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A WELL POSED RIEMANN PROBLEM FOR THE p–SYSTEM AT A JUNCTION

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Abstract. This work is devoted to the solution to Riemann Problems for the p-system at a junction, the main goal being the extension to the case of an ideal junction of the classical results that hold in the standard case.

1. **Introduction.** This note is devoted to the study of the Riemann Problem for the p-system at a junction connecting n ducts. We see this as a preliminary step for the study of the p-system on a network of ducts. Indeed, in the theory of conservation laws the solution to Riemann Problems in [17] is preliminary to the general existence result [15] and of key importance for the subsequent well posedness [5, 6].

Similarly, in the case of phase transitions or chemical reactions, the choice of the *"good"* solutions to Riemann Problems [1, 11, 19, 20, 21, 22] opens the way to well posedness results [9, 10].

Concerning traffic flows in urban areas, conservation laws on networks are considered in [7, 8, 12, 13, 14] and in each of these papers the choice of the solutions to Riemann Problems at junctions has a key role.

We consider the p-system in Eulerian coordinates, i.e.

$$\left\{\begin{array}{lcl} \partial_t \rho + \partial_x q & = & 0 \\ \partial_t q + \partial_x \left(\dfrac{q^2}{\rho} + p(\rho) \right) & = & 0\,. \end{array}\right. \tag{1}$$

where, $t \in \mathbb{R}^+$ (we denote $\mathbb{R}^+ = [0, +\infty[$ and $\mathring{\mathbb{R}}^+ =]0, +\infty[$) is time, $x \in \mathbb{R}^+$ is the abscissa along a duct with uniform cross section, the junction being at $x = 0$. $\rho \in \mathbb{R}^+$ is the density of the fluid, $q \in \mathbb{R}$ is the linear momentum density, i.e. $q = \rho v$, $v \in \mathbb{R}$ being the fluid speed, and $p = p(\rho)$ is a pressure law satisfying

(P): $p \in \mathbf{C^2}\left(\mathbb{R}^+; \mathbb{R}^+\right)$ and for all $\rho > 0$, $p'(\rho) > 0$, $p''(\rho) \geq 0$.

A typical example is the γ–pressure law $p(\rho) = k\rho^\gamma$ with $k > 0$ and $\gamma \geq 1$.

Our aim is to extend to the case of junctions among n ducts the properties of the solution to the standard Riemann Problem on the real line. In particular, differently from other works in the literature [2, 3],

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1. in any duct, the sign of the fluid speed is *not* fixed *a priori*;
2. the Riemann Problem is well posed.

From the physical point of view, the definition of solution to Riemann Problems at a junction that we present here is based on the conservation of mass, on the increase of entropy and on the requirement of well posedness. In general, the linear momentum is *not* conserved at the junction.

More precisely, it turns out that well posedness requires the equality of the *dynamic pressure*, i.e. of the flow of the linear momentum:

$$P(\rho, q) = \frac{q^2}{\rho} + p(\rho)\,. \tag{2}$$

When the variations in the fluid speed are small, this condition is approximated by the condition found, for instance, in [2, 3], namely the equality of pressures at the junction.

The second principle of thermodynamics enters this framework through a suitable extension of the entropy condition based on the *entropy flow*

$$F(\rho, q) = q \cdot \left(\frac{q^2}{2\rho^2} + \int_{\rho_*}^{\rho} \frac{p'(r)}{r}\, dr \right)\,, \tag{3}$$

where $\rho_* > 0$ is a suitable constant.

With these choices, we obtain the local well posedness of Riemann Problems for any number of ducts, under some restriction on the initial states.

In this paper, we prove the existence and uniqueness of some stationary solutions to the Riemann Problem and of their perturbations. These results are global in space-time and local in the subsonic region of the state space (ρ, q). In the supersonic region, uniqueness may fail, as we show in Example 1. In other words, we show that the presence of supersonic states is not compatible with *local* results in the (ρ, q) space. Indeed, small perturbations of stationary supersonic solutions may result in large waves exiting the junction. On the other hand, these waves have small speed, so that the $\mathbf{L^1}$ continuous dependence apparently still holds.

From the point of view of numerical applications, the theorems and the examples below show that the integration of Riemann Problems at junctions amounts to solve a (half-) Riemann Problem along each duct together with an algebraic system at the junction. The latter system is nonlinear and contains one unknown for each duct. The present theory ensures that this problem yields a unique solution which is stable with respect to the $\mathbf{L^1}$ norm. Similar perturbation results, but related to other equations, are found in [16, 18].

2. **Solution to the Riemann problem.**

Consider n-ducts having sections α_1, ..., α_n, filled with the same fluid and all having one entry fixed at a junction. In each duct, the state of the fluid is described by the p-system (1), the pressure law being the same in all ducts.

Assigning at time $t = 0$ an initial state $(\bar\rho(x), \bar q_i(x))$ in each duct we have what, in the present setting, we call *Riemann Problem at the Junction*:

$$\left\{ \begin{array}{lrcl} \partial_t \rho_i + \partial_x q_i = 0, & t & \in & \mathbb{R}^+ \\ \partial_t q_i + \partial_x \left(\dfrac{q_i^2}{\rho_i} + p(\rho_i) \right) = 0, & x & \in & \mathbb{R}^+ \\ & i & \in & \{1, \ldots, n\} \\ (\rho_i, q_i)(0, x) = (\bar\rho_i, \bar q_i)\,, & (\bar\rho_i, \bar q_i) & \in & \mathring{\mathbb{R}}^+ \times \mathbb{R}\,. \end{array} \right. \tag{4}$$

Differently from the case of traffic flow [7, 8, 13, 14] and similarly to the standard Riemann Problem on the line [4, Chapter 5], we do not distinguish between *incoming* and *outgoing* ducts. Indeed, the direction of the fluid in the ducts is *not* assigned *a priori*, but it results from the thermodynamic states of all the ducts. Therefore, throughout our study, we consider all ducts oriented by a space variable *decreasing* towards the junction. In other words, each duct is modeled by a copy of $\mathbb{R}^+$ and the junction is at $x = 0$.

We now define the solution to Riemann Problems at junctions. Below we write (ρ, q) for the n-tuple of pairs $((\rho_1, q_1), \ldots, (\rho_n, q_n)) \in (\mathbb{R}^+ \times \mathbb{R})^n$.

Definition 1. A *solution* to the Riemann Problem (4) is a self-similar function $(\rho, q) \in \mathbf{BV}\left(\mathbb{R}^+ \times \mathbb{R}^+; (\mathring{\mathbb{R}}^+ \times \mathbb{R})^n\right)$ such that

(RP0): For $i = 1, \ldots, n$, the map $(t, x) \mapsto (\rho_i, q_i)(t, x)$ coincides with the restriction to $x \in \mathring{\mathbb{R}}^+$ of the Lax solution to the standard Riemann Problem

$$\left\{\begin{array}{l} \partial_t \rho_i + \partial_x q_i = 0 \\ \partial_t q_i + \partial_x \left(\dfrac{q_i^2}{\rho_i} + p(\rho_i)\right) = 0 \\ (\rho_i, q_i)(0, x) = \left\{\begin{array}{ll} (\bar\rho_i, \bar q_i) & \text{if } x > 0 \\ \lim\limits_{x \to 0+} (\rho_i, q_i)(t, x) & \text{if } x < 0\,. \end{array}\right. \end{array}\right.$$

(RP1): Mass is conserved at the junction, i.e. for all $t > 0$

$$\sum_{i=1}^n \alpha_i \cdot \lim_{x \to 0+} q_i(t, x) = 0\,.$$

(RP2): All ducts have the same dynamic pressure at the junction, i.e. there exists a positive constant P_* such that for all $i = 1. \ldots, n$ and all $t > 0$

$$\lim_{x \to 0+} \alpha_i \cdot \left(\frac{q_i^2(t, x)}{\rho_i(t, x)} + p\left(\rho_i(t, x)\right)\right) = P_*\,.$$

(RP3): Entropy may not decrease, i.e. for all $t > 0$

$$\sum_{i=1}^n \alpha_i \cdot \lim_{x \to 0+} F\left(\rho(t, x), q(t, x)\right) \le 0\,.$$

Above, by *self-similar* we mean that for all $i = 1, \ldots, n$, all $x \in \mathring{\mathbb{R}}^+$, all $t \in \mathring{\mathbb{R}}^+$ and all $\lambda > 0$, we have that $(\rho, q)(\lambda t, \lambda x) = (\rho, q)(t, x)$.

Condition **(RP0)** is a necessary consequence of our choice to *extend* the standard Riemann solver. The conservation of mass stated in **(RP1)** is mandatory. As stated in the Introduction, **(RP2)** is motivated by the well posedness of the Cauchy Problem; see Section 2.1. Moreover, it is consistent with some engineering models, see for example [23, § 6.3.2]. Finally, the entropy inequality **(RP3)** can be justified through the usual distributional arguments. We only note that this condition is essentially motivated by the usual entropy inequalities [4, § 4.4] and is not related to any maximization principle as in the case of traffic, see [8, 13, 14].

We stress that neither the number of waves in the solution along the ducts nor their characteristic family are restricted by the definition above. Moreover, the above definition of solution obviously does not depend on the numbering of the ducts.

We consider below the case of ducts all having the same section, i.e. $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 1$.

First, we show that Definition 1 extends the standard notion in the case of 2 ducts.

Proposition 1. *Let $n = 2$. The map $(t,x) \mapsto (\rho_*, q_*)(t,x)$ is the standard Lax solution to* (1) *with data*

$$(\rho, q)(0,x) = \left\{ \begin{array}{lccc} (\bar\rho_1, -\bar q_1) & \textit{if} & x & < & 0 \\ (\bar\rho_2, \bar q_2) & \textit{if} & x & > & 0 \end{array} \right.$$

if and only if the map

$$(t,x) \mapsto \left\{ \begin{array}{lccc} (\rho_*, -q_*)(t,-x) & \textit{if} & x & \le & 0 \\ (\rho_*, q_*)(t,x) & \textit{if} & x & \ge & 0 \end{array} \right. \tag{5}$$

solves (4) *with data $(\bar\rho_1, \bar q_1)$ and $(\bar\rho_2, \bar q_2)$ in the sense of Definition 1.*

Proof. To check **(RP0)**, use the symmetry $x \to -x$ along the duct 1. **(RP1)** is immediate. Consider **(RP2)**. Then, two cases are possible: if the solution (5) is smooth along $x = 0$, then **(RP2)**, obviously hold. On the contrary, if (5) has a jump discontinuity along $x = 0$, then **(RP2)** is nothing but the Rankine-Hugoniot relation along this stationary jump. □

Let, as usual, $\lambda_i(\rho, q)$ be the i-th characteristic speed; see (21). It is natural to

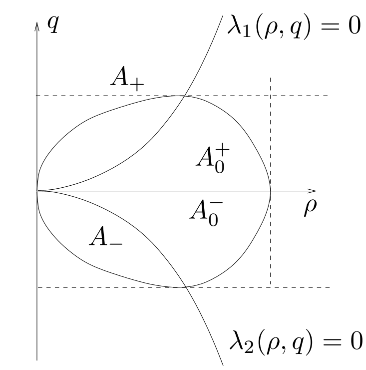

FIGURE 1. The regions A_-, $A_0^\pm$, A_+ and a level curve of the dynamic pressure.

partition the state space $\mathbb{R}^+ \times \mathbb{R}$ in the sets

$$\begin{array}{lcl} A_- & = & \{\mathbb{R}^+ \times \mathbb{R} \colon \lambda_2(\rho,q) < 0\} \\ A_0^- & = & \{\mathbb{R}^+ \times \mathbb{R} \colon \lambda_2(\rho,q) \ge 0 \text{ and } q \le 0\} \\ A_0^+ & = & \{\mathbb{R}^+ \times \mathbb{R} \colon \lambda_1(\rho,q) \le 0 \text{ and } q \ge 0\} \\ A_+ & = & \{\mathbb{R}^+ \times \mathbb{R} \colon \lambda_1(\rho,q) > 0\} \end{array} \qquad A_0 = A_0^- \cup A_0^+ \tag{6}$$

shown in Figure 1. We shall often refer to A_0 as the *subsonic* region.

A first simple example showing that **(RP3)** does indeed restrict the set of admissible solutions is in the next proposition.

Proposition 2. *Let **(P)** hold and fix $n \in \mathbb{N}$, with $n \ge 1$, and $P_* \in \mathring{\mathbb{R}}^+$. Consider n initial data $(\bar\rho_i, \bar q_i) \in A_+$ satisfying $P_* = P(\bar\rho_1, \bar q_1) = \cdots = P(\bar\rho_n, \bar q_n)$. For* *every $j = n+1, \ldots, 2n$ consider the state $(\bar\rho_j, \bar q_j) \in A_0$ such that $\bar q_j = -\bar q_{j-n}$ and $P_* = P(\bar\rho_j, \bar q_j)$. Then, the points $((\bar\rho_1, \bar q_1), \ldots, (\bar\rho_{2n}, \bar q_{2n}))$ satisfy*

$$\sum_{i=1}^{n} \bar q_i = 0 \quad \textit{and} \quad P(\bar\rho_i, \bar q_i) = P_* \quad \textit{but} \quad \sum_{i=1}^{n} F_i(\bar\rho_i, \bar q_i) > 0\,.$$

Proof. It is sufficient to prove the statement in the case $n = 1$. Consider the points $(\bar\rho_1, \bar q_1) \in A_+$ and $(\bar\rho_2, -\bar q_1) \in A_0$ such that $P(\bar\rho_2, -\bar q_1) = P(\bar\rho_1, \bar q_1)$. Then, $F(\bar\rho_1, \bar q_1) + F(\bar\rho_2, -\bar q_1) = F(\bar\rho_1, \bar q_1) - F(\bar\rho_2, \bar q_1) > 0$ by Lemma 2. □

A more interesting example is the following.

Proposition 3. *Let **(P)** hold. Consider $n+1$ ducts (with $n \geq 2$) and assign the initial states (ρ_0, q_0) and $(\rho_1, q_1), (\rho_2, q_2), \ldots, (\rho_n, q_n)$ such that*

$$q_0 \neq 0\,, \quad \sum_{i=1}^{n} q_i = -q_0 \quad \textit{and} \quad P(\rho_i, q_i) = P(\rho_0, q_0) \textit{ for } i = 1, \ldots, n\,.$$

Then

1. *If $(\rho_0, q_0) \in A_-$ and $(\rho_i, q_i) \in A_+$, then $\sum_{i=0}^{n} F(\rho_i, q_i) > 0$.*
2. *If $(\rho_0, q_0) \in A_+$ and $(\rho_i, q_i) \in A_-$, then $\sum_{i=0}^{n} F(\rho_i, q_i) < 0$.*
3. *If $(\rho_0, q_0) \in A_0^-$ and $(\rho_i, q_i) \in A_+$, then $\sum_{i=0}^{n} F(\rho_i, q_i) > 0$.*
4. *If $(\rho_0, q_0) \in A_0^+$ and $(\rho_i, q_i) \in A_-$, then $\sum_{i=0}^{n} F(\rho_i, q_i) < 0$.*
5. *If $(\rho_0, q_0) \in A_0^-$ and $(\rho_i, q_i) \in A_0^+$, then $\sum_{i=0}^{n} F(\rho_i, q_i) > 0$.*
6. *If $(\rho_0, q_0) \in A_0^+$ and $(\rho_i, q_i) \in A_0^-$, then $\sum_{i=0}^{n} F(\rho_i, q_i) < 0$.*
7. *If $(\rho_0, q_0) \in A^+$ and $(\rho_i, q_i) \in A_0^-$, then $\sum_{i=0}^{n} F(\rho_i, q_i) > 0$.*
8. *If $(\rho_0, q_0) \in A^-$ and $(\rho_i, q_i) \in A_0^+$, then $\sum_{i=0}^{n} F(\rho_i, q_i) < 0$.*

The proof is deferred to Section 4.

A condition different from **(RP2)** appears in the literature (see for instance [2, 3]), namely that the (trace of the) pressure at the junction is the same along the different ducts:

(RP2′): All ducts have the same pressure at the junction, i.e. there exists a $p_* > 0$ such that for $i = 1 \ldots, n$ and $t > 0$ $\lim_{x \to 0+} \alpha_i\, p\left(\rho_i(t,x)\right) = p_*$.

By **(P)**, in the case of ducts all having the same cross section, **(RP2')** reduces to require that the densities along all ducts have the same trace at the junction.

The Riemann Solver obtained replacing **(RP2)** with **(RP2')** leads to solutions to Cauchy Problems that fail to depend continuously in $\mathbf{L^1}$ on the initial data. Here, by *Cauchy Problem at a Junction* we mean the problem

$$\left\{\begin{array}{lrcl} \partial_t \rho_i + \partial_x q_i = 0 & t & \in & \mathbb{R}^+ \\ \partial_t q_i + \partial_x \left(\dfrac{q_i^2}{\rho_i} + p(\rho_i)\right) = 0 & x & \in & \mathbb{R}^+ \\ & i & \in & \{1, \ldots, n\} \\ (\rho_i, q_i)(0,x) = (\bar\rho_i, \bar q_i)(x) & (\bar\rho_i, \bar q_i) & \in & \mathbf{BV}\big(\mathbb{R}^+; \mathring{\mathbb{R}}^+ \times \mathbb{R}\big). \end{array}\right. \tag{7}$$

The next example is devoted to a family of Cauchy Problems whose solutions can be constructed piecing together solutions to Riemann problems. First, Definition 1 is used and continuous dependence holds. Secondly, **(RP2)** is replaced with **(RP2')** and continuous dependence is lost.

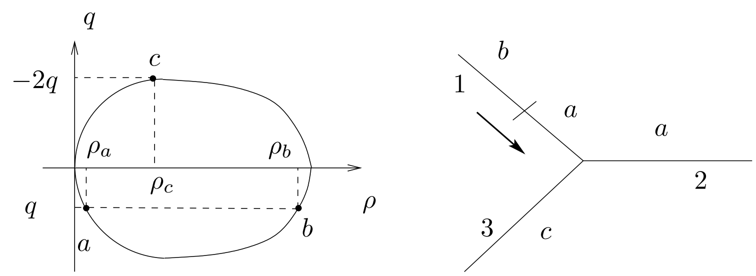

FIGURE 2. Notation for the example in Paragraph 2.1.

2.1. **Continuous dependence with (RP2).** Consider a junction with 3 ducts and a pressure satisfying **(P)**. Define the states $a \equiv (\rho_a, q)$, $b \equiv (\rho_b, q)$ and $c \equiv (\rho_c, -2q)$ such that

$$q < 0,\ \rho_a < \rho_c < \rho_b,\ \lambda_1(c) = 0 \ \text{ and } \ P(a) = P(b) = P(c), \tag{8}$$

see Figure 2. Consider the Cauchy Problem (7) with $n = 3$ and initial data

$$(\bar\rho_1, \bar q_1)(x) = \left\{ \begin{array}{l} b \text{ if } x \in [\varepsilon, +\infty[\\ a \text{ if } x \in [0, \varepsilon[\end{array} \right. \qquad \begin{array}{rcl} (\bar\rho_2, \bar q_2)(x) & = & a \\ (\bar\rho_3, \bar q_3)(x) & = & c\,. \end{array} \tag{9}$$

The Cauchy Problem (7)–(9) consists of a standard Riemann Problem along the first duct, whose Lax solution is stationary, and by a Riemann Problem at the junction.

Proposition 4. *Let p satisfy **(P)** and let a, b, c satisfy* (8). *Then, according to Definition 1, the Riemann Problem at the junction* (4) *with*

$$(\bar\rho_1, \bar q_1) = a\,, \quad (\bar\rho_2, \bar q_2) = a\,, \quad (\bar\rho_3, \bar q_3) = c\,. \tag{10}$$

is solved in the sense of Definition 1 by the stationary solution. The same holds with the data

$$(\bar\rho_1, \bar q_1) = b\,, \quad (\bar\rho_2, \bar q_2) = a\,, \quad (\bar\rho_3, \bar q_3) = c\,. \tag{11}$$

Proof. Note first that both the stationary solutions satisfy **(RP0)** obviously, **(RP1)** and **(RP2)** by (8). In the case of (4)–(10), **(RP3)** holds by 4. in Proposition 3. Concerning (4)–(11),

$$F(a) + F(b) + F(c) = \frac{q}{2} \cdot \left(\varphi(\rho_a) + \varphi(\rho_b) - 2\varphi(\rho_c)\right)\,,$$

where φ is the convex function defined in Lemma 1. By 4 in Lemma 1, the term in parenthesis is positive and, since $q < 0$, we have that **(RP3)** holds, too. □

A solution $(\rho^\varepsilon, q^\varepsilon)$ to the Cauchy Problem (7)–(9) is thus obtained piecing together the solutions to the Riemann Problems:

$$(\rho_1^\varepsilon, q_1^\varepsilon)(t,x) = \left\{ \begin{array}{l} b \text{ if } x \in [\varepsilon, +\infty[\,, \\ a \text{ if } x \in [0, \varepsilon[\,, \end{array} \right. \quad (\rho_2^\varepsilon, q_2^\varepsilon)(t,x) = a\,, \quad (\rho_3^\varepsilon, q_3^\varepsilon)(t,x) = c\,.$$

As $\varepsilon \to 0$, the limit of the solution $(\rho^\varepsilon, q^\varepsilon)$ to (7)–(9) solves the limit problem (4)–(10), the limits being in the $\mathbf{L^1}$ norm.

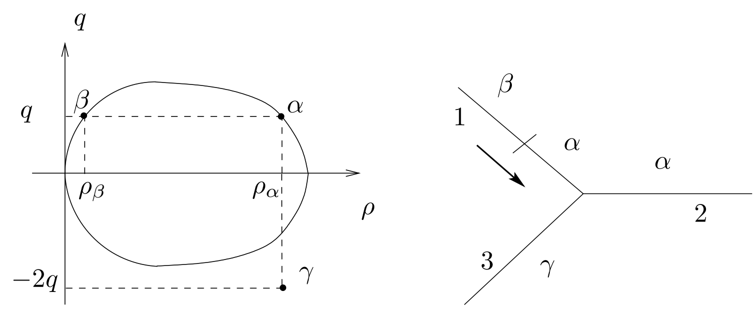

FIGURE 3. Notation for the example in Paragraph 2.1.

2.2. **Lack of continuous dependence with (RP2').** A situation analogous to the one considered above but with **(RP2')** replacing **(RP2)** leads to a lack of continuous dependence. Indeed, consider a junction with $n = 3$ ducts and a pressure satisfying **(P)**. Define the states $\alpha \equiv (\rho_\alpha, q)$, $\beta \equiv (\rho_\beta, q)$ and $\gamma \equiv (\rho_\gamma, -2q)$ such that

$$q > 0\,, \quad \rho_\beta < \rho_\alpha = \rho_\gamma\,, \quad P(\alpha) = P(\beta)\,, \quad p(\alpha) = p(\gamma) \tag{12}$$

see Figure 3. Introduce the initial datum

$$(\rho_1, q_1)(x) = \left\{ \begin{array}{l} \beta \text{ if } x \in [\varepsilon, +\infty[\\ \alpha \text{ if } x \in [0, \varepsilon[\end{array} \right. \qquad \begin{array}{rcl} (\rho_2, q_2)(x) & = & \alpha \\ (\rho_3, q_3)(x) & = & \gamma\,. \end{array} \tag{13}$$

Proposition 5. *Let p satisfy **(P)** and let α, β, γ satisfy* (12)*. Then, the stationary solution satisfies **(RP0)**, **(RP1)**, **(RP2')** and **(RP3)** for the Riemann Problem* (4) *with data*

$$(\bar\rho_1, \bar q_1) = \alpha\,, \quad (\bar\rho_2, \bar q_2) = \alpha\,, \quad (\bar\rho_3, \bar q_3) = \gamma\,. \tag{14}$$

On the other hand, the Riemann Problem with initial data

$$(\bar\rho_1, \bar q_1) = \beta\,, \quad (\bar\rho_2, \bar q_2) = \alpha\,, \quad (\bar\rho_3, \bar q_3) = \gamma\,. \tag{15}$$

does not admit a stationary solution.

Proof. Note first that the stationary solution with initial data (14) satisfy **(RP0)** obviously, **(RP1)** and **(RP2')**.

Consider **(RP3)**. We have

$$2F(\alpha) + F(\gamma) = 2q\frac{q^2}{\rho_\alpha^2} - 2q\frac{4q^2}{\rho_\gamma^2} = \frac{-6q^3}{\rho_\alpha^2} < 0\,.$$

So **(RP3)** is satisfied.

By **(RP2')** and **(P)**, in each duct, the trace of solutions to a Riemann Problem must have the same density. Moreover the traces of solution could not belong to the region A_- and hence to A_+. Therefore in the duct 1 and 2, the traces of solutions must coincide and so we conclude that the stationary solution is unique.

The Riemann Problem with initial data (15) does not admit a stationary solution, since the points α, β and γ do not satisfy **(RP2')**. □

Therefore, as $\varepsilon \to 0$ the $\mathbf{L^1}$ limit of the solution to (7)–(13) does *not* solve the limit Cauchy Problem (7) with initial datum

$$(\rho_1^0, q_1^0)(x) = \beta\,, \quad (\rho_2^0, q_2^0)(x) = \alpha\,, \quad (\rho_3^0, q_3^0)(x) = \gamma\,. \tag{16}$$

3. **Well posedness of the Riemann problem.** We now investigate the well posedness of (4), beginning with the uniqueness of some stationary solutions. The proofs below are deferred to Section 4.

Below, by *"local solution to* (4)*"* we mean a solution to (4) defined on $(\mathbb{R}^+ \times \mathbb{R}^+)^n$ with range in a small neighborhood of $(\bar\rho, \bar q)$.

Theorem 1. *Let p satisfy **(P)**. Fix $n \in \mathbb{N}$ with $n \geq 2$, a positive P_* and initial data $(\bar\rho_1, \bar q_1), \ldots, (\bar\rho_n, \bar q_n) \in \mathring{\mathbb{R}}^+ \times \mathbb{R}$ such that*

$$\begin{aligned} &\sum_{i=1}^n \bar q_i = 0 \\ &P(\bar\rho_i, \bar q_i) = P_* \qquad\qquad \textit{for } i = 1, \ldots, n \\ &\sum_{i=1}^n F(\bar\rho_i, \bar q_i) \leq 0. \end{aligned} \tag{17}$$

Then, the following implications hold:

1. *The stationary function $(\rho_i, q_i)(t, x) = (\bar\rho_i, \bar q_i)$ solves* (4) *according to Definition 1.*
2. *If $(\bar\rho_1, \bar q_1) \in A_-$, then the trace at the junction of the dynamic pressure of any local solution to* (4) *is P_*.*
3. *If at most one of the initial states is in the interior of A_+ and there exists a state in A_-, then $(\rho_i, q_i)(t, x) = (\bar\rho_i, \bar q_i)$ is the* unique *local solution to* (4)*.*
4. *If $(\bar\rho_i, \bar q_i) \in A_0$ for all i, then the stationary solution $(\rho_i, q_i)(t, x) = (\bar\rho_i, \bar q_i)$ is the unique solution to* (4) *attaining values in A_0.*

Above, the special role of the first duct is only fictitious, since for Definition 1 it is invariant with respect to the numbering of the ducts.

If at least two states are in A_+, then the uniqueness of the local solution may fail.

Example 1. Let $n = 6$ and p satisfy **(P)**. Fix $(\rho_*, q_*) \in A_+$ and choose the following initial states:

$$\begin{array}{ccccccccc} & & (\bar\rho_1, \bar q_1) & = & (\bar\rho_2, \bar q_2) & = & (\rho_*, q_*) & \in & A_+ \\ (\bar\rho_3, \bar q_3) & = & (\bar\rho_4, \bar q_4) & = & (\bar\rho_5, \bar q_5) & = & (\rho_*, -q_*) & \in & A_- \\ & & & & (\bar\rho_6, \bar q_6) & = & (\tilde\rho, q_*) & \in & A_0^+ \end{array} \tag{18}$$

so that $P(\rho_*, q_*) = P(\tilde\rho, q_*)$. Note that (17) holds. Indeed, the former two conditions are immediate, moreover

$$\sum_{l=1}^6 F(\bar\rho_l, \bar q_l) = F(\rho_*, -q_*) + F(\tilde\rho, q_*) < 0$$

by Lemma 2. Hence, the stationary solution solves (4). A further one parameter family of stationary solutions is as follows. Let $\varepsilon > 0$ and call $\rho_\varepsilon^+, \rho_\varepsilon^-$ those densities in a neighborhood of ρ_* defined by $P(\rho_\varepsilon^+, q_* + \varepsilon) = P(\rho_\varepsilon^-, q_* - \varepsilon) = P(\rho_*, q_*)$ and $(\rho_\varepsilon^+, q_* + \varepsilon), (\rho_\varepsilon^-, q_* - \varepsilon) \in A_+$. Call $(\rho^+, q^+)(t, x)$, respectively $(\rho^-, q^-)(t, x)$, the solutions to the standard Riemann problem for (1) with initial data

$$\begin{aligned} (\rho, q)(0, x) &= \begin{cases} (\rho_\varepsilon^+, q_* + \varepsilon) & \text{if } x < 0 \\ (\bar\rho_1, \bar q_1) & \text{if } x > 0 \end{cases} \quad \text{respectively} \\ (\rho, q)(0, x) &= \begin{cases} (\rho_\varepsilon^-, q_* - \varepsilon) & \text{if } x < 0 \\ (\bar\rho_2, \bar q_2) & \text{if } x > 0\,. \end{cases} \end{aligned}$$

Then, the function

$$\begin{array}{rclll} (\rho_1, q_1)(t,x) & = & (\rho^+, q^+)(t,x) & \text{for } x & \geq & 0 \\ (\rho_2, q_2)(t,x) & = & (\rho^-, q^-)(t,x) & \text{for } x & \geq & 0 \\ (\rho_i, q_i)\,(t,x) & = & (\bar\rho_i, \bar q_i) & \text{for } i & = & 3, \ldots, 6 \end{array}$$

is a non stationary local solution to (4)-(18) according to Definition 1.

We are now ready to state the well posedness of all Riemann Problems (4) with subsonic data in a neighborhood of a stationary entropic solution.

Theorem 2. *Let p satisfy **(P)**. Fix $n \in \mathbb{N}$ with $n \geq 2$ and a positive dynamic pressure P_*. Choose n initial states satisfying*

$$(\bar\rho_i, \bar q_i) \in A_0, \ \sum_{i=1}^{n} \bar q_i = 0, \ P(\bar\rho_i, \bar q_i) = P_* \quad \text{and} \quad \sum_{i=1}^{n} F(\bar\rho_i, \bar q_i) < 0. \tag{19}$$

Then, for every $C > 0$, there exists $\delta > 0$ such that the Riemann problem (4) *for all n-tuples of initial states $(\widetilde\rho, \widetilde q) \in \left(\mathring{\mathbb{R}}^+ \times \mathbb{R}\right)^n$ with $|\bar\rho_i - \widetilde\rho_i| + |\bar q_i - \widetilde q_i| < \delta$, admits a unique solution $(\rho, q) = (\rho, q)(t,x)$ in the sense of Definition 1 satisfying*

$$|\rho_i(t,x) - \bar\rho_i| + |q_i(t,x) - \bar q_i| < C, \tag{20}$$

for every $i \in \{1, \ldots, n\}$, $t \in \mathbb{R}^+$, and $x \in \mathbb{R}^+$.

In the above result, the assumption that all initial states belong to the subsonic region is necessary for the strong stability (20). If only one of the initial states is in A_-, then this stability fails, as the next example shows.

Example 2. Fix a pressure p satisfying **(P)** and consider a junction with 3 ducts. Fix the states $a \equiv (\rho_a, q)$, $b \equiv (\rho_b, q)$ and $c \equiv (\rho_c, -2q)$, with $a \in A_-$, $b, c \in A_0$ and all having the same dynamic pressure; see Figure 4.

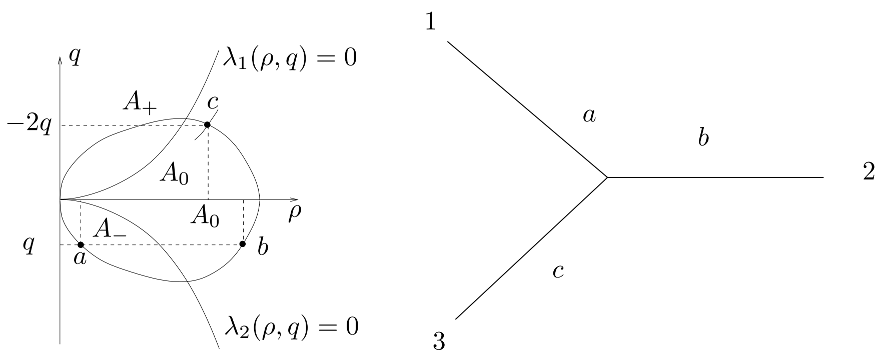

Figure 4. Notation for Example 2.

The same arguments used in Proposition 4 ensure that if c is sufficiently near to the boundary with A_+, then $F(a) + F(b) + F(c) < 0$. By Theorem 1, the constant map attaining values a, b and c on the ducts is the unique stationary solution to the Riemann Problem at the junction. Consider the Riemann Problem with data a, b, c', where c' is near to c. Then, a solution (ρ, q) to (4) satisfying (20) is constant and coincides with a along the first tube. Hence, $P(\rho_i, q_i) = P(a)$, so that by Lemma 1, $(\rho_2, q_2) \equiv b$. However, if c' is not on the 2-Lax curve exiting c and the trace of the dynamic pressure at the junction along the third tube is $P(a)$, then **(RP1)** may not hold.

4. **Technical details.** First, this paragraph recalls basic formulæ of the p-system (1) valid for a pressure satisfying **(P)**. Throughout, $c(\rho) = \sqrt{p'(\rho)}$ denotes the sound speed. Let λ_i be the i-eigenvalue corresponding to the i-right eigenvector r_i of the Jacobian of the flow $f(\rho, q) = \left[q \;\; q^2/\rho + p(\rho)\right]^T$. We have

$$\begin{array}{rclrcl} \lambda_1(\rho,q) &=& \frac{q}{\rho} - c(\rho) & \lambda_2(\rho,q) &=& \frac{q}{\rho} + c(\rho) \\ r_1(\rho,q) &=& \left[\begin{array}{c}\rho \\ q-\rho c(\rho)\end{array}\right] & r_2(\rho,q) &=& \left[\begin{array}{c}\rho \\ q+\rho c(\rho)\end{array}\right] \\ \nabla\lambda_1\cdot r_1 &=& -c(\rho)-\rho c'(\rho) & \nabla\lambda_2\cdot r_2 &=& c(\rho)+\rho c'(\rho)\,. \end{array} \tag{21}$$

The $1,2$-shock and the $1,2$-rarefaction curves exiting (ρ_o, q_o) have equations

$$\begin{array}{ll} S_1(\rho;\rho_o,q_o) = \dfrac{\rho}{\rho_o}q_o - \sqrt{\dfrac{\rho}{\rho_o}(\rho-\rho_o)\left(p(\rho)-p(\rho_o)\right)} & \text{for } \rho \geq \rho_o \\ S_2(\rho;\rho_o,q_o) = \dfrac{\rho}{\rho_o}q_o - \sqrt{\dfrac{\rho}{\rho_o}(\rho-\rho_o)\left(p(\rho)-p(\rho_o)\right)} & \text{for } \rho \leq \rho_o \\ R_1(\rho;\rho_o,q_o) = \dfrac{\rho}{\rho_o}q_o - \rho\displaystyle\int_{\rho_o}^{\rho}\dfrac{c(r)}{r}\,dr & \text{for } \rho \leq \rho_o \\ R_2(\rho;\rho_o,q_o) = \dfrac{\rho}{\rho_o}q_o + \rho\displaystyle\int_{\rho_o}^{\rho}\dfrac{c(r)}{r}\,dr & \text{for } \rho \geq \rho_o\,. \end{array}$$

The speed of $1,2$-shock waves between (ρ_o, q_o) and the state at density ρ are

$$\Lambda_1(\rho,\rho_o) = \frac{q_o}{\rho_o} - \sqrt{\frac{\rho}{\rho_o}\cdot\frac{p(\rho)-p(\rho_o)}{\rho-\rho_o}} \quad \Lambda_2(\rho,\rho_o) = \frac{q_o}{\rho_o} + \sqrt{\frac{\rho}{\rho_o}\cdot\frac{p(\rho)-p(\rho_o)}{\rho-\rho_o}}\,.$$

The (forward) $1,2$-Lax curves have expressions

$$\begin{array}{rcl} L_1(\rho;\rho_o,q_o) &=& \left\{\begin{array}{ll} R_1(\rho;\rho_o,q_o), & \text{if } \rho<\rho_o, \\ S_1(\rho;\rho_o,q_o), & \text{if } \rho>\rho_o, \end{array}\right. \\ L_2(\rho;\rho_o,q_o) &=& \left\{\begin{array}{ll} S_2(\rho;\rho_o,q_o), & \text{if } \rho<\rho_o, \\ R_2(\rho;\rho_o,q_o), & \text{if } \rho>\rho_o, \end{array}\right. \end{array}$$

see Figure 5, left, while the reversed $1,2$-Lax curves exiting $(\bar\rho, \bar q)$ are

$$\begin{array}{rcl} L_1^-(\rho;\bar\rho,\bar q) &=& \left\{\begin{array}{ll} \dfrac{\rho}{\bar\rho}\bar q + \sqrt{\dfrac{\rho}{\bar\rho}(\bar\rho-\rho)\left(p(\bar\rho)-p(\rho)\right)}, & \rho<\bar\rho, \\ \dfrac{\rho}{\bar\rho}\bar q - \rho\displaystyle\int_{\bar\rho}^{\rho}\dfrac{c(r)}{r}dr, & \rho>\bar\rho, \end{array}\right. \\ L_2^-(\rho;\bar\rho,\bar q) &=& \left\{\begin{array}{ll} \dfrac{\rho}{\bar\rho}\bar q + \sqrt{\dfrac{\rho}{\bar\rho}(\rho-\bar\rho)\left(p(\rho)-p(\bar\rho)\right)}, & \rho>\bar\rho, \\ \dfrac{\rho}{\bar\rho}\bar q - \rho\displaystyle\int_{\rho}^{\bar\rho}\dfrac{c(r)}{r}dr, & \rho<\bar\rho, \end{array}\right. \end{array} \tag{22}$$

see Figure 5, right.

The next lemma underlines properties of the Lax curves and of the level sets of the dynamic pressure that are of use in the sequel.

Lemma 1. *Let p satisfy* **(P)**. *Then the following assertions hold:*

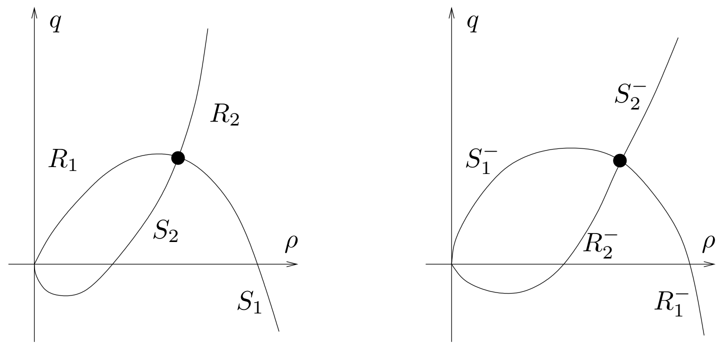

FIGURE 5. Left, the forward Lax curve and, right, the backward ones.

1. *For any $P_* > 0$, the curve $P(\rho, q) = P_*$ is the boundary of a bounded convex set; its tangents are vertical at $(0,0)$ and at $(\widehat{\rho}, 0)$, where $\widehat{\rho} = p^{-1}(P_*)$; they are horizontal at $(\check{\rho}, \pm\check{q})$, where $\lambda_1(\check{\rho}, -\check{q}) = 0$ and $\lambda_2(\check{\rho}, \check{q}) = 0$; see Figure 1.*
2. *For all $(\bar{\rho}, \bar{q}) \in A_0$, if $\left(\rho, L_2^-(\rho; \bar{\rho}, \bar{q})\right) \in A_0$ then $\frac{\partial L_2^-}{\partial \rho}(\rho; \bar{\rho}, \bar{q}) \geq 0$.*
3. *For all $(\bar{\rho}, \bar{q}) \in A_0 \cup A_+$, the map*
$$\begin{array}{rccc} \Pi: & \mathring{\mathbb{R}}^+ & \to & \mathring{\mathbb{R}}^+ \\ & \rho & \mapsto & P(\rho, L_2^-(\rho; \bar{\rho}, \bar{q})) \end{array} \tag{23}$$
is strictly increasing.
4. *For a fixed $P_* > 0$ denote by $\widehat{\rho} = p^{-1}(P_*)$. For some $\rho_* \in \,]0, \widehat{\rho}]$, define $\varphi\colon \,]0, \widehat{\rho}] \mapsto \mathbb{R}$ by*
$$\varphi(\rho) = \frac{P_* - p(\rho)}{\rho} + 2\int_{\rho_*}^{\rho} \frac{p'(r)}{r}\, dr\,.$$
Then, denoting by $(\check{\rho}, \check{q})$ the point where $P(\check{\rho}, \check{q}) = P_*$ and $\lambda_2(\check{\rho}, \check{q}) = 0$, φ strictly decreases on $[0, \check{\rho}]$, it strictly increases on $[\check{\rho}, \widehat{\rho}]$ and is strictly convex.

Proof. The point 1. is immediate and, hence, omitted. Consider 2. Then, if $\rho < \bar{\rho}$,
$$\frac{\partial L_2^-}{\partial \rho}(\rho; \bar{\rho}, \bar{q}) = \frac{\bar{q}}{\bar{\rho}} + \int_{\bar{\rho}}^{\rho} \frac{c(r)}{r}\, dr + c(\rho) = \lambda_2(\rho, L_2^-(\rho; \bar{\rho}, \bar{q})) \geq 0\,.$$
On the other hand, if $\rho > \bar{\rho}$,
$$\frac{\partial L_2^-}{\partial \rho}(\rho; \bar{\rho}, \bar{q}) = \frac{\bar{q}}{\bar{\rho}} + \frac{(2\rho - \bar{\rho})\,(p(\rho) - p(\bar{\rho})) + \rho(\rho - \bar{\rho})p'(\rho)}{2\bar{\rho}\sqrt{\frac{\rho}{\bar{\rho}}(\rho - \bar{\rho})\,(p(\rho) - p(\bar{\rho}))}} > 0\,.$$
To prove 3., fix $(\bar{\rho}, \bar{q}) \in A_0 \cup A_+$, consider the function Π at (23) and verify that $\Pi'(\rho) > 0$ for every $\rho > 0$. Preliminarily, set $q = L_2^-(\rho; \bar{\rho}, \bar{q})$ and introduce a function ψ such that
$$L_2^-(\rho; \bar{\rho}, \bar{q}) = \frac{\bar{q}}{\bar{\rho}}\rho + \psi(\rho)\,.$$
Compute:
$$\begin{aligned} \Pi'(\rho) &= \partial_\rho P(\rho, q) + \partial_q P(\rho, q) \cdot \partial_\rho L_2^-(\rho; \bar{\rho}, \bar{q}) \\ &= \left(\frac{\bar{q}}{\bar{\rho}} + \frac{\psi(\rho)}{\rho}\right) \cdot \left(\frac{\bar{q}}{\bar{\rho}} - \frac{\psi(\rho)}{\rho} + 2\psi'(\rho)\right) + p'(\rho). \end{aligned} \tag{24}$$

By (22), if $\bar\rho \geq \rho$ then $\psi'(\rho) = \frac{\psi(\rho)}{\rho} + c(\rho)$, so that

$$\Pi'(\rho) = \left(\frac{\bar q}{\bar\rho} + \frac{\psi(\rho)}{\rho} + c(\rho)\right)^2 \geq 0\,.$$

We are thus left with the case $\bar\rho \leq \rho$. Consider the right hand side in (24) as a function of $\bar q/\bar\rho$, then

$$\partial_{(\bar q/\bar\rho)}\left(\Pi'(\rho)\right) = 2\left(\frac{\bar q}{\bar\rho} + \psi'(\rho)\right)$$

By (22)

$$\begin{aligned}
\frac{\psi(\rho)}{\rho} &= \sqrt{\frac{\rho-\bar\rho}{\rho\bar\rho}\left(p(\rho)-p(\bar\rho)\right)} = \left(\sqrt{\frac{\rho}{\bar\rho}} - \sqrt{\frac{\bar\rho}{\rho}}\right)\sqrt{\frac{p(\rho)-p(\bar\rho)}{\rho-\bar\rho}}\,,\\
\psi'(\rho) &= \frac{1}{2}\left(\frac{\psi(\rho)}{\rho} + \sqrt{\frac{\rho}{\bar\rho}}\sqrt{\frac{p(\rho)-p(\bar\rho)}{\rho-\bar\rho}} + \sqrt{\frac{\rho}{\bar\rho}}\sqrt{\frac{\rho-\bar\rho}{p(\rho)-p(\bar\rho)}}p'(\rho)\right)\,.
\end{aligned}$$

By the convexity required in **(P)**,

$$p'(\rho) \geq \frac{p(\rho)-p(\bar\rho)}{\rho-\bar\rho} \geq p'(\bar\rho) \quad \text{and} \quad c(\rho) \geq \sqrt{\frac{p(\rho)-p(\bar\rho)}{\rho-\bar\rho}} \geq c(\bar\rho).$$

Therefore the summands in the expression of ψ' satisfy

$$\begin{aligned}
\sqrt{\frac{\rho}{\bar\rho}}\sqrt{\frac{p(\rho)-p(\bar\rho)}{\rho-\bar\rho}} &\geq c(\bar\rho)\\
\sqrt{\frac{\rho}{\bar\rho}}\sqrt{\frac{\rho-\bar\rho}{p(\rho)-p(\bar\rho)}}p'(\rho) &\geq \frac{1}{c(\rho)}\left(c(\rho)\right)^2 \geq c(\bar\rho)
\end{aligned}$$

proving that $\psi'(\rho) \geq c(\bar\rho)$ and

$$\partial_{(\bar q/\bar\rho)}\Pi'(\rho) \geq 0\,. \tag{25}$$

We are left to verify that $\Pi'(\rho)_{|\bar q/\bar\rho = -c(\bar\rho)} \geq 0$. Indeed:

$$\begin{aligned}
\Pi'(\rho)_{|\frac{\bar q}{\bar\rho}=-c(\bar\rho)} &= \left(-c(\bar\rho) + \frac{\psi(\rho)}{\rho}\right)\cdot\left(-c(\bar\rho) - \frac{\psi(\rho)}{\rho} + 2\psi'(\rho)\right) + p'(\rho)\\
&= \left(-c(\bar\rho) + \left(\sqrt{\frac{\rho}{\bar\rho}} - \sqrt{\frac{\bar\rho}{\rho}}\right)\sqrt{\frac{p(\rho)-p(\bar\rho)}{\rho-\bar\rho}}\right)\\
&\quad\cdot\left(-c(\bar\rho) + \sqrt{\frac{\rho}{\bar\rho}}\sqrt{\frac{p(\rho)-p(\bar\rho)}{\rho-\bar\rho}} + \sqrt{\frac{\rho}{\bar\rho}}\sqrt{\frac{\rho-\bar\rho}{p(\rho)-p(\bar\rho)}}p'(\rho)\right)\\
&\quad+p'(\rho).
\end{aligned}$$

Introducing for notational simplicity the quantities

$$\bar c = c(\bar\rho)\,, \qquad c = c(\rho)\,, \qquad \alpha = \sqrt{\frac{\rho}{\bar\rho}} \quad \text{and} \quad v = \sqrt{\frac{p(\rho)-p(\bar\rho)}{\rho-\bar\rho}}$$

that satisfy

$$c \geq v \geq \bar c \quad \text{and} \quad \alpha \geq 1\,, \tag{26}$$

we obtain

$$\begin{aligned}\Pi'(\rho)_{|\frac{\bar q}{\bar\rho}=-c(\bar\rho)} &= \left(\left(\alpha-\frac{1}{\alpha}\right)v-\bar c\right)\left(\alpha v+\frac{\alpha}{v}c^2-\bar c\right)+c^2\\ &= \left((\alpha v-\bar c)^2\right)+\frac{1}{\alpha v}\left(\alpha^2c^2-v^2\right)\cdot(\alpha v-\bar c)\,.\end{aligned}$$

By (26), the latter expression is the sum of two non negative summands, completing the proof of 2.

Concerning 4, compute the first derivative $\varphi'(\rho)=\frac{p(\rho)+\rho p'(\rho)-P_*}{\rho^2}$. It vanishes at $\check\rho$, is negative for $\rho<\check\rho$ and positive for $\rho>\check\rho$. The second derivative $\varphi''(\rho)=2\frac{P_*-p(\rho)}{\rho^3}+\frac{p''(\rho)}{\rho}$ is strictly positive. □

The next lemma classifies those stationary solutions in the case of $n=2$ ducts that are entropic. With respect to standard treatment, we only note that here the number of waves is not *a priori* selected.

Lemma 2. *Let **(P)** hold. Then, the following implications hold:*

1. $\left.\begin{array}{rcl}(\rho_+,q)&\in&A_+\\(\rho_-,-q)&\in&A_0^-\\P(\rho_+,q)&=&P(\rho_-,-q)\end{array}\right\}\Rightarrow F(\rho_+,q)+F(\rho_-,-q)>0\,,$

2. $\left.\begin{array}{rcl}(\rho_+,q)&\in&A_+\\(\rho_-,-q)&\in&A_-\\P(\rho_+,q)&=&P(\rho_-,-q)\end{array}\right\}\Rightarrow \left\{\begin{array}{l}F(\rho_+,q)+F(\rho_-,-q)=0\,,\\ \rho_+=\rho_-\,,\end{array}\right.$

3. $\left.\begin{array}{rcl}(\rho_+,q)&\in&A_0^+\\(\rho_-,-q)&\in&A_-\\P(\rho_+,q)&=&P(\rho_-,q)\end{array}\right\}\Rightarrow F(\rho_+,q)+F(\rho_-,-q)<0\,,$

4. $\left.\begin{array}{rcl}(\rho_+,q)&\in&A_0^+\\(\rho_-,-q)&\in&A_0^-\\P(\rho_+,q)&=&P(\rho_-,-q)\end{array}\right\}\Rightarrow \left\{\begin{array}{l}F(\rho_+,q)+F(\rho_-,-q)=0\,,\\ \rho_-=\rho_+\,.\end{array}\right.$

Proof. The second and fourth relations are immediate. To prove the first one, note preliminarily that

$$P(\rho_+,q)=P(\rho_-,q)\Leftrightarrow\frac{q^2}{\rho_+}-\frac{q^2}{\rho_-}=p(\rho_-)-p(\rho_+)\,.$$

Since $q>0$, we prove that $\frac{2}{q}\left(F(\rho_+,q)+F(\rho_-,-q)\right)>0$. We have

$$\begin{aligned}2\frac{F(\rho_+,q)+F(\rho_-,-q)}{q} &= \frac{q^2}{\rho_+^2}-\frac{q^2}{\rho_-^2}-2\int_{\rho_+}^{\rho_-}\frac{p'(r)}{r}\,dr\\ &= \frac{\rho_-+\rho_+}{\rho_-\rho_+}\left(p(\rho_-)-p(\rho_+)\right)-2\int_{\rho_+}^{\rho_-}\frac{p'(r)}{r}\,dr\end{aligned}$$

Define the function $h(\rho)=\dfrac{\rho+\rho_+}{\rho_+\rho}\left(p(\rho)-p(\rho_+)\right)-2\displaystyle\int_{\rho_+}^{\rho}\frac{p'(r)}{r}\,dr$. It is clear that $h(\rho_+)=0$. Moreover $h'(\rho)=\frac{\rho^2p'(\rho)-\rho\rho_+p'(\rho)-\rho_+p(\rho)+\rho_+p(\rho_+)}{\rho^2\rho_+}$ and $h'(\rho_+)=0$. The sign of the derivative of h is the same as the sign of the numerator $g(\rho)=\rho^2p'(\rho)-\rho\rho_+p'(\rho)-\rho_+p(\rho)+\rho_+p(\rho_+)$. We have $g(\rho_+)=0$ and $g'(\rho)=p''(\rho)\rho(\rho-\rho_+)+2p'(\rho)(\rho-\rho_+)$. If $\rho>\rho_+$, then $g'(\rho)>0$, which implies that $g(\rho)>0$. This permits to conclude that $h(\rho)>0$ if $\rho>\rho_+$ and the lemma is proved.

The third implication follows, since $F(\rho,-q)=-F(\rho,q)$. □

The following corollary is an immediate consequence of Lemma 2 and we omit its proof.

Corollary 1. *Let **(P)** hold. Let $(\rho_-, q), (\rho_+, q)$ be such that $P(\rho_-, q) = P(\rho_+, q)$ and $\rho_- < \rho_+$. Then,*

$$\begin{array}{lccccccc} (1) & q & > & 0 & \Rightarrow & F(\rho_-, q) & > & F(\rho_+, q)\,, \\ (2) & q & < & 0 & \Rightarrow & F(\rho_-, q) & < & F(\rho_+, q)\,. \end{array}$$

Consider now Proposition 3, which extends Lemma 2 to cases with many ducts.

Proof of Proposition 3. In each of the cases above, let $(\tilde\rho, \tilde q)$ be such that $P(\tilde\rho, \tilde q) = P(\rho_0, q_0)$, $\tilde q = -q_0$ and $(\tilde\rho, \tilde q)$ belongs to the same region as (ρ_i, q_i) for $i = 1, \ldots, n$.

1. Note that $\tilde\rho > \max_{i=1,\ldots,n} \rho_i$ and $F(\rho_0, q_0) + F(\tilde\rho, \tilde q) = 0$, by Lemma 2. Hence, by 4 in Lemma 1,

$$\begin{array}{rclrcl} \displaystyle\min_{i=1,\ldots,n} \varphi(\rho_i) & > & \varphi(\tilde\rho)\,, & \displaystyle\sum_{i=1}^n \frac{q_i}{\tilde q} \varphi(\rho_i) & > & \varphi(\tilde\rho)\,, \\ \displaystyle\sum_{i=1}^n q_i\, \varphi(\rho_i) & > & \tilde q\, \varphi(\tilde\rho)\,, & \displaystyle\sum_{i=1}^n F(\rho_i, q_i) & > & F(\tilde\rho, \tilde q)\,, \\ \displaystyle\sum_{i=1}^n F(\rho_i, q_i) & > & -F(\rho_0, q_0)\,, & \displaystyle\sum_{i=0}^n F(\rho_i, q_i) & > & 0\,. \end{array}$$

2. Using step 1. above:

$$\sum_{i=0}^n F(\rho_i, q_i) = \sum_{i=0}^n -F(\rho_i, -q_i) = -\sum_{i=0}^n F(\rho_i, -q_i) < 0\,.$$

3. Let $(\widehat\rho_0, q_0) \in A_-$ be such that $P(\widehat\rho_0, q_0) = P(\rho_0, q_0)$. By (2) of Corollary 1 and step 1. above

$$\sum_{i=0}^n F(\rho_i, q_i) > F(\widehat\rho_0, q_0) + \sum_{i=1}^n F(\rho_i, q_i) > 0\,.$$

4. Using step 3. above:

$$\sum_{i=0}^n F(\rho_i, q_i) = \sum_{i=0}^n -F(\rho_i, -q_i) = -\sum_{i=0}^n F(\rho_i, -q_i) < 0\,.$$

5. Now, $\tilde\rho < \min_{i=1,\ldots,n} \rho_i$. Similar computations as in step 1. still hold.
6. Using step 5. above:

$$\sum_{i=0}^n F(\rho_i, q_i) = \sum_{i=0}^n -F(\rho_i, -q_i) = -\sum_{i=0}^n F(\rho_i, -q_i) < 0\,.$$

7. Using the map φ defined in 4 of Lemma 1 and its properties,

$$\begin{aligned}
\sum_{i=0}^{n} F(\rho_i, q_i) &= \frac{q_0}{2} \cdot \left(\frac{P_* - p(\rho_0)}{\rho_0} - \sum_{i=1}^{n} \frac{|q_i|}{q_0} \varphi(\rho_i) \right) \\
&> \frac{q_0}{2} \cdot \left(\frac{P_* - p(\rho_0)}{\rho_0} - \varphi\left(\max_{i=1,\dots,n} \rho_i \right) \right) \\
&\geq \frac{q_0}{2} \cdot \left(\frac{P_* - p(\rho_0)}{\rho_0} - \varphi(\rho_*) \right) \\
&= \frac{q_0}{2} \cdot \left(\frac{P_* - p(\rho_0)}{\rho_0} - \frac{P_*}{\rho_*} + \frac{p(\rho_0)}{\rho_0} - \int_{\rho_0}^{\rho_*} \frac{p(r)}{r^2}\, dr \right) \\
&= \frac{q_0}{2} \cdot P_* \cdot \left(\frac{1}{\rho_0} - \frac{1}{\rho_*} \right) - \int_{\rho_0}^{\rho_*} \frac{p(r)}{r^2}\, dr \\
&\geq \frac{q_0}{2} \cdot P_* \cdot \left(\frac{1}{\rho_0} - \frac{1}{\rho_*} - \int_{\rho_0}^{\rho_*} \frac{1}{r^2}\, dr \right) \\
&= 0\,.
\end{aligned}$$

8. Use the previous step and the symmetry $F(\rho, -q) = -F(\rho, q)$.

This concludes the proof. □

Proof of Theorem 1. Let $(t, x) \mapsto (\tilde\rho, \tilde q)(t, x)$ be a solution to (4) in the sense of Definition 1 and let $\tilde P$ be the corresponding value of the dynamic pressure at the junction.

1. Given n states satisfying (17), it is immediate to verify that they yield a solution to (4).
2. Necessarily, $(\tilde\rho_1, \tilde q_1) = (\bar\rho_1, \bar q_1)$ by the signs of the characteristic speeds; hence $\tilde P = P_*$.
3. By 3., we deduce that $\tilde P = P_*$. If $(\bar\rho_i, \bar q_i) \in A_0$, then automatically the solution for the i-th duct is the stationary one. We conclude in the case no states is in A_+. Assume therefore that $(\bar\rho_1, \bar q_1)$ belongs to A_+. Conditions **(RP1)** and **(RP2)** imply that also for this duct the solution is the stationary one.
4. Assume that there exists another solution with values in A_0 and call its trace at the junction $(\tilde\rho, \tilde q)$, with $P_{**} = P(\tilde\rho_1, \tilde q_1)$. If $P_{**} > P_*$, then by 3 and 2 in Lemma 1, $\tilde q_i > \bar q_i$ for all i, hence $\sum_{i=1}^n \tilde q_i > 0$. Similarly, if $P_{**} < P_*$, then $\tilde q_i < \bar q_i$ for all i and $\sum_{i=1}^n \tilde q_i < 0$.

This concludes the proof. □

Proof of Theorem 2. By Definition 1, solving (4) means finding an n-tuple of pairs (ρ, q) such that

$$\begin{cases}
L_2(\widetilde\rho_i; \rho_i, q_i) = \widetilde q_i & i = 1, \dots, n \\
\sum_{i=1}^n q_i = 0 & \\
P(\rho_i, q_i) = P(\rho_{i-1}, q_{i-1}) & i = 2, \dots, n \\
\sum_{i=1}^n F(\rho_i, q_i) \leq 0. &
\end{cases}$$

Using the reversed Lax curve (22) of the second family, the above system reduces to the following n equations

$$\begin{cases}
\sum_{i=1}^n L_2^-(\rho_i; \widetilde\rho_i, \widetilde q_i) = 0 & \\
P\left(\rho_i, L_2^-(\rho_i; \widetilde\rho_i, \widetilde q_i)\right) = P\left(\rho_{i-1}, L_2^-(\rho_{i-1}; \widetilde\rho_{i-1}, \widetilde q_{i-1})\right) & i = 2, \dots, n
\end{cases}$$

in the unknown variables $\rho_1, \ldots, \rho_n$, coupled with the entropy inequality

$$\sum_{i=1}^{n} F\left(\rho_i, L_2^-(\rho_i; \tilde{\rho}_i, \tilde{q}_i)\right) \leq 0.$$

To apply the Implicit Function Theorem, compute preliminarily

$$\begin{aligned}
\frac{\partial P\left(\rho, L_2^-(\rho; \bar{\rho}, \bar{q})\right)}{\partial \rho}_{|\rho=\bar{\rho}} &= \frac{\partial}{\partial \rho}\left(\frac{\left(L_2^-(\rho; \bar{\rho}, \bar{q})\right)^2}{\rho} + p(\rho)\right)_{|\rho=\bar{\rho}} \\
&= \frac{2\,\bar{q}\,\lambda_2(\bar{\rho}, \bar{q})\,\bar{\rho} - \bar{q}^2}{\bar{\rho}^2} + p'(\bar{\rho}) \\
&= \left(\lambda_2(\bar{\rho}, \bar{q})\right)^2 .
\end{aligned}$$

We are thus lead to consider the matrix

$$\begin{bmatrix}
\lambda_2(\bar{\rho}_1, \bar{q}_1) & \lambda_2(\bar{\rho}_2, \bar{q}_2) & \lambda_2(\bar{\rho}_3, \bar{q}_3) & \cdots & \lambda_2(\bar{\rho}_{n-1}, \bar{q}_{n-1}) & \lambda_2(\bar{\rho}_n, \bar{q}_n) \\
\lambda_2(\bar{\rho}_1, \bar{q}_1)^2 & -\lambda_2(\bar{\rho}_2, \bar{q}_2)^2 & 0 & \cdots & 0 & 0 \\
0 & \lambda_2(\bar{\rho}_2, \bar{q}_2)^2 & -\lambda_2(\bar{\rho}_3, \bar{q}_3)^2 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & \lambda_2(\bar{\rho}_{n-1}, \bar{q}_{n-1})^2 & -\lambda_2(\bar{\rho}_n, \bar{q}_n)^2
\end{bmatrix}$$

whose determinant, by Lemma 3, is

$$(-1)^{n+1} \cdot \left(\prod_{i=1}^{n} \lambda_2(\bar{\rho}_i, \bar{q}_i)\right) \cdot \sum_{i=1}^{n} \prod_{j \neq i} \lambda_2(\bar{\rho}_j, \bar{q}_j)$$

which is non zero if $(\bar{\rho}_i, \bar{q}_i) \in A_0$. An application of the Implicit Function Theorem and of the fact that the entropy is continuous along the Lax curves of the second family, completes the proof. $\square$

Lemma 3. *Let $n \in \mathbb{N}$, $n \geq 2$, and $a_1, \ldots, a_n$ be in $\mathbb{R}$. Then*

$$\det \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 & 1 \\
a_1 & -a_2 & 0 & \cdots & 0 & 0 \\
0 & a_2 & -a_3 & \cdots & 0 & 0 \\
0 & 0 & a_3 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & a_{n-1} & -a_n
\end{bmatrix} = (-1)^{n+1} \sum_{i=1}^{n} \prod_{j \neq i} a_j$$

A simple induction argument, which we omit, suffices to prove this lemma.

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