NETWORKS AND HETEROGENEOUS MEDIA
©American Institute of Mathematical Sciences
Volume **3**, Number **4**, December **2008**
doi:10.3934/nhm.2008.3.863
pp. **863–879**

ANALYSIS OF A REACTION-DIFFUSION SYSTEM MODELING PREDATOR-PREY WITH PREY-TAXIS

Mostafa Bendahmane

Departamento de Ingenieria Matematica
Universidad de Concepcion
Casilla 160-C, Concepcion, Chile

(Communicated by Kenneth Karlsen)

Abstract. In this paper, we consider a system of nonlinear partial differential equations modeling the Lotka Volterra interactions of preys and actively moving predators with prey-taxis and spatial diffusion. The interaction between predators are modelized by the statement of a food pyramid condition. We establish the existence of weak solutions by using Schauder fixed-point theorem and uniqueness via duality technique. This paper is a generalization of the results obtained in [2].

1. **Introduction.** This work is devoted to the mathematical analysis of a predators-preys system in a heterogeneous spatial domain. We are interested to a n-predators $\times$ m-preys system with prey-taxis, logistic growth for the preys population and a Holling type II functional response to predation. Prey-taxis is a kind of density-dependent cross-diffusion and it is a direct movement of predators in response to a variation of preys. The cross-diffusion expresses the population fluxes of one specie due to the presence of the others species. The concept of cross-diffusion was studied by Levin [10], Levin and Segel, [11], Okubo [15], Mimura and Murray [13], Mimura and Kawasaki [12], Mimura and Yamaguti [14] and many others authors. In passing, we mention that in [4] (see also [7]) the authors have considered the interaction of two species assuming that both species attract the other by some devise.

In this paper, we assume that the spatial dispersal of the prey is pure diffusion and the spatial-temporal variations of the predator's velocities are determined by the prey gradient. At each point and each instant, predators attack preys following the familiar Lotka-Volterra interaction.

Let us first consider a spatial and bounded domain Ω in $\mathbb{R}^N$, $N > 1$ with boundary $\partial\Omega$. Our state variables $u = u(t,x)$ and $v(t,x)$ represent the predator and prey populations densities respectively at time t and position x. We assume the spatial habitat to be heterogeneous. We are led to consider spatially dependent decay rates $a(t,x)$ for predators, growth rates $r(t,x)$, the conversion rate from prey to predator $e(t,x)$, density dependent mortality rates $k(t,x)|v|^{\sigma-1}$ for preys and the predation rate $\frac{p(t,x)v}{1+q(t,x)v}$. Next, $e(t,x)$ is being the conversion rate from prey to predator. Last, $d_u > 0$ and $d_v > 0$ are diffusion rates of predator and prey

2000 *Mathematics Subject Classification.* Primary: 35K57, 35K55, 92B05.
Key words and phrases. reaction-diffusion system, predator-prey, prey-taxis, Schauder fixed-point, weak solution, existence, uniqueness.

respectively. Then, proceeding as in [2], the dynamic of the predator-prey system with prey-taxis is governed by the system of semilinear equations

$$\begin{cases} \partial_t v - d_v \Delta v = r(t,x)v - k(t,x)|v|^{\sigma-1}v - \dfrac{p(t,x)v}{1+q(t,x)v}u, \\ \partial_t u - d_u \Delta u + \operatorname{div}(u\chi(u)\nabla v) = -a(t,x)u + e(t,x)\dfrac{p(t,x)v}{1+q(t,x)v}u,, \end{cases} \tag{1}$$

in $(0,T)\times\Omega$. This system is augmented with Neumann boundary condition on $(0,T)\times\partial\Omega$

$$\nabla u \cdot \eta = 0, \qquad \nabla v \cdot \eta = 0, \tag{2}$$

where η is the outward normal to Ω on $\partial\Omega$, and an initial distribution

$$u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ x \in \Omega. \tag{3}$$

In the model above, the predators are attracted by the preys and χ denotes their prey-tactic sensitivity. We assume that there exists a maximal density (the threshold) of predators u_m, such that $\chi(u_m) = 0$. Intuitively, this amounts to a switch to repulsion at high densities, sometimes referred to as volume–filling effect or prevention of overcrowding (see [8, 5]). This threshold condition has a clear biological interpretation: the predators stop to accumulate at a given point of Ω after their density attains certain threshold values and the prey-tactic cross diffusion $g(u) = u\chi(u)$ vanishes identically when $u \geq u_m$.

The problem (1) was studied recently from well-posedness (existence and uniqueness) point view in [2] with the logistic case ($\sigma = 2$).

Our model that governs the dynamics of a n-predators and m-preys system in a heterogeneous spatial domain is the following reaction-diffusion-advection system, for $i = 1,...,m$ and $j = 1,...,n$, and for some $\sigma_i > 1$,

$$\begin{cases} \partial_t v_i - d_{v_i}\Delta v_i = r_i(t,x)v_i - k_i(t,x)|\vartheta|^{\sigma_i-1}v_i - \displaystyle\sum_{j=1}^{n} h_{i,j}(t,x,u_j,v_i), \\ \\ \partial_t u_j - d_{u_j}\Delta u_j + \operatorname{div}(u_j\chi_j(u_j)\nabla\vartheta) \\ \qquad = \displaystyle\sum_{i=1}^{m} e_i(t,x)h_{i,j}(t,x,u_j,v_i) - \sum_{k=j+1}^{n} C_{k,j}(t,x,u_k,u_j) \\ \qquad\quad + \displaystyle\sum_{k=1}^{j-1} d_{k,j}(t,x)C_{k,j}(t,x,u_k,u_j) - a_j(t,x)u_j, \end{cases} \tag{4}$$

in $(0,T)\times\Omega$, together with Neumann boundary conditions on $(0,T)\times\partial\Omega$

$$\nabla v_i \cdot \eta = 0, \qquad \nabla u_j \cdot \eta = 0. \tag{5}$$

Last, an initial distribution is assumed at $t = 0$

$$v_i(x,0) = v_{i,0}(x),\ u_j(x,0) = u_{j,0}(x),\ x \in \Omega, \tag{6}$$

for $i = 1,...,m$ and $j = 1,...,n$. Herein

$$\vartheta = \sum_{i=1}^{m} v_i, \tag{7}$$

is the total prey population,

$$h_{i,j}(t,x,u,v) = \frac{p_{i,j}(t,x)v}{1+q_{i,j}(t,x)v}u \tag{8}$$

is the Holling type II functional response to predation on prey species i from predator species j and

$$C_{k,j}(t,x,u,v) = \frac{c_{k,j}(t,x)v}{1+f_{k,j}(t,x)v}u. \tag{9}$$

In system (4), all functions are nonnegative. Here for $i = 1, ..., m$ and $j = 1, ..., n$; d_{v_i}, d_{u_j}, χ_j, r_i, k_i, e_i, $p_{i,j}$, $q_{i,j}$ and a_j have similar properties to d_v, d_u, χ, r, k, e, p, q and a. Last, the coefficients $c_{k,j}$, $f_{k,j}$ and $d_{k,j}$ describe interactions between predator species k and j. In the model (4), we assume that there exists a maximal density of predators $u_{j,m}$, such that $\chi_j(u_{j,m}) = 0$ for $j = 1, ..., n$.

Note that for $\sigma_i = 2$ and $\chi_j = 0$, system (4) is still a standard Lotka-Volterra system; in this case $h_{i,j}$ has similar properties to $C_{k,j}$.

When cross-diffusion is ignored, this model is similar to these in [1], [6] and [3] in connection with ecological models.

The basic hypothesis is a food pyramid condition on

$$B_j(t,x,u_j) = -\sum_{k=j+1}^{n} \frac{c_{k,j}(t,x)u_j}{1+f_{k,j}(t,x)u_j}u_k + \sum_{k=1}^{j-1} d_{k,j}(t,x)\frac{c_{k,j}(t,x)u_j}{1+f_{k,j}(t,x)u_j}u_k; \tag{10}$$

such a statement is motivated by the fact that given n species living isolated in a certain region one may arrange them in an arithmetic sequence and in such a way so that the ith species may feed on any jth species ($j \le i$) and may not feed on any kth species($k > i$). Consequently, the growth of the ith species should be bounded in terms of the available food, that is the magnitude of the jth species. This last statement is the content of the food pyramid condition. In passing we want to mention that when food pyramid condition is ignored, $\sigma_i = 2$ and $n = m = 1$ our model is similar to this in [2].

In this work, the basic hypothesis are the food pyramid condition to modelize competition of n species of predators, a logistic growth for preys, Holling type II functional response and a prey-taxis (cross-diffusion) term to modelize interactions between predators and preys.

Comparing to [2], the novelty in this work is the additional food pyramid condition (10) and the lower-order term div $(u_j\chi_j(u_j)\nabla\vartheta)$, where $\vartheta = \sum_{i=1}^m v_i$. The main difficulty in studying the system (4) is due to the strong coupling of the equations. Note that standard parabolic theory is not directly aplicable to the reaction-diffusion system (4), (5) and (6) due to the prey-taxis term (div $(u_j\chi_j(u_j)\nabla\vartheta)$). We will show that this term satisfies the appropriate growth conditions due to its special form and the available regularity for v_i for $i = 1, \dots, m$.

The remaining part of this paper is organized as follows. Section 2 is devoted to presenting a notion of weak solutions (Definition 2.1) and stating the main convergence theorem (Theorem 2.1). In Section 3 we prove existence of solutions to the approximate problem (18). The existence result to (4)-(6) is proved in Section 4. We conclude the paper in Section 5 by proving uniqueness of weak solutions.

2. Main result.

Before stating our main result, we collect some preliminary material, including relevant notations and conditions imposed on the data of our problem. Let Ω be a bounded, open subset of $\mathbb{R}^N$ ($N \ge 2$) with a smooth boundary $\partial\Omega$; η is the unit outward normal to Ω on $\partial\Omega$. Next, $|\Omega|$ is the N dimensional Lebesgue measure of Ω. We denote by $H^1(\Omega)$ the Sobolev space of functions $u : \Omega \to \mathbb{R}$ for which $u \in L^2(\Omega)$ and $\nabla u \in L^2(\Omega; R^N)$. For $1 \le p \le +\infty$, $\|\cdot\|_{L^p(\Omega)}$ is the usual norm in $L^p(\Omega)$; then

$$L^p_+(\Omega) = \{u : \Omega \longrightarrow \mathbb{R}_+ \text{ measurable and } \int_\Omega |u(x)|^p dx < +\infty\},$$
$$L^\infty_+(\Omega) = \{u : \Omega \longrightarrow \mathbb{R}_+ \text{ measurable and } \sup_{x\in\Omega} |u(x)| < +\infty\}.$$

If X is a Banach space, $a < b$ and $1 \le p \le +\infty$, $L^p(a, b; X)$ denotes the space of all measurable functions $u : (a, b) \longrightarrow X$ such that $\| u(\cdot) \|_X$ belongs to $L^p(a, b)$.

Next T is a positive number and

$$Q_T = (0, T) \times \Omega,\ \Sigma_T = (0, T) \times \partial\Omega.$$

Our basic requirement is

$$\sigma_i > 1, \tag{11}$$
$$a_j,\ e_i,\ r_i \text{ and } k_i \in L^\infty_+(Q_T), \tag{12}$$
$$p_{i,j},\ q_{i,j},\ c_{k,j},\ f_{k,j} \text{ and } d_{k,j} \in L^\infty_+(Q_T), \tag{13}$$
$$0 \le e_i(t, x) \le \overline{e},\ 0 < r_0 \le r_i(t, x) \le \overline{r},\ 0 < k_0 \le k_i(t, x) \le \overline{k}, \tag{14}$$
$$0 \le p_{i,j}(t, x) \le \overline{p},\ 0 < q_0 \le q_{i,j}(t, x) \le \overline{q},\ 0 \le c_{k,j}(t, x) \le \overline{c}, \tag{15}$$
$$0 < f_0 \le f_{k,j}(t, x) \le \overline{f},\ 0 \le d_{k,j}(t, x) \le \overline{d} \text{ a.e. } (t, x) \in Q_T, \tag{16}$$

for $i = 1, \dots, m$, $j = 1, \dots, n$ and $k = 1, \dots, j-1, j+1, \dots, n$. Finally, we assume that the function χ_j in (4) satisfies

$$\chi_j \in C^1([0, u_{j,m}]) \text{ and } \chi_j(u_{j,m}) = 0 \text{ for } j = 1, \dots, n. \tag{17}$$

Recall that $u_{j,m}$ is the maximal density of jth predator.

Now we give the definition of a weak solution for nonlinear parabolic systems of type (4) with no-flux boundary (5) and initial condition (6). Then, we supply our main result.

Definition 2.1. A weak solution of (4), (5) and (6), is a set of functions $((u_j)_{1\le j\le n}, (v_i)_{1\le i\le m})$ such that, for $i = 1, \dots, m$ and $j = 1, \dots, n$:

$$\begin{aligned}
&u_j \in L^\infty_+(Q_T) \cap L^2(0, T; H^1(\Omega)) \cap C(0, T, L^2(\Omega)),\\
&\partial_t u_j \in L^2(0, T; (H^1(\Omega))'), \quad u_j(0) = u_{j,0},\\
&v_i \in L^\infty_+(Q_T) \cap L^p(0, T; W^{2,p}(\Omega)) \cap C(0, T, L^2(\Omega)), \text{ for all } p > 1,\\
&\partial_t v_i \in L^2(Q_T), \quad v_i(0) = v_{i,0},
\end{aligned}$$

and, for all $\varphi_i, \psi_j \in L^2(0, T; H^1(\Omega))$,

$$\begin{aligned}
&\iint_{Q_T} \partial_t v_i\, \varphi_i\, dx\, dt + d_{v_i} \iint_{Q_T} \nabla v_i \cdot \nabla\varphi_i\, dx\, dt\\
&= \iint_{Q_T} (r_i(t, x)v_i - k_i(t, x)\, |\vartheta|^{\sigma_i-1} v_i)\varphi_i\, dx\, dt - \iint_{Q_T} \sum_{j=1}^n h_{i,j}(t, x, u_j, v_i)\varphi_i\, dx\, dt,\\
&\int_0^T \langle \partial_t u_j, \psi_j\rangle\, dt + \iint_{Q_T} (d_{u_j} \nabla u_j - u_j\chi_j(u_j)\nabla\vartheta) \cdot \nabla\psi_j\, dx\, dt\\
&\qquad = \iint_{Q_T} \left(\sum_{i=1}^m e_i(t, x)h_{i,j}(t, x, u_j, v_i) - \sum_{k=j+1}^n C_{k,j}(t, x, u_k, u_j)\right)\psi_j dxdt\\
&\qquad\qquad + \iint_{Q_T} \left(\sum_{k=1}^{j-1} d_{k,j}(t, x)C_{k,j}(t, x, u_k, u_j) - a_j(t, x)u_j\right)\psi_j dxdt,
\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^1(\Omega)$ and $(H^1(\Omega))'$.

Our main result is the following existence and uniqueness theorem for weak solutions.

Theorem 2.1. *Assume (11)-(17) hold. If $u_{j,0},\ v_{i,0} \in L^\infty_+(\Omega)$ with $u_{j,0} \le u_{j,m}$ a.e. in Ω for $i = 1, \dots, m$ and $j = 1, \dots, n$, then there exists a unique weak solution of (4)-(6) in the sense of Definition 2.1.*

The proof of Theorem 2.1 is based on introducing approximation systems to which we can apply the Shauder fixed-point theorem. To prove convergence to weak solutions of the approximate solutions we use monotonicity and compactness methods. We prove first existence of solutions to the approximate problem of (18) by applying the Schauder fixed-point theorem (in an appropriate functional setting), deriving a priori estimates, and then passing to the limit in the approximate solutions using monotonicity and compactness arguments. Having proved existence to the system (18), the goal is to send the regularization parameter ε to zero in sequences of such solutions to fabricate weak solutions of the original systems (4)-(6). Again convergence is achieved by priori estimates and compactness arguments.

3. **Existence of solutions for the approximate problems.**

This section is devoted to proving existence of solutions to the approximate problem of (4). The existence proof is based on the Shauder fixed-point theorem, a priori estimates, and the compactness method. The approximation systems read, for $i = 1, \dots, m$ and $j = 1, \dots, n$

$$
\begin{cases}
\partial_t v_i - d_{v_i} \Delta v_i = r_i(t,x) v_i - K_{i,\varepsilon}(t,x,v_i,\vartheta) - \displaystyle\sum_{j=1}^{n} h_{i,j,\varepsilon}(t,x,u_j,v_i) \text{ in } Q_T, \\[2ex]
\partial_t u_j - d_{u_j} \Delta u_j + \operatorname{div}(u_j \chi_j(u_j) \nabla \vartheta) = \\
\qquad \displaystyle\sum_{i=1}^{m} e_i(t,x) h_{i,j,\varepsilon}(t,x,u_j,v_i) - \sum_{k=j+1}^{n} C_{k,j,\varepsilon}(t,x,u_k,u_j) \\
\qquad\qquad + \displaystyle\sum_{k=1}^{j-1} d_{k,j}(t,x) C_{k,j,\varepsilon}(t,x,u_k,u_j) - a_j(t,x) u_j \quad \text{in } Q_T, \\[2ex]
\nabla v_i \cdot \eta = \nabla u_j \cdot \eta = 0 \text{ on } \Sigma_T, \quad v_i(\cdot,0) = v_{i,0}(\cdot),\ u_j(\cdot,0) = u_{j,0}(\cdot), \in \Omega,
\end{cases}
\tag{18}
$$

for each fixed $\varepsilon > 0$. Herein

$$
h_{i,j,\varepsilon}(\cdot,\cdot,r_1,r_2) = \frac{h_{i,j}(\cdot,\cdot,r_1,r_2)}{1 + \varepsilon\,|h_{i,j}(\cdot,\cdot,r_1,r_2)|}, \quad C_{k,j,\varepsilon}(\cdot,\cdot,r_1,r_2) = \frac{C_{k,j}(\cdot,\cdot,r_1,r_2)}{1 + \varepsilon\,|C_{k,j}(\cdot,\cdot,r_1,r_2)|},
$$

and

$$
K_{i,\varepsilon}(\cdot,\cdot,r_1,r_2) = \frac{k_i(\cdot,\cdot)|r_2|^{\sigma_i - 1} r_1}{1 + \varepsilon\,|k_i(\cdot,\cdot)|r_2|^{\sigma_i - 1} r_1|}
$$

for a.e. $r_1, r_2 \in \mathbb{R}$, for $i = 1, \dots, m$, $j = 1, \dots, n$ and $k = 1, \dots, j-1, j+1, \dots, n$.

First, a natural extension of nonlinear functions, i.e. $h_{i,j}, C_{k,j}$, is made in order to ensure the nonnegativity of solutions. We need to extend the function $F =$

$h_{i,j,\varepsilon}$, $C_{k,j,\varepsilon}$ so that it becomes measurable on Q_T, continuous with respect to u and v. We do this by setting

$$F(t,x,u,v) = \begin{cases} F(t,x,u,0) & \text{if} \quad u \geq 0, v < 0, \\ F(t,x,0,v) & \text{if} \quad u < 0, v \geq 0, \\ F(t,x,0,0) & \text{if} \quad u < 0, v < 0, \end{cases}$$

for $i = 1, ..., m,\ j = 1, ..., n$ and $k = 1, ..., j-1, j+1, ..., m$. In what follows, we define the following new variables $(\tilde{u}_j, \tilde{v}_j)$ by setting $u_j = e^{\lambda t}\tilde{u}_j$ and $v_i = e^{\lambda t}\tilde{v}_i$, where $\lambda > 0$ is a constant satisfying: for $i = 1, ..., m$ and $j = 1, ..., n$

$$\lambda - \sum_{i=1}^{m} \left| e_i(t,x)\frac{p_{i,j}(t,x)}{q_{i,j}(t,x)} \right| - \sum_{k=j+1}^{n} \left| d_{k,j}(t,x)\frac{c_{k,j}(t,x)}{f_{k,j}(t,x)} \right| \geq 0, \quad \lambda - r_i(t,x) \geq 0, \tag{19}$$

for a.e. $(t,x) \in Q_T$. Then (u_j, v_i) satisfies (18) with g_j, and the right hand side functions $F_{j,\varepsilon}$, $G_{i,\varepsilon}$ replaced by (recall that $g_j(u) = u\chi_j(u)$)

$$\begin{cases} g_j(u_j) = e^{\lambda t}u_j\chi_j(e^{\lambda t}u_j), \\ F_{i,\varepsilon}(v_1, \dots, v_m, u_1, \dots, u_n) = r_i(t,x)u_i - \lambda\, v_i - K_{i,\varepsilon}(t,x,v_i,e^{\lambda t}\vartheta) \\ \qquad\qquad - \displaystyle\sum_{j=1}^{n} e^{-\lambda t}h_{i,j,\varepsilon}(t,x,e^{\lambda t}u_j, e^{\lambda t}v_i), \\ G_{j,\varepsilon}(v_1, \dots, v_m, u_1, \dots, u_n) = -a_j(t,x)u_j - \lambda u_j \\ \qquad\qquad + \displaystyle\sum_{i=1}^{m} e_i(t,x)e^{-\lambda t}h_{i,j,\varepsilon}(t,x,e^{\lambda t}u_j, e^{\lambda t}v_i) \\ \qquad\qquad - \displaystyle\sum_{k=j+1}^{n} e^{-\lambda t}C_{k,j,\varepsilon}(t,x,e^{\lambda t}u_k, e^{\lambda t}u_j), \\ \qquad\qquad + \displaystyle\sum_{k=1}^{j-1} d_{k,j}(t,x)e^{-\lambda t}C_{k,j,\varepsilon}(t,x,e^{\lambda t}u_k, e^{\lambda t}u_j) \end{cases} \tag{20}$$

for $i = 1, ..., m$ and $j = 1, ..., n$. Observe that for $j = 1, ..., n$ the function $g_j(u)$ vanish if $u \geq e^{-\lambda t}u_{j,m}$ for almost $t \in (0,T)$ (recall that $u_{j,m}$ is the maximal density of jth predators).

Remark 3.1. Note that from (12)-(16), we deduce

$$\left| e_i(t,x)\frac{p_{i,j}(t,x)}{q_{i,j}(t,x)} \right| \leq \overline{e}\,\frac{\overline{p}}{q_0} \text{ and } \left| d_{k,j}(t,x)\frac{c_{k,j}(t,x)}{f_{k,j}(t,x)} \right| \leq \overline{d}\,\frac{\overline{c}}{f_0},$$

for $i = 1, ..., m,\ j = 1, ..., n$ and $k = 1, ..., j-1, j+1, ..., m$. This implies that the expressions under the sums in (19) are bounded.

3.1. **Existence result to the fixed problem.** In this subsection, we omit the dependence of the solutions on the parameter ε. We prove, for each fixed $\varepsilon > 0$, the existence of solutions to the fixed problem (22)-(23), by applying the Schauder fixed-point theorem. Since we use Schauder fixed-point theorem, we need to introduce the following closed subset of the Banach space $L^2(Q_T, \mathbb{R}^n)$:

$$\mathcal{A} = \{U = (u_1, \dots, u_n) \in L^2(Q_T, \mathbb{R}^n) : 0 \leq u_j(t,x) \leq e^{-(\lambda-\beta)t}u_{j,m}, \\ \text{for a.e.}\,(t,x) \in Q_T, \qquad j = 1, \dots, n\}. \tag{21}$$

Herein β is a positive constant to be fixed in Lemma 3.3 below.
With $\overline{U} = (\overline{u}_1, \dots, \overline{u}_n) \in \mathcal{A}$ fixed, let v_i be the unique solution of the parabolic

problem

$$\begin{cases} \partial_t v_i - d_{v_i}\Delta v_i = F_{i,\varepsilon}(v_1,\ldots,v_m,\overline{u}_1,\ldots,\overline{u}_n), & \text{in } Q_T, \\ \nabla v_i\cdot\eta = 0 \text{ on } \Sigma_T, \quad v_i(x,0)=v_{i,0}(x), & \text{for } x\in\Omega, \end{cases} \tag{22}$$

for $i = 1,\ldots,m$. Given the function v_i, let u_j be the unique solution of the quasilinear parabolic problem

$$\begin{cases} \partial_t u_j - d_{u_j}\Delta u_j + \operatorname{div}(g_j(u_j)\nabla\vartheta) = G_{j,\varepsilon}(v_1,\ldots,v_m,u_1,\ldots,u_n), & \text{in } Q_T, \\ \nabla u_j\cdot\eta = 0 \text{ on } \Sigma_T, \quad u_j(x,0)=u_{j,0}(x)\geq 0, & \text{for } x\in\Omega, \end{cases} \tag{23}$$

for $j = 1,\ldots,n$. In (22)- (23), $v_{i,0}$ and $u_{j,0}$ are functions satisfying the hypothesis of Theorem 2.1 for $j = 1,\ldots,n$ and $i = 1,\ldots,m$.

Note that from the definition of $K_{i,\varepsilon}$ and $h_{i,j,\varepsilon}$, we get: for $i = 1,...,m$ and $j = 1,...,n$

$$K_{i,\varepsilon}(t,x,v_i,e^{\lambda t}\vartheta)\,\mathrm{sign}(v_i)\geq 0 \text{ and } \left|h_{i,j,\varepsilon}(t,x,e^{\lambda t}\overline{u}_j,e^{\lambda t}v_i)\right|\leq C\left|\overline{u}_j\right|,$$

for some constant $C > 0$.

Observe that for any fixed $\overline{U}\in\mathcal{A}$, problem (22) is uniformly parabolic, so we have immediately (see [9]):

Lemma 3.1. *If $v_{i,0}\in L^\infty_+(\Omega)$, then (22) has a unique solution $v_i\in L^\infty_+(Q_T)\cap L^p(0,T;W^{2,p}(\Omega))\cap C(0,T;L^2(\Omega))$, for all $p > 1$, satisfying $v_i(t,x)\geq 0$ for a.e. $(t,x)\in Q_T$ and*

$$\begin{aligned} &\|v_i\|_{L^\infty(Q_T)}\leq C,\\ &\|v_i\|_{L^2(0,T;H^1(\Omega))}\leq C,\\ &\|\partial_t v_i\|_{L^2(Q_T)}\leq C, \end{aligned} \tag{24}$$

where $C > 0$ is a constant which depends only on $\|v_{i,0}\|_{L^\infty(\Omega)}$, $\|r_i\|_{L^\infty(\Omega)}$, $\|k_i\|_{L^\infty(\Omega)}$, $\|p_{i,j}\|_{L^\infty(\Omega)}$, $\|q_{i,j}\|_{L^\infty(\Omega)}$, and $|Q_T|$ for $i = 1,\ldots,m$ and $j = 1,\ldots,n$.

Remark 3.2. Note that the first estimate in (24) follows from the comparison principle (see e.g. [3]). Multiplying the first equation in (22) by v_i and integrating over Q_T, we get the second estimate. Since the right-hand side of (22) is bounded, we deduce from classical results on L^p regularity the third estimate in (24).

We have the following lemma for problem (23):

Lemma 3.2. *If $u_{j,0}\in L^\infty_+(\Omega)$, then, for any $\varepsilon > 0$, there exists a unique weak solution $u_j\in L^\infty_+(Q_T)\cap L^2(0,T;H^1(\Omega))$ to problem (23) for $j = 1,\ldots,n$.*

We refer to [9] for the existence and the uniqueness proofs.

3.2. **The fixed-point method.** Now, we introduce a map $\text{Ł}:\mathcal{A}\to\mathcal{A}$ such that $\text{Ł}(\overline{U}) = (u_1,\ldots,u_n)$, where u_j solves (23) for $j = 1,\ldots,n$, *i.e.*, Ł is the solution operator of (23) associated with the coefficient $\overline{u}_j$ and the solution v_i coming from (22). By using the Schauder fixed-point theorem, we prove that the map Ł has a fixed point for (22)-(23).

First, let us show that Ł is a continuous mapping. For this, we let $(\overline{U}_\ell)_\ell$ be a sequence in $\mathcal{A}$ and $\overline{U}\in\mathcal{A}$ be such that $\overline{U}_\ell\to\overline{U}$ in $L^2(Q_T,\mathbb{R}^n)$ as $\ell\to\infty$. Define $U_\ell = \text{Ł}(\overline{U}_\ell)$, *i.e.*, $u_{j,\ell}$ is the solution of (23) associated with $\overline{u}_{j,\ell}$ and the solution $v_{i,\ell}$ of (22) for $j = 1,\ldots,n$ and $i = 1,\ldots,m$. The goal is to show that U_ℓ converges to $\text{Ł}(\overline{U})$ in $L^2(Q_T,\mathbb{R}^n)$. Next, we need the following lemma:

Lemma 3.3. *The solutions $u_{j,\ell}$ to problem (23) satisfy: for $j = 1,\ldots,n$*

(i) *There exists a constant* $\gamma \geq 0$ *such that*

$$0 \leq u_{j,\ell}(t,x) \leq e^{\gamma t} u_{j,m} \textit{ for a.e. } (t,x) \in Q_T.$$

(ii) *The sequence* $(u_{j,\ell})_\ell$ *is bounded in* $L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$.
(iii) *The sequence* $(u_{j,\ell})_\ell$ *is relatively compact in* $L^2(Q_T)$.

Proof. (i) For the function g_j, we choose a Lipschitz continuous extension $\tilde{g}_j$ of g_j satisfying

$$\tilde{g}_j(s) = \begin{cases} g_j(s) & \text{if } 0 \leq s \leq e^{-\lambda t} u_{j,m}, \\ g_j(0) = 0 & \text{if } s \leq 0, \\ g_j(e^{\lambda t} u_{j,m}) = 0 & \text{if } s \geq e^{-\lambda t} u_{j,m}, \end{cases} \tag{25}$$

for $t \in (0,T)$. We then replace the equation in (23) by

$$\partial_t u_{j,\ell} - d_{u_j} \Delta u_{j,\ell} + \operatorname{div}\left(\tilde{g}(u_{j,\ell}) \nabla \vartheta_\ell\right) = G_{j,\varepsilon}(v_{1,\ell}, \ldots, v_{m,,\ell}, u_{1,,\ell}, \ldots, u_{n,\ell}) \text{ in } Q_T. \tag{26}$$

Multiplying this equation by $-u_{j,\ell}^- = \dfrac{u_{j,\ell} - |u_{j,\ell}|}{2}$ and integrating over Ω, the result is

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \int_\Omega \left|u_{j,\ell}^-\right|^2 dx + d_{u_j} \int_\Omega \left|\nabla u_{j,\ell}^-\right|^2 dx \\ = - \int_\Omega \tilde{g}(u_{j,\ell}) \nabla \vartheta_\ell \cdot \nabla u_{j,\ell}^- \, dx - \int_\Omega (a_j + \lambda) \left|u_{j,\ell}^-\right|^2 dx \\ - \int_\Omega \sum_{i=1}^m e_i(t,x) e^{-\lambda t} h_{i,j,\varepsilon}(t,x,e^{\lambda t} u_{j,\ell}, e^{\lambda t} v_{i,\ell}) u_{j,\ell}^- \, dx \\ + \int_\Omega \sum_{k=j+1}^n e^{-\lambda t} C_{k,j,\varepsilon}(t,x,e^{\lambda t} u_{k,\ell}, e^{\lambda t} u_{j,\ell}) u_{j,\ell}^- \, dx \\ - \int_\Omega \sum_{k=1}^{j-1} d_{k,j}(t,x) e^{-\lambda t} C_{k,j,\varepsilon}(t,x,e^{\lambda t} u_{k,\ell}, e^{\lambda t} u_{j,\ell}) u_{j,\ell}^- \, dx.. \end{aligned}$$

According to the positivity of the second term of the left-hand side, and since $\tilde{g}_j(s) = 0$ and $h_{i,j,\varepsilon}(\cdot,\cdot,s,s') = 0$ for $s \leq 0$, $s' \in \mathbb{R}$, and $C_{k,j,\varepsilon}(\cdot,\cdot,s,s') = 0$, for $s' \leq 0$, $s \in \mathbb{R}$, we get

$$\frac{1}{2}\frac{d}{dt} \int_\Omega \left|u_{j,\ell}^-\right|^2 dx \leq 0 \text{ for } j = 1, \ldots, n.$$

Since the data $u_{j,0}$ is is nonnegative, we deduce that $u_{j,\ell}^- = 0$ for $j = 1, \ldots, n$.

Now, we multiply (26) by $(u_{j,\ell} - e^{-(\lambda-\beta)t}u_{j,m})^+$ for $t \in (0,T)$ and integrate over Ω (recall that β is defined in the subset $\mathcal{A}$), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \left|(u_{j,\ell} - e^{-(\lambda-\beta)t}u_{j,m})^+\right|^2 dx \\
&\qquad + \int_\Omega (\beta-\lambda)e^{-(\lambda-\beta)t}u_{j,m}(u_{j,\ell} - e^{-(\lambda-\beta)t}u_{j,m})^+ \, dx \\
&\qquad\quad + d_{u_j}\int_\Omega \left|\nabla(u_{j,\ell} - e^{-(\lambda-\beta)t}u_{j,m})^+\right|^2 dx \\
&= \int_\Omega \tilde{g}_j(u_{j,\ell})\nabla\vartheta_\ell \cdot \nabla(u_{j,\ell} - e^{-(\lambda-\beta)t}u_{j,m})^+ \, dx \\
&\quad - \int_\Omega a_j u_{j,\ell}(u_{j,\ell} - e^{-(\lambda-\beta)t}u_{j,m})^+ \, dx - \int_\Omega \lambda u_{j,\ell}(u_{j,\ell} - e^{-(\lambda-\beta)t}u_{j,m})^+ \, dx \\
&\quad + \int_\Omega \sum_{i=1}^m e_i(t,x)e^{-\lambda t}h_{i,j,\varepsilon}(t,x,e^{\lambda t}u_{j,\ell},e^{\lambda t}v_{i,\ell})(u_{j,\ell} - e^{-(\lambda-\beta)t}u_{j,m})^+ \, dx \\
&\quad - \int_\Omega \sum_{k=j+1}^n e^{-\lambda t}C_{k,j,\varepsilon}(t,x,e^{\lambda t}u_{k,\ell},e^{\lambda t}u_{j,\ell})(u_{j,\ell} - e^{-(\lambda-\beta)t}u_{j,m})^+ \, dx \\
&\quad + \int_\Omega \sum_{k=1}^{j-1} d_{k,j}(t,x)e^{-\lambda t}C_{k,j,\varepsilon}(t,x,e^{\lambda t}u_{k,\ell},e^{\lambda t}u_{j,\ell})(u_{j,\ell} - e^{-(\lambda-\beta)t}u_{j,m})^+ \, dx.
\end{aligned}$$

Observe that for $j = 1,\dots,n$, $i = 1,\dots,n$ and $k = 1,\dots,j-1,j+1,\dots,n$

$$h_{i,j,\varepsilon}(\cdot,\cdot,r,s) \le \frac{p_{i,j}(\cdot,\cdot)}{q_{i,j}(\cdot,\cdot)}s \text{ and } C_{k,j,\varepsilon}(\cdot,\cdot,r,s) \le \frac{c_{k,j}(\cdot,\cdot)}{f_{k,j}(\cdot,\cdot)}s,$$

for $r, s \in \mathbb{R}$. This implies

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \left|(u_{j,\ell} - e^{-(\lambda-\beta)t}u_{j,m})^+\right|^2 dx \\
&\qquad + \int_\Omega (\beta-\lambda)e^{-(\lambda-\beta)t}u_{j,m}(u_{j,\ell} - e^{-(\lambda-\beta)t}u_{j,m})^+ \, dx \\
&\qquad\quad + \int_\Omega \left(\lambda - \sum_{i=1}^m e_i(t,x)\frac{p_{i,j}(t,x)}{q_{i,j}(t,x)} - \sum_{k=1}^{j-1} d_{k,j}(t,x)\frac{c_{k,j}(t,x)}{f_{k,j}(t,x)}\right) \\
&\qquad\qquad\qquad \times u_{j,\ell}(u_{j,\ell} - e^{-(\lambda-\beta)t}u_{j,m})^+ \, dx \\
&\le \int_\Omega \tilde{g}_j(u_{j,\ell})\nabla\vartheta_\ell \cdot \nabla(u_{j,\ell} - e^{-\gamma t}u_{j,m})^+ \, dx.
\end{aligned} \tag{27}$$

Then for $\beta \ge \lambda$, we have $\tilde{g}(u_{j,\ell}) = 0$ for $u_{j,\ell} \ge e^{-(\lambda-\beta)t}u_{j,m}$. Finally, by the choice of λ in (19) we deduce from (27)

$$\frac{d}{dt}\int_\Omega \left|(u_{j,\ell} - e^{-(\lambda-\beta)t}u_{j,m})^+\right|^2 dx \le 0. \tag{28}$$

Using that $u_{j,0} \le u_{j,m}$ in Ω, we conclude from this $u_{j,\ell}(t,\cdot) \le e^{-(\lambda-\beta)t}u_{j,m}$ in Ω for all $t \in (0,T)$.

(ii) We multiply the equation (26) by $u_{j,\ell}$ and integrate over Ω, yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |u_{j,\ell}|^2\,dx + d_{u_j}\int_\Omega |\nabla u_{j,\ell}|^2\,dx \\
&\qquad = \int_\Omega g_j(u_{j,\ell})\nabla\vartheta_\ell\cdot\nabla u_{j,\ell}\,dx - \int_\Omega (a_j+\lambda)\,|u_{j,\ell}|^2\,dx \\
&\qquad\quad + \int_\Omega \sum_{i=1}^m e_i(t,x)e^{-\lambda t}h_{i,j,\varepsilon}(t,x,e^{\lambda t}u_{j,\ell},e^{\lambda t}v_{i,\ell})u_{j,\ell}\,dx \\
&\qquad\quad - \int_\Omega \sum_{k=j+1}^n e^{-\lambda t}C_{k,j,\varepsilon}(t,x,e^{\lambda t}u_{k,\ell},e^{\lambda t}u_{j,\ell})u_{j,\ell}\,dx \\
&\qquad\quad + \int_\Omega \sum_{k=1}^{j-1} d_{k,j}(t,x)e^{-\lambda t}C_{k,j,\varepsilon}(t,x,e^{\lambda t}u_{k,\ell},e^{\lambda t}u_{j,\ell})u_{j,\ell}\,dx.
\end{aligned}\tag{29}$$

Exploiting the boundedness of $u_{j,\ell}$, $u_{k,\ell}$ and $v_{i,\ell}$ for $j=1,\dots,n$, $i=1,\dots,n$ and $k=1,\dots,j-1,j+1,\dots,n$, and using Young inequality, we deduce from (29)

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_{j,\ell}|^2\,dx + C_1\int_\Omega |\nabla u_{j,\ell}|^2\,dx \le C_2, \tag{30}$$

for some constants $C_1, C_2 > 0$ independent of ℓ. This completes the proof of (ii).

(iii) Finally, we multiply the equation (23) by $\varphi_j \in L^2(0,T;H^1(\Omega))$ and we use the boundedness of $u_{j,\ell}$, $u_{k,\ell}$ and $v_{i,\ell}$, the result is

$$\begin{aligned}
\left|\int_0^T \langle \partial_t u_{j,\ell},\varphi_j\rangle\,dt\right| &\le d_{u_j}\|\nabla u_{j,\ell}\|_{L^2(Q_T)}\|\nabla\varphi_j\|_{L^2(Q_T)} \\
&\quad + \|g_j(u_{j,\ell})\|_{L^\infty(Q_T)}\|\nabla\vartheta_\ell\|_{L^2(Q_T)}\|\nabla\varphi_j\|_{L^2(Q_T)} \\
&\quad + C_3\sum_{j=1}^n \|u_{j,\ell}\|_{L^2(Q_T)}\|\varphi_j\|_{L^2(Q_T)} \\
&\le C_4\|\varphi_j\|_{L^2(0,T;H^1(\Omega))},
\end{aligned}\tag{31}$$

for some constants $C_3, C_4 > 0$ independent of ε. We obtain the bound

$$\|\partial_t u_{j,\ell}\|_{L^2(0,T;(H^1(\Omega))')} \le C, \tag{32}$$

for $j=1,\dots,n$. Then, (iii) is a consequence of (ii) and the uniform boundedness of $(\partial_t u_{j,\ell})_\ell$ in $L^2(0,T;(H^1(\Omega)')$ for $j=1,\dots,n$. □

Now we have the following classical result (see [9]).

Lemma 3.4. *There exists a function $v_i \in L^2(0,T;H^1(\Omega))$ such that the sequence $(v_{i,\ell})_\ell$ converges strongly to v_i in $L^2(0,T;H^1(\Omega))$ for $i=1,\dots,m$.*

From Lemmata 3.2, 3.3 and 3.4, there exist functions $u_j, v_i \in L^2(0,T;H^1(\Omega))$ such that, up to extracting subsequences if necessary, for $j=1,\dots,n$ and $i=1,\dots,m$

$$u_{j,\ell}\to u_j \text{ in } L^2(Q_T) \text{ strongly}, \qquad v_{i,\ell}\to v_i \text{ in } L^2(0,T;H^1(\Omega)) \text{ strongly},$$

and from this the continuity of Ł on $\mathcal{A}$ follows.

We observe that, from Lemma 3.3, Ł$(\mathcal{A})$ is bounded in the set

$$\mathcal{E} = \left\{u \in L^2(0,T;H^1(\Omega,\mathbb{R}^n)) \;:\; \partial_t u \in L^2(0,T;(H^1(\Omega,\mathbb{R}^n))')\right\}. \tag{33}$$

By the results of [18], $\mathcal{E} \hookrightarrow L^2(Q_T, \mathbb{R}^n)$ is compact, thus Ł is compact. Now, by the Schauder fixed point theorem, the operator Ł has a fixed point $U_\varepsilon = (u_{1,\varepsilon}, \dots, u_{n,\varepsilon})$ such that $Ł(U_\varepsilon) = U_\varepsilon$. Then there exists a solution $(u_{j,\varepsilon}, v_{i,\varepsilon})$ of

$$\begin{cases} \iint_{Q_T} \partial_t v_{i,\varepsilon}\, \varphi_i \, dx\, dt + d_{u_j} \iint_{Q_T} \nabla v_{i,\varepsilon} \cdot \nabla \varphi_i \, dx\, dt \\ \qquad\qquad = \iint_{Q_T} F_{i,\varepsilon}(v_{1,\varepsilon}, \dots, v_{m,\varepsilon}, u_{1,\varepsilon}, \dots, u_{n,\varepsilon})\, \varphi_i \, dx\, dt, \\ \int_0^T \langle \partial_t u_{j,\varepsilon}, \psi_j \rangle \, dt + \iint_{Q_T} (d_{v_i} \nabla u_{j,\varepsilon} - g_j(u_{j,\varepsilon})\vartheta_\varepsilon) \cdot \nabla \psi_j \, dx\, dt \\ \qquad\qquad = \iint_{Q_T} G_{j,\varepsilon}(v_{1,\varepsilon}, \dots, v_{m,\varepsilon}, u_{1,\varepsilon}, \dots, u_{n,\varepsilon})\, \psi_j \, dx\, dt, \end{cases} \tag{34}$$

for all $\varphi_i, \psi_j \in L^2(0,T; H^1(\Omega))$, for $i = 1, \dots, m$ and $j = 1, \dots, n$.

4. **Existence of weak solutions.** We have shown in Section 3 that the problem (18) admits a solution $(u_{j,\varepsilon}, v_{i,\varepsilon})$. The goal in this section is to send the regularization parameter ε to zero in sequences of such solutions to obtain weak solutions of the original system (4)-(6). Note that, for each fixed $\varepsilon > 0$, we have shown the existence of a solution $(u_{j,\varepsilon}, v_{i,\varepsilon})$ to (18) such that for $j = 1, \dots, n$

$$0 \le u_{j,\varepsilon}(t,x) \le e^{\gamma t} u_{j,m} \quad \text{and} \quad 0 \le v_{i,\varepsilon}(t,x), \quad \gamma > 0, \tag{35}$$

for a.e. $(t,x) \in Q_T$.

Using the first equation of (18) and (35), it is easy to see that the estimates of (24) are independent of ε:

$$\begin{aligned} &\|v_{i,\varepsilon}\|_{L^\infty(Q_T)} + \|v_{i,\varepsilon}\|_{L^\infty(0,T;L^2(\Omega))} \le C, \\ &\|v_{i,\varepsilon}\|_{L^2(0,T;H^1(\Omega))} \le C, \end{aligned} \tag{36}$$

and from classical results on L^p regularity, we have

$$\|\partial_t v_{i,\varepsilon}\|_{L^p(Q_T)} + \|v_{i,\varepsilon}\|_{L^p(0,T;W^{2,p}(\Omega))} \le C, \qquad 1 \le p < \infty,$$

for $i = 1, \dots, m$, where $C > 0$ is a constant independent of ε.

Taking $\varphi_j = u_{j,\varepsilon}$ as a test function in (34) and using the estimates (36), we obtain

$$\sup_{0 \le t \le T} \int_\Omega |u_{j,\varepsilon}(t,x)|^2 \, dx + C_5 \iint_{Q_T} |\nabla u_{j,\varepsilon}|^2 \, dx\, dt \le C_6, \tag{37}$$

for some constants $C_5, C_6 > 0$ independent of ε,

Working exactly as the proof of *(iii)* in Lemma 3.3, we get

$$\|\partial_t u_{j,\varepsilon}\|_{L^2(0,T;(H^1(\Omega))')} \le C. \tag{38}$$

Then, by (36)-(38) and standard compactness results (see [18]) we can extract subsequences, which we do not relabel, such that, as ε goes to 0,

$$\begin{aligned} &u_{j,\varepsilon} \to u_j \text{ and } v_{i,\varepsilon} \to v_i \text{ weakly-}\star \text{ in } L^\infty(Q_T), \\ &u_{j,\varepsilon} \to u_j \text{ strongly in } L^2(Q_T) \text{ and weakly in } L^2(0,T; H^1(\Omega)), \\ &v_{i,\varepsilon} \to v_i \text{ strongly in } L^2(Q_T) \text{ and weakly in } L^2(0,T; H^1(\Omega)), \\ &\partial_t u_{j,\varepsilon} \to \partial_t u_j \text{ weakly in } L^2(0,T; (H^1(\Omega))'), \\ &\partial_t v_{i,\varepsilon} \to \partial_t v_i \text{ weakly in } L^2(Q_T). \end{aligned} \tag{39}$$

With this and the weak-$\star$ convergence of $u_{j,\varepsilon}$ to u_j in $L^\infty(Q_T)$, we obtain

$$u_{j,\varepsilon} \to u_j \text{ strongly in } L^p(Q_T)\ , j = 1, \dots, n,$$

for $1 \leq p < \infty$. Similarly, $v_{i,\varepsilon} \to v$ strongly in $L^p(Q_T)$ for $1 \leq p < \infty$. With this we have for $i = 1, \dots, m$ and $j = 1, \dots, n$

$$\begin{aligned} F_{i,\varepsilon}(v_{1,\varepsilon}, \dots, v_{m,\varepsilon}, u_{1,\varepsilon}, \dots, u_{n,\varepsilon}) &\to F_i(v_1, \dots, v_m, u_1, \dots, u_n), \\ G_{j,\varepsilon}(v_{1,\varepsilon}, \dots, v_{m,\varepsilon}, u_{1,\varepsilon}, \dots, u_{n,\varepsilon}) &\to G_j(v_1, \dots, v_m, u_1, \dots, u_n), \end{aligned} \tag{40}$$

almost everywhere in Q_T and strongly in $L^p(Q_T)$ for $1 \leq p < \infty$. Herein

$$\begin{cases} F_i(v_1, \dots, v_m, u_1, \dots, u_n) = & -r_i(t,x)u_i - \lambda\, v_i - k_i(t,x)|e^{\lambda t}\vartheta|^{\sigma_i - 1}v_i \\ & -\displaystyle\sum_{j=1}^{n} e^{-\lambda t} h_{i,j}(t,x,e^{\lambda t}u_j, e^{\lambda t}v_i) \\ G_j(v_1, \dots, v_m, u_1, \dots, u_n) = & -a_j(t,x)u_j - \lambda u_j \\ & +\displaystyle\sum_{i=1}^{m} e_i(t,x)e^{-\lambda t} h_{i,j}(t,x,u_j,v_i) \\ & -\displaystyle\sum_{k=j+1}^{n} e^{-\lambda t} C_{k,j}(t,x,e^{\lambda t}u_k, e^{\lambda t}u_j), \\ & +\displaystyle\sum_{k=1}^{j-1} d_{k,j}(t,x)e^{-\lambda t} C_{k,j}(t,x,e^{\lambda t}u_k, e^{\lambda t}u_j). \end{cases} \tag{41}$$

To pass to the limit in (34) as $\varepsilon \to 0$, we need the following lemma.

Lemma 4.1. *The sequence $(v_{i,\varepsilon})_\varepsilon$ converges strongly to v_i in $L^2(0,T;H^1(\Omega))$ for $i = 1, \dots, m$.*

Proof. Subtracting the relations satisfied by $(u_{j,\varepsilon}, v_{i,\varepsilon})$ and (u_j, v_i), we have

$$\begin{cases} \partial_t(v_{i,\varepsilon} - v_i) - d_{v_i}(\Delta v_{i,\varepsilon} - \Delta v_i) \\ \qquad = F_{i,\varepsilon}(v_{1,\varepsilon}, \dots, v_{m,\varepsilon}, u_{1,\varepsilon}, \dots, u_{n,\varepsilon}) - F_i(v_1, \dots, v_m, u_1, \dots, u_n) \text{ in } Q_T, \\ \frac{\partial v_{i,\varepsilon}}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0 \text{ on } \Sigma_T, \quad (v_{i,\varepsilon} - v)(x,0) = 0, \text{ for } x \in \Omega. \end{cases} \tag{42}$$

Multiplying this equation by $v_{i,\varepsilon} - v_i$ and integrating over Ω, we get

$$\begin{aligned} &\frac{d}{dt}\int_\Omega |v_{i,\varepsilon} - v_i|^2 \, dx + d_{v_i}\int_\Omega |\nabla(v_{i,\varepsilon} - v_i)|^2 \, dx \\ &\quad = \int_\Omega \Big(F_{i,\varepsilon}(v_{1,\varepsilon}, \dots, v_{m,\varepsilon}, u_{1,\varepsilon}, \dots, u_{n,\varepsilon}) - F_i(v_1, \dots, v_m, u_1, \dots, u_n)\Big) \\ &\qquad\qquad \times (v_{i,\varepsilon} - v)\, dx. \end{aligned} \tag{43}$$

Using Young's inequality and integrating the inequality (43) over $(0,T)$ we obtain from (43)

$$\begin{aligned} &d_{v_i}\iint_{Q_T} |\nabla(v_{i,\varepsilon} - v)|^2 \, dx\, dt \\ &\quad \leq \frac{1}{2}\iint_{Q_T} |F_{i,\varepsilon}(v_{1,\varepsilon}, \dots, v_{m,\varepsilon}, u_{1,\varepsilon}, \dots, u_{n,\varepsilon}) - F_i(v_1, \dots, v_m, u_1, \dots, u_n)|^2 \, dx\, dt \\ &\qquad + \frac{1}{2}\iint_{Q_T} |v_{i,\varepsilon} - v_i|^2 \, dx\, dt.. \end{aligned} \tag{44}$$

Finally using the strong convergence in $L^2(Q_T)$ and in $L^p(Q_T)$, $p \geq 1$, of $v_{i,\varepsilon}$ and $F_{i,\varepsilon}(v_{1,\varepsilon}, \dots, v_{m,\varepsilon}, u_{1,\varepsilon}, \dots, u_{n,\varepsilon})$ to v_i and $F_i(v_1, \dots, v_m, u_1, \dots, u_n)$, respectively, we deduce from (44) the strong convergence of the sequence $(\nabla v_{i,\varepsilon})_\varepsilon$ to ∇v_i in $L^2(Q_T)$. This completes the proof of the lemma. □

Our final goal is to prove that the limit function $((u_j)_{1\le j\le n},(v_i)_{1\le i\le m})$ constructed in (39) and in Lemma 4.1 constitute a weak solution of the system (4), (5) and (6). Let $\varphi_j \in L^2(0,T;H^1(\Omega))$ be a test function in (34). By (39) and (40) it is clear that as $\varepsilon\to 0$

$$\int_0^T \langle \partial_t u_{j,\varepsilon},\varphi_j\rangle\,dt \to \int_0^T \langle \partial_t u_j,\varphi_j\rangle\,dt,$$
$$\iint_{Q_T} \nabla u_{j,\varepsilon}\cdot\nabla\varphi_j\,dx\,dt \to \iint_{Q_T} \nabla u_j\cdot\nabla\varphi_j\,dx\,dt$$
$$\iint_{Q_T} G_{j,\varepsilon}(v_{1,\varepsilon},\dots,v_{m,\varepsilon},u_{1,\varepsilon},\dots,u_{n,\varepsilon})\,\varphi_j\,dx\,dt \to \iint_{Q_T} G_j(v_1,\dots,v_m,u_1,\dots,u_n)\,\varphi_j\,dx\,dt.$$

Since $g_j(u_{j,\varepsilon})$ is bounded in $L^\infty(Q_T)$ (g_j is continuous), and, by Lemma 4.1, $v_{i,\varepsilon}\to v$ in $L^2(0,T;H^1(\Omega))$, we also have that, as $\varepsilon\to 0$,

$$\iint_{Q_T} g_j(u_{j,\varepsilon})\nabla\vartheta_\varepsilon\cdot\nabla\varphi_j\,dx\,dt \to \iint_{Q_T} g_j(u_j)\nabla\vartheta\cdot\nabla\varphi_j\,dx\,dt,$$

for $j=1,\dots,n$. Thus we have identified u_j as the second component of a solution of (4), (5) and (6) for $j=1,\dots,n$. Reasoning the same lines as above, we would identify v_i as the first component of a solution for $i=1,\dots,m$.

5. **Uniqueness of weak solutions.**

In this section we prove uniqueness of weak solutions to our systems by using duality technique (see e.g. [17]), thereby completing the well-posedness analysis.

First, we consider $(v_{i,1},v_{i,2})$ and $(u_{j,1},v_{2,j})$ two solutions of the system (4)-(6) for $i=1,\dots,m$ and $j=1,\dots,n$. We set $V_i=v_{i,1}-v_{i,2}$ and $U_i=u_{1,i}-u_{2,i}$, then V_i and U_i satisfy

$$\begin{cases} \partial_s V_i - d_{v_i}\Delta V_i \\ \quad = F_i(v_{1,1},\dots,v_{m,1},u_{1,1},\dots,u_{n,1}) - F_i(v_{1,2},\dots,v_{m,2},u_{1,2},\dots,u_{n,2}) \text{ in } Q_T,\\ \\ \partial_s U_j - d_{u_j}\Delta U_j + \operatorname{div}(u_{j,1}\chi_j(u_{j,1})\nabla\vartheta_1 - u_{j,2}\chi_j(u_{j,2})\nabla\vartheta_2)\\ \quad = G_j(v_{1,1},\dots,v_{m,1},u_{1,1},\dots,u_{n,1}) - G_j(v_{1,2},\dots,v_{m,2},u_{1,2},\dots,u_{n,2}) \text{ in } Q_T,\\ \\ \nabla v_i\cdot\eta = \nabla u_j\cdot\eta = 0 \text{ on } \Sigma_T, \qquad V_i(x,0)=U_j(x,0)=0 \text{ for } x\in\Omega, \end{cases} \tag{45}$$

for $i=1,\dots,m$ and $j=1,\dots,n$. Now, we define the function φ_j solution of the problem

$$-\Delta\varphi_j(t,\cdot) = U_j(t,\cdot) \text{ in } \Omega \text{ and } \frac{\partial\varphi_j(t,\cdot)}{\partial\eta} = 0 \text{ on } \partial\Omega, \qquad j=1,\dots,n, \tag{46}$$

for a.e. $t\in(0,T)$. Since $u_{j,1}$ and $u_{j,2}$ are bounded, then we get from the theory of linear elliptic equations, the existence, uniqueness and regularity of solution φ_j satisfying

$$\varphi_j \in C([0,T];H^2(\Omega)) \text{ with } \int_\Omega \varphi_j(t,\cdot)\,dx = 0, \text{ for } j=1,\dots,n.$$

Note that from the boundary condition of φ_j in (46) and $U_j(0,\cdot)=0$ we deduce that

$$\nabla\varphi_j(0,\cdot)=0 \text{ in } L^2(\Omega) \text{ for } j=1,\dots,n. \tag{47}$$

Multiplying the second equation in (45) by $\psi_j \in L^2(0,T;H^1(\Omega))$ and integrating over $Q_t:=(0,t)\times\Omega$, we get

$$\begin{aligned}
&\int_0^t \langle \partial_s U_j,\psi_j\rangle\,ds + d_{u_j}\iint_{Q_t}\nabla U_j\cdot\nabla\psi_j\,dx\,ds\\
&=\iint_{Q_t}\Big(u_{j,1}\chi_j(u_{j,1})\nabla\vartheta_1 - u_{j,2}\chi_j(u_{j,2})\nabla\vartheta_2\Big)\cdot\nabla\psi_j\,dx\,ds\\
&\quad+\iint_{Q_t}\Big(F_i(v_{1,1},\dots,v_{m,1},u_{1,1},\dots,u_{n,1})\\
&\qquad\qquad -F_i(v_{1,2},\dots,v_{m,2},u_{1,2},\dots,u_{n,2})\Big)\psi_j\,dx\,ds.
\end{aligned}\tag{48}$$

Since $\varphi_j \in L^2(0,T;H^1(\Omega))$ we can take $\psi_j=\varphi_j$ in (48) and we obtain from (46) and (47)

$$\begin{aligned}
2\int_0^t \langle \partial_s U_j,\varphi_j\rangle\,ds &= -2\int_0^t \langle \partial_s\Delta\varphi_j,\varphi_j\rangle\,ds\\
&=\int_\Omega |\nabla\varphi_j(t,x)|^2\,dx - \int_\Omega |\nabla\varphi_j(0,x)|^2\,dx\\
&=\int_\Omega |\nabla\varphi_j(t,x)|^2\,dx,
\end{aligned}\tag{49}$$

and

$$\begin{aligned}
&\int_0^t \langle \partial_s U_j,\varphi_j\rangle\,ds - d_{u_j}\iint_{Q_t} U_j\Delta\varphi_j\,dx\,ds\\
&=\iint_{Q_t}\Big(u_{j,1}\chi_j(u_{j,1}) - u_{j,2}\chi_j(u_{j,2})\Big)\nabla\vartheta_1\cdot\nabla\varphi_j\,dx\,ds\\
&\quad+\iint_{Q_t} u_{j,2}\chi_j(u_{j,2})\nabla\mathcal{V}\cdot\nabla\varphi_j\,dx\,ds\\
&\qquad+\iint_{Q_t}\Big(F_i(u_{1,1},\dots,u_{m,1},,v_{1,1},\dots,v_{n,1}\Big)\\
&\qquad\qquad -F_i(u_{1,2},\dots,u_{m,2},v_{1,2},\dots,v_{n,2}))\varphi_j\,dx\,ds,
\end{aligned}\tag{50}$$

where $\mathcal{V}=\vartheta_1-\vartheta_2$. Since $u_{j,1}$, $u_{j,2}$ and $v_{i,1}$, $v_{i,2}$ are bounded, then there exist a constant $C>0$ depending on $\|u_{j,1}\|_{L^\infty(\Omega)}$, $\|u_{j,2}\|_{L^\infty(\Omega)}$, $\|v_{i,1}\|_{L^\infty(\Omega)}$, $\|v_{i,2}\|_{L^\infty(\Omega)}$, $\|p_{i,j}\|_{L^\infty(\Omega)}$, $\|q_{i,j}\|_{L^\infty(\Omega)}$, $\|c_{k,j}\|_{L^\infty(\Omega)}$ and $\|f_{k,j}\|_{L^\infty(\Omega)}$ such that

$$\begin{aligned}
&|h_{i,j}(\cdot,\cdot,u_{j,1},v_{i,1}) - h_{i,j}(\cdot,\cdot,u_{j,2},v_{i,2})|\\
&\qquad\qquad + |C_{k,j}(\cdot,\cdot,u_{k,1},u_{j,1}) - C_{k,j}(\cdot,\cdot,u_{k,2},u_{j,2})|\\
&\quad\le C(|u_{j,1}-u_{j,2}| + |v_{i,1}-v_{i,2}| + |u_{k,1}-u_{k,2}| + |u_{j,1}-u_{j,2}|),
\end{aligned}\tag{51}$$

for $i=1,\dots,m$, $j=1,\dots,n$ and $k=1,\dots,j-1,j+1,\dots,n$.

Using (46), (51), Hölder's, Young's, Sobolev poincaré's inequalities yields from (50)

$$\begin{aligned}
&\int_0^t \langle \partial_s U_j, \varphi_j \rangle \, ds \\
\leq \quad & -d_{u_j} \iint_{Q_t} |U_j|^2 \, dx \, ds + \frac{d_{u_j}}{4} \iint_{Q_t} |U_j|^2 \, dx \, ds \\
& +C_7 \int_0^t \|\nabla \vartheta_1\|^2_{L^\infty(\Omega)} \|\nabla \varphi_j\|^2_{L^2(\Omega)} \, ds \\
& +\frac{1}{2n} \int_0^t \sum_{i=1}^m d_{v_i} \|\nabla V_i\|^2_{L^2(\Omega)} \, ds + C_8 \int_0^t \|\nabla \varphi_j\|^2_{L^2(\Omega)} \, ds \\
& +\frac{d_{u_j}}{4} \iint_{Q_t} |U_j|^2 \, dx \, ds + C_9 \int_0^t \|\nabla \varphi_j\|^2_{L^2(\Omega)} \, ds \\
& +\frac{d_{u_j}}{8} \iint_{Q_t} |U_j|^2 \, dx \, ds + C_{10} \iint_{Q_t} \sum_{i=1}^m |V_i|^2 \, dx \, ds \\
& +C_{11} \int_0^t \|\nabla \varphi_j\|^2_{L^2(\Omega)} \, ds + \frac{1}{8n} \int_{Q_t} \sum_{k=1}^n d_{u_k} |U_k|^2 \, dx \, ds \\
& +\frac{d_{u_j}}{8} \iint_{Q_t} |U_j|^2 \, dx \, ds + C_{12} \int_0^t \|\nabla \varphi_j\|^2_{L^2(\Omega)} \, ds \\
= \quad & -\frac{d_{u_j}}{4} \iint_{Q_t} |U_j|^2 \, dx \, ds \\
& +\frac{1}{2n} \int_0^t \sum_{i=1}^m d_{v_i} \|\nabla V_i\|^2_{L^2(\Omega)} \, ds \\
& + \int_0^t \Big(C_7 \|\nabla \vartheta_1\|^2_{L^\infty(\Omega)} + C_8 + C_9 + C_{11} + C_{12} \Big) \|\nabla \varphi_j\|^2_{L^2(\Omega)} \, ds \\
& +C_{10} \sum_{i=1}^m \iint_{Q_t} |V_i|^2 \, dx \, ds + \frac{1}{8n} \iint_{Q_t} \sum_{k=1}^n d_{u_k} |U_k|^2 \, dx \, ds,
\end{aligned} \tag{52}$$

for some constants $C_7, C_8, C_9, C_{10}, C_{11}, C_{12} > 0$.

Now multiplying the first equation in (45) by V_i and integrating over Q_t, we obtain from Young's inequality and (51)

$$\begin{aligned}
&\frac{1}{2} \int_0^t \frac{d}{ds} \int_\Omega |V_i|^2 \, dx \, ds \\
&\quad = -d_{v_i} \iint_{Q_t} |\nabla V_i|^2 \, dx \, ds + \iint_{Q_t} r_i(t,x) |V_i|^2 \, dx \, ds \\
&\qquad - \iint_{Q_t} k_i(t,x) \Big(|\vartheta_1|^{\sigma_i - 1} v_{i,1} - |\vartheta_2|^{\sigma_i - 1} v_{i,2} \Big) (v_{i,1} - v_{i,2}) \, dx \, ds \\
&\qquad - \sum_{j=1}^n \iint_{Q_t} (h_{i,j}(t,x,u_{j,1},v_{i,1}) - h_{i,j}(t,x,u_{j,2},v_{i,2})) V_i \, dx \, ds \\
&\quad \leq -d_{v_i} \iint_{Q_t} |\nabla V_i|^2 \, dx \, ds + \iint_{Q_t} r_i(t,x) |V_i|^2 \, dx \, ds \\
&\qquad +C_{13} \iint_{Q_t} |V_i|^2 \, dx \, ds \\
&\qquad +\frac{1}{4m} \iint_{Q_t} \sum_{j=1}^n d_{u_j} |U_j|^2 \, dx \, ds + C_{14} \iint_{Q_t} |V_i|^2 \, dx \, ds
\end{aligned} \tag{53}$$

$$\le -d_{u_i} \iint_{Q_t} |\nabla V_i|^2 \, dx\, ds + (C_{13} + C_{14} + C_{15}) \iint_{Q_t} |V_i|^2 \, dx\, ds + \frac{1}{4m} \iint_{Q_t} \sum_{j=1}^{n} d_{u_j} |U_j|^2 \, dx\, ds,$$

for some constants $C_{13}, C_{14}, C_{15} > 0$.

Finally, we deduce from (49), (52) and (53)

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \sum_{i=1}^{m} |V_i(t,x)|^2 \, dx + \int_\Omega \sum_{j=1}^{n} |\nabla \varphi_j(t,x)|^2 \, dx \\
&= \frac{1}{2}\int_\Omega \sum_{i=1}^{m} |V_i(t,x)|^2 \, dx + 2\sum_{j=1}^{n} \int_0^t \langle \partial_s U_j, \varphi_j \rangle \, ds \\
&\le 2\int_0^t \Big(C_7 \|\nabla \vartheta_1\|_{L^\infty(\Omega)} + C_8 + C_9 + C_{11} + C_{12} \Big) \sum_{j=1}^{n} \|\nabla \varphi_j\|^2_{L^2(\Omega)} \, ds \\
&+ (2nC_{10} + C_{13} + C_{14} + C_{15}) \iint_{Q_t} \sum_{i=1}^{m} |V_i|^2 \, dx\, ds
\end{aligned} \tag{54}$$

Using $\nabla \vartheta_1 \in L^p(0,T;L^\infty(\Omega))$ for all $p > 1$ and Gronwall's lemma to conclude from (54)

$$V_i = 0 \text{ and } \nabla \varphi_j = 0, \; i = 1, \dots, m, \; j = 1, \dots, n,$$

almost everywhere in Q_T, ensuring the uniqueness of weak solutions.

Acknowledgment. This research is supported by FONDECYT projects 1070682.

REFERENCES

[1] N. D. Alikakos, *An application of the invariance principle to reaction-diffusion equations*, J. Differential Equations, **33** (1979), 201–225.

[2] B. E. Ainsebaa, M. Bendahmane and Ahmed Noussair, *A reaction-diffusion system modeling predator-prey with prey-taxis*, Nonlinear Analysis: Real World Applications, **9** (2008), 2086–2105.

[3] B. E. Ainseba , W. E. Fitzgibbon and M. Langlais, *An application of homogenization techniques to population dynamics models*, Communications on Pure and Applied Analysis, **1** (2002), 19–33.

[4] S. Aly and M. Farkas, *Prey-predator in patchy environment with cross diffusion*, Differential Equations Dynam. Systems, **13** (2005), 311–321.

[5] M. Bendahmane, K. H. Karlsen and J. M. Urbano, *On a two-sidedly degenerate chemotaxis model with volume-filling effect*, Math. Models Methods Appl. Sci., **17** (2007), 783–804.

[6] S. Williams and P. L. Chow, *Non-linear reaction-diffusion models for interacting populations*, J. Math. Anal. Appl., **62** (1978), 157–169.

[7] M. Farkas, *Two ways of modelling cross-diffusion*, Proceedings of the Second World Congress of Nonlinear Analysts, Part 2 (Athens, 1996). Nonlinear Anal., **30** (1997), 1225–1233.

[8] T. Hillen and K. Painter, *Global existence for a parabolic chemotaxis model with prevention of overcrowding*, Adv. in Appl. Math., **26** (2001), 280–301.

[9] O. A. Ladyzhenskaya, V. Solonnikov and N. Ural'ceva, "Linear and Quasi-Linear Equations of Parabolic Type," Transl. AMS 23, Providence, 1968.

[10] S. A. Levin, *A more functional response to predator prey stability*, Am. Nat., **108** (1977), 207–228.

[11] S. A. Levin and L. A. Segel, *Hypothesis for origin of planktonic patchiness*, Nature, **259** (1976), 659–659.

[12] M. Mimura and K. Kawasaki, *Spatial segregation in competitive interaction-diffusion equations*, J. Math. Biol., **9** (1980), 49–64.

[13] M. Mimura and J. D. Murray, *On a diffusion prey-predator model which exhibits patchiness*, J. Theor. Biol., **75** (1978), 249–252.

[14] M. Mimura and M. Yamaguti, *Pattern formation in interacting and diffusive systems in population biology*, Adv. Biophys., **15** (1982), 19–65.

[15] A. Okubo, "Diffusion and Ecological Problems: Mathematical Models," An extended version of the Japanese edition, Ecology and diffusion. Translated by G. N. Parker, Biomathematics, 10. Springer-Verlag, Berlin-New York, 1980.

[16] A. Okubo and H. C. Chiang, *An analysis of the kinematics of swarming of Anarete pritchardi Kim (Diptera: Cecidomyiidae)*, Res. Popul. Ecol. (Kyoto), **16** (1974), 1–42.

[17] J. Rulla, *Weak solutions to Stefan Problems with prescribed convection*, SIAM J. Math. Anal., **18** (1987), 1784–1800.

[18] J. Simon, *Compact sets in the space* $L^p(0,T;B)$, Ann. Mat. Pura Appl. (4), **146** (1987), 65–96.

Received September 2007; revised January 2008.

E-mail address: mostafab@ing-mat.udec.cl