemporal oscillation.** Theorems 4.2 and 4.3 show that there are two critical parameters  $\lambda_c$  and  $\Lambda_c$ , such that

- if  $\lambda_c < \Lambda_c$ , the patterns formed by biological organisms are steady states, as exhibited by many experimental results, and
- if  $\Lambda_c < \lambda_c$ , a spatial-temporal oscillatory behavior takes place.

For the case with rich nutrient,

$$u_1^* = 1, \quad u_3^* = \infty.$$

In this situation,  $\lambda_c$  in (4.11) is reduced to (3.8), and obviously we have that

$$\lambda_c < \Lambda_c \quad \text{for both (4.10) and (4.35),}$$

and the dynamic transition and pattern formation are determined by Theorems 3.1 and 3.2. Hence there is no spatiotemporal oscillations for the rich nutrient case, and the time periodic oscillation of chemotaxis occurs only for the case where the nutrient is moderately supplied.

In particular, if  $\mu, r \cong 0$ , and

$$\delta^2 u_1^{*2} (1 + u_3^*)^2 < \alpha \delta_0,$$

then for  $\Lambda_c$  defined by (4.10) and (4.35), we have

$$\Lambda_c < \lambda_c.$$

In this case, a spatial-temporal oscillation pattern are expected for  $\lambda > \Lambda_c$ .

**5.3. Transition types.** One of the most important aspects of the study for phase transitions is to determine the transition types for a given system. The main theorems in this article provide precise information on the transition types. In all cases, types are precisely determined by the sign of some non dimensional parameters; see  $b$ ,  $b_0$  and  $b_1$  respectively in the main theorems. Hence a global phase diagram can be obtained easily by setting the related parameter to be zero.

For example, when  $\Omega = (0, L_1)$  is one-dimensional or when  $K = (K_1, 0)$  (resp.  $K = (0, K_2)$ ), the parameter  $b$  in (3.14) can be simplified into the following form

$$b = 2 \left[ -3\mu\rho_K + 9\alpha - \frac{(2\mu\rho_K + \alpha)(2\mu\lambda_K^2 + 28\alpha\rho_K + 4\alpha - \mu\rho_K)}{(\mu\rho_{2K} + 2\alpha)(\rho_{2K} + 1) - \rho_{2K}\lambda_c} \right]. \tag{5.4}$$

For a non-growth system,  $\alpha = 0, K = (1, 0), \lambda_c = \mu(\rho_K + 1)$ . Then, (5.4) becomes

$$b = \frac{\mu}{3}(1 - 20\lambda_1), \quad \lambda_1 = \frac{\pi^2}{L_1^2}, \tag{5.5}$$

and  $\lambda = \frac{ar_1x}{r_2k_2}$ , with  $a = \frac{1}{|\Omega|} \int_{\Omega} u_1 dx$ . It follows from (5.5) that

$$b \begin{cases} < 0 & \text{if } L_1 < 2\sqrt{5}\pi, \\ > 0 & \text{if } L_1 > 2\sqrt{5}\pi. \end{cases} \quad (5.6)$$

By Theorems 3.1 and 3.2, the phase transition of (3.3) and (3.1) from  $(u, \lambda) = (u^*, \lambda_c)$  is continuous if the length scale  $L_1$  of  $\Omega$  is less than  $2\sqrt{5}\pi$ , and jump if  $L_1$  is bigger than  $2\sqrt{5}\pi$ .

In addition, when we take

$$\chi(u) = \frac{\chi_1 u_1}{(\beta + u_2)^2}$$

as the chemotaxis function, by Remark 4.1, the parameter  $b$  of (5.5) is replaced by

$$b_1 = \frac{\mu}{3} \left( 1 - \frac{20\pi^2}{L_1^2} \right) + \frac{4\kappa\mu^2\pi^2}{L_1^2}$$

with

$$\kappa = \frac{k_2}{\beta\chi + k_2\lambda_c}, \quad \text{and } \lambda_c = \mu \left( \frac{\pi^2}{L_1^2} + 1 \right).$$

The above conclusion amounts to saying that for a non-growth system, the parameter

$$\lambda = \frac{r_1\chi}{r_2k_2}a, \quad \text{with } a = \frac{1}{|\Omega|} \int_{\Omega} u_1 dx,$$

is proportional to the average density  $a$  of initial condition of  $u_1$  ( $u_1$  is conservation). Hence, the biological individual is in a homogenous distribution state provided

$$\frac{1}{|\Omega|} \int_{\Omega} \varphi dx < \frac{r_2k_2}{r_1\chi} \mu \left( \frac{\pi^2}{L_1^2} + 1 \right), \quad \varphi = u_1(0),$$

and the bacteria will aggregate to form numbers of high density regions provided

$$\frac{1}{|\Omega|} \int_{\Omega} \varphi dx > \frac{r_2k_2}{r_1\chi} \mu \left( \frac{\pi^2}{L_1^2} + 1 \right). \quad (5.7)$$

Moreover, under the condition (5.7), if the scale  $L_1$  of  $\Omega$  is smaller than some critical value  $L_c$  (in (5.6)  $L_c = 2\sqrt{5}\pi$ ), i.e.  $L_1 < L_c$ , the continuous transition implies that there is only one high density region of bacteria to be formed, and if  $L_1 > L_c$  then the jump transition expects a large number of high density regions to appear.

**5.4. Pattern formation.** As mentioned before, the pattern formation behavior is dictated by the dynamic transition of the system. In this article, we studied the formation of two type patterns—the lamella and the rectangular patterns, although the approach can be generalized to study the formation of other more complex patterns.

For a growth system, the critical parameter  $\lambda_c$  takes its value at some eigenvalue  $\rho_K$  of  $-\Delta$  for  $K = (K_1, K_2)$ , as shown by (3.11) and (4.11). From the pattern formation point of view, for the Type-I transition, the patterns described by the transition solutions in the three main theorems are either lamella or rectangular:

$$\begin{array}{ll} \text{lamella pattern} & \text{for } K_1K_2 = 0, \\ \text{rectangular pattern} & \text{for } K_1K_2 \neq 0. \end{array}$$

In the case where  $b > 0$ , the system undergoes a more drastic change. As  $\lambda^* < \lambda < \lambda_c$ , the homogeneous state, the new patterns  $v_2^\lambda$  and  $v_4^\lambda$  are metastable. For

$\lambda > \lambda_c$ , the system undergoes transitions to more complex patterns away from the basic homogeneous state.

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