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Research article

An efficacious improvement of the Anuj transformation method for solving higher-order fractional differential equations with constant coefficients

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Abstract: In this study, we obtained the fractional formula of the Anuj transformation, which is utilized to acquire an accurate outcome for linear fractional differential equations (LFDEs). It is employed for Riemann-Liouville's and Caputo's fractional derivatives. To do this, we started with developing the Anuj transform of basic functions in mathematics and then examined its primary properties, which may be used in solving different mathematical models, especially fractional differential equations. We then proceeded to present the exact solution for a particular case of a fractional differential equation. We explored four numerical challenges and presented a thorough solution for each to illustrate how the studied transform may be useful. The findings revealed that the newly recommended transformation and the specific solutions that have been supplied are more effective and straightforward in solving mathematical models. The obtained formula has been utilized to solve different cases of fractional differential equations and reach a precise solution. The outcomes have been expressed using two-dimensional graphs.

Keywords: fractional derivatives; Anuj transform; inverse Anuj transform; linear fractional differential equations

1. Introduction

Studying fractional derivatives and integrals is fundamental to solving fractional differential equations. Fractional differential equations are effective in simulating complicated physical processes such as diffusion, viscoelasticity, and wave propagation. The application of fractional differential equations in control systems, signal processing, and mechanical systems has gained momentum owing to their ability to capture non-local and memory-dependent phenomena. Fractional differential equations have developed as a strong tool for modeling biological phenomena such as population dynamics, bio-molecular interactions, and neural signaling. In all, the applications of fractional differential equations cover many domains and give a better insight into the behavior of complex systems [1, 2]. They may be utilized in the broadening of scientific and technical domains, such as chemistry, biology, finance, fluid mechanics, abnormal diffusion, viscoelasticity, and more. In most applications, a set of integro-differential and integro equations with singularities yields fractional differential equations [3, 4]. Fractional differential equations may already be solved using an array of published analytical or numerical approaches, including [5]. Since integral transforms have advantages over other mathematical methods for solving models related to basic science, modeling, and special physical engineering, including simplicity, and the capacity to generate results without requiring boring calculations, they are currently the approach of choice for investigators [6–8].

The Kamal transformation is one of the unique integral transformations that was created to solve ordinary differential equations, including first- and higher-order differential equations and fractional differential equations [9]. Aboodh [10], Sumudu [11], Anuj [12], Aggarwal, and other specialists addressed common issues with a variety of integral transformations such as the Rishi transformation [13, 14]. Sawi transformations are some examples of contemporary developments in fractional differential equations and integral transforms [15, 16], the Bessel collocation method to solve Fredholm–Volterra integro-fractional differential equations of multi-high order in the Caputo sense [17], the existence and uniqueness of the solution for a fractional-order Regge problem [18], the asymptotic behavior of eigenvalues and eigenfunctions of the fractional Regge problem [19], the Anuj transform to solve Volterra integral equations of the first kind [20], and the Kamal transform to solve fractional ordinary differential equations [21]. Researchers have created and developed several methods to solve various mathematical models. These include: the Bernoulli sub-ODE and its improved version, methods related to trigonometric, and hyperbolic trigonometric functions with various improvements and modifications [22–25] the modified rational $\sin-\cos$ technique, the $(1/G')$ expansion scheme, the ansatz approach, and conjugate direction methods [26–29]. The existence and uniqueness of solutions for Hadamard implicit fractional differential equations with extended Hadamard fractional integro-differential boundary conditions were established using the contraction concept of the Banach and Leray-Schauder fixed point theorems [30]. A coupled system of nonlinear impulsive Langevin equations with four Hilfer fractional-order derivatives was examined by employing the methodologies of nonlinear functional analysis [31]. The jerk-type fractional differential equations in the method of Hadamard and Caputo fractional derivatives with detached boundary conditions were studied in [32], and the corresponding solutions of ordinary differential equations with variable coefficients were obtained using the Anuj transform in [33].

By developing a novel integral transform called the Anuj transform, which has certain characteristics necessary for fractional calculus and fractional differential equations, our current effort aims to solve fractional differential equations. One can observe that the Anuj transform that offered here, is better than the other transforms that have previously been produced, as it addresses the difficulties properly and does not need complicated calculations. Both the popular and widely used Laplace transform and the Anuj transformation are dualistic. We want to create a fractional formula for additional transformations, as is done with most transformations. We have acquired the fractional formula of the transformation and used it to analyze several problems. A number of other transformations for fractional derivatives and integrals were also devised. This study introduces the Anuj transform, a new integral transformation designed to solve ordinary, partial, and fractional differential equations. It provides exact solutions more efficiently than existing transforms, including the Laplace transform. A fractional formula was also derived to handle fractional integrals and derivatives using this transform.

This investigation inquiry ia organized as follows: Section 1 is restricted for reviewing the literature connected to the Anuj transform technique and assessing it in a simple summary. The important and universal characteristics of fractional calculus are illustrated in Section 2. The main properties of the suggested technique have been provided in Section 3. Anuj transformations of fractional integrals and fractional derivatives have been presented in Section 4. Section 5 is specialized for certain interesting cases. The acquired findings have been described in Section 6. Finally, the closing remarks are offered in Section 7.

2. Fundamental properties of fractional calculus

We have provided remark, definitions, theorems, and propositions for the various fraction types that are necessary for our goal in this section.

Remark 2.1. In this article, we have dealt with $C^n[a,b]$ space, so we have the following mathematical statement:

$$f(t) \in C^n\ [0,b],\ {}_0^C D_x^{\alpha} f(t) \in C^n[0,b],$$

where $C[a,b]$ is the space of all continuous functions from closed interval $[a,b]$ to $\mathbf{R}$.

Definition 2.1. (See [2, 5]) The Riemann-Liouville fractional integral of fractional order $\beta > 0$ is:

$$I_x^{\beta} f(x) = \frac{1}{\Gamma(\beta)} \int_a^x (x-u)^{\beta-1} f(u)\, du, -\infty \leq a < x < \infty, \tag{2.1}$$

such that gamma function $\Gamma(\beta)$ is defined by:

$$\Gamma(\beta) = \int_0^{\infty} x^{\beta-1} e^{-x} dx.$$

Definition 2.2. (See [2, 5, 7]) The fractional derivative of order β in the sense of Riemann-Liouville where $\beta > 0$ and $\mu = \lceil \beta \rceil$ is defined as:

$$ {}_{a}^{RL}D_{x}^{\beta} f(x) = \frac{1}{\Gamma(\mu-\beta)} \frac{d^{\mu}}{dx^{\mu}} \int_a^x (x-u)^{\mu-\beta-1} f(u)\, du. \quad (2.2)$$

Definition 2.3. (See [5,17]) The Liouville-Caputo fractional derivative of order β where $\beta > 0$, $n = \lceil \beta \rceil$, and $f(x)$ is a differentiable function, for all $x \in [a, \infty)$, is defined as follows:

$$ {}_{a}^{LC}D_{x}^{\beta} f(x) = \frac{1}{\Gamma(n-\beta)} \int_a^x (x-u)^{n-\beta-1} \left(\frac{d}{du}\right)^n f(u)\, du. \quad (2.3)$$

In the rest of this section, we want to demonstrate some of the introductory characteristics of fractional calculus [2, 5, 18, 19]:

(1) Fractional operators are linear (integral and differential).

(2) The following represents the definition of composition between two Riemann-Liouville integrations of orders e and b:

$$ {}_{a}I_{s}^{e}\, {}_{a}I_{s}^{b} f(s) = {}_{a}I_{s}^{b}\, {}_{a}I_{s}^{e} f(s) = I_{t}^{e+b} f(s). \quad (2.4)$$

(3) For $l \geq \beta$ and $f(s) \in C[a, b]$, and for every element $s \in [a, b]$, then

$$ {}_{a}^{RL}D_{s}^{l} \left({}_{a}I_{s}^{\beta} f(t) \right) = {}_{a}^{RL}D_{s}^{l-\beta} f(s),$$

the relation is done.

(4) The Liouville-Caputo operator of order α is defined as follows in terms of composition between fractional differentiating and fractional integrating.

$$ {}_{a}^{LC}D_{s}^{\beta} \left({}_{a}I_{s}^{\beta} f(s) \right) = f(s). \quad (2.5)$$

(5) The Liouville-Caputo operator of order α, between fractional (integration and differentiation) composition and $m = \lceil \alpha \rceil$, is defined as:

$$ {}_{a}I_{t}^{\alpha} \left({}_{a}^{LC}D_{t}^{k} f(t) \right) = f(t) - \sum_{k=0}^{m-1} \frac{(t-a)^k}{k!} f^{(k)}(a). \quad (2.6)$$

In general,

$$ {}_{a}^{LC}D_{t}^{k} \left({}_{a}I_{t}^{\alpha} f(t) \right) \neq {}_{a}I_{t}^{\alpha} \left({}_{a}^{LC}D_{t}^{k} f(t) \right).$$

(6) Using the differential and fractional integral of the Liouville-Caputo operator of function s^m, $m \geq 0$, we obtain:

$$ {}_{a}I_{t}^{\alpha} s^m = \frac{\Gamma(1+m)}{\Gamma(1+m+\alpha)} s^{m+\alpha} \quad (2.7)$$

and

$$ {}_{a}^{LC}D_{t}^{k} s^m = \frac{\Gamma(1+m)}{\Gamma(1+m-\alpha)} s^{m-\alpha}. \quad (2.8)$$

3. Basic features of Anuj transformations

Here, we define the Anuj transform and its characteristics. We provide an overview of the fractional formula of the new transform by several further investigations [12, 20].

Definition 3.1. Anuj transforms (See [12]) The piece-wise continuous function that is exponentially ordered and has the Anuj transform function $f(u)$ defined in the interval $[0, \infty)$ is as follows:

$$A\{f(u)\} = p^2 \int_0^{\infty} f(u) e^{-\left(\frac{1}{p}\right)u} du = F(p), \; p > 0. \quad (3.1)$$

Property 1: [20] Some basic functions and their Anuj transformations have been organized in the Table 1:

Table 1. Anuj transformations for some basic functions.

$f(t),\ t > 0$	$A\{f(t)\} = F(p)$	$f(t),\ t > 0$	$A\{f(t)\} = F(p)$
1	p^3	$\sin kt$	$\frac{kp^4}{1+k^2p^2}$
e^{kt}	$\frac{p^3}{1-kp}$	$\cos kt$	$\frac{p^3}{1+k^2p^2}$
$t^{\beta},\ \beta \in \mathbf{N}$	$p^{\beta+3}\ \beta!$	$\sinh kt$	$\frac{kp^4}{1-k^2p^2}$
$t^{\beta},\ \beta > -1,\ \beta \in \mathbf{R}$	$p^{\beta+3}\ \Gamma(\beta+1)$	$\cosh kt$	$\frac{p^3}{1-k^2p^2}$

Property 2: [12] The inverse Anuj transformations for some fundamental functions have been addressed in the following (Table 2):

Table 2. The inverse Anuj transformations for some well-known functions.

$A\{f(t)\} = F(p)$	$f(t),\ t > 0$	$A\{f(t)\} = F(p)$	$f(t),\ t > 0$
p^3	1	$\frac{p^3}{1+k^2p^2}$	$\cos kt$
$\frac{p^3}{1-kp}$	e^{kt}	$\frac{kp^4}{1-k^2p^2}$	$\sinh kt$
$p^{\beta+3}$	$\frac{t^\beta}{\Gamma(\beta+1)},\ \beta > -1, \beta \in \mathbf{R}$	$\frac{p^3}{1-k^2p^2}$	$\cosh kt$
$p^{\beta+3}$	$\frac{t^\beta}{\beta!},\ \beta \in \mathbf{N}$	$\frac{kp^4}{1+k^2p^2}$	$\sin kt$

Property 3. (See [12]) (Convolution theorem) let $A\{k(x)\} = K(p)$ and $A\{l(x)\} = L(p)$, and then

$$A\{k(x) * l(x)\} = \frac{1}{p^2} K(p)L(p),$$

such that * shows the convolution of L and l, and

$$k(x) * l(x) = \int_0^t k(x-u)l(u)du.$$

Property 4. (See [12]) Both the Anuj transform and its inverse are linear operators:

(i) The Anuj transformation is a linear operator.

$$A\left\{\sum_{i=0}^{n} a_i f_i(t)\right\} = \sum_{i=0}^{n} a_i A\{f_i(t)\},$$

such that a_i are arbitrary constants.

(ii) The inverse Anuj transformation is also a linear operator.

If $f_i(t) = A^{-1}\{F_i(p)\}$, then

$$A^{-1}\left\{\sum_{i=0}^{n} a_i F_i(p)\right\} = \sum_{i=0}^{n} a_i A^{-1}\{F_i(p)\},$$

such that a_i are arbitrary constants.

4. Anuj transformation of fractional integrals and fractional derivatives

In the following section, we provide the modified fractional formula of the Anuj transform linked to fractional integrals and fractional derivatives by combining the existing convolution rule with the existing properties.

Property 5. (See [12]) The Anuj transformation for the function $f(t)$ that has an integer-order derivative is defined as follows:

$$A\{f^n(t)\} = \frac{1}{p^n} F(p) - \sum_{k=0}^{n-1} p^{2-k} f^{n-k-1}(0). \tag{4.1}$$

The proof is omitted, as it is easy to prove by using mathematical induction.

4.1. Anuj transformation of fractional integrals

Theorem 4.1. If $\alpha \in [n-1, n)$, the fractional integral by applying the Anuj transform is:

$$A[I^\alpha f(t)] = p^\alpha F(p). \tag{4.2}$$

Proof.

$$\begin{aligned} A\{I^\alpha f(t)\} &= A\{D^{-\alpha} f(t)\}, \\ &= \frac{1}{\Gamma(\alpha)} A\left\{\int_0^t (t-u)^{\alpha-1} f(u)du\right\}, \\ &= \frac{1}{\Gamma(\alpha)} \frac{1}{p^2} A\left\{t^{\alpha-1}\right\} F(p), \\ &= \frac{1}{\Gamma(\alpha)} \frac{1}{p^2} p^{\alpha+2} \Gamma(\alpha) F(p), \\ &= p^\alpha F(p). \end{aligned}$$

□

4.2. Anuj transformation of fractional derivatives

The Anuj transform for derivatives with integer numbers has been described by Eq (4.1). We aim to refine and develop the Anuj transform for fractional derivatives of both sorts (Caputo and Reimann). The primary key to this improvement is the indicated formula (4.1). To reach this goal, we first start with the following theorems:

Theorem 4.2. If $f(t)$ is a function and $F(p)$ is an Anuj transform for the Reimann-Liouville fractional derivative with order α, then:

$$A[D^\alpha f(t)] = p^{-\alpha} F(p) - \sum_{k=0}^{n-1} p^{2-k} D^{\alpha-k-1}(f(0)). \tag{4.3}$$

Proof. By using the Riemann-Liouville fractional derivative and the corresponding fractional integral property, the following is achieved:

$$A\left[D^{\alpha}f(t)\right]=A\left\{D^{n}I^{n-\alpha}f(t)\right\},$$

and from the properties of the Anuj transform for derivatives, we have:

$$A\left\{f^{n}(t)\right\}=\frac{1}{p^{n}}F(p)-\sum_{k=0}^{n-1}p^{2-k}f^{n-k-1}(0),$$

so,

$$\begin{aligned}A\left\{D^{\alpha}f_{(t)}\right\}&=A\left\{D^{n}I^{n-\alpha}f(t)\right\},\\&=\frac{1}{p^{n}}A\left\{I^{n-\alpha}f(t)\right\}-\sum_{k=0}^{n-1}p^{2-k}\frac{d^{n-k-1}}{dt^{n-k-1}}I^{n-\alpha}f(0),\\&=\frac{1}{p^{n}}p^{n-\alpha}F(p)-\sum_{k=0}^{n-1}p^{2-k}D^{\alpha-k-1}(f(0)),\\&=p^{-\alpha}F(p)-\sum_{k=0}^{n-1}p^{2-k}D^{\alpha-k-1}(f(0)).\end{aligned}$$

□

Theorem 4.3. Let $f(t)$ be a function and $F(p)$ be an Anuj transform. Then the Anuj transform for the Caputo fractional derivative of order α is:

$$A\left\{{}^{c}D_{t}^{\alpha}f(t)\right\}=p^{-\alpha}\cdot F(p)-\sum_{k=0}^{n-1}p^{n-\alpha-k+2}f^{n-k-1}(0).$$

Proof. Using the Caputo fractional derivative and the related fractional integral, one obtains:

$$A\left\{{}^{c}D_{t}^{\alpha}f(t)\right\}=A\left\{I^{n-\alpha}D^{n}f(t)\right\},$$

so,

$$\begin{aligned}A\left\{{}^{c}D_{t}^{\alpha}f(t)\right\}&=A\left\{I^{n-\alpha}D^{n}\cdot f(t)\right\},\\&=p^{n-\alpha}\left\{\frac{1}{p^{n}}F(p)-\sum_{k=0}^{n-1}p^{2-k}f^{n-k-1}(0)\right\},\\&=p^{-\alpha}F(p)-\sum_{k=0}^{n-1}p^{n-\alpha-k+2}f^{n-k-1}(0).\end{aligned}$$

□

A comparison among formulas of the well-known transformations of Laplace, Rishi, and Anuj has been introduced in Table 3 on the following page [14, 17]:

5. Illustrative examples

This section contains four fractional problems that illustrate how to use the Anuj transform to identify the exact (analytic) solution of the provided multiple high-order linear fractional differential equations in the sense of Liouville-Caputo and Riemann-Liouville.

Example 5.1. For Riemann-Liouville's fractional derivative, consider the following linear fractional differential equation:

$$\begin{aligned}{}_{0}^{R}D_{t}^{1.3}y(t)+2\frac{d}{dt}y(t)&=\frac{6}{\Gamma(2.7)}t^{1.7}\\&\quad-\frac{2}{\Gamma(1.7)}t^{0.7}+6t^{2}-4t,\end{aligned}\tag{5.1}$$

where the following initial conditions are provided:

$$y(0)=0,\ \left[{}_{0}^{R}D_{t}^{0.3}(y(t))\right]_{t=0}=0,\ \text{and}\ \left[{}_{0}^{R}D_{t}^{-0.7}(y(t))\right]_{t=0}=0.$$

Solution: By applying the Anuj transform to both sides, where Theorem 4.2 is considered, one achieves this form:

$$\begin{aligned}&A\left\{{}^{R}D_{t}^{1.3}y(t)\right\}+A\left\{2\frac{d}{dt}y(t)\right\}\\&=A\left\{\frac{6}{\Gamma(2.7)}t^{1.7}-\frac{2}{\Gamma(1.7)}t^{0.7}+6t^{2}-4t\right\}.\end{aligned}\tag{5.2}$$

By apply Theorem 4.2 and Property 1, we get:

$$\begin{aligned}&p^{-1.3}Y(p)-\sum_{k=0}^{1}p^{2-k}\left[{}_{0}^{RL}D_{t}^{1.3-k-1}y(t)\right]_{t=0}\\&+2\left(\frac{1}{p}Y(p)-\sum_{k=0}^{0}(p)^{2-k}y^{(1-1-k)}(0)\right)\\&=\frac{6}{\Gamma(2.7)}(p)^{4.7}\Gamma(2.7)-\frac{2}{\Gamma(1.7)}(p)^{3.7}\Gamma(1.7)+12p^{5}-4p^{4},\\&p^{-1.3}Y(p)-(p)^{2}\left[{}_{0}^{RL}D_{t}^{0.3}y(t)\right]_{t=0}-p\left[{}_{0}^{RL}D_{t}^{-0.7}y(t)\right]_{t=0}\\&+\frac{2}{p}Y(p)-2p^{2}y(0)=6p^{4.7}-2p^{3.7}+12p^{5}-4p^{4},\end{aligned}$$

and

$$\begin{aligned}&Y(p)\left(p^{-0.3}+2\right)\frac{1}{p}=6p^{4.7}-2p^{3.7}+12p^{5}-4p^{4},\\&Y(p)\left(p^{-0.3}+2\right)=6p^{5.7}+12p^{6}-2p^{4.7}-4p^{5},\\&Y(p)=\frac{1}{p^{-0.3}+2}\left(6p^{6}\left(p^{-0.3}+2\right)-2p^{5}\left(p^{-0.3}+2\right)\right),\\&=\frac{1}{p^{-0.3}+2}\left(6p^{6}-2p^{5}\right)\left(p^{-0.3}+2\right),\end{aligned}$$

Table 3. A comparison table among formulas of Laplace, Rishi, and Anuj transformations.

Transformations	Laplace	Rishi	Anuj
n^{th} derivative	$L\{f^n(x)\} = s^n F(s) - \sum_{k=0}^{n-1} s^k f^{n-1-k}(0)$	$\mathcal{R}\left\{f^{(m)}(t)\right\} = (\frac{q}{p})^m \mathcal{F}(q,p) - \sum_{k=0}^{m-1} \left(\frac{q}{p}\right)^{k-1} f^{(m-1-k)}(0)$	$A\{f^n(t)\} = \frac{1}{p^n} F(p) - \sum_{k=0}^{n-1} p^{2-k} f^{n-k-1}(0)$
Fractional integral	$L[I^\alpha f(t)] = \frac{F(s)}{s^\alpha}$	$\mathcal{R}\{I_t^\alpha f(t)\} = \left(\frac{p}{q}\right)^\alpha \mathcal{F}(q,p)$	$A[I^\alpha f(t)] = p^\alpha F(p)$
Reimann-Liouville fractional derivative	$L[D_t^\alpha f(x)] = s^\alpha F(s) - \sum_{k=0}^{m-1} s^k D_t^{\alpha-k-1} f(0)$	$\mathcal{R}\left\{{}_0^{RL} D_t^\alpha f(t)\right\} = \left(\frac{q}{p}\right)^\alpha \mathcal{F}(q,p) - \sum_{k=0}^{m-1} \left(\frac{q}{p}\right)^{k-1} \left[{}_0^{RL} D_t^{\alpha-k-1} f(t)\right]_{t=0}$	$A[D^\alpha f(t)] = p^{-\alpha} \cdot F(p) - \sum_{k=0}^{n-1} p^{2-k} D^{\alpha-k-1}(f(0))$
Caputo fractional derivative	$L[{}^c D_a^\alpha f(x)] = s^\alpha F(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(0)$	$\mathcal{R}\left\{{}_0^C D_t^\alpha f(t)\right\} = \left(\frac{q}{p}\right)^\alpha \mathcal{F}(q,p) - \sum_{k=0}^{m-1} \left(\frac{p}{q}\right)^{m-k-\alpha+1} f^{(m-k-1)}(0)$	$A[{}^c D_t^\alpha f(t)] = p^{-\alpha} \cdot F(p) - \sum_{k=0}^{n-1} p^{n-\alpha-k+2} f^{n-k-1}(0)$

hence

$$Y(p) = 6p^6 - 2p^5 \cdot$$

Taking the inverse Anuj transformation on both sides,

$$A^{-1}\{Y(p)\} = A^{-1}\left\{6p^6 - 2p^5\right\},$$

the exact solution is:

$$y(t) = t^2(t-1).$$

Example 5.2. (See [21]) Examine the fractional derivative of Riemann-Liouville's linear fractional differential equation, which is as follows:

$$ {}_0^{RL} D_t^{\frac{1}{2}} f(t) + f(t) = \frac{1}{2}t + \frac{\sqrt{t}}{\sqrt{\pi}}, \tag{5.3}$$

with the provided initial condition:

$$\left[{}_0^{RL} D_t^{\frac{-1}{2}} f(t)\right]_{t=0} = 0.$$

Solution: By using the Anuj transform on both sides of the above model, and considering Theorem 4.2, one can acquire the following:

$$A\left\{{}^{RL}D_t^{\frac{1}{2}} f(t)\right\} + A\{f(t)\} = A\left\{\frac{1}{2}t + \frac{\sqrt{t}}{\sqrt{\pi}}\right\}. \tag{5.4}$$

By applying Theorem 4.2, Property 1, and Definition 3.1 we can state:

$$\begin{aligned} A\left\{{}_0^{RL} D_t^{\frac{1}{2}} f(t)\right\} &= p^{-\frac{1}{2}}\mathcal{F}(p) - \sum_{k=0}^{0} p^{2-k}\left[{}_0^{RL}D_t^{\frac{1}{2}-k-1} f(t)\right]_{t=0} \\ &= A\left\{\frac{1}{2}t + \frac{\sqrt{t}}{\sqrt{\pi}}\right\}, \\ &= \frac{1}{2}p^4 + \frac{1}{\sqrt{\pi}} p^{\frac{7}{2}} \Gamma(\frac{3}{2}), \end{aligned}$$

where $\mathcal{R}\{f(t)\} = \mathcal{F}(p) \cdot$ The above equation can be expressed as follows:

$$\begin{aligned} p^{-\frac{1}{2}}\mathcal{F}(p) - p^2\left[{}_0^{RL}D_t^{-\frac{1}{2}} f(t)\right]_{t=0} + \mathcal{F}(p) &= \frac{1}{2}p^4 + \frac{1}{\sqrt{\pi}}p^{\frac{7}{2}}\Gamma(\frac{3}{2}), \\ \mathcal{F}(p)\left(\frac{1}{\sqrt{p}} + 1\right) &= \frac{1}{2}p^4 + \frac{1}{2}p^{\frac{7}{2}}, \\ &= \frac{1}{2}p^4\left(1 + \frac{1}{\sqrt{p}}\right), \\ \mathcal{F}(p) &= \frac{1}{2}p^4 \cdot \end{aligned}$$

So, from the inverse Anuj transform, we obtain:

$$A^{-1}\{\mathcal{F}(p)\} = A^{-1}\left\{\frac{1}{2}p^4\right\},$$

which implies that the exact solution is as follows:

$$f(t) = \frac{1}{2}t.$$

The analytic exact solutions that are obtained above are illustrated by the following two-dimensional graphs, where $t \in [0, 2]$ (see Figures 1 and 2).

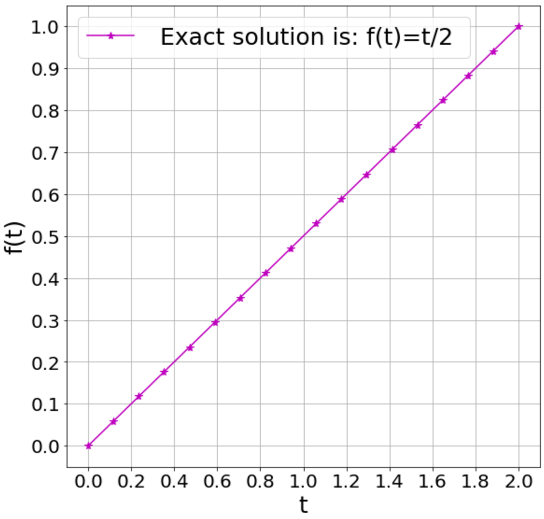

Figure 1. Two-dimensional graph representing the obtained exact solution for Example 5.1.

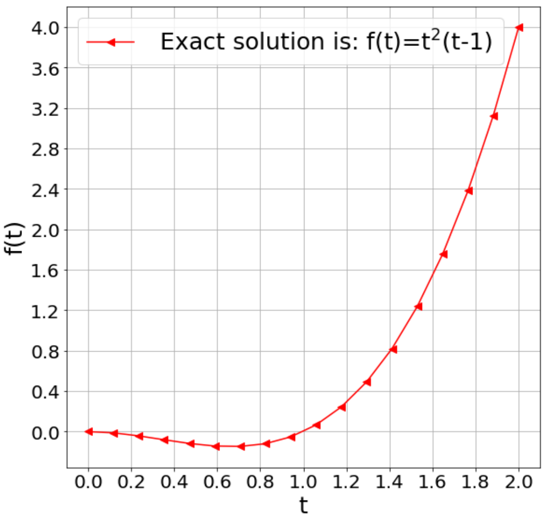

Figure 2. Two-dimensional graph represents the obtained exact solution for Example 5.2.

Remark 5.1. Kamal and other scholars demonstrated the previous example in [14, 21]. Both techniques produced analytic exact solutions.

Example 5.3. For the Caputo fractional derivative [14], consider the following linear fractional differential equation:

$$ {}_0^C D_t^{1.2} f(t) + {}_0^C D_t^{0.2} f(t) + f(t) = \frac{\Gamma(2.5)}{\Gamma(1.3)} t^{0.3} + \frac{\Gamma(2.5)}{\Gamma(2.3)} t^{1.3} + t^{1.5} + 1, $$

where $f(0) = 1,\ f'(0) = 0$ are given as initial conditions.

Solution: After applying the Anuj transform on both sides, one can obtain the following:

$$ A\left\{ {}_0^C D_t^{1.2} f(t) + {}_0^C D_t^{0.2} f(t) \right\} + A\{f(t)\} = A\left\{ \frac{\Gamma(2.5)}{\Gamma(1.3)} t^{0.3} + \frac{\Gamma(2.5)}{\Gamma(2.3)} t^{1.3} + t^{1.5} + 1 \right\}. \tag{5.5} $$

By applying Theorem 4.2, we get:

$$ p^{-1.2}\mathcal{F}(p) - \sum_{k=0}^{1} p^{2-1.8-k+2} f^{(1-k)}(0) + p^{-0.2}\mathcal{F}(p) - \sum_{k=0}^{0} p^{1-0.8-k+2} f^{(-k)}(0) + \mathcal{F}(p) = \frac{\Gamma(2.5)}{\Gamma(1.3)} p^{3.3}\Gamma(1.3) + \frac{\Gamma(2.5)}{\Gamma(2.3)} p^{4.3}\Gamma(2.3) + \Gamma(2.5)\, p^{4.5} + p^{3}. $$

When simplifying the above statement, one ends up with:

$$ p^{-1.2}\mathcal{F}(p) - p^{2.2} f'(0) - p^{1.2} f(0) + p^{-0.2}\mathcal{F}(p) - p^{2.2} f(0) + \mathcal{F}(p) = \Gamma(2.5)\, p^{3.3} + \Gamma(2.5)\, p^{4.3} + \Gamma(2.5)\, p^{4.5} + p^{3}. $$

After including the starting conditions, one obtains:

$$ \begin{aligned} \mathcal{F}(p)\left(p^{-1.2} + p^{-0.2} + 1\right) &= \Gamma(2.5)\, p^{3.3} + \Gamma(2.5)\, p^{4.3} + \Gamma(2.5)\, p^{4.5} + p^{3} + p^{1.2} + p^{2.2} \\ &= \Gamma(2.5)\, p^{4.5}\left(p^{-1.2} + p^{-0.2} + 1\right) + p^{3}\left(p^{-1.2} + p^{-0.2} + 1\right) \\ &= \left(p^{-1.2} + p^{-0.2} + 1\right)\left[\Gamma(2.5)\, p^{4.5} + p^{3}\right], \end{aligned} $$

which implies that

$$ \mathcal{F}(p) = \Gamma(2.5)\, p^{4.5} + p^{3}. $$

Taking the inverse Anuj transform on both sides, we get:

$$ A^{-1}\{\mathcal{F}(p)\} = A^{-1}\left\{ \Gamma(2.5)\, p^{4.5} + p^{3} \right\}, $$

so the desired solution takes the following form:

$$ f(t) = 1 + t\sqrt{t}. $$

Example 5.4. Examine the fractional differential equation that is linear and provided by

$$ {}_0^C D_t^{1.9} f(t) + f(t) = \frac{2}{\Gamma(1.1)} t^{0.1} + t^{2} + 1, \tag{5.6} $$

where $f(0) = 1,\ f'(0) = 0$.

Solution: Applying the Anuj transform on both sides, we obtain:

$$A\left\{{}_{0}^{C}D_{t}^{1.9}f(t)\right\}+A\{f(t)\}=A\left\{\frac{2}{\Gamma(1.1)}t^{0.1}+t^{2}+1\right\}. \quad (5.7)$$

If Theorem 4.2 is considered, then one achieves:

$$\begin{aligned}&p^{-1.9}\mathcal{F}(p)-\sum_{k=0}^{1}p^{2-1.9-k+2}f^{(1-k)}(0)+\mathcal{F}(p)\\&=\frac{2}{\Gamma(1.1)}p^{3.1}\Gamma(1.1)+p^{5}\Gamma(3)+p^{3},\\&p^{-1.9}\mathcal{F}(p)-p^{2.1}f'(0)-p^{1.1}f(0)+\mathcal{F}(p)\\&=2p^{3.1}+2p^{5}+p^{3}.\end{aligned}$$

When starting conditions are entered, we obtain:

$$\begin{aligned}\mathcal{F}(p)\left(p^{-1.9}+1\right)&=2p^{3.1}+2p^{5}+p^{3}+p^{1.1},\\\mathcal{F}(p)&=\frac{1}{p^{-1.9}+1}\left(2p^{5}\left(p^{-1.9}+1\right)+p^{3}\left(p^{-1.9}+1\right)\right)\\&=\left(\frac{1}{p^{-1.9}+1}\right)\left(p^{-1.9}+1\right)\left(2p^{5}+p^{3}\right)\\&=p^{3}+2p^{5}.\end{aligned}$$

Finally, from the inverse Anuj transformations, we obtain:

$$A^{-1}\{\mathcal{F}(p)\}=A^{-1}\left\{p^{3}+2p^{5}\right\},$$

so, the obtained solution is:

$$f(t)=1+t^{2}.$$

The analytic exact solutions that are obtained above are illustrated by the following two-dimensional graphs, where $t\in[0,2]$ (see Figures 3 and 4).

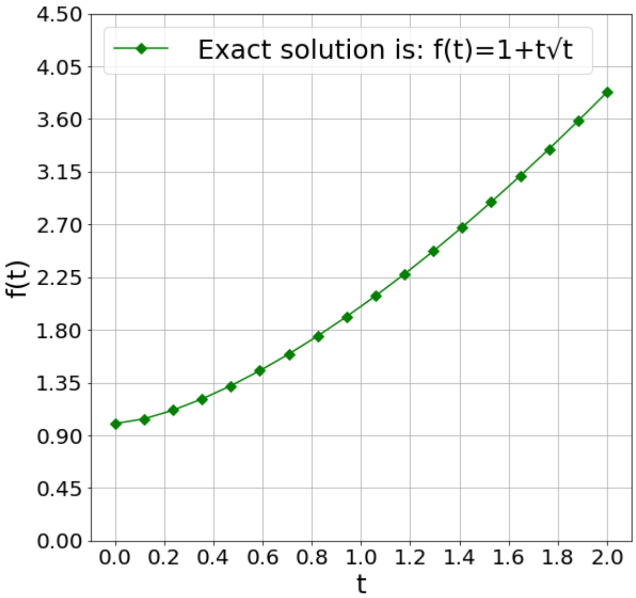

Figure 3. Two-dimensional graph representing the obtained exact solution for Example 5.3.

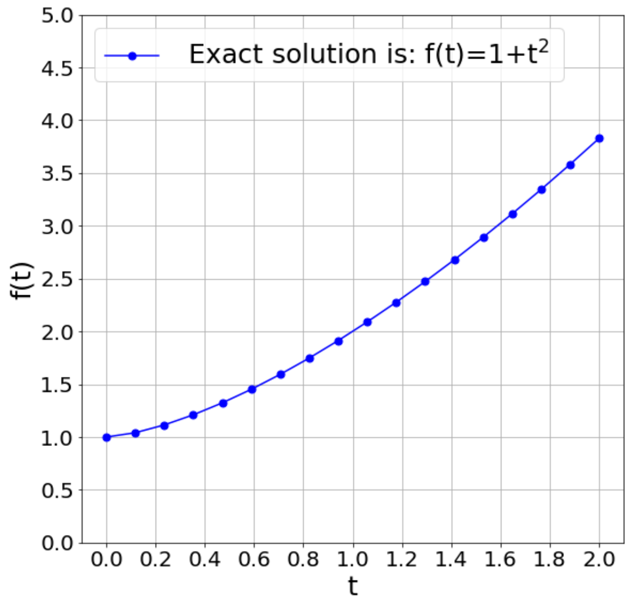

Figure 4. Two-dimensional graph representing the obtained exact solution for Example 5.4.

6. Results and discussion

The effectiveness and reliability of this strategy are supported by the evidence that it produces distinct classes of functions in all four examples discussed in this article. These functions consist of a linear function, a quadratic function, a cubic function, and a nonlinear polynomial. Furthermore, all the results obtained have been verified and inserted into the associated fractional differential equations; they are fully fulfilled. However, earlier works focused on various types of integral transformations. Out of these methods, the Anuj transformation has been used to solve ordinary/partial differential equations and to handle linear Volterra integral equations of the first class. The primary objective of this work is to hold the fractional differential equations with varying formulations. We have presented a table containing a brief comparison among well-known transformations by Laplace, Rishi, and Anuj; see Table 3.

7. Conclusions

Currently, the Anuj transformation method is an acceptable, well-organized tool for solving various formations of fractional differential equations (FDEs). This investigation successfully derived some new formulas for the Anuj transform, which will be helpful for upcoming research. The process starts with converting the linear fractional differential models into a more simplified form. Our findings are dependable; they express the worth, trust, and effectiveness of the applied method. The functionality of the applied method is not restricted to solving fractional models; it is also applicable in reducing the orders. With more study and refinement, the applied method can become an even more powerful and useful tool for dealing with these more complex FDEs. The outcomes of these fractional models indicate that the proposed Anuj transformation supplies remarkable results without necessitating boring computations. Ultimately, the Anuj transformation approach is found to be an excellent method to handle FDEs.

Author contributions

The authors investigated the research model, developed applications, and performed calculations. All authors contributed the writing of the paper and equally to the assessment of the results. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors announce they have not used artificial intelligence (AI) tools in the preparation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest in this paper.

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