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Volume **1**, Number **4**, December **2006** pp. **639–673**

HYPERBOLIC-ELLIPTIC MODELS FOR WELL-RESERVOIR FLOW

Steinar Evje*,** and Kenneth H. Karlsen*

*Centre of Mathematics for Applications, University of Oslo
P.O. Box 1053, Blindern, N–0316 Oslo, Norway
**International Research Institute of Stavanger, University of Stavanger
P.O. Box 8046, N–4068 Stavanger, Norway

Abstract. We formulate a hierarchy of models relevant for studying coupled well-reservoir flows. The starting point is an integral equation representing unsteady single-phase 3-D porous media flow and the 1-D isothermal Euler equations representing unsteady well flow. This 2×2 system of conservation laws is coupled to the integral equation through natural coupling conditions accounting for the flow between well and surrounding reservoir. By imposing simplifying assumptions we obtain various hyperbolic-parabolic and hyperbolic-elliptic systems. In particular, by assuming that the fluid is incompressible we obtain a hyperbolic-elliptic system for which we present existence and uniqueness results. Numerical examples demonstrate formation of steep gradients resulting from a balance between a local nonlinear convective term and a non-local diffusive term. This balance is governed by various well, reservoir, and fluid parameters involved in the non-local diffusion term, and reflects the interaction between well and reservoir.

1. Introduction.

We are interested in coupled well-reservoir flow modeling. For that purpose we consider a model composed of a hyperbolic system of two conservation laws corresponding to the isothermal Euler equations with source terms, and an integral equation. It results from coupling a transient well flow model with a transient reservoir model and is given on the following form.

$$\begin{aligned} \partial_t(\rho) + \partial_x(\rho u) &= \frac{1}{\eta}\rho q_{\mathrm{V}}, && \eta > 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p(\rho) &= q_{\mathrm{F}}, && q_{\mathrm{F}} = q_{\mathrm{F}}(\rho, u), \\ p_0 - p(x,t) &= \int_0^t \int_0^1 H^r(x, x', t-t') q_{\mathrm{V}}(x', t')\, dx'dt', \end{aligned} \tag{1}$$

for $x \in [0,1]$. Here, ρ, u, and $p(\rho)$ are, respectively, the mass density, fluid velocity, and pressure, whereas q_{V} represents volumetric flow rate accounting for flow between well and reservoir. Thus, the unknown variables are ρ, u, and q_{V}. Moreover, p_0 which we assume to be constant, is initial reservoir pressure whereas η is a small known constant parameter characterizing the well volume relatively the pore volume

2000 *Mathematics Subject Classification.* Primary: 35L65, 35L60; Secondary: 35K40.

Key words and phrases. Non-local conservation law, coupled well-reservoir flow, advanced well, hyperbolic-elliptic model, entropy weak solution, existence, uniqueness.

This research is supported by an Outstanding Young Investigators Award from the Research Council of Norway. Steinar Evje is grateful to Ove Sævareid for helpful discussions.

associated with the reservoir. The q_{F} term represents friction between fluid and wall, and we have assumed that the well is horizontal so that gravitation can be neglected. Finally, the kernel $H^r(x,x',t-t')$ is characteristic for the reservoir under consideration as well as the geometry of the well-path. Typical applications of such a model might be processes in conjunction with drilling, production, or injection scenarios.

Advanced oil-well designs of increasing sophistication are now routinely used throughout the industry. Complex wellbore trajectories combined with devices for downhole measurements and regulations provide an overwhelming amount of available data and operational flexibility. The challenge of identifying and utilising significant information might well be regarded as a bottleneck of current operations. Transients of interest will typically arise from production start-up or shut-down of a single well, or adjustment of one or several downhole valves in an advanced completion. The perturbations induced across different zones or laterals of the same well or between entirely different wells reflect characteristic behaviour of the reservoir. In this context there is a need for an improved understanding of coupled well-reservoir dynamics. This serves as our motivation for studying the well-reservoir model (1).

Transients in wellbore flow typically operates on time scales ranging from seconds to minutes whereas the more relevant part of the reservoir dynamics will be the compression waves, typically having relaxation times in the order of hours. Within the petroleum engineering literature there has been some focus on modeling of coupled well-reservoir flows relevant for production scenarios where main focus is on prediction of reservoir inflow. For that purpose it is reasonable to consider a steady well model, see for example [25]–[27], [33]–[35]. However, by starting with the model (1) we intend to take a broader approach in the sense that we include transient effects both from well and reservoir.

We may study various simplified versions of the well-reservoir model (1). For instance, we can impose the following assumptions: (i) consider a straight line well-path geometry, (ii) account only for a steady-state response from the reservoir, (iii) apply an approximation argument for the kernel function $H^r(x,x',t-t')$. Then we arrive at a well-reservoir model on the form

$$\begin{aligned} \partial_t(\rho) + \partial_x(\rho u) &= \rho(A - Bp(\rho) + p(\rho)_{xx}), \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p(\rho) &= q_{\mathrm{F}}(\rho, u), \end{aligned} \tag{2}$$

for appropriate chosen constants $A, B > 0$. This indicates that the impact from the reservoir on the well-flow dynamic through the volumetric flow rate q_{V} imposes a regularization effect in the isothermal Euler model. Investigations of this model will be addressed somewhere else.

However, as a first step in order to get some understanding of basic underlying mechanisms present in the well-reservoir model (1), we take a step further and assume that the fluid, both in the well and reservoir, is incompressible. In addition, for simplicity reasons only, we consider the model on the whole real axis instead of the bounded domain $[0,1]$. We then get a scalar conservation law with a non-local diffusion term on the form

$$\begin{aligned} &\partial_t u + \partial_x(u^2) = -\partial_x p, \\ &p_0 - p(x,t) = \varepsilon \int_{-\infty}^{+\infty} G^r(x,x') u_{x'}(x',t)\, dx' = \varepsilon G^r * u_x, \end{aligned} \tag{3}$$

with

$$G^r(x,x') = \frac{r^2}{\sqrt{(x-x')^2 + r^2}}, \qquad r > 0, \tag{4}$$

and

$$\varepsilon = \frac{\mu D}{4\rho k}, \tag{5}$$

where μ is fluid viscosity, k is permeability, D is a characteristic time, r the well radius, and ρ denotes the constant fluid density. We may write (3) on the following form

$$\begin{aligned} \partial_t u + \partial_x(u^2) = \varepsilon G^r_x * u_x = \varepsilon G^r_{xx} * u, \qquad \varepsilon, r > 0, \\ u(x,0) = u_0(x). \end{aligned} \tag{6}$$

The mission of this paper, in light of the preceding discussion, is three-folded.

- We present the background needed for deriving the dynamic, coupled well-reservoir model (1) which takes into account the transients of the well flow as well as the transients of the reservoir flow. In particular, this model contains as a special case the "steady well-unsteady reservoir" model previously studied within the petroleum science literature [25].
- We identify various simplified versions of the well-reservoir model (1) by imposing appropriate assumptions. Examples are given by (2) and (3). The motivation for this is to search for models more amenable to mathematical analysis, and still able to capture one aspect or another of the more general model (1).
- Having identified the incompressible well-reservoir model (3), we provide a mathematical framework appropriate for exploring its mathematical properties. We also present numerical calculations demonstrating characteristic behaviour like formation of discontinuities.

Regarding the mathematical analysis of the well-reservoir model (3), a main observation is that the form of this model bears similarities to the so-called radiating gas model [14, 12, 28] as well as a Burger-Poisson type of model studied in [10]. Motivated by this, we propose a notion of entropy weak solutions that allows for discontinuities and provide existence and uniqueness results. The framework we use is fairly general and might be applied for more general models than (3) obtained by taking into account effects which are included in the original well-reservoir model (1) but not in (3). A main difference between our model problem (3) and the models studied in [14, 10] is that the involved kernel (4) does not correspond to a differential operator. This additional information is explicitly used, for example, in travelling wave analysis performed for the radiating gas model [14, 15, 16, 24, 21, 30] and the Burger-Poisson type model [10] mentioned above. Thus, such techniques may not directly apply to our model problem.

To be more specific about the mathematical results, first, we provide a local existence result for smooth solutions of (3). Then we provide global existence results under various regularity on initial data. More precisely, we prove that there exists a unique entropy weak solution for initial data

$$u_0(x) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}). \tag{7}$$

Then, we prove that there exists at least one weak solution for initial data

$$u_0(x) \in L^2(\mathbb{R}) \cap L^4(\mathbb{R}). \tag{8}$$

An interesting by-product of this analysis is that it allows us to explore the dependence on the well-radius r. More precisely, we observe that as the well radius r goes to zero, the entropy weak solution of (6) converges to the entropy weak solution of the conservation law $u_t + (u^2)_x = 0$.

The remaining part of the paper is organized as follows: In Section 2 we give a more detailed description of the underlying ideas which lead to the system (1) as well as the simplified variants (2) and (3). In Section 3 we identify links between the incompressible well-reservoir model (3) and related models known from the literature and give some motivation for the framework we shall use to obtain well-posedness. In particular, the notion of weak solution and entropy weak solution are introduced. In Section 4 an existence and uniqueness result are given for solutions in L^∞ whereas existence is proved in a L^2 setting in Section 5. Finally, in Section 6 we show some numerical results and illustrate characteristic behaviour of the balance of the local convective term and the non-local diffusive term appearing in (3).

2. **Mathematical models for single-phase reservoir and well flow.** In this section we first set up relevant single-phase models for reservoir flow and well (pipe) flow, respectively. Then, following the line of previous studies within the petroleum science literature [25]–[27], [33]–[35], we formulate coupled well-reservoir models. More precisely, in Section 2.1 we identify a transient reservoir model by using a density formulation whereas in Section 2.2 we use a pressure formulation for the same model. Then, in Section 2.3 we describe a basic well flow model (compressible and incompressible). Section 2.4, 2.5, and 2.6 are devoted to a discussion of compressible coupled well-reservoir flow models as well as incompressible variants, corresponding to the flow models (1), (2), and (3).

2.1. **Reservoir flow: Compressible fluid flow via a density formulation.** We consider the flow of a compressible single-phase fluid in a 3D reservoir. Darcy's law gives us

$$\mathbf{U} = -\frac{\mathbf{K}}{\mu}(\nabla p - g\mathbf{g}).$$

The continuity equation for flow in porous medium is given in the form

$$\frac{\partial \phi\rho}{\partial t} + \nabla\cdot(\rho\mathbf{U}) = Q_{\text{mass}}(\mathbf{x}, t), \qquad \mathbf{x}\in\Omega.$$

The unknown are p pressure, ρ density (which is a function of p), ϕ porosity, μ viscosity. Here we have also included a source term $Q_{\text{mass}}(\mathbf{x}, t)$ which accounts for the mass flow through wells. These two equations may be combined to give a dynamic equation

$$\frac{\partial \phi\rho}{\partial t} = \nabla\cdot\Big[\rho\frac{\mathbf{K}}{\mu}(\nabla p - \rho\mathbf{g})\Big] + Q_{\text{mass}}(\mathbf{x}, t). \tag{9}$$

We assume that $\mathbf{K} = \text{diag}(k_x, k_y, k_z)$ is a diagonal tensor. Moreover, we assume that the the fluid has constant compressibility c, i.e.,the density is given by an equation of state of the form

$$\rho = \rho(p) = \rho_0\exp[c(p - p_0)], \qquad c = \rho^{-1}\partial\rho/\partial p. \tag{10}$$

In this case, since $\nabla\rho = c\rho\nabla p$, we see that (9) takes the form

$$\frac{\partial \phi\rho}{\partial t} = \nabla\cdot\Big[\frac{\mathbf{K}}{c\mu}\left(\nabla\rho - c\rho^2\mathbf{g}\right)\Big] + Q_{\text{mass}}(\mathbf{x}, t), \qquad \mathbf{x}\in\Omega. \tag{11}$$

This type of equation enters the discussion when gas expands in a porous medium and in pressure tests used in oil production. In the following we will neglect the gravity term (as in horizontal flow).

Let $\mathbf{X}_w(s) = (x_w(s), y_w(s), z_w(s))$ with $s \in [0,1]$ (dimensionless) be a parametrization of the line Γ_w representing the well path with $\mathbf{X}'_w(s)$ continuous on $[0,1]$. Let α denote the arc-length function defined by

$$\alpha(s) = \int_0^s \|\mathbf{X}'_w(u)\|\,du, \qquad \|\mathbf{x}\| = \sqrt{x^2+y^2+z^2}. \tag{12}$$

We assume that the length of the well path is L_w, i.e., $\alpha(1) = \int_0^1 \|\mathbf{X}'_w(s)\|\,ds = L_w$.

The source term $Q_{\text{mass}}(\mathbf{x},t)$ represents a delta function singularity along the well path Γ_w given by

$$Q_{\text{mass}}(\mathbf{x},t) = \int_{\Gamma_w} q_{\text{M}}(\alpha,t)\delta(\mathbf{x}-\mathbf{X}_w(\alpha))\,d\alpha, \qquad q_{\text{M}} = \rho q_{\text{V}}, \tag{13}$$

where $\delta(\mathbf{x})$ is a three-dimensional Dirac function $\delta(\mathbf{x}) = \delta(x)\delta(y)\delta(z)$, $q_{\text{M}}(\alpha,t)$ is the mass flow rate per unit wellbore length and $q_{\text{V}}(\alpha,t)$ the volumetric influx or efflux rate per unit wellbore length. By this we mean that $Q_{\text{mass}}(\mathbf{x},t)$ is a distribution with the property that

$$\int Q_{\text{mass}}(\mathbf{x},t)\phi(\mathbf{x})\,d\mathbf{x} = \int_{\Gamma_w} q_{\text{M}}(\alpha,t)\phi(\mathbf{X}_w(\alpha))\,d\alpha, \tag{14}$$

for any smooth test function $\phi(\mathbf{x})$. Then the line integral with respect to arc length along Γ_w appearing on the right hand side of (14) is evaluated as follows

$$\begin{aligned}\int_{\Gamma_w} q_{\text{M}}(\alpha,t)\phi(\mathbf{X}_w(\alpha))\,d\alpha &= \int_0^1 q_{\text{M}}(s,t)\phi(\mathbf{X}_w(s))\|\mathbf{X}'_w(s)\|\,ds \\ &= L_w \int_0^1 q_{\text{M}}(s,t)\phi(\mathbf{X}_w(s))\,ds,\end{aligned} \tag{15}$$

if we consider a well with a straight line geometry, since $\|\mathbf{X}'_w(s)\| = L_w$. In the following we restrict ourselves to this well geometry.

Generally, the model equation (11) is subject to initial and boundary conditions given by

$$\rho = \rho_0 = \rho(p_0), \qquad \text{at } t = 0, \quad (p_0 \text{ is the initial reservoir pressure}) \tag{16}$$

$$\rho = \rho_R \text{ or } \frac{\partial \rho}{\partial n} = q_R, \qquad \text{for } \mathbf{x} \in \partial\Omega. \tag{17}$$

In this work we shall assume that the medium is isotropic, i.e., $k_x = k_y = k_z = k$. The corresponding density equation takes the form

$$\phi\frac{\partial \rho}{\partial t} - \frac{k}{c\mu}\Big[\frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} + \frac{\partial^2 \rho}{\partial z^2}\Big] = Q_{\text{mass}}(\mathbf{x},t), \tag{18}$$

where $(\mathbf{x},t) = (x,y,z,t) \in \Omega \times [0,T]$. In the following we assume that Ω is a cube of length L. It is convenient to introduce dimensionless variables in space and time on the form [25]

$$\hat{x} = \frac{x}{L}, \qquad \hat{y} = \frac{y}{L}, \qquad \hat{z} = \frac{z}{L}, \qquad \hat{t} = t\frac{k}{L^2 c\mu\phi} = \frac{t}{D}, \tag{19}$$

where L is the characteristic length of the reservoir domain such that our domain of interest will have length one and $D = \frac{L^2 c\mu\phi}{k}$ is a characteristic length of the reservoir time period. We also introduce a non-dimensional density $\hat{\rho}$ and mass flow rate $\hat{q}_{\mathrm{M}}$ defined by

$$\hat{\rho} = \frac{\rho}{\bar{\rho}}, \qquad \hat{q}_{\mathrm{M}} = \frac{q_{\mathrm{M}}}{\bar{q}_{\mathrm{M}}}, \tag{20}$$

where $\bar{\rho}$ is a characteristic density whereas $\bar{q}_{\mathrm{M}}$ is the characteristic mass flow rate given by

$$\bar{q}_{\mathrm{M}} = \frac{\text{total reservoir fluid mass}}{\text{reservoir time} \cdot \text{well length}} = \frac{L^3 \phi \bar{\rho}}{D \cdot L_w} = \frac{Lk\bar{\rho}}{L_w c\mu}. \tag{21}$$

In terms of the new variables (19) and (20) the model (18) takes the following form for $\hat{\rho} = \hat{\rho}(\hat{\mathbf{x}}, \hat{t})$

$$\frac{\partial \hat{\rho}}{\partial \hat{t}} - \Big[\frac{\partial^2 \hat{\rho}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{\rho}}{\partial \hat{y}^2} + \frac{\partial^2 \hat{\rho}}{\partial \hat{z}^2}\Big] = \frac{L^2 c\mu}{k\bar{\rho}} Q_{\mathrm{mass}}(L\hat{\mathbf{x}}, t) = \frac{c\mu}{Lk\bar{\rho}} Q_{\mathrm{mass}}(\hat{\mathbf{x}}, t) \stackrel{\mathrm{def}}{:=} \hat{Q}_{\mathrm{mass}}(\hat{\mathbf{x}}, \hat{t}), \tag{22}$$

for $(\hat{\mathbf{x}}, \hat{t}) \in \hat{\Omega} \times [0, \hat{T}]$ where $\hat{T} = \frac{T}{D}$. Here we have used that

$$\begin{aligned} Q_{\mathrm{mass}}(L\hat{\mathbf{x}}, t) &= \int_{\Gamma_w} q_{\mathrm{M}}(\alpha, t) \delta(L[\hat{\mathbf{x}} - \hat{\mathbf{X}}_w(\alpha)])\, d\alpha \\ &= \frac{1}{L^3} \int_{\Gamma_w} q_{\mathrm{M}}(\alpha, t) \delta(\hat{\mathbf{x}} - \hat{\mathbf{X}}_w(\alpha))\, d\alpha = \frac{1}{L^3} Q_{\mathrm{mass}}(\hat{\mathbf{x}}, t), \end{aligned}$$

since $\delta(L\hat{\mathbf{x}}) = \delta(L\hat{x})\delta(L\hat{y})\delta(L\hat{z}) = \frac{1}{L^3}\delta(\hat{\mathbf{x}})$. Moreover, in view of (14) and (15), the meaning of the source term $\hat{Q}_{\mathrm{mass}}(\hat{\mathbf{x}}, \hat{t})$ in (22) is

$$\int \hat{Q}_{\mathrm{mass}}(\hat{\mathbf{x}}, t) \phi(\hat{\mathbf{x}})\, d\hat{\mathbf{x}} = \int_0^1 \hat{q}_{\mathrm{M}}(s, \hat{t}) \phi(\hat{\mathbf{X}}_w(s))\, ds, \qquad \hat{q}_{\mathrm{M}} = \frac{L_w c\mu}{Lk\bar{\rho}} q_{\mathrm{M}} = \frac{q_{\mathrm{M}}}{\bar{q}_{\mathrm{M}}}, \tag{23}$$

in accordance with (20) and (21). In the following, if nothing else is said, we work with the above dimensionless variables although this distinction is not expressed explicitly in the notation.

Regarding the solution of (22) and (23), we note that generally, when smart well systems are used (which involve a number of wells with any number of laterals of arbitrary configuration), the source term of (22) can have a rather complicated impact on the solution [25]. Following in the footsteps of [25]–[27], [33]–[35] we assume that each well and lateral is represented by a line source or sink. This leads to an integral representation of the model (22) and (23) on the form

$$\begin{aligned} \rho_0(\mathbf{x}) - \rho(\mathbf{x}, t) &= \int_0^t \int_\Gamma G(\mathbf{x}, \mathbf{x}', t - t') Q_{\mathrm{mass}}(\mathbf{x}', t')\, d\mathbf{x}' dt', \quad \mathbf{x} \in \Omega, t \in [0, T], \\ &= \int_0^t \int_0^1 G(\mathbf{x}, \mathbf{X}_w(s'), t - t') q_{\mathrm{M}}(s', t')\, ds' dt'. \end{aligned} \tag{24}$$

Note that in this formulation a positive mass flowrate q_{M} represents radial inflow and is associated with a pressure drop $p < p_0$ which leads to a corresponding drop in density $\rho < \rho_0$.

Moreover, G is the fundamental solution of the heat equation in $\Omega = [0, 1]^3$ whose specific form depend on the boundary conditions (Dirichlet or Neumann). The integral representation above is flexible and may be applied to reservoir problems with complex well configurations. Successful applications of this approach have been reported by Economides et al [6] and Ouyang et al [27], see also references therein.

Next, we follow [9], and let $G(\mathbf{x}, \mathbf{x}', t - t')$ be the Green's function for the heat equation in 3D where outer boundary conditions have been neglected (i.e., the free-space kernel is considered), given by

$$G(\mathbf{x}, \mathbf{x}', t - t') = \frac{1}{[4\pi(t - t')]^{3/2}} \exp\Big[-\frac{\|\mathbf{x} - \mathbf{x}'\|^2}{4(t - t')}\Big], \qquad t > t'. \tag{25}$$

For short-time well-reservoir processes this seems to be a natural simplification since it takes time before the impact from the boundaries is actuated.

By setting $\mathbf{x} = \mathbf{X}_w(s) + \mathbf{r}_w$ for $s \in [0, 1]$ in (24), we note that $q_{\mathrm{M}}(s', t')$ satisfies the integral equation

$$\begin{aligned} \Delta\rho(\mathbf{X}_w(s) + \mathbf{r}_w, t) &= \int_0^t \int_\Gamma G(\mathbf{X}_w(s) + \mathbf{r}_w, \mathbf{x}', t - t') Q_{\mathrm{mass}}(\mathbf{x}', t')\, d\mathbf{x}' dt', \\ &= \int_0^t \int_0^1 G(\mathbf{X}_w(s) + \mathbf{r}_w, \mathbf{X}_w(s'), t - t') q_{\mathrm{M}}(s', t')\, ds' dt'. \end{aligned} \tag{26}$$

Here $\Delta\rho(\mathbf{X}_w(s) + \mathbf{r}_w, t) = \rho_0(\mathbf{X}_w(s) + \mathbf{r}_w) - \rho(\mathbf{X}_w(s) + \mathbf{r}_w, t)$ represents the change in density at the well boundary, i.e., a radial displacement $\mathbf{r}_w$ away from the well centerline Γ_w described by $\mathbf{X}_w$ and such that this radial displacement is equal to the wellbore radius $r_w = \|\mathbf{r}_w\|$. Equation (26) is an integral equation of first kind, Fredholm in space and Volterra in time. For later use, we observe the following identity

$$\begin{aligned} B(\mathbf{x}, t; t_1, t_2) &= \int_{t_1}^{t_2} G(\mathbf{x}, \mathbf{x}', t - t')\, dt', \qquad t > t' \in [t_1, t_2] \\ &= \frac{1}{4\pi\|\mathbf{x} - \mathbf{x}'\|} \cdot \left[\mathrm{erf}\Big(\frac{\|\mathbf{x} - \mathbf{x}'\|}{2\sqrt{t - t_2}}\Big) - \mathrm{erf}\Big(\frac{\|\mathbf{x} - \mathbf{x}'\|}{2\sqrt{t - t_1}}\Big)\right]. \end{aligned} \tag{27}$$

Remark 2.1. From a numerical point of view one should note that it is in general very challenging to solve the model (22) and (23) accurately due to the delta function singularity. For a discussion of this issue in various contexts, as well as description of some proposed techniques for handling this problem, we refer to [20, 5, 7, 8] and references therein.

2.2. **Reservoir flow: Compressible and incompressible fluid flow via a pressure formulation.** Assuming that the compressibility is weak we may take ρ outside the nabla operator on the right hand side of (9), i.e., we neglect a term on the form $\frac{c\rho}{\mu}\nabla p \cdot (\mathbf{K}\nabla p)$. In addition, we assume the porosity is constant. Then, in view of (10), we obtain the pressure equation

$$c\phi \frac{\partial p}{\partial t} - \nabla \cdot \Big[\frac{\mathbf{K}}{\mu}(\nabla p - \rho \mathbf{g})\Big] = \frac{Q_{\mathrm{mass}(\mathbf{x},t)}}{\rho} = Q_{\mathrm{vol}}(\mathbf{x}, t), \tag{28}$$

where

$$Q_{\mathrm{vol}}(\mathbf{x}, t) = \int_{\Gamma_w} q_{\mathrm{V}}(\alpha, t)\delta(\mathbf{x} - \mathbf{X}_w(\alpha))\, d\alpha. \tag{29}$$

The two equations (9) and (28) are often used in reservoir engineering [1]. Again, we consider the transformed variables (19) together with a non-dimensional pressure $\hat{p}$ and volumetric flow rate $\hat{q}_{\mathrm{V}}$ defined by

$$\hat{p} = \frac{p}{\bar{p}}, \qquad \hat{q}_{\mathrm{V}} = \frac{q_{\mathrm{V}}}{\bar{q}_{\mathrm{V}}}, \tag{30}$$

where $\bar{p}$ is a characteristic reservoir pressure whereas $\bar{q}_{\mathrm{V}}$ is the characteristic volumetric flow rate given by

$$\bar{q}_{\mathrm{V}} = \frac{\text{total pore volume}}{\text{reservoir time} \cdot \text{well length}} \cdot \bar{p}c = \frac{L^3\phi\cdot}{D \cdot L_w} \cdot \bar{p}c = \frac{Lk\bar{p}}{L_w\mu}. \quad (31)$$

Assuming isotropic medium and neglecting the gravitation term, the pressure equation (28) takes the form

$$\frac{\partial \hat{p}}{\partial \hat{t}} - \Big[\frac{\partial^2 \hat{p}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{p}}{\partial \hat{y}^2} + \frac{\partial^2 \hat{p}}{\partial \hat{z}^2}\Big] = \frac{L^2\mu}{k\bar{p}} Q_{\mathrm{vol}}(L\hat{\mathbf{x}}, t) = \frac{\mu}{kL\bar{p}} Q_{\mathrm{vol}}(\hat{\mathbf{x}}, t) \overset{\mathrm{def}}{:=} \hat{Q}_{\mathrm{vol}}(\hat{\mathbf{x}}, \hat{t}), \quad (32)$$

where the meaning of $\hat{Q}_{\mathrm{vol}}(\hat{\mathbf{x}}, \hat{t})$, in light of (14) and (15), is

$$\int \hat{Q}_{\mathrm{vol}}(\hat{\mathbf{x}}, t)\phi(\hat{\mathbf{x}})\, d\hat{\mathbf{x}} = \int_0^1 \hat{q}_{\mathrm{V}}(s, \hat{t})\phi(\hat{\mathbf{X}}_w(s))\, ds, \qquad \hat{q}_{\mathrm{V}} = \frac{L_w\mu}{Lk\bar{p}} q_{\mathrm{V}} = \frac{q_{\mathrm{V}}}{\bar{q}_{\mathrm{V}}}. \quad (33)$$

Following the approach as described above for the density equation we arrive at the following integral equation, where $q_{\mathrm{V}}(s', t')$ and $p(s, t)$ now are non-dimensional variables

$$\begin{aligned} \Delta p(\mathbf{X}_w(s) + \mathbf{r}_w, t) &= \int_0^t \int_\Gamma G(\mathbf{X}_w(s) + \mathbf{r}_w, \mathbf{x}', t - t') q_{\mathrm{V}}(\mathbf{x}', t')\, d\mathbf{x}' dt', \\ &= \int_0^t \int_0^1 G(\mathbf{X}_w(s) + \mathbf{r}_w, \mathbf{X}_w(s'), t - t') q_{\mathrm{V}}(s', t')\, ds' dt', \end{aligned} \quad (34)$$

where $\Delta p(\mathbf{X}_w(s)+\mathbf{r}_w, t) = p_0(\mathbf{X}_w(s)+\mathbf{r}_w) - p(\mathbf{X}_w(s)+\mathbf{r}_w, t)$ for $(s, t) \in [0, 1] \times [0, T]$.

Assuming that the fluid is incompressible, the temporal term in (28) vanishes, i.e., we have

$$-\nabla \cdot \Big[\frac{\mathbf{K}}{\mu}(\nabla p - \rho \mathbf{g})\Big] = \frac{Q_{\mathrm{mass}(\mathbf{x},t)}}{\rho} = Q_{\mathrm{vol}}(\mathbf{x}, t), \quad (35)$$

where Q_{vol} is given by (29). Now, we consider the transformed variables (19) (only the spatial variables are relevant) together with a non-dimensional pressure $\hat{p}$ and volumetric flow rate $\hat{q}_{\mathrm{V}}$ defined by (30) and (31). Assuming isotropic medium and neglecting the gravitation term, the pressure equation (35) takes the form

$$-\Big[\frac{\partial^2 \hat{p}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{p}}{\partial \hat{y}^2} + \frac{\partial^2 \hat{p}}{\partial \hat{z}^2}\Big] = \hat{Q}_{\mathrm{vol}}(\hat{\mathbf{x}}, \hat{t}), \quad (36)$$

where $\hat{Q}_{\mathrm{vol}}$ is defined by (33). Following the approach as described above for the density equation, we arrive at the following integral equation where $q_{\mathrm{V}}(s', t)$ and $p(s, t)$ now are non-dimensional variables

$$\begin{aligned} \Delta p(\mathbf{X}_w(s) + \mathbf{r}_w, t) &= \int_\Gamma G(\mathbf{X}_w(s) + \mathbf{r}_w, \mathbf{x}') Q_{\mathrm{vol}}(\mathbf{x}', t)\, d\mathbf{x}', \qquad s \in [0, 1], \\ &= \int_0^1 G(\mathbf{X}_w(s) + \mathbf{r}_w, \mathbf{X}_w(s')) q_{\mathrm{V}}(s', t)\, ds', \end{aligned} \quad (37)$$

where $\Delta p(\mathbf{X}_w(s) + \mathbf{r}_w, t) = p_0(\mathbf{X}_w(s) + \mathbf{r}_w) - p(\mathbf{X}_w(s) + \mathbf{r}_w, t)$. Here the kernel G is the Green's function associated with the pressure equation

$$-\Delta p = \delta(\mathbf{x} - \mathbf{X}_w) \quad (38)$$

in 3D. That is,

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi\|\mathbf{x} - \mathbf{x}'\|}. \quad (39)$$

This may be seen from the fact that the heat kernel $G(\mathbf{x}, \mathbf{x}', t-t')$ is related to the Green function to the Laplace equation (39), let's denote is as $K(\mathbf{x}, \mathbf{x}')$, through the relation (see for example [11, 31])

$$K(\mathbf{x}, \mathbf{x}') = \int_0^\infty G(\mathbf{x}, \mathbf{x}', t-t')\, dt,$$

that is, for the free-space kernel (25), as observed in (27), we get

$$\int_{t_1}^{t_2} G(\mathbf{x}, \mathbf{x}', t-t') dt = \frac{1}{4\pi\|\mathbf{x}-\mathbf{x}'\|}\left[\operatorname{erf}\left(\frac{\|\mathbf{x}-\mathbf{x}'\|}{2\sqrt{t_2}}\right) - \operatorname{erf}\left(\frac{\|\mathbf{x}-\mathbf{x}'\|}{2\sqrt{t_1}}\right)\right],$$

which tends to $\frac{1}{4\pi\|\mathbf{x}-\mathbf{x}'\|}$ as $(t_1, t_2) \to (0, +\infty)$.

2.3. **Well flow: Compressible and incompressible fluid flow.** The purpose of this section is to present a basic well-type model for a compressible fluid as well as the corresponding model for an incompressible fluid.

Compressible fluid. A single-phase, compressible, isothermal and unsteady well flow model is given on the form

$$\begin{aligned} \partial_t(A\rho_w) + \partial_\alpha(A\rho_w u) &= q_{\mathrm{M}} = \rho_w q_{\mathrm{V}} \\ \partial_t(A\rho_w u) + \partial_\alpha(A\rho_w u^2) + A\partial_\alpha p_w &= -A\rho_w g \sin\theta - \tau_w S, \end{aligned} \tag{40}$$

where α is the arc-length variable associated with the well path Γ_w defined in (12) and t is the time variable. Here ρ_w is the fluid density, u the fluid velocity, $p_w = p(\rho_w)$ the pressure, q_{M} mass flow rate per unit wellbore length whereas q_{V} represents volumetric flux per unit wellbore length. Moreover, τ_w represents wall fraction shear rate given by

$$\tau_w = \frac{1}{2} f \rho_w u|u|,$$

where f is the Fanning factor and $A = \pi r_w^2$ is the pipe cross-sectional area and $S = 2\pi r_w$ is the pipe perimeter for a well of radius r_w. In addition, the well model is subject to the following initial data

$$p(\alpha, 0) = p_0(\alpha), \qquad u(\alpha, 0) = u_0(\alpha). \tag{41}$$

Introducing a characteristic time according to (19) as well as applying (12), which corresponds to $\alpha(s) = L_w s$ for $s \in [0, 1]$, we see that the model (40) can be written as

$$\begin{aligned} \partial_{\hat{t}}(A\rho_w) + \partial_s\left(\frac{AD}{L_w}\rho_w u\right) &= Dq_{\mathrm{M}} \\ \partial_{\hat{t}}(A\rho_w u) + \partial_s\left(\frac{AD}{L_w}\rho_w u^2\right) + \frac{AD}{L_w}\partial_s p_w &= -AD\rho_w g \sin\theta - \tau_w SD, \end{aligned} \tag{42}$$

for $(s, \hat{t}) \in [0, 1] \times [0, \hat{T}]$. In order to be consistent with the reservoir model, we hereafter neglect the gravity term and write the model on the following form (skipping the "hat" notation)

$$\begin{aligned} \partial_t(\rho_w) + \partial_s(\overline{a}\rho_w u) &= \overline{b} q_{\mathrm{M}}, \qquad \overline{a} = \frac{D}{L_w}, \quad \overline{b} = \frac{D}{A} \\ \partial_t(\rho_w u) + \partial_s(\overline{a}\rho_w u^2) + \overline{a}\partial_s p_w &= -S\overline{b}\frac{1}{2} f \rho_w u|u|, \end{aligned} \tag{43}$$

where $(s,t) \in [0,1] \times [0,T]$. Note that A represents the well cross-sectional area $A = \pi r_w^2$ where the well radius r_w is related to the non-dimensional well radius r by $r_w = Lr$. Next, we introduce non-dimensional variables as follows:

$$\hat{u} = \frac{u}{\bar{u}}, \qquad \hat{p}_w = \frac{p_w}{\bar{p}}, \qquad \hat{\rho}_w = \frac{\rho_w}{\bar{\rho}}, \qquad \hat{q}_{\mathrm{M}} = \frac{q_{\mathrm{M}}}{\bar{q}_{\mathrm{M}}} \tag{44}$$

where $\bar{p}$ is the characteristic pressure introduced in (30) and $\bar{\rho}$ and $\bar{q}_{\mathrm{M}}$ are the characteristic density and mass flow rate used in (20). The characteristic fluid velocity $\bar{u}$ is chosen to be

$$\bar{u} = \frac{1}{\bar{a}} = \frac{L_w}{D}, \tag{45}$$

where D is a characteristic time. If the well model is coupled to a time-dependent reservoir model as described in Section 2.1 and 2.2, D is given by (19), i.e.,

$$D = \frac{L^2 c\mu\phi}{k}, \tag{46}$$

which is a characteristic length of the time period associated with the reservoir. If we are interested only in a steady response from the reservoir, i.e. we consider the model (36), we may choose D as a characteristic time period associated with the well flow dynamic. In terms of the non-dimensional variables (44), (45), and (46), the model (43) takes the form

$$\begin{aligned} \partial_t(\hat{\rho}_w) + \partial_s(\hat{\rho}_w \hat{u}) &= \frac{1}{\nu}\hat{q}_{\mathrm{M}}, \qquad \nu = \frac{\text{total well volume}}{\text{total pore volume}} = \frac{L_w A}{L^3 \phi} \\ \partial_t(\hat{\rho}_w \hat{u}) + \partial_s(\hat{\rho}_w \hat{u}^2) + h_0 \partial_s \hat{p}_w &= -\frac{L_w}{r_w} f \hat{\rho}_w \hat{u}|\hat{u}|, \qquad h_0 = \frac{\bar{p}}{\bar{\rho}\bar{u}^2}. \end{aligned} \tag{47}$$

Remark 2.2. A more natural non-dimensional form of the well model when we are interested in the well-reservoir process under the whole lifespan of the reservoir, i.e., a typical production scenario is to replace the characteristic fluid velocity (45) with the following one

$$\tilde{u} = \frac{1}{\nu}\bar{u}. \tag{48}$$

In terms of the corresponding non-dimensional variables, the model (43) now takes the form

$$\begin{aligned} \nu\partial_t(\hat{\rho}_w) + \partial_s(\hat{\rho}_w \hat{u}) &= \hat{q}_{\mathrm{M}}, \qquad \nu = \frac{\text{total well volume}}{\text{total pore volume}} = \frac{L_w A}{L^3 \phi} \\ \nu\partial_t(\hat{\rho}_w \hat{u}) + \partial_s(\hat{\rho}_w \hat{u}^2) + h_0 \partial_s \hat{p}_w &= -\frac{L_w}{r_w} f \hat{\rho}_w \hat{u}|\hat{u}|, \qquad h_0 = \frac{\bar{p}}{\bar{\rho}\tilde{u}^2}. \end{aligned} \tag{49}$$

In this light it is a reasonable assumption to neglect the temporal terms of the well model for coupled well-reservoir modeling where focus is on reservoir transients and not the well transients, see for example [25]–[27], [33]–[35].

Incompressible fluid. We assume that the fluid is incompressible, i.e. ρ_w is constant. In view of (43) we then obtain the following equations

$$\begin{aligned} \partial_s(\bar{a}u) &= \bar{b}q_{\mathrm{V}}, \qquad \bar{a} = \frac{D}{L_w}, \quad \bar{b} = \frac{D}{A} \\ \partial_t(u) + \partial_s(\bar{a}u^2) + \frac{\bar{a}}{\rho_w}\partial_s p_w &= -S\bar{b}\frac{1}{2} f u|u|, \end{aligned} \tag{50}$$

In addition to the nondimensional volumetric flow rate $\hat{q}_{\rm V}$ given by (31), we introduce a nondimensional fluid velocity $\hat{u}$ and pressure $\hat{p}_w$ given by (44) and (45), where D is a characteristic time for the well flow dynamic which must be specified, e.g. by (46). In terms of non-dimensional variables the model (50) takes the form

$$\begin{aligned} \partial_s(\hat{u}) = \frac{1}{k_0}\hat{q}_{\rm V}, \qquad \frac{1}{k_0} = \frac{\bar{q}_{\rm V} L_w}{\bar{u}A} = \frac{Lk\bar{p}D}{L_w \mu A}, \\ \partial_t(\hat{u}) + \partial_s(\hat{u}^2) + h_0\partial_s\hat{p}_w = -S\bar{b}\frac{\bar{u}}{2}f\hat{u}|\hat{u}| = -\frac{L_w f}{r_w}\hat{u}|\hat{u}|, \qquad h_0 = \frac{\bar{p}}{\rho_w \bar{u}^2}. \end{aligned} \tag{51}$$

2.4. **Coupled Well-Reservoir flow: Compressible fluid.** The plan is now to follow along the same line as [25]–[27], [33]–[35] in order to obtain *coupled* well-reservoir models. In view of the density and pressure-based reservoir models (26) and (34), it seem convenient to formulate corresponding density and pressure-based coupled models.

Variant I. Let $\rho_w(s,t)$ be the fluid density associated with the well flow model (47) whereas $\rho(\mathbf{X}_w(s) + \mathbf{r}_w, t)$ is the fluid density described by the density-based reservoir model (26) along the well path. If we assume that the fluid is entering or leaving the wellbore through the porous pipe wall such as in open-hole horizontal well situations, then it is reasonable that $\rho_w(s,t)$ and $\rho(\mathbf{X}_w(s) + \mathbf{r}_w, t)$ are linked through the relation

$$\rho_w(s,t) = \rho(\mathbf{X}_w(s) + \mathbf{r}_w, t) \stackrel{\rm def}{:=} \rho(s,t). \tag{52}$$

This results in the following coupled well-reservoir model

$$\begin{aligned} \partial_t(\rho) + \partial_s(\rho u) = \frac{1}{\nu}q_{\rm M}, \qquad \nu = \frac{L_w A}{L^3\phi} \\ \partial_t(\rho u) + \partial_s(\rho u^2) + \partial_s P(\rho) = q_{\rm F}, \qquad P(\rho) = h_0 p_w(\rho), \qquad h_0 = \frac{\bar{p}}{\bar{\rho}\bar{u}^2}, \\ \rho_0 - \rho(s,t) = \int_0^t \int_0^1 G(\mathbf{X}_w(s) + \mathbf{r}_w, \mathbf{X}_w(s'), t-t')q_{\rm M}(s',t')\, ds'dt', \end{aligned} \tag{53}$$

with $q_{\rm F} = -\frac{L_w}{r_w}f\rho u|u|$ and where we have assumed that initial density ρ_0 is a constant. In this model, the density $\rho = \rho(P, q_{\rm M})$ is pointwise (locally) related to the pressure P, whereas it is related to the mass rate $q_{\rm M}$ in a non-local manner (via a functional).

Variant II. A closely related well-reservoir model is obtained by coupling the well model (47) with the pressure-based reservoir model (34) using the assumption

$$p_w(\rho(s,t)) = p(\mathbf{X}_w(s) + \mathbf{r}_w, t) \stackrel{\rm def}{:=} p(s,t). \tag{54}$$

Noting that (21) and (31) gives us $\bar{q}_{\rm M} = \frac{\bar{\rho}}{\bar{p}c}\bar{q}_{\rm V}$, we get a model on the form

$$\begin{aligned} \partial_t(\rho) + \partial_s(\rho u) = \frac{1}{\eta}\rho q_{\rm V}, \qquad \eta = \frac{\nu}{\bar{p}c}, \\ \partial_t(\rho u) + \partial_s(\rho u^2) + \partial_s P(\rho) = q_{\rm F}, \qquad P(\rho) = h_0 p_w(\rho), \qquad h_0 = \frac{\bar{p}}{\bar{\rho}\bar{u}^2}, \\ P_0 - P(s,t) = h_0\int_0^t \int_0^1 G(\mathbf{X}_w(s) + \mathbf{r}_w, \mathbf{X}_w(s'), t-t')q_{\rm V}(s',t')\, ds'dt', \end{aligned} \tag{55}$$

with $q_{\rm F} = -\frac{L_w}{r_w}f\rho u|u|$ and where we have assumed that initial pressure p_0 is a constant. In this formulation the pressure $P = P(\rho, q_{\rm V})$ is related to the density

ρ in a local manner whereas its relation to the volumetric rate q_{V} is non-local (functional dependence). We note that this model corresponds to the model problem (1) presented in Section 1.

2.5. **A simplified "compressible well-incompressible reservoir" model.** In order to explore some aspects of the well-reservoir model (55), we here propose a simplified variant by neglecting the transient response from the reservoir. In other words, we treat the reservoir fluid as an incompressible fluid. In view of (37) and (39) we obtain a well-reservoir model on the form

$$\begin{aligned}
&\partial_t(\rho) + \partial_s(\rho u) = \frac{1}{\eta}\rho q_{\mathrm{V}}, \qquad \eta = \frac{\nu}{\bar{p}c}, \qquad \nu = \frac{L_w A}{L^3\phi},\\
&\partial_t(\rho u) + \partial_s(\rho u^2) + \partial_s P(\rho) = q_{\mathrm{F}}, \qquad P(\rho) = h_0 p_w(\rho), \quad h_0 = \frac{\bar{p}}{\bar{\rho}\bar{u}^2},\\
&P_0 - P(s,t) = \int_0^1 H^r(s,s') q_{\mathrm{V}}(s',t)\,ds',
\end{aligned} \tag{56}$$

where $q_{\mathrm{F}} = -\frac{L_w}{r_w} f\rho u|u|$ and

$$H^r(s,s') = h_0 G(\mathbf{X}_w(s) + \mathbf{r}, \mathbf{X}_w(s')), \qquad G(\mathbf{x},\mathbf{x}') = \frac{1}{4\pi\|\mathbf{x}-\mathbf{x}'\|}. \tag{57}$$

Thus, (55) becomes a hyperbolic-elliptic type of model instead of a hyperbolic-parabolic. Next, we may seek more insight into characteristic properties of this model by specifying a well geometry. For that purpose we assume that the well-line is described by a straight line placed in the center of the unit box and given by

$$\begin{aligned}
&\mathbf{X}_w(s') = \Big([1-s']a + s'b, \frac{1}{2}, \frac{1}{2}\Big), \qquad \mathbf{X}_w(s) + \mathbf{r} = \Big([1-s]a + sb, \frac{1}{2} + r, \frac{1}{2}\Big),\\
&\text{with} \qquad \mathbf{r} = (0, r, 0), \quad r = \frac{r_w}{L}, \quad b - a = \frac{L_w}{L},
\end{aligned} \tag{58}$$

for $s, s' \in [0,1]$ and constants $a < b$ in $(0,1)$ where r is the dimensionless well radius and the dimensionless length of the well is $b - a = L_w/L$. It is convenient to introduce the dimensionless radius $\bar{r}$ defined by

$$\bar{r} = \frac{r_w}{L_w}, \tag{59}$$

which implies that $r = \bar{r}\frac{L_w}{L} = \bar{r}(b-a)$. We then arrive at the following expression for the kernel $H^r(s,s')$ in (57).

$$\begin{aligned}
H^r(s,s') &= h_0 G(\mathbf{X}_w(s) + \mathbf{r}, \mathbf{X}_w(s'))\\
&= h_0 \frac{1}{4\pi\sqrt{(b-a)^2(s-s')^2 + (b-a)^2\bar{r}^2}}.\\
&= \frac{Lh_0}{L_w}\frac{1}{4\pi\sqrt{(s-s')^2 + \bar{r}^2}} = \varepsilon_1 \frac{1}{\sqrt{\big(\frac{s-s'}{\bar{r}}\big)^2 + 1}}, \qquad \varepsilon_1 = \frac{h_0 L}{4\pi\bar{r}L_w}.
\end{aligned} \tag{60}$$

In order to get a better understanding of the interaction between well and reservoir present in the model (56) we may consider the following approximation of the kernel function H^r:

$$H^r(s) = \varepsilon_1 \frac{1}{\sqrt{(s/\bar{r})^2 + 1}} \approx \varepsilon_1 \frac{1}{\sqrt{\exp(\frac{2\alpha}{\bar{r}}|s|)}} = \varepsilon_1 e^{-\frac{\alpha}{\bar{r}}|s|} = \varepsilon_1 K^{r,\alpha}(s), \tag{61}$$

for some choice of $\alpha > 0$ that might depend on $\bar{r}$. This corresponds to the approximation

$$h^{\bar{r},\alpha}(x) = \frac{1}{\sqrt{\exp(\alpha|(x/\bar{r})|)}} \approx \frac{1}{\sqrt{1+(x/\bar{r})^2}} = g^{\bar{r}}(x), \qquad x \in (-\delta, +\delta),$$

for some $\delta > 0$. For a case with $\bar{r} = 0.001$, and $\alpha = 0.2, 0.5$ and $\alpha = 1.0$, see Fig. 1 for a comparison of these two functions. Note that the role of the parameter α is to determine to what extent the convolution has a local effect or a more global effect. "Small" values for α implies that the kernel $h^{\bar{r},\alpha}(x)$ is centered around a larger interval of zero, see Fig. 1. "Large" values for α implies that $h^{\bar{r},\alpha}(x)$ is centered around a smaller interval of zero, i.e., the convolution operator is more localized. Regarding the approximation (61) we note that, from the point of view of applications, we may argue that there is naturally room for various choices for the kernel function since this represents the unknown reservoir. In fact, we are satisfied with a kernel that are able to represent *some* characteristic information about the reservoir which surrounds the well.

Next, we observe that $K^{r,\alpha}(s,s')$ satisfy the equation,

$$\lambda^2 K^{r,\alpha} - K^{r,\alpha}_{ss} = 2\lambda\delta(s-s')e^{-\lambda|s-s'|}, \qquad \lambda = \frac{\alpha}{\bar{r}}.$$

Observing from (56), where we now make use of the approximation (61), that

$$P_0 - P = H^r * q_{\mathrm{V}} \approx \varepsilon_1 K^{r,\alpha} * q_{\mathrm{V}}, \qquad \varepsilon_1 = \frac{h_0 L}{4\pi \bar{r} L_w}$$

it follows that

$$\begin{aligned} \lambda^2(P_0 - P) + P_{ss} &= \varepsilon_1(\lambda^2 K^{r,\alpha} - K^{r,\alpha}_{ss}) * q_{\mathrm{V}} \\ &= 2\lambda\varepsilon_1\delta(s-s')e^{-\lambda|s-s'|} * q_{\mathrm{V}} = 2\lambda\varepsilon_1 q_{\mathrm{V}}. \end{aligned} \tag{62}$$

That is,

$$q_{\mathrm{V}} = \frac{1}{2\varepsilon_1\lambda}\Big(\lambda^2(P_0 - P) + P_{ss}\Big) = \frac{1}{2\varepsilon_1\lambda}\Big(A - BP(\rho) + CP(\rho)_{ss}\Big).$$

Inserting this in the continuity equation of (56) we obtain a model on the form

$$\begin{aligned} \partial_t\rho + \partial_s(\rho u) &= \frac{1}{\varepsilon}\Big(A\rho - B\rho P(\rho) + \rho P(\rho)_{ss}\Big), \qquad \varepsilon = 2\varepsilon_1\lambda\eta, \\ \partial_t(\rho u) + \partial_s(\rho u^2) + \partial_s P(\rho) &= q_{\mathrm{F}}, \end{aligned} \tag{63}$$

where $A, B > 0$ are given by

$$A = \lambda^2 P_0, \qquad B = \lambda^2.$$

We note that this model corresponds to the model problem (2) mentioned in Section 1.

Remark 2.3. We may consider the above models (56) and (63) as approximative models that still are able to give some insight into characteristic behavior possessed by the original well-reservoir models (53) and (55). Hopefully, we should be able to demonstrate that the simplified models are able to capture one aspect or another of the more general ones. The simplified model may allow us to draw rigorous conclusions that explain rather satisfactorily some aspects of specific physical situations which may also be observed experimentally.

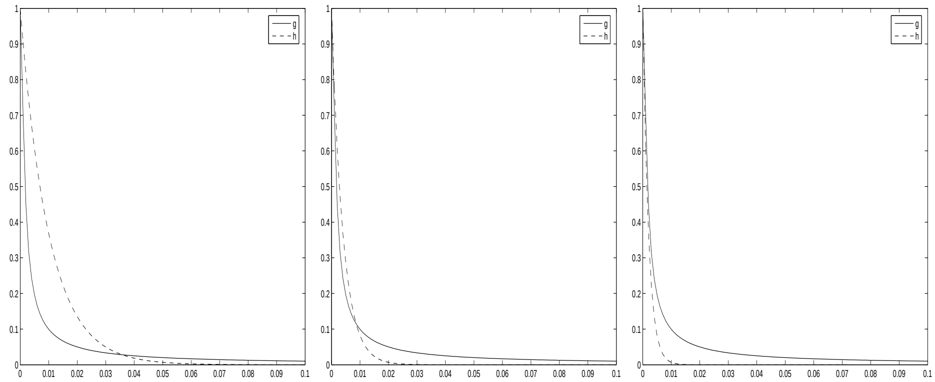

FIGURE 1. Plot of the functions g and h for various α with $r = 0.001$. Left: $\alpha = 0.2$. Middle: $\alpha = 0.5$. Right: $\alpha = 1.0$.

2.6. **Coupled Well-Reservoir flow: Incompressible fluid.** We take a step further and impose another simplification in model (56) by assuming that the well fluid is incompressible, i.e. $\bar{\rho} = \rho = \rho_w = \text{const}$. In other words, we replace the compressible well model in (56) by the incompressible well model (51) which yields the following simplified well-reservoir model

$$\begin{aligned} &\partial_s(u) = \frac{1}{k_0} q_{\mathrm{V}}, \qquad \frac{1}{k_0} = \frac{Lk\bar{p}D}{L_w \mu A}, \qquad A = \pi r_w^2 = \pi(\bar{r}L_w)^2, \\ &\partial_t(u) + \partial_s(u^2) + \partial_s P = q_{\mathrm{F}}, \qquad P = h_0 p, \quad h_0 = \frac{\bar{p}}{\bar{\rho}\bar{u}^2}, \quad q_{\mathrm{F}} = -\frac{1}{\bar{r}} f u|u|, \\ &P_0 - P(s,t) = \int_0^1 H^r(s,s') q_{\mathrm{V}}(s',t)\, ds'. \end{aligned} \tag{64}$$

In view of (60), we have that

$$H^r(s,s') = \frac{h_0 L}{4\pi \bar{r}^2 L_w} \frac{\bar{r}^2}{\sqrt{(s-s')^2 + \bar{r}^2}} = \frac{h_0 L}{4\pi \bar{r}^2 L_w} G^r(s,s'), \tag{65}$$

where the kernel G^r is defined as

$$G^r(s,s') = \frac{\bar{r}^2}{\sqrt{(s-s')^2 + \bar{r}^2}}, \qquad \bar{r} = \frac{r_w}{L_w}. \tag{66}$$

Inserting the first equation of (64) in the integral equation of (64), we get

$$\begin{aligned} P_0 - P(s,t) &= \int_0^1 H^r(s,s') q_{\mathrm{V}}(s',t)\, ds' \\ &= \frac{h_0 k_0 L}{4\pi \bar{r}^2 L_w} \int_0^1 G^r(s,s') u_{s'}\, ds' = \varepsilon \int_0^1 G^r(s,s') u_{s'}\, ds', \end{aligned}$$

where

$$\varepsilon = \frac{h_0 k_0 L}{4\pi \bar{r}^2 L_w} = \frac{\mu D}{4\bar{p}k}. \tag{67}$$

Thus, the model (64) is equivalently written on the form

$$\begin{aligned} &\partial_t u + \partial_s(u^2) = -\partial_s P + q_{\mathrm{F}}, \\ &P_0 - P(s,t) = \varepsilon \int_0^1 G^r(s,s') u_{s'}(s',t)\, ds' = \varepsilon G^r * u_s, \end{aligned} \tag{68}$$

where ε and G^r are given, respectively, by (67) and (66). This model corresponds to the model problem (3)–(5), presented in Section 1 but where we now, for simplicity, have replaced the finite domain $[0,1]$ by the real axis as well as neglected the friction term q_{F}.

Remark 2.4. The well-reservoir interaction is clearly reflected through the model (68) which involves a balance between a local convective force and a non-local diffusive force. By letting the permeability go to zero (i.e., the flow between well and reservoir must also go to zero) we see from (67) that ε becomes large. Consequently, an initial disturbance in the fluid velocity, e.g. a Gaussian pulse, is quickly damped to zero due to a strong (non-local) diffusive force, see Section 6. On the other hand, by letting k becomes large, the fluid is allowed to flow with low resistance between well and reservoir. For this case, ε becomes small and the convective force becomes the more dominating one.

Remark 2.5. By making use of the approximation (62) we see that the model (68) (without friction term) takes the form

$$\begin{aligned} &\partial_t u + \partial_s(u^2) = -\partial_s P, \\ &\lambda^2(P_0 - P) + P_{ss} = s_0 u_s, \qquad \lambda = \frac{\alpha}{\bar{r}}, \qquad s_0 = 2\lambda\varepsilon_1 k_0 = 2\lambda\varepsilon\bar{r} = 2\alpha\varepsilon. \end{aligned} \tag{69}$$

From the first equation of (69) we formally obtain the following two equations:

$$\begin{aligned} &\lambda^2 u_t + \lambda^2 (u^2)_s + \lambda^2 P_s = 0 \\ &-u_{tss} - (u^2)_{sss} - P_{sss} = 0. \end{aligned} \tag{70}$$

From the second equation of (69) we also obtain the equation

$$-P_{sss} + \lambda^2 P_s = -s_0 u_{ss}. \tag{71}$$

Summing the two equations in (70) and using (71), we arrive at the equation

$$u_t + (u^2)_s - c_0 u_{tss} - c_0 (u^2)_{sss} - s_1 u_{ss} = 0, \qquad c_0 = \frac{1}{\lambda^2}, \qquad s_1 = c_0 s_0 = \frac{s_0}{\lambda^2}. \tag{72}$$

We may write it on the form

$$u_t + (u^2)_s - c_0 u_{tss} - 2c_0(u_s^2 + u u_{ss})_s = s_1 u_{ss}, \tag{73}$$

or the form

$$u_t + (u^2)_s - c_0 u_{tss} = 6c_0 u_s u_{ss} + 2c_0 u u_{sss} + c_0 s_0 u_{ss}. \tag{74}$$

Remark 2.6. We note that by letting the compressibility c go to zero in the equation of state (10) such that $\rho \to \rho_0 = \text{constant}$, then the model (63) formally is reduced to the incompressible model (69), alternatively (72). It would be interesting to explore this limit in a rigorous mathematical sense.

In the remaining part of this paper we focus exclusively on the model problem (4)–(6). We are interested in general existence and uniqueness results that apply for our model problem, which might be considered as a simplest possible approximation to the more general well-reservoir model (1). In the next section we first present some motivation for the solution concept to be used, together with a local existence result. Global existence results are then presented in Section 4 and 5.

3. **Preliminaries.**

In Section 3.1 we relate our model problem to other non-local conservation laws. This section also serves as motivation for the solution concept introduced in Section 3.2. Finally, in Section 3.3. we also include a local existence result.

3.1. **Relation to some other models.**

As a first approach, it is instructive to compare our model problem (4)–(6) with similar non-local conservation laws already explored in the literature, however, within different contexts. Here we will mention two of them to which it seems particularly relevant to relate our model equation.

Fellner and Schmeiser [10] studied a Burgers-Poisson type of model on the form

$$\partial_t u + \partial_x(\frac{1}{2}u^2) = -\partial_x p, \qquad p(x,t) = \int_{-\infty}^{+\infty} H(x,x')u(x',t)\,dx' = H * u, \tag{75}$$

with

$$H(x,x') = \frac{1}{2}e^{-|x-x'|}. \tag{76}$$

Alternatively, we may write (75) on the form

$$\partial_t u + \partial_x(\frac{1}{2}u^2) = -H * u_x = -H_x * u. \tag{77}$$

Due to the fact that the kernel $H(x,x')$ corresponds to the operator $(1-\partial_{xx}^2)$, (77) can be written on the form

$$u_t + uu_x = -p_x, \qquad -p_{xx} + p = u. \tag{78}$$

Another model which has attracted much attention more lately is the so-called radiating gas model [28, 12, 14, 15, 16, 24, 21, 18, 30, 17, 19]. This model is obtained by replacing $p = H * u$ by $p = -H * u_x$ in (75). That is, we get the equation

$$\partial_t u + \partial_x(\frac{1}{2}u^2) = -\partial_x p, \quad p(x,t) = -\int_{-\infty}^{+\infty} H(x,x')u_{x'}(x',t)\,dx' = -H * u_x. \tag{79}$$

As before, we may write (79) on the form

$$\partial_t u + \partial_x(\frac{1}{2}u^2) = H_x * u_x = H_{xx} * u = [H-\delta] * u = H * u - u, \tag{80}$$

where δ represents the Dirac delta function. Again, since the convolution kernel $H(x,x')$ corresponds to the operator $(1-\partial_{xx}^2)$, (80) can be written on the form

$$u_t + uu_x = -p_x, \qquad -p_{xx} + p = -u_x. \tag{81}$$

It is instructive to observe that the three models (6), (77), and (80) can all be written on the form

$$u_t + f(u)_x = L_i u = G_i * u_x, \qquad i = 1,2,3, \tag{82}$$

where $G_i(x,x')$ corresponds to the following different choices

$$\begin{aligned} G_1(x,x') &= \varepsilon G_x^r(x,x') && \text{(well reservoir)}, \\ G_2(x,x') &= -H(x,x') && \text{(Burgers Poisson)}, \\ G_3(x,x') &= H_x(x,x') && \text{(radiating gas)}. \end{aligned} \tag{83}$$

The plots in Fig. 2 (compare left and right plot) show that the kernels corresponding to the well-reservoir model and the radiating gas model, respectively, bear strong

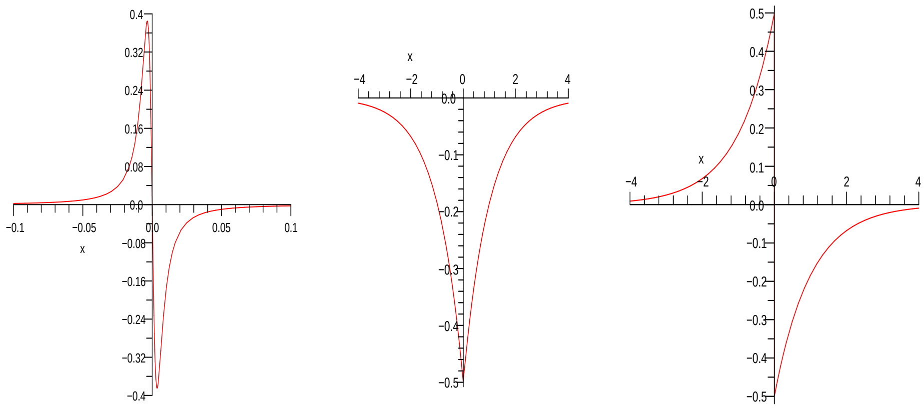

FIGURE 2. Plot of kernels corresponding to **Left:** well-reservoir model $G_1(x)$; **Middle:** Burger Poisson $G_2(x)$, **Right:** radiating gas $G_3(x)$.

similarities. Thus, we may expect to see (at least numerically) that the two models possess similar properties. However, as far as mathematical results are concerned we should bear in mind that the the radiating gas model possesses a particularly nice structure since the right hand side also can be written on the form

$$L_3 u = G_3 * u_x = H_{xx} * u = H * u - u, \qquad H \geq 0, \qquad \int_{\mathbb{R}} H(x)\,dx = 1, \tag{84}$$

where the L_3 operator now can be shown to be a L^1-contractive operator. This feature strongly hang on the special form of the right hand side given by (84).

Remark 3.1. One important difference between the models (6), (77), and (80) is that the two last ones can be written as hyperbolic-elliptic coupled systems, corresponding to (78) and (81), which involve no convolution operator. In general, we cannot expect the kernel G^r involved in (6) to correspond to a differential operator. The reformulations (78) and (81) are, for instance, explicitly used in travelling wave analysis, see [10, 15, 16, 24].

Remark 3.2. A common feature of the above three models (6), (77), and (80), written on the form (82) and (83), is that the right hand side can be written on the form $G_i * u_x = G_{i,x} * u$. This contrasts other nonlinear dispersive models like the Camassa-Holm and Degasperis-Proces models which involve nonlinear terms respectively on the form $H * (\frac{3}{2}u^2 + \frac{1}{2}(u_x)^2)$ and $H * (\frac{3}{2}u^2)$, where H is given by (76). This makes it considerably more delicate to obtain a priori estimates for these models, see for example [2, 3].

3.2. **Solution concept.** In [21] it is shown that for the radiating gas model (80) there are initial data such that the corresponding solution to the Cauchy problems develop discontinuities in finite time. Similarly, for the Burgers-Poisson equation (77) numerical results indicate that the model features wave breaking in finite time [10]. In view of the similarity between (77), (80), and (6), we may expect that the non-local diffusion term $L_1 u = \varepsilon G^r_{xx} * u$ appearing in (6) in general cannot prevent shock formation. Numerical simulations in Section 6 also indicate that one must expect loss of regularity. Thus, it is reasonable to use weak solution concepts similar to those that has been used for models (77) and (80).

Definition 3.1. (Weak solution) We call a function $u : (0,T) \times \mathbb{R} \to \mathbb{R}$ a weak solution of (6) provided

i) $u \in L^\infty\left((0,T); L^2(\mathbb{R})\right)$, and

ii) $\partial_t u + \partial_x(u^2) + \partial_x p =$ in $\mathcal{D}'((0,T)\times\mathbb{R})$, that is, $\forall \phi \in C_0^\infty([0,T)\times\mathbb{R})$ there holds the equation

$$\int_0^T \int_{\mathbb{R}} \Big(u\partial_t\phi + u^2\partial_x\phi - \partial_x p\phi \Big)\, dx\, dt + \int_{\mathbb{R}} u_0(x)\phi(x,0) = 0, \tag{85}$$

where

$$p_0 - p(x,t) = \varepsilon G_x^r * u = \varepsilon \int_{\mathbb{R}} G_x^r(x,x')u(x',t)\, dx'.$$

Definition 3.2. (Entropy weak solution) We call a function $u : (0,T)\times\mathbb{R}\to\mathbb{R}$ an entropy weak solution of (6) provided

i) $u \in L^\infty((0,T)\times\mathbb{R}) \cap C\left([0,T]; L^1(\mathbb{R})\right)$ for any $T > 0$, and

ii) for any convex C^2 entropy $\eta : \mathbb{R}\to\mathbb{R}$ with corresponding entropy flux $q : \mathbb{R} \to \mathbb{R}$ defined by $q'(u) = 2u\eta'(u)$ there holds the inequality

$$\partial_t \eta(u) + \partial_x q(u) + \eta'(u)\partial_x p \le 0, \quad \text{in } \mathcal{D}'((0,T)\times\mathbb{R}),$$

that is, $\forall\phi \in C_0^\infty([0,T)\times\mathbb{R})$, $\phi \ge 0$, there holds the inequality

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} \Big(\eta(u)\partial_t\phi + q(u)\partial_x\phi - \eta'(u)\partial_x p\phi \Big)\, dx\, dt + \int_{\mathbb{R}} \eta(u_0(x))\phi(x,0)\, dx \ge 0, \tag{86}$$

where

$$p_0 - p(x,t) = \varepsilon G_x^r * u = \varepsilon \int_{\mathbb{R}} G_x^r(x,x')u(x',t)\, dx'.$$

In the next section we shall repeatedly apply the following well known result.

Lemma 3.1 (Young's inequality)**.** *Suppose* $1 \le p, q \le \infty$ *and* $1/r = 1/p + 1/q - 1 \ge 0$. *If* $f \in L^p$ *and* $g \in L^q$, *then* $f * g \in L^r$ *and*

$$\|f * g\|_r \le \|f\|_p \|g\|_q.$$

Note that for the special case $r = p$ *and* $q = 1$ *we get*

$$\|f * g\|_p \le \|f\|_p \|g\|_1.$$

We also note that we have the following relations relevant for the kernel G^r:

$$G_x^r = \frac{-r^2[x-x']}{\Big([x-x']^2 + r^2\Big)^{3/2}}, \qquad G_{xx}^r = \frac{r^2\Big(\sqrt{2}[x-x'] - r\Big)\Big(\sqrt{2}[x-x'] + r\Big)}{\Big([x-x']^2 + r^2\Big)^{5/2}}. \tag{87}$$

Particularly, we observe that

$$\begin{aligned} &\int_{\mathbb{R}} G_{xx}^r\, dx = 0, \\ &\|G_x^r\|_{L^1(\mathbb{R})} = 2\int_{-\infty}^0 G_x^r\, dx = 2r, \\ &\|G_{xx}^r\|_{L^1(\mathbb{R})} = -4\int_0^{r/\sqrt{2}} G_{xx}^r\, dx = \frac{8}{3\sqrt{3}} \le 2. \end{aligned} \tag{88}$$

Moreover,

$$\|G_x^r\|_{L^\infty(\mathbb{R})} = G_x^r(r/\sqrt{2}) = \frac{2}{3\sqrt{3}}, \qquad \|G_{xx}^r\|_{L^\infty(\mathbb{R})} = |G_{xx}^r(0)| = \frac{1}{r}. \tag{89}$$

3.3. **A local existence result.** Along the line of [10] we can obtain the following local existence result for the model problem (6).

Theorem 3.1 (Local strong solution)**.** *Assume $u_0 \in H^k(\mathbb{R})$ with $k > \frac{3}{2}$. Then, there exists a time $T > 0$ and a unique solution*

$$u \in L^\infty\left((0,T); H^k(\mathbb{R})\right) \cap C\left([0,T]; H^{k-1}(\mathbb{R})\right) \stackrel{def}{:=} X,$$

of (6)*.*

Proof. For completeness we include the proof of this theorem. We first define a map S_T as follows: for any function $v \in B_T$, with

$$B_T := \{w \in X : \sup_{t\in[0,T]} \|w(\cdot,t)\|_{H^k(\mathbb{R})} \le 2\|u_0\|_{H^k(\mathbb{R})}\},$$

let the image $S_T(v)$ be the unique solution u of

$$\partial_t u + \partial_x(u^2) = \varepsilon G_{xx}^r * v, \qquad \varepsilon, r > 0, \qquad u(x,0) = u_0(x). \tag{90}$$

Step 1. We must show that S_T is a mapping $B_T \to B_T$ for some choice of $T > 0$. We take the derivative ∂_x^α for $\alpha \le k$ to (90) which yields

$$(\partial_x^\alpha u)_t + \partial_x^\alpha(2uu_x) = \varepsilon G_{xx}^r * \partial_x^\alpha v.$$

Then we multiply with $\partial_x^\alpha u$ and integrate in space and obtain

$$\frac{1}{2}\left(\int_{\mathbb{R}} [\partial_x^\alpha u]^2\,dx\right)_t + \int_{\mathbb{R}} \partial_x^\alpha(2uu_x)\partial_x^\alpha u\,dx = \varepsilon\int_{\mathbb{R}} [G_{xx}^r * \partial_x^\alpha v]\partial_x^\alpha u\,dx. \tag{91}$$

The second term on the left hand side is treated as follows. First, we see that the product rule gives

$$\partial_x^\alpha(2uu_x) = 2u\partial_x^{\alpha+1}u + 2\sum_{l=1}^{\alpha}\binom{\alpha}{l}\partial_x^l u\partial_x^{\alpha+1-l}u,$$

so we have to deal with a term on the form

$$\int_{\mathbb{R}} \partial_x^\alpha(2uu_x)\partial_x^\alpha u\,dx = 2\int_{\mathbb{R}} u\partial_x^{\alpha+1}u\partial_x^\alpha u\,dx + 2\int_{\mathbb{R}}\sum_{l=1}^{\alpha}\binom{\alpha}{l}\partial_x^l u\partial_x^{\alpha+1-l}u\partial_x^\alpha u\,dx. \tag{92}$$

The first term on the right hand side of (92) is estimated as follows:

$$\begin{aligned}&\left|\int_{\mathbb{R}} u\partial_x^{\alpha+1}u\cdot\partial_x^\alpha u\,dx\right| \\ &= \frac{1}{2}\left|\int_{\mathbb{R}} u\partial_x\left([\partial_x^\alpha u]^2\right)\,dx\right| \le \frac{1}{2}\|u_x\|_{L^\infty}\|u\|_{H^\alpha}^2 \le \frac{1}{2}\|u_x\|_{L^\infty}\|u\|_{H^k}^2.\end{aligned} \tag{93}$$

The second term on the right hand side of (92) is estimated as follows:

$$\begin{aligned}
&\left|\int_{\mathbb{R}} \sum_{l=1}^{\alpha} \binom{\alpha}{l} \partial_x^l u \partial_x^{\alpha+1-l} u \cdot \partial_x^\alpha u \, dx\right| \\
&\leq C \sum_{l=1}^{\alpha} \int_{\mathbb{R}} \left|\partial_x^l u \partial_x^{\alpha+1-l} u \cdot \partial_x^\alpha u\right| \, dx \leq C \sum_{l=1}^{\alpha} \|\partial_x^l u \partial_x^{\alpha+1-l} u\|_{L^2} \|\partial_x^\alpha u\|_{L^2} \\
&\leq CD \sum_{l=1}^{\alpha} \Big(\|u_x\|_{L^\infty} \|u\|_{H^\alpha} + \|u\|_{H^\alpha} \|u_x\|_{L^\infty} \Big) \|\partial_x^\alpha u\|_{L^2} \\
&\leq 2\alpha CD \Big(\|u\|_{H^\alpha} \|u_x\|_{L^\infty} \Big) \|u\|_{H^\alpha} \leq 2\alpha CD \|u\|_{H^\alpha}^2 \|u_x\|_{L^\infty},
\end{aligned} \tag{94}$$

where we have applied the following interpolation estimate [10]

$$\|(\partial_x^{l-1} f_x)(\partial_x^{\alpha-l} g_x)\|_{L^2} \leq D \left(\|f_x\|_{L^\infty} \|g\|_{H^\alpha} + \|f\|_{H^\alpha} \|g_x\|_{L^\infty} \right). \tag{95}$$

Consequently, in view of (93) and (94), we get

$$\left| \int_{\mathbb{R}} \partial_x^\alpha (2uu_x) \cdot \partial_x^\alpha u \, dx \right| \leq E \|u\|_{H^k}^2 \|u_x\|_{L^\infty} \leq E' \|u\|_{H^k}^3, \tag{96}$$

by using the Sobolev imbedding result $W^{1,\infty}(\mathbb{R}) \hookrightarrow H^k(\mathbb{R})$ for $k > 3/2$.

For the right hand side of (91) we get

$$\left| \int_{\mathbb{R}} [G_{xx}^r * \partial_x^\alpha v] \partial_x^\alpha u \, dx \right| \leq \|G_{xx}^r * \partial_x^\alpha v\|_{L^2} \|\partial_x^\alpha u\|_{L^2} \leq \|G_{xx}^r\|_{L^1} \|v\|_{H^k} \|u\|_{H^k}. \tag{97}$$

Thus, in view of (91), (96), and (97), we get

$$\|u\|_{H^k} \frac{d}{dt} \|u\|_{H^k} \leq c \|u\|_{H^k} (\|v\|_{H^k} + \|u\|_{H^k}^2),$$

or

$$\frac{d}{dt} \|u\|_{H^k} \leq c(\|v\|_{H^k} + \|u\|_{H^k}^2).$$

For T small enough, a comparison principle shows that $\|u(\cdot,t)\|_{H^k} \leq 2\|u_0\|_{H^k}$ for $t \in [0,T]$. Since $u \in C\left([0,T]; H^{k-1}(\mathbb{R})\right)$ we may conclude that $S_T : B_T \to B_T$.

Step 2. We shall show that S_T is a contraction with respect to the topology in $C\left([0,T]; H^{k-1}(\mathbb{R})\right)$ in the sense that

$$\|S_T(v_1) - S_T(v_2)\|_{H^{k-1}(\mathbb{R})} < \|v_1 - v_2\|_{H^{k-1}(\mathbb{R})}$$

for two elements v_1, v_2 in B_T. Setting $u_i = S_T(v_i)$ for $i = 1, 2$ and $u = u_1 - u_2$ and $v = v_1 - v_2$, we get an equation for the difference u on the form

$$\partial_t u + 2u\partial_x u_1 + 2u_2 \partial_x u = \varepsilon G_{xx}^r * v, \qquad u(x, t=0) = 0.$$

We proceed as in the step above and apply the operator ∂_x^α and then take the L^2-scalar product with $\partial_x^\alpha u$:

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \left(\int_{\mathbb{R}} [\partial_x^\alpha u]^2 \, dx \right) + \int_{\mathbb{R}} \partial_x^\alpha (2uu_{1,x}) \partial_x^\alpha u \, dx + \int_{\mathbb{R}} \partial_x^\alpha (2u_2 u_x) \partial_x^\alpha u \, dx \\
= \varepsilon \int_{\mathbb{R}} [G_{xx}^r * \partial_x^\alpha v] \partial_x^\alpha u \, dx.
\end{aligned} \tag{98}$$

Now we must deal with the following term:

$$\int_{\mathbb{R}} \partial_x^\alpha (2uu_{1,x}) \partial_x^\alpha u \, dx = 2\int_{\mathbb{R}} u \partial_x^{\alpha+1} u_1 \partial_x^\alpha u \, dx + 2\int_{\mathbb{R}} \sum_{l=1}^{\alpha} \binom{\alpha}{l} \partial_x^l u \partial_x^{\alpha+1-l} u_1 \partial_x^\alpha u \, dx. \tag{99}$$

The first term on the right hand side of (99) is estimated as follows:

$$\begin{aligned} &\left| \int_{\mathbb{R}} u \partial_x^{\alpha+1} u_1 \cdot \partial_x^\alpha u \, dx \right| \\ &\le \|u\|_{L^\infty} \|u_1\|_{H^{\alpha+1}} \|u\|_{H^\alpha} \le \|u\|_{L^\infty} \|u_1\|_{H^k} \|u\|_{H^{k-1}} \le \|u_1\|_{H^k} \|u\|_{H^{k-1}}^2, \end{aligned} \tag{100}$$

by choosing that $\alpha \le k-1$ and using the embedding $L^\infty(\mathbb{R}) \hookrightarrow H^{k-1}(\mathbb{R})$ for $k > 3/2$. In other words, at this point we are forced to reduce the order of differentiation by one. Moreover, the second term on the right hand side of (99) is estimated as follows:

$$\begin{aligned} &\left| \int_{\mathbb{R}} \sum_{l=1}^{\alpha} \binom{\alpha}{l} \partial_x^l u \partial_x^{\alpha+1-l} u_1 \cdot \partial_x^\alpha u \, dx \right| \\ &\le C \sum_{l=1}^{\alpha} \int_{\mathbb{R}} \left| \partial_x^l u \partial_x^{\alpha+1-l} u_1 \cdot \partial_x^\alpha u \right| \, dx \le C \sum_{l=1}^{\alpha} \|\partial_x^l u \partial_x^{\alpha+1-l} u_1\|_{L^2} \|\partial_x^\alpha u\|_{L^2} \\ &\le CD \sum_{l=1}^{\alpha} \Big(\|u\|_{L^\infty} \|u_1\|_{H^{\alpha+1}} + \|u\|_{H^\alpha} \|u_{1,x}\|_{L^\infty} \Big) \|\partial_x^\alpha u\|_{L^2} \\ &\le \alpha CD \Big(\|u\|_{L^\infty} \|u_1\|_{H^k} + \|u\|_{H^{k-1}} \|u_{1,x}\|_{L^\infty} \Big) \|u\|_{H^{k-1}} \\ &\le 2\alpha CD \|u\|_{H^{k-1}}^2 \|u_1\|_{H^k}, \end{aligned} \tag{101}$$

where we have applied the interpolation estimate (95) in the following way

$$\|(\partial_x^{(l+1)-1} u_{2,x})(\partial_x^{(\alpha+1)-(l+1)} u_{1,x})\|_{L^2} \le D \left(\|u_{2,x}\|_{L^\infty} \|u_1\|_{H^{\alpha+1}} + \|u_2\|_{H^{\alpha+1}} \|u_{1,x}\|_{L^\infty} \right),$$

with $u_{2,x} = u$. Consequently, in view of (100) and (101), we get

$$\left| \int_{\mathbb{R}} \partial_x^\alpha (2uu_{1,x}) \cdot \partial_x^\alpha u \, dx \right| \le E \|u_1\|_{H^k} \|u\|_{H^{k-1}}^2 \le 2E \|u_0\|_{H^k} \|u\|_{H^{k-1}}^2. \tag{102}$$

Similarly, we get

$$\left| \int_{\mathbb{R}} \partial_x^\alpha (2u_2 u_x) \cdot \partial_x^\alpha u \, dx \right| \le E \|u_2\|_{H^k} \|u\|_{H^{k-1}}^2 \le 2E \|u_0\|_{H^k} \|u\|_{H^{k-1}}^2. \tag{103}$$

The right hand side of (98) is estimated as in (97) and we get

$$\frac{d}{dt} \|u\|_{H^{k-1}} \le c(\|u\|_{H^{k-1}} + \|v\|_{H^{k-1}}),$$

and we conclude that

$$\|u\|_{H^{k-1}} < \|v\|_{H^{k-1}}$$

for sufficient small T, i.e., S_T is a strict contraction. □

4. Global existence theory in $L^1 \cap L^\infty$.

Theorem 4.1 (Well-posedness in $L^1 \cap L^\infty$)**.** *Assume that* (7) *holds. Then there exists an entropy weak solution to* (6) *in the sense of Definition 3.2. Moreover, for any (fixed) $T > 0$, let $u, v : (0,T) \times \mathbb{R} \to \mathbb{R}$ be two entropy weak solutions with initial data $u_0, v_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, respectively. Then for any $t \in (0,T)$*

$$\|u(\cdot,t) - v(\cdot,t)\|_{L^1(\mathbb{R})} \le K_T \|u_0 - v_0\|_{L^1(\mathbb{R})}, \tag{104}$$

with

$$K_T = (1 + 2\varepsilon T e^{2\varepsilon T}).$$

As a consequence, there is at most one entropy weak solution to (6)*. The entropy weak solution u satisfies the following estimates for any $t \in (0,T)$:*

$$\|u(\cdot,t)\|_{L^1(\mathbb{R})} \le C_1(T, \|u_0\|_{L^1(\mathbb{R})}) \tag{105}$$

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \le C_2(r, T, \|u_0\|_{L^\infty(\mathbb{R})}, \|u_0\|_{L^1(\mathbb{R})}). \tag{106}$$

If $u_0 \in BV(\mathbb{R})$, then u also satisfies

$$\|u(\cdot,t)\|_{BV(\mathbb{R})} \le C_3(T, \|u_0\|_{BV(\mathbb{R})}) \tag{107}$$

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \le C_3(T, \|u_0\|_{BV(\mathbb{R})}). \tag{108}$$

Furthermore, for all $t_1, t_2 \in [0,T]$,

$$\|u(\cdot,t_2) - u(\cdot,t_1)\|_{L^1(\mathbb{R})} \le C_4(T, \|u_0\|_{L^\infty}, \|u_0\|_{L^1}, \|u_0\|_{BV})|t_2 - t_1|. \tag{109}$$

Proof. In view of Theorem 4.2, the existence result and various estimates (105)–(109) hold for $u_0 \in BV \cap L^1 \cap L^\infty$ whereas the stability (uniqueness) result (104) holds for $u_0 \in L^1 \cap L^\infty$, due to Theorem 4.3. Next, for $u_0 \in L^\infty \cap L^1$ we can find a sequence u_0^k in BV such that $u_0^k \to u_0$ as $k \to \infty$. Then the L^1-stability result implies that the corresponding entropy weak solution sequence $u^k \in L^\infty((0,T) \times \mathbb{R}) \cap C([0,T]; L^1(\mathbb{R}))$ with initial data u_0^k is a Cauchy sequence relatively $L^1(\mathbb{R})$-norm which yields a subsequence converging to $u \in L^\infty((0,T) \times \mathbb{R}) \cap C([0,T]; L^1(\mathbb{R}))$. Clearly, u inherits the estimates (105) and (106) from u^k. □

For the existence results presented below we will follow the usual procedure and consider the following viscous approximation

$$\begin{aligned} &\partial_t u^\mu + \partial_x f(u^\mu) = \varepsilon G^r_{xx} * u^\mu + \mu u^\mu_{xx}, \quad \mu > 0, \quad f(u) = u^2, \\ &u^\mu(x,0) = u^\mu_0(x). \end{aligned} \tag{110}$$

4.1. Estimates.

In this section we derive a priori estimates. First, we want to bound u^μ in L^1. For that purpose, we need to make the assumptions that

$$u_0, u_0^\mu \in L^1(\mathbb{R}), \qquad \|u_0^\mu\|_{L^1(\mathbb{R})} \le \|u_0\|_{L^1(\mathbb{R})}. \tag{111}$$

Lemma 4.1 (L^1-estimate)**.** *Under the assumption of* (111)*, for each $T > 0$ there is a constant $C(T, \|u_0\|_1)$ such that the following estimates hold:*

$$\|u^\mu(t)\|_{L^1(\mathbb{R})} \le C(T, \|u_0\|_1), \tag{112}$$

for $t \in (0,T)$.

Proof. Let $\eta \in C^2(\mathbb{R})$ and $q : \mathbb{R} \to \mathbb{R}$ be such that $q'(u) = f'(u)\eta'(u)$. Multiplying (110) by $\eta'(u^\mu)$ and using the chain rule we arrive at

$$\eta(u^\mu)_t + q(u^\mu)_x = \eta'(u^\mu)\varepsilon G^r_{xx} * u^\mu + \mu\eta(u^\mu)_{xx} - \mu(u^\mu_x)^2\eta''(u^\mu). \tag{113}$$

Identifying $\eta(.)$ with $|.|$ (modulo an approximation argument), and then integrating over $x \in \mathbb{R}$ yields

$$\frac{d}{dt}\int_{\mathbb{R}} |u^\mu| \le \varepsilon \int_{\mathbb{R}} |G^r_{xx} * u^\mu|\,dx \le \varepsilon \|G^r_{xx}\|_{L^1(\mathbb{R})}\|u^\mu\|_{L^1(\mathbb{R})} \le 2\varepsilon\|u^\mu\|_{L^1(\mathbb{R})}, \tag{114}$$

by an application of Young's inequality and (88). Gronwall's lemma then gives

$$\|u^\mu\|_{L^1(\mathbb{R})} \le e^{2\varepsilon t}\|u_0\|_{L^1(\mathbb{R})},$$

which gives us (112). □

Next, we derive BV estimates. For that purpose, we need to make the assumptions that

$$u_0, u_0^\mu \in BV(\mathbb{R}), \qquad \|u_0^\mu\|_{BV(\mathbb{R})} \le \|u_0\|_{BV(\mathbb{R})}. \tag{115}$$

We here use standard arguments and let ρ be a mollifier. Then we define the mollification of u_0 to be

$$u_0^\mu = (u_0\chi_\mu) * \rho_\mu,$$

where $\rho_\mu(x) = \frac{1}{\mu}\rho(\frac{x}{\mu})$ and $\chi_\mu(x) = 1$ for $|x| \le 1/\mu$ and 0 otherwise. In particular, we see that $\|\partial_{xx}u_0^\mu\|_{L^1} \le \frac{1}{\mu}\|\partial_x u_0^\mu\|_{L^1} \le \frac{1}{\mu}\|u_0\|_{BV}$.

Lemma 4.2 (BV-estimate)**.** *Under the assumption of* (115)*, for each $T > 0$ there is a constant $C(T, \|u_0\|_{BV})$ such that the following estimates hold:*

$$\|u^\mu(t)\|_{BV(\mathbb{R})} \le C(T, \|u_0\|_{BV}), \tag{116}$$

for $t \in (0, T)$.

Proof. Let $v^\mu = u^\mu_x$. Differentiating (110) with respect to x yields the following equation

$$\partial_t v^\mu + \partial_x(f'(u^\mu)v^\mu) = \varepsilon G^r_{xx} * v^\mu + \mu v^\mu_{xx}, \qquad \mu > 0, \qquad f(v) = v^2. \tag{117}$$

Let η be a function $\eta \in C^2(\mathbb{R})$. Multiplying (117) by $\eta'(v^\mu)$ and using the chain rule we arrive at

$$\begin{aligned}\eta(v^\mu)_t + (f'(u^\mu)v^\mu\eta'(v^\mu))_x - f'(u^\mu)v^\mu\eta''(v^\mu)v^\mu_x \\ = \eta'(v^\mu)\varepsilon G^r_{xx} * v^\mu + \mu\eta(v^\mu)_{xx} - \mu(v^\mu_x)^2\eta''(v^\mu).\end{aligned}$$

Identifying $\eta(.)$ with $|.|$ (modulo an approximation argument), and then integrating over $x \in \mathbb{R}$ yields

$$\frac{d}{dt}\int_{\mathbb{R}} |v^\mu| \le \varepsilon \int_{\mathbb{R}} |G^r_{xx} * v^\mu|\,dx \le \varepsilon \|G^r_{xx}\|_{L^1(\mathbb{R})}\|v^\mu\|_{L^1(\mathbb{R})} \le 2\varepsilon\|v^\mu\|_{L^1(\mathbb{R})},$$

by an application of Young's inequality and (88). Here we also have used the fact that $v^\mu\eta''(v^\mu) = 0$ by an approximation argument where η'' is an approximation to the delta-function. Gronwall's lemma then gives

$$\|v^\mu\|_{L^1(\mathbb{R})} \le e^{2\varepsilon t}\|v_0\|_{L^1(\mathbb{R})},$$

which gives us (116). □

Lemma 4.3 (L^∞-estimate)**.** *Under the assumption* (115)*, for each $T > 0$ there is a constant $C(T, \|u_0\|_{BV})$ such that the following estimate hold:*

$$\|u^\mu(t)\|_{L^\infty(\mathbb{R})} \le C(T, \|u_0\|_{BV}), \tag{118}$$

for $t \in (0, T)$*. Moreover, under the assumption* (111) *there is a constant* $C(T, \|u_0\|_{L^\infty(\mathbb{R})}, \|u_0\|_{L^1(\mathbb{R})})$ *such that the following estimate hold:*

$$\|u^\mu(t)\|_{L^\infty(\mathbb{R})} \leq C(r, T, \|u_0\|_{L^\infty(\mathbb{R})}, \|u_0\|_{L^1}), \tag{119}$$

Proof. Estimate (118) follows directly from the estimate

$$|u^\mu(x,t)| \leq \int_{\mathbb{R}} |\partial_x u^\mu(y,t)|\, dy \leq C(T, \|u_0\|_{BV}),$$

where we have applied the previous lemma. Estimate (119) follows from the maximum principle

$$|u^\mu(x,t)| \leq \|u_0^\mu\|_{L^\infty(\mathbb{R})} + \varepsilon t \|G^r_{xx} * u^\mu\|_{L^\infty(\mathbb{R}\times(0,T))}.$$

Now we observe that

$$|G^r_{xx} * u^\mu(x,t)| \leq \|G^r_{xx}\|_{L^\infty(\mathbb{R})} \|u^\mu(t)\|_{L^1(\mathbb{R})} \leq \frac{1}{r} C(T, \|u_0\|_{L^1}),$$

in view of Lemma 4.1 and (89), from which (119) follows. □

Lemma 4.4 (BV-estimate in time)**.** *Under the assumption of* (115)*, for each* $T > 0$ *there is a constant* $C(T, \|u_0\|_{L^1}, \|u_0\|_{BV})$ *such that the following estimates hold:*

$$\|\partial_t u^\mu(t)\|_{L^1(\mathbb{R})} \leq C(T, \|u_0\|_{L^1}, \|u_0\|_{BV}), \tag{120}$$

for $t \in (0, T)$*.*

Proof. We follow the same approach as in Lemma 4.2, where $v^\mu = u^\mu_t$, and we end up with an inequality

$$\|v^\mu\|_{L^1(\mathbb{R})} \leq e^{2\varepsilon t} \|v_0^\mu\|_{L^1(\mathbb{R})}.$$

From this we get the estimate

$$\begin{aligned} \|\partial_t u^\mu(t)\|_{L^1(\mathbb{R})} &\leq e^{2\varepsilon t} \|\partial_t u_0^\mu\|_{L^1(\mathbb{R})} \\ &\leq e^{2\varepsilon t} \Big(2\|u_0^\mu\|_{L^\infty} \|\partial_x u_0^\mu\|_{L^1} + 2\varepsilon \|u_0^\mu\|_{L^1} + \mu \|\partial_{xx} u_0^\mu\|_{L^1} \Big). \end{aligned}$$

In view of the comments which follow after (115), the result of the lemma follows. □

Remark 4.1. Note that the above L^∞ estimates (118) and (119) are not sharp enough to ensure that we can demonstrate threshold for the breakdown of solutions (i.e. formation of discontinuity) along the line of [21]. Such results hang on the time independent results

$$\min_{x\in\mathbb{R}} u_0(x) \leq u(x,t) \leq \max_{x\in\mathbb{R}} u_0(x),$$

whereas the estimates of Lemma 4.3 involves a constant on the form $e^{2\varepsilon T}$. Numerical results in Section 6, however, clearly indicate that discontinuities can form. This reflects that sharper estimates than those obtained above seem to hold for the model (6).

4.2. **Existence of BV entropy weak solutions.**

Theorem 4.2 (Existence of solution in BV). *Assume that $u_0 \in BV \cap L^1$. Then there exists at least one entropy weak solution in BV to* (6) *which satisfies the estimates* (105)–(109).

Proof. We assume that the approximating solutions $\{u^\mu\}_{\mu>0}$ is chosen such that (111) and (115) hold. Then, in view of the a priori estimates of Section 4.1, it follows by standard arguments that there exists a function $u \in L^\infty\left((0,T)\times\mathbb{R}\right) \cap C\left([0,T]; L^1(\mathbb{R})\right)$ and a sequence $\{\mu_k\}$ tending to zero as $k \to \infty$ such that

$$u^{\mu_k} \to u \text{ in } L^1_{\text{loc}}((0,T)\times\mathbb{R}), \qquad u^{\mu_k} \to u \text{ a.e. in } (0,T)\times\mathbb{R},$$
$$\text{and} \qquad u^{\mu_k} \to u \text{ a.e. in } C\left([0,T]; L^1_{\text{loc}}(\mathbb{R})\right),$$

for all $T > 0$. Moreover, the a priori estimates in Section 4.1 imply immediately that the limit function u satisfy the estimates (105)–(109). Finally, to show that u is an entropy weak solution we rely on standard limit operations, see also Lemma 5.7 for relevant details. □

4.3. **L^1-stability and uniqueness of entropy weak solutions.** Now, L^1 stability (and thus uniqueness) of entropy weak solutions can be shown relying on a straightforward adaption of Kruzkov's device of doubling the variables.

Theorem 4.3 (L^1 stability). *Let u, v be two entropy weak solutions of* (6) *with corresponding initial data u_0, v_0 satisfying* (7). *Fix any $T > 0$. Then*

$$\|u(\cdot,t) - v(\cdot,t)\|_{L^1(\mathbb{R})} \le K_T \|u_0 - v_0\|_{L^1(\mathbb{R})}, \qquad t \in [0,T], \tag{121}$$

with K_T given in Theorem 4.1.

Proof. By standard arguments it suffices to work with the entropy inequality (86) with Kruzkov entropies/entropy fluxes given by

$$\eta(u) = |u-k|, \qquad q(u) = \operatorname{sgn}(u-k)\left(u^2 - k^2\right), \qquad k \in \mathbb{R}. \tag{122}$$

We set $Q_T = (0,T)\times\mathbb{R}$, and let $\psi(t,x,s,y)$ be a positive C^∞ function with compact support. Since u, v are entropy weak solutions according to (86) with (122), we find by standard arguments

$$\begin{aligned}\iint_{Q_T\times Q_T} &\Big(|u(x,t) - v(y,s)|(\partial_t\psi + \partial_s\psi) + \\ &\operatorname{sgn}(u(x,t) - v(y,s))\Big[u(x,t)^2 - v(y,s)^2\Big](\partial_x\psi + \partial_y\psi)\Big)\,dt\,dx\,ds\,dy \\ \ge -\varepsilon &\iint_{Q_T\times Q_T} \Big|[G^r_{xx} * u](x,t) - [G^r_{yy} * v](y,s)\Big|\psi\,dt\,dx\,ds\,dy.\end{aligned} \tag{123}$$

Next, we let $h \in C^\infty(Q_T)$ be such that

$$\operatorname{supp}(h) \subset [-1,1], \qquad 0 \le h \le 1, \qquad \int_{\mathbb{R}} h(x)\,dx = 1.$$

For $\delta > 0$, define

$$h_\delta(x) := \frac{1}{\delta} h(\frac{x}{\delta}),$$

Consider a $C^\infty(Q_T)$ function ω with compact support, and define

$$\psi_\delta(t,x,s,y) = \omega(\frac{t+s}{2}, \frac{x+y}{2}) h_\delta(\frac{t-s}{2}) h_\delta(\frac{x-y}{2}).$$

With $\psi = \psi_\delta$ as the choice of test function and using a standard argument which only require that

$$u, v, G^r_{xx} * u, G^r_{yy} * v \in L^1_{\mathrm{loc}}((0,T)\times\mathbb{R}),$$

we can let δ go to zero in (123) which gives

$$\begin{aligned}\iint_{Q_T} \Big(|u-v|\partial_t\omega + \operatorname{sgn}(u-v)\Big[u^2 - v^2\Big]\partial_x\omega\Big)\,dt\,dx \\ \geq -\varepsilon \iint_{Q_T} \Big|G^r_{xx} * [u-v]\Big|\omega\,dt\,dx.\end{aligned} \tag{124}$$

By standard arguments choosing $\omega(x,t) = \omega_1(t)\omega_2(x)$, and letting ω_2 tend to the function that is identically one, we obtain

$$\iint_{Q_T} |u(x,t) - v(x,t)|\omega_{1,t}\,dt\,dx + \varepsilon \iint_{Q_T} \Big|G^r_{xx} * [u(x,t) - v(x,t)]\Big|\omega_1\,dt\,dx \geq 0. \tag{125}$$

Letting $\omega_1(t) = \chi_{[0,t]}$, and noting that for $t \in (0,T)$

$$\begin{aligned}\int_{\mathbb{R}} |G^r_{xx} * [u(x,t) - v(x,t)]|\,dx &\leq \|G^r_{xx}\|_{L^1(\mathbb{R})}\|u(\cdot,t) - v(\cdot,t)\|_{L^1(\mathbb{R})} \\ &\leq 2\|u(\cdot,t) - v(\cdot,t)\|_{L^1(\mathbb{R})},\end{aligned}$$

we conclude from (125) that

$$\|u(\cdot,t) - v(\cdot,t)\|_{L^1(\mathbb{R})} \leq \|u_0 - v_0\|_{L^1(\mathbb{R})} + 2\varepsilon\int_0^t \|u(\cdot,\tau) - v(\cdot,\tau)\|_{L^1(\mathbb{R})}\,d\tau.$$

The result then follows by using Gronwall's lemma. □

5. **Global existence theory in L^2.** In this section we prove existence of at least one weak solution to (6) under assumption (8) in which we are outside the BV/L^∞ framework. Since no L^∞ bound is available we can only prove that this weak solution satisfies the entropy inequality for convex C^2 entropies η possessing a bounded second order derivative η''.

Theorem 5.1 (Existence in L^2)**.** *Suppose* (8) *holds. Then there exists a function u which is a weak solution of* (6) *in the sense of Definition 3.1. That is,*

$$u \in L^\infty\left((0,T); L^2(\mathbb{R})\right), \qquad \text{for any } T > 0,$$

which solves the Cauchy problem (6) *and* (8) *in* $\mathcal{D}'([0,T)\times\mathbb{R})$.

Proof. This follows directly from the Lemmas 5.6 and 5.7. □

For the initial data we assume that

$$u_0 \in L^2(\mathbb{R}), \tag{126}$$

and

$$u_0^\mu \in H^s(\mathbb{R}),\ \ s \geq 2, \qquad \|u_0^\mu\|_{L^2(\mathbb{R})} \leq \|u_0\|_{L^2(\mathbb{R})}, \qquad u_0^\mu \to u_0 \quad \text{in } L^2(\mathbb{R}). \tag{127}$$

5.1. **Estimates.**

Lemma 5.1 (energy estimate)**.** *Under the assumption of* (126) *and* (127)*, for each* $T>0$ *there is a constant* $C(T,\|u_0\|_2)$ *such that the following estimates hold:*

$$\|u^\mu(t)\|_{L^2(\mathbb{R})} \le C(T,\|u_0\|_2), \qquad \sqrt{\mu}\|\partial_x u^\mu\|_{L^2((0,T)\times\mathbb{R})} \le C(T,\|u_0\|_2), \tag{128}$$

for $t\in(0,T)$*.*

Proof. First we derive a uniform $L^2(\mathbb{R})$ bound for the approximate solutions. Multiplying (110) by u^μ and integrating in $x\in\mathbb{R}$, we arrive at

$$\frac{d}{dt}\int_{\mathbb{R}} \frac{|u^\mu|^2}{2}\,dx + \mu\int_{\mathbb{R}} |\partial_x u^\mu|^2\,dx = \varepsilon\int_{\mathbb{R}} u^\mu G^r_{xx} * u^\mu\,dx$$

Applying Holder's inequality, followed by an application of Young's inequality gives

$$\begin{aligned}\int_{\mathbb{R}} u^\mu G^r_{xx} * u^\mu\,dx &\le \|u^\mu\|_{L^2(\mathbb{R})}\|G^r_{xx} * u^\mu\|_{L^2(\mathbb{R})}\\ &\le \|u^\mu\|^2_{L^2(\mathbb{R})}\|G^r_{xx}\|_{L^1(\mathbb{R})} \le 2\|u^\mu\|^2_{L^2(\mathbb{R})} = 4\int_{\mathbb{R}}\frac{|u^\mu|^2}{2}\,dx.\end{aligned}$$

By Gronwall's inequality we get

$$\int_{\mathbb{R}} \frac{|u^\mu(t)|^2}{2}\,dx + e^{4\varepsilon t}\mu\int_0^t\int_{\mathbb{R}} |\partial_x u^\mu|^2\,dx\,dt \le e^{4\varepsilon t}\int_{\mathbb{R}}\frac{|u_0^\mu|^2}{2}\,dx.$$

Thus, we conclude that for all $T>0$, there exists $C(T,\|u_0\|_2)$ such that

$$\|u^\mu(t)\|^2_{L^2(\mathbb{R})} + \mu\int_0^T\int_{\mathbb{R}} |u^\mu_x(x,t)|^2\,dx\,dt \le C(T,\|u_0\|_2), \qquad t\in(0,T). \tag{129}$$

□

Next, we derive a L^p estimate. That is, assume that

$$u_0, u_0^\mu \in L^p(\mathbb{R}), \qquad \|u_0^\mu\|_{L^p(\mathbb{R})} \le \|u_0\|_{L^p(\mathbb{R})}, \qquad p\ge 1. \tag{130}$$

Lemma 5.2 (L^p-estimate)**.** *Under the assumption of* (130)*, for each* $T>0$ *there is a constant* $C(T,\|u_0\|_p)$ *such that the following estimates hold:*

$$\|u^\mu(t)\|_{L^p(\mathbb{R})} \le C(T,\|u_0\|_p), \tag{131}$$

for $t\in(0,T)$*.*

Proof. The starting point is (113), however, now we associated $\eta(\cdot)$ with the function $|\cdot|^p$. Consequently, $\eta'(\cdot) = p|\cdot|^{p-1}\mathrm{sgn}(\cdot)$ and (114) is replaced by

$$\frac{d}{dt}\int_{\mathbb{R}} |u^\mu|^p\,dx \le p\varepsilon\int_{\mathbb{R}} |u^\mu|^{p-1}|G^r_{xx} * u^\mu|\,dx. \tag{132}$$

Moreover, for the right hand side of (132) we observe that setting $g=|u|^{p-1}\in L^{p'}$ and $h=|G^r_{xx} * u|\in L^{q'}$ with $p'=p/(p-1)$ and $q'=p$ the Holder inequality gives us $\int |gh| \le ||g||_{p'}||h||_{q'}$, that is,

$$\begin{aligned}\int_{\mathbb{R}} |u|^{p-1}|G^r_{xx} * u|\,ds &\le \Big(\int_{\mathbb{R}} |u|^p\,ds\Big)^{\frac{p-1}{p}} \cdot \Big(\int_{\mathbb{R}} |G^r_{xx} * u|^p\,ds\Big)^{\frac{1}{p}}\\ &= \|u\|^{p-1}_{L^p(\mathbb{R})}\cdot\|G^r_{xx} * u\|_{L^p(\mathbb{R})}.\end{aligned}$$

Moreover,

$$\|G^r_{xx} * u\|_{L^p(\mathbb{R})} \le \|u\|_{L^p(\mathbb{R})}\|G^r_{xx}\|_{L^1(\mathbb{R})} \le 2\|u\|_{L^p(\mathbb{R})},$$

by application of Young's inequality and (88). Thus, (132) is replaced by

$$\frac{d}{dt}\int_{\mathbb{R}} |u^\mu|^p \, dx \le 2p\varepsilon \|u^\mu\|^p_{L^p(\mathbb{R})}, \tag{133}$$

and Gronwall's lemma then gives

$$\|u^\mu\|^p_{L^p(\mathbb{R})} \le e^{2p\varepsilon t} \|u_0\|^p_{L^p(\mathbb{R})},$$

which gives us (131).

□

5.2. **Existence of weak solutions.** We shall only make use of the estimates involved in Lemma 5.1 and 5.2. Along the same line as in [3] we rely on Schonbek's L^p version [29] of the compensated compactness method [32] to obtain strong convergence of a subsequence of viscosity approximations. We shall also make use of the following lemma [22] which avoids assumption of strict convexity of the flux function.

Lemma 5.3. *Let Ω be a bounded open subset of $\mathbb{R}_+ \times \mathbb{R}$. Let $f \in C^2(\mathbb{R})$ satisfy*

$$|f(u)| \le C|u|^{s+1}, \quad u \in \mathbb{R}, \qquad |f'(u)| \le C|u|^2 \quad u \in \mathbb{R},$$

for some $s \ge 0$, and $f''(u) \ne 0$ a.e. in $\mathbb{R}$. Then define functions $I_l, f_l, F_l : \mathbb{R} \to \mathbb{R}$ as follows:

$$\begin{aligned} I_l \in C^2(\mathbb{R}), \qquad |I_l(u)| \le |u|, \quad |I_l'(u)| \le 2 \quad \text{for } u \in \mathbb{R} \\ |I_l(u)| \le |u| \quad \text{for } |u| \le l, \\ I_l(u) = 0 \quad \text{for } |u| \ge 2l, \end{aligned}$$

and

$$f_l(u) = \int_0^u I_l'(s) f'(s)\, ds, \qquad F_l(u) = \int_0^u f_l'(s) f'(s)\, ds.$$

Suppose $\{u_n\}_{n=1}^\infty \subset L^{2(s+1)}(\Omega)$ is such that the two sequences

$$\{\partial_t I_l(u_n) + \partial_x f_l(u_n)\}_{n=1}^\infty, \qquad \{\partial_t f_l(u_n) + \partial_x F_l(u_n)\}_{n=1}^\infty$$

of distributions belong to a compact subset of $H^{-1}_{loc}(\Omega)$, for each $l > 0$.

Then there exists a subsequence of $\{u_n\}_{n=1}^\infty$ that converges to a limit function $u \in L^{2(s+1)}(\Omega)$ strongly in $L^r(\Omega)$ for any $1 \le r < 2(s+1)$.

The following lemma of Murat [23] will also be used.

Lemma 5.4. *Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \ge 2$. Suppose the sequence $\{\mathcal{L}_n\}_{n=1}^\infty$ of distributions is bounded in $W^{-1,\infty}(\Omega)$. Suppose also that*

$$\mathcal{L}_n = \mathcal{L}_n^1 + \mathcal{L}_n^2,$$

where $\{\mathcal{L}_n^1\}_{n=1}^\infty$ lies in a compact subset of $H^{-1}_{loc}(\Omega)$ and $\{\mathcal{L}_n^1\}_{n=1}^\infty$ lies in a bounded subset of $\mathcal{M}_{loc}(\Omega)$. Then $\{\mathcal{L}_n\}_{n=1}^\infty$ lies in a compact subset of $H^{-1}_{loc}(\Omega)$.

The proof of Theorem 5.1 follows basically from the next two lemmas. First, we have the following result.

Lemma 5.5. *Assume* (8) *holds. Then there exists a subsequence $\{u^{\mu_k}\}_{k=1}^\infty$ of $\{u^\mu\}_{\mu>0}$ and a limit function u such that*

$$u \in L^\infty\left((0,T); L^2(\mathbb{R})\right) \cap L^\infty\left((0,T); L^4(\mathbb{R})\right), \quad \forall T > 0 \tag{134}$$

such that

$$u^{\mu_k} \to u \text{ in } L^p((0,T)\times\mathbb{R}), \quad \forall T > 0, \quad \forall p \in [1,4). \tag{135}$$

Proof. Let $\eta : \mathbb{R} \to \mathbb{R}$ be a any convex C^2 entropy function that is compactly supported, and let $q : \mathbb{R} \to \mathbb{R}$ be the corresponding entropy flux defined by $q'(u) = \eta'(u)2u$. We then claim that

$$\partial_t \eta(u^\mu) + \partial_x q(u^\mu) = \mathcal{L}^1_\mu + \mathcal{L}^2_\mu, \tag{136}$$

for some distributions $\mathcal{L}^1_\mu$ and $\mathcal{L}^2_\mu$ that satisfy

$$\begin{aligned} &\mathcal{L}^1_\mu \to 0 \quad \text{in } H^{-1}((0,T)\times\mathbb{R}) \ , \\ &\mathcal{L}^2_\mu \ \text{ is uniformly bounded in } \mathcal{M}((0,T)\times\mathbb{R}). \end{aligned} \tag{137}$$

Indeed, by (113) we have

$$\eta(u^\mu)_t + q(u^\mu)_x = [\mu\eta(u^\mu)_{xx}] + [\eta'(u^\mu)\varepsilon G^r_{xx} * u^\mu - \mu(u^\mu_x)^2\eta''(u^\mu)] = \mathcal{L}^1_\mu + \mathcal{L}^2_\mu.$$

In light of (128) we have

$$\begin{aligned} \|\mu\eta(u^\mu)_x\|_{L^2((0,T)\times\mathbb{R})} &\le \sqrt{\mu}\|\eta'\|_\infty C(T,\|u_0\|_2) \to 0 \quad \text{as } \mu \to 0, \\ \|\mu\eta''(u^\mu)(u^\mu_x)^2\|_{L^1((0,T)\times\mathbb{R})} &\le \|\eta''\|_\infty C(T,\|u_0\|_2)^2, \\ \|\eta'(u^\mu)\varepsilon G^r_{xx} * u^\mu\|_{L^1((0,T)\times\mathbb{R})} &\le \|\eta'\|_\infty 2\varepsilon T e^{2\varepsilon T}\|u_0\|_{L^1}, \end{aligned} \tag{138}$$

where we have used the calculations in (114) for the last estimate. Thus, (136) and (137) follow. In view of Lemma 5.4 we conclude that $\partial_t\eta(u^\mu) + \partial_x q(u^\mu)$ is compact in $H^{-1}_{\text{loc}}((0,T)\times\mathbb{R})$.

Now we want to apply this approach in combination with Lemma 5.3. First, we observe that $\{u^\mu\}_{\mu>0} \subset L^2((0,T)\times\mathbb{R}) \cap L^4((0,T)\times\mathbb{R})$ (in view of Lemma 5.2) and that

$$\{\partial_t I_l(u^\mu) + \partial_x f_l(u^\mu)\}_{\mu>0}, \qquad \{\partial_t f_l(u^\mu) + \partial_x F_l(u^\mu)\}_{\mu>0},$$

satisfy estimates similar to (138), thus, are compact in $H^{-1}_{\text{loc}}((0,T)\times\mathbb{R})$ for each fixed $l > 0$, by application of Lemma 5.4. Hence, the assumptions of Lemma 5.3 are satisfied with $s = 1$ and we can conclude that there exists a subsequence $\{u^{\mu_k}\}_{k=1}^\infty$ that converges to a limit function $u \in L^2((0,T)\times\mathbb{R}) \cap L^4((0,T)\times\mathbb{R})$ strongly in $L^r((0,T)\times\mathbb{R})$ for any $1 \le r < 4$. □

Lemma 5.6 (Weak solution)**.** *Assume that* (8) *holds. Then the limit function u from Lemma 5.5 is a weak solution of* (6) *in the sense of* (85)*.*

Proof. We only have to note multiply (110) with a test function ϕ, integrate in space and time, apply integration by parts, and then take the limit $k \to \infty$. In view of Lemma 5.5 and the convergence result (135), it follows that

$$\begin{aligned} \int_0^T \int_{\mathbb{R}} u^{\mu_k}\phi_t \, dx\, dt &\to \int_0^T \int_{\mathbb{R}} u\phi_t \, dx\, dt, \\ \int_0^T \int_{\mathbb{R}} (u^{\mu_k})^2\phi_x \, dx\, dt &\to \int_0^T \int_{\mathbb{R}} u^2\phi_x \, dx\, dt. \end{aligned}$$

For the $\partial_x p_\mu = -G^r_{xx} * u^{\mu_k}$ term we have that

$$\begin{aligned}
&\|G^r_{xx} * (u^{\mu_k} - u)\|^p_{L^p((0,T)\times\mathbb{R})} \\
&= \int_0^T \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |G^r_{xx}(u^{\mu_k}(x',t) - u(x',t))| \, dx' \Big)^p \, dx \, dt \\
&\le \int_0^T \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |G^r_{xx}|^{(p-1)/p} \Big| (G^r_{xx})^{1/p} (u^{\mu_k}(x',t) - u(x',t)) \Big| \, dx' \Big)^p \, dx \, dt \\
&\le \int_0^T \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |G^r_{xx}|^1 \, dx' \Big)^{p-1} \Big(\int_{\mathbb{R}} |G^r_{xx}|^1 \, |u^{\mu_k}(x',t) - u(x',t)|^p \, dx' \Big) \, dx \, dt \\
&\le \|G^r_{xx}\|^{p-1}_{L^1(\mathbb{R})} \|G^r_{xx}\|_{L^1(\mathbb{R})} \int_0^T \int_{\mathbb{R}} |u^\mu(x',t) - u(x',t))|^p \, dx' \, dt \\
&\le \|G^r_{xx}\|^p_{L^1(\mathbb{R})} \|u^{\mu_k} - u\|^p_{L^p((0,T)\times\mathbb{R})} \le 2^p \|u^{\mu_k} - u\|^p_{L^p((0,T)\times\mathbb{R})} \to 0,
\end{aligned} \tag{139}$$

as $\mu \to 0$ where we use that $G^r_{xx}(x,x')^{1/p}(u^{\mu_k}(x',t) - u(x',t)) \in L^p(\mathbb{R})$ and $G^r_{xx}(x,x')^{(p-1)/p} \in L^{p/(p-1)}(\mathbb{R})$ since $G^r_{xx} \in L^1(\mathbb{R})$. Consequently,

$$\int_0^T \int_{\mathbb{R}} G^r_{xx} * u^{\mu_k} \phi \, dx \, dt \to \int_0^T \int_{\mathbb{R}} G^r_{xx} * u\phi \, dx \, dt.$$

□

Corollary 5.1. *Assume that* (8) *holds. Let* u^μ_r *denote the viscous approximation* (110) *used in Lemma 5.6. Then there is a limit function* $\bar{u}$ *such that*

$$u^\mu_r \to \bar{u} \text{ in } L^\infty\left((0,T); L^2(\mathbb{R})\right), \qquad \text{as } \mu, r \to 0, \tag{140}$$

and $\bar{u}$ *is a weak solution of the equation*

$$\partial_t \bar{u} + \partial_x(\bar{u}^2) = 0, \qquad \bar{u}(x,0) = u_0(x). \tag{141}$$

Proof. All the estimates used in Lemma 5.5 are independent of the r parameter. Thus (140) follows. In order to conclude that the limit $\bar{u}$ is a weak solution of (141), we only have to check the convergence of the term

$$\int_0^T \int_{\mathbb{R}} (G^r_{xx} * u^\mu)\phi \, dx \, dt = \int_0^T \int_{\mathbb{R}} (G^r_x * u^\mu)_x \phi \, dx \, dt = -\int_0^T \int_{\mathbb{R}} (G^r_x * u^\mu)\phi_x \, dx \, dt.$$

Since, for $1/p + 1/q = 1$,

$$\begin{aligned}
\Big| \int_0^T \int_{\mathbb{R}} (G^r_x * u^\mu)\phi_x \, dx \, dt \Big| &\le \|G^r_x * u^\mu\|_{L^p((0,T)\times\mathbb{R})} \|\phi_x\|_{L^q((0,T)\times\mathbb{R})} \\
&\le \|G^r_x\|_{L^1(\mathbb{R})} \|u^\mu\|_{L^p((0,T)\times\mathbb{R})} \|\phi_x\|_{L^q((0,T)\times\mathbb{R})} \\
&\le 2r \|u^\mu\|_{L^p((0,T)\times\mathbb{R})} \|\phi_x\|_{L^q((0,T)\times\mathbb{R})} \to 0, \qquad \text{as } r \to 0,
\end{aligned}$$

by using (139) and (88).

□

Lemma 5.7 (Entropy weak solution)**.** *Assume that* (8) *holds. Then the limit function* u *from Lemma 5.5 is an entropy weak solution of* (6) *in the sense that it satisfies the entropy inequality* (86) *for any convex entropy* $\eta : \mathbb{R} \to \mathbb{R}$ *with* η'' *bounded and corresponding entropy flux* $q : \mathbb{R} \to \mathbb{R}$ *defined by* $q'(u) = 2\eta'(u)u$.

Proof. Let (η, q) be as in the lemma. In view of (113) we have

$$\eta(u^{\mu_k})_t + q(u^{\mu_k})_x \leq \eta'(u^{\mu_k})\varepsilon G^r_{xx} * u^{\mu_k} + \mu\eta(u^{\mu_k})_{xx}, \qquad \text{in } \mathcal{D}'([0,T)\times\mathbb{R}). \quad (142)$$

The assumptions on (η, q) imply that

$$|\eta(u)| = O(1+u^2), \qquad |\eta'(u)| = O(1+u), \qquad |q(u)| = O(1+u^3).$$

Consequently, in light of the convergence (135) of Lemma 5.5 we conclude that

$$\int_0^T\int_{\mathbb{R}} \eta(u^{\mu_k})\phi_t\,dx\,dt \to \int_0^T\int_{\mathbb{R}} \eta(u)\phi_t\,dx\,dt,$$
$$\int_0^T\int_{\mathbb{R}} q(u^{\mu_k})\phi_x\,dx\,dt \to \int_0^T\int_{\mathbb{R}} q(u)\phi_x\,dx\,dt.$$

By using the calculation (139), we also see that

$$\int_0^T\int_{\mathbb{R}} \eta'(u^{\mu_k})G^r_{xx} * u^{\mu_k}\phi\,dx\,dt \to \int_0^T\int_{\mathbb{R}} \eta'(u)G^r_{xx} * u\phi\,dx\,dt.$$

□

Corollary 5.2. *Assume that* (8) *holds. Let* u^μ_r *denote the viscous approximation* (110) *used in Lemma 5.6 with* $\mu = O(r^d)$ *for* $d < 2$*. Then there is a limit function* $\bar{u}$ *such that*

$$u^\mu_r \to \bar{u} \text{ in } L^\infty\left((0,T); L^2(\mathbb{R})\right), \qquad \text{as } r \to 0, \quad (143)$$

and $\bar{u}$ *is an entropy weak solution of the equation* (141) *in the sense of*

$$\partial_t\eta(\bar{u}) + \partial_x q(\bar{u}) \leq 0 \qquad \text{in } \mathcal{D}'([0,T)\times\mathbb{R}),$$

for (η, q) *defined as in Lemma 5.7.*

Proof. This follows by the same arguments as in Corollary 5.1. We only have to check the convergence of the term

$$\int_0^T\int_{\mathbb{R}} \eta'(u^{\mu_k})(G^r_{xx} * u^{\mu_k})\phi\,dx\,dt = \int_0^T\int_{\mathbb{R}} \eta'(u^{\mu_k})(G^r_x * u^{\mu_k}_x)\phi\,dx\,dt,$$

where we no longer can move one derivative over to the test function ϕ and instead must rely on the L^2 estimate of u^μ_x in (128). That is,

$$\begin{aligned}
&\Big|\int_0^T\int_{\mathbb{R}} \eta'(u^{\mu_k})(G^r_x * u^{\mu_k}_x)\phi\,dx\,dt\Big| \\
&\leq \|\eta'\|_{L^\infty(\mathbb{R}}\|G^r_x * u^{\mu_k}_x\|_{L^2((0,T)\times\mathbb{R})}\|\phi\|_{L^2((0,T)\times\mathbb{R})} \\
&\leq \|\eta'\|_{L^\infty(\mathbb{R}}\|G^r_x\|_{L^1(\mathbb{R})}\|u^\mu_x\|_{L^2((0,T)\times\mathbb{R})}\|\phi\|_{L^2((0,T)\times\mathbb{R})} \\
&\leq 2C(T,\|u_0\|_2)\frac{r}{\sqrt{\mu_k}}\|\eta'\|_{L^\infty(\mathbb{R}}\|u^\mu\|_{L^p((0,T)\times\mathbb{R})}\|\phi\|_{L^2((0,T)\times\mathbb{R})} \to 0,
\end{aligned}$$

as $\mu = O(r^d)$ with $d < 2$ and by using (139) with $p = 2$, (128), and (88).

□

6. **Numerical examples.** In this section we illustrate characteristic behavior of solutions to the well-reservoir model (3)–(5) by performing some numerical experiments. To solve the model we use the second order relaxed scheme [13] for the discretization of the convective flux. The pressure flux (non-local term) is discretized in a straightforward manner as explained below.

Discretization approach. We consider a straightforward discretization of the model (6). That is, we consider a discrete scheme on the form

$$\frac{u_j^{k+1}-u_j^k}{\Delta t}+\frac{1}{\Delta x}\Big(F_{j+1/2}^k-F_{j-1/2}^k\Big)=\varepsilon\sum_{i=1}^N\int_{x_{i-1/2}}^{x_{i+1/2}}G_x^r(x_j,x')\Big(\frac{u_{i+1/2}^{k+1}-u_{i-1/2}^{k+1}}{\Delta x}\Big)\,dx',$$

$$u_{1/2}=u_{\text{in}},\quad u_{i+1/2}=\frac{1}{2}\Big(u_i+u_{i+1}\Big),\quad (i=2,\dots,N-1),\quad u_{N+1/2}=u_{\text{out}},$$

where $G_x^r(x,x')$ is given by (87). We note that $G_x^r(x,x')=-G_{x'}^r(x,x')$ and define

$$\Delta_i(x_j):=\int_{x_{i-1/2}}^{x_{i+1/2}}G_x^r(x_j,x')\,dx'=-\int_{x_{i-1/2}}^{x_{i+1/2}}G_{x'}^r(x_j,x')\,dx',$$

where G^r is given by (87). In other words

$$\Delta_i(x_j)=-\Big(G^r(x_j,x_{i+1/2})-G^r(x_j,x_{i-1/2})\Big),$$

and we see that we may rewrite as follows

$$u_j^{k+1}-\varepsilon\lambda\sum_{i=1}^N\Delta_i(x_j)[u_{i+1/2}^{k+1}-u_{i-1/2}^{k+1}]=u_j^k-\lambda\Big(F_{j+1/2}^k-F_{j-1/2}^k\Big),\qquad \lambda=\frac{\Delta t}{\Delta x},$$

where $F_{j+1/2}^k$ represents the second order flux of the relaxed scheme as described in [13]. Further algebraic manipulation gives

$$\begin{aligned}u_j^{k+1}&+\frac{\varepsilon\lambda}{2}u_1^{k+1}D_{3/2}(x_j)+\frac{\varepsilon\lambda}{2}\sum_{i=2}^{N-1}u_i^{k+1}D_i(x_j)+\frac{\varepsilon\lambda}{2}u_N^{k+1}D_{N-1/2}(x_j)\\&=u_j^k-\lambda\Big(F_{j+1/2}^k-F_{j-1/2}^k\Big)+\varepsilon\lambda u_{\text{out}}\Delta_N(x_j)-\varepsilon\lambda u_{\text{in}}\Delta_1(x_j),\quad \text{for } j=1,\dots,N,\end{aligned}$$

where

$$D_{i+1/2}(x_j)=\Delta_{i+1}(x_j)-\Delta_i(x_j),\qquad D_i(x_j)=\Delta_{i+1}(x_j)-\Delta_{i-1}(x_j).$$

The resulting discrete system we solve is on the form $\mathbf{A}\mathbf{x}=\mathbf{b}$. Here the $\mathbf{A}$ matrix is given by

$$\mathbf{A}=\mathbf{A}_1+\mathbf{A}_2,$$

with

$$\mathbf{A}_1=\mathbf{I},$$

and

$$\mathbf{A}_2=\frac{\varepsilon\lambda}{2}\begin{pmatrix}D_{3/2}(x_1)&D_2(x_1)&\dots&D_{N-1}(x_1)&D_{N-1/2}(x_1)\\D_{3/2}(x_2)&D_2(x_2)&\dots&D_{N-1}(x_2)&D_{N-1/2}(x_2)\\\vdots&\vdots&\vdots&\vdots&\vdots\\D_{3/2}(x_{N-1})&D_2(x_{N-1})&\dots&D_{N-1}(x_{N-1})&D_{N-1/2}(x_{N-1})\\D_{3/2}(x_N)&D_2(x_N)&\dots&D_{N-1}(x_N)&D_{N-1/2}(x_N)\end{pmatrix}.$$

Moreover,

$$\mathbf{x}=\mathbf{u},\qquad \mathbf{b}=(\dots,b_j,\dots)^T,$$

with

$$b_j=u_j^k-\lambda\Big(F_{j+1/2}^k-F_{j-1/2}^k\Big)-\varepsilon\lambda u_{\text{out}}\Delta_N(x_j)+\varepsilon\lambda u_{\text{in}}\Delta_1(x_j).$$

In the following we consider as initial data a Gaussian pulse on the form

$$u_0(x)=5\exp(-100(x-0.5)^2),$$

together with the boundary data $u_{\text{in}}=u_{\text{out}}=0$.

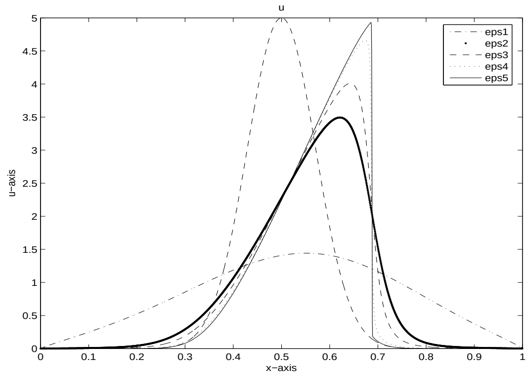

FIGURE 3. Plot of solutions at time $T = 0.02$ with $r = 10^{-4}$ for different choices of ε corresponding to $\varepsilon_1 = 10^7$, $\varepsilon_2 = 10^6$, $\varepsilon_3 = 5 \cdot 10^5$, $\varepsilon_4 = 10^5$, and $\varepsilon_5 = 0$. Loss of regularity is seen for $\varepsilon > 0$.

Example 1. First, we consider an example with well radius $r = 10^{-4}$ and time $T = 0.02$ and a grid with $N = 1600$ cells. We explore the behavior for a varying diffusion parameter ε which has a clear physical meaning since the parameter ε given by (5) is composed of different well and reservoir parameters, thus, representing a balance of different forces. In Fig. 3 plots are shown for $\varepsilon_1 = 10^7$, $\varepsilon_2 = 10^6$, $\varepsilon_3 = 5 \cdot 10^5$, $\varepsilon_4 = 10^5$, and $\varepsilon_5 = 0$. We demonstrate the steepening of the gradient, i.e., wave breaking in finite time, for $\varepsilon > 0$. In particular, this justifies the need for working with weak and entropy weak solutions in the sense of Definitions 3.1 and 3.2.

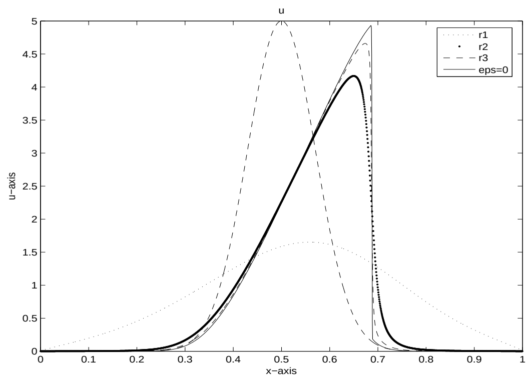

FIGURE 4. Plot of solutions at time $T = 0.02$ with $\varepsilon = 10^5$ for various choices of well radius corresponding to $r_1 = 10^{-3}$, $r_2 = 0.5 \cdot 10^{-3}$, and $r_3 = 10^{-4}$. The solution of the hyperbolic conservation law ($\varepsilon = 0$) is also included. The plots reflect convergence toward hyperbolic conservation law as r tends to zero.

Example 2. In this example we keep the parameter ε fixed, $\varepsilon = 10^5$. Again we compute solutions after $T = 0.02$ on a grid of $N = 1600$ cells. In Fig. 4 we compare solutions for different choices of the well radius r corresponding to $r_1 = 10^{-3}$, $r_2 = 0.5 \cdot 10^{-3}$, and $r_3 = 10^{-4}$. The pure hyperbolic case $\varepsilon = 0$ is also included for comparison, and we observe how the solution is approaching to the hyperbolic solution as r tends to zero.

As a final remark we note that the numerical simulations do not indicate that $\|u\|_\infty$ and $\|u\|_{\mathrm{BV}}$ increase with time with a factor e^{ct}. In other words, we may expect that sharper estimates should be possible (under some appropriate assumptions/modifications) similar to those that have been shown for the radiating gas model (80).

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Received for publication September 2006.

E-mail address: steinar.evje@irisresearch.no
E-mail address: kennethk@math.uio.no