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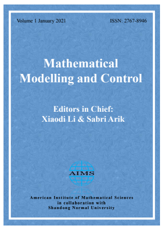

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Research article

Estimation of generalized fractal dimensions for diamond and square fractal networks using neighborhood degree-based topological indices

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Abstract: In many disciplines, including biology, image and signal processing, chemistry, sociology, medical imaging, and physics, self-similar networks describe and explain complicated systems with hierarchical or recursive structures. Self-similar network theory is one of the important areas in mathematics, which can be used to model real-world issues. Due to its universal applications, researchers have shown interest in self-similar networks. In this case, topological indices are used as numerical quantities that transform complex self-similar network structures into numerical values. We can discuss the intricate architecture of diamond fractal networks (DFNs) and square fractal networks (SFNs) by using the generalized fractal dimensions (GFD), which is newly defined by using different types of neighborhood degree-based topological indices. In this context, the neighborhood degree-based topological indices, namely the third neighborhood degree-based index developed by De (ND_e), the neighborhood version of the hyper-Zegreb index, the neighborhood forgotten topological index, the Sanskruti index, and the neighborhood inverse sum index are derived and computed for the representative networks. Moreover, the multifractal dimension measures are calculated for all indices from the general form of the obtained neighborhood degree-based topological indices. In addition, the comparison graphs of all indices and the generalized fractal dimensions are shown and geometrically discussed for the aggregate structure of the aforementioned networks with respect to all indices for each iteration ($k \geq 3$). Multifractal GFD spectral curves are also compared graphically with all indices at each iteration for the networks considered, and we analyze the complexity level of the networks at each iteration.

Keywords: fractal analysis; generalized fractal dimensions; diamond and square fractal networks; neighborhood degree; topological indices

1. Introduction

Fractal geometry is considered to be the most crucial tool when it comes to describing complex objects, as it provides a useful method. Fractal geometry is an advanced approach for characterizing complex natural structures by their self-similar properties. The concept was originally explored by Mandelbrot in 1975, and the term “fractal” is derived from the Latin word “fractus”, which means broken or fractured. Mandelbrot made it clear that traditional geometric approaches were insufficient to explain the nature of mountains, clouds, and other complex objects. The complexity of heterogeneous natural objects, which are usually irregular and fragmented in shape, cannot be adequately explained using traditional dimensional measures. Therefore, the fractal was developed as a non-linear geometrical approach to explain objects with a non-integer dimension, which are referred to as the fractal dimension. The measurement and categorization of shape and texture can be improved by applying the concept of the fractal dimension. Fractals, according to Mandelbrot’s mathematical definition, are sets whose Hausdorff dimension is strictly greater than their topological dimension. Fractal theory has a wide range of applications in various areas, such as biology, economics, geology, physics, and computer science [1–5].

There is significant interest in complex networks across many disciplines. The small-world, scale-free, and community properties of complex networks have been extensively analyzed. Multifractals are objects with more complexity. Multifractals are not perfect self-repetitions but are rich in localized geometric patterns. A single fractal dimension cannot characterize the irregularity of geometric patterns, since the scaling factor measured across the object may be varied. Multifractal analysis (MFA) is a helpful generalization of fractal analysis that may be used to systematically analyze the spatial heterogeneity of theoretical and experimental fractal structures. MFA provides generalized fractal dimensions $D(q)$ under many distortion factors q, and it is characterized by generalized fractal dimensions (GFD) or Renyi fractal dimensions. Therefore, GFD are used to analyze, classify, and evaluate complex self-similar networks. GFD are extremely relevant for use in fields such as chemistry, physics, and network analysis because they highlight the multifractal character of some systems and provide a more sophisticated and in-depth method of investigating them [6, 7].

The application of chemical graph theory (CGT) represents a relation between discrete mathematics and chemical graph theory. A graph G is a pair of ordered sets of vertices and edges $(V(G), E(G))$. In CGT, a molecule is presented as a graph, with the atoms as vertices and the chemical bonds as edges. The topological index is an important tool that CGT uses to predict different traits and activities. While searching for the boiling points of alkanes, the renowned chemist Harold Wiener discovered the first topological index in 1947, which is referred to as the Wiener index [8]. Topological indices are used to describe the design of the molecular structure and to predict its features and activities, such as its entropy, stability, boiling point, acentric factor, molar refraction, etc., and they are also used in quantitative structure-property relationship (QSPR) and quantitative structure-activity relationship (QSAR) analysis. In the literature, there are thousands of indices that have been created in CGT. Researchers are currently focusing on topological indices based on the vertex's neighborhood degree sum [9, 10]. To characterize the complexity level of self-similar networks considered in this research, multifractal measures are described by topological indices. Moreover, the spectra of GFD is constructed for certain complex self-similar networks using the general form obtained by some neighborhood degree-based topological indices. The resulting GFD values are compared by all indices at each iteration.

This paper is structured into five sections, with Section 1 as the introduction. Section 2 provides some of the fundamental concepts, definitions, and methods. Section 3 describes the derivation of neighborhood degree-based topological indices for diamond fractal networks (DFNs) and square fractal networks (SFNs). Section 4 presents the results of the study and includes a discussion of those results. Finally, the conclusions are presented in Section 5.

2. Mathematical methods

This section discusses the fractal dimension, Renyi entropy, GFD, and the basic structures of DFNs and SFNs.

2.1. Fractal dimension

The complexity, irregularity, or self-similarity of a fractal object or set can be quantified by its "fractal dimension". It measures how the object's details or the structure of the item changes as it zooms in or out. A fractal dimension can be a number that is not an integer, indicating the set's fractal nature. Higher values for the fractal dimension indicate greater complexity and self-similarity of the object. Depending on the properties of the fractal, there are different methods to calculate the fractal dimensions, namely the box-counting dimension, the Hausdorff dimension, the similarity dimension, etc. The number of self-similar copies N and the scaling ratio ε are two very important parameters of the fractal dimension. In this case, the fractal dimension may be called the similarity dimension [3, 5, 7]. The similarity dimension is defined as

$$D = \lim_{\varepsilon \to 0} \frac{log\,(N)}{log\left(\frac{1}{\varepsilon}\right)}. \tag{2.1}$$

2.2. Renyi entropy

Renyi entropy was first introduced by Alfred Renyi, a Hungarian mathematician. Renyi entropy is another term for the generalized Shannon entropy of a given probability distribution [11–13]. Moreover, the contribution of Renyi entropy to the development of information theory is substantial. Renyi entropy also provides a way to explain the GFD [14–17]. Renyi entropy is defined as

$$R_q = \frac{1}{1-q} ln\left(\sum_{i=1}^{N} p_i^q\right), \tag{2.2}$$

where $q \in \mathbb{R}\,\&\,q \neq 1$ is the order. Moreover, the given $p_i \in [0,1]$ is the probability of x_i, $i \in \{1,2,...,N\}$.

2.3. Generalized fractal dimensions

In the fields of non-linear analysis and statistics, Renyi entropies play a crucial role as measurements of uncertainty or unpredictability. A range of fractal dimension indices (Renyi fractal dimensions or GFD) can also be computed. In multifractal theory, the underlying concept is the GFD [18, 19]. In this section, the multifractal analysis using GFD through neighborhood degree-based topological indices is proposed to determine the size and properties of self-similar fractal networks [20].

Proposed GFD for self-similar networks

The following construction is used to introduce the probability distribution through the general formula of neighborhood degree-based topological indices. Let N be the number of sub-partitions in self-similar fractal networks, and each sub-partition is computed by neighborhood degree-based topological indices. Furthermore, let ε be the size of self-similar fractal networks. The probability p_i of the estimated neighborhood degree-based topological indices containing the i^{th} sub-partition of the self-similar structure graph is defined as

$$p_i = \frac{G_i}{G},\ i = 1,2,...,N,$$

where G_i is the value obtained by the estimated neighborhood degree-based topological indices for the i^{th} sub-partition and G is the total value obtained by the estimated neighborhood degree-based topological indices for the total structure.

The Renyi fractal dimensions or GFD of order $q \in (-\infty,\infty)$ such that $q \neq 1$ for the known probability distribution of the given self-similar fractal networks can be defined as

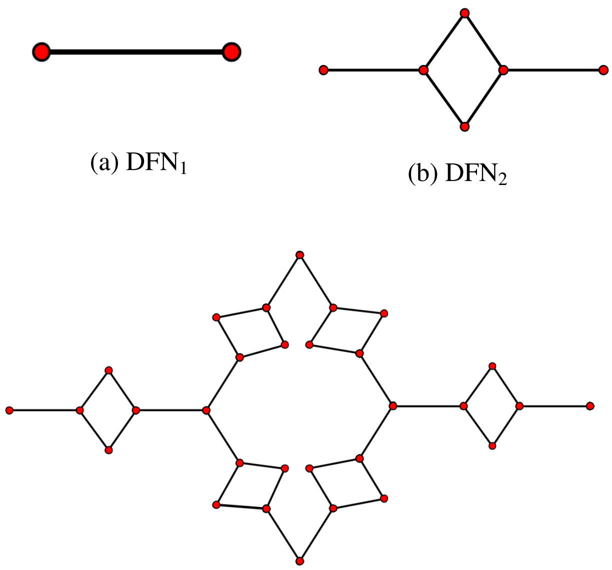
$$D_q = \lim_{\varepsilon\to 0} \frac{1}{q-1} \frac{ln\left(\sum_{i=1}^{N} p_i^q\right)}{ln\,\varepsilon}. \tag{2.3}$$

Here, D_q is the generalized Renyi entropy.

2.4. Construction of self-similar fractal networks

2.4.1. Diamond fractal networks

The construction of DFNs is illustrated in Figure 1, which can be considered to be a well-documented fractal. A diamond fractal is created by using an iterative process starting with a unit line segment at the beginning of the set in the first iteration. The unit line segment is divided into three parts, and the middle third part is removed. The middle third part is then replaced with four equal segments, each one-third in length, which join two equal segments at the top and two equal segments at the bottom to form a diamond structure. This eventually creates a second iteration that has a diamond shape. The next step is the third iteration, where the middle third in each of the six segments is removed and replaced with four equal segments in each portion as in the second iteration.



(a) DFN_1 (b) DFN_2

(c) DFN_3

Figure 1. Three iterations of a diamond fractal network.

In the k^{th} iteration of the diamond fractal network (DFN_k), the total length of all the line segments of the pre-fractal iteration is to $\frac{6}{3}(DFN_{k-1})$, where DFN_{k-1} is the total length of all the line segments in the previous step. This process is repeated for an infinite number of times to create a DFN. The geometric shapes depicted in Figure 1 are known as the branched Koch curve [21, 22] and hence, its similarity dimension is,

$$D = \frac{log\, 6}{log\, 3} = 1.63093.$$

The third iteration of the DFN is divided into two partitions A and B, as shown in Figure 2. Here, A has two sub-partitions G_1 and G_2, and B has four sub-partitions G_3, G_4, G_5 and G_6. Similarly, each iteration is divided into two partitions (say, A and B), which are further subdivided into six sub-partitions, namely G_1, G_2, G_3, G_4, G_5, and G_6. Using Eq (2.3), the GFD for DFN$_k$ ($k \geq 3$) is defined as,

$$D_q = \lim_{\varepsilon \to 0} \frac{1}{q-1} \frac{ln\left((\frac{G_1}{G})^q + (\frac{G_2}{G})^q + (\frac{G_3}{G})^q + (\frac{G_4}{G})^q + (\frac{G_5}{G})^q + (\frac{G_6}{G})^q \right)}{ln\, \varepsilon}. \tag{2.4}$$

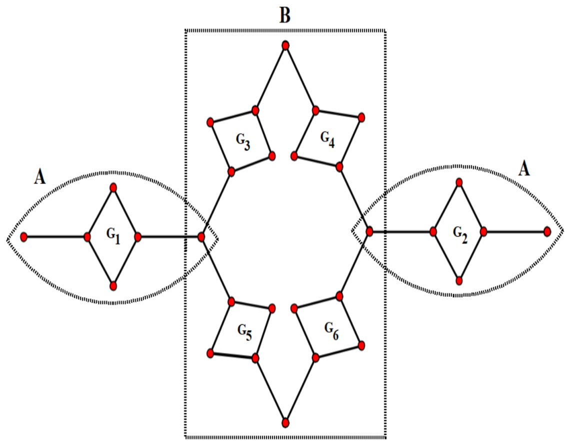
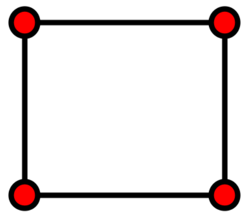


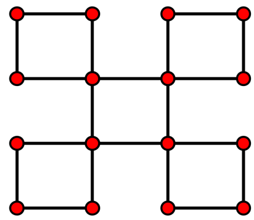
Figure 2. Partitions and sub-partitions in the DFN$_3$ structure.

2.4.2. Square fractal networks

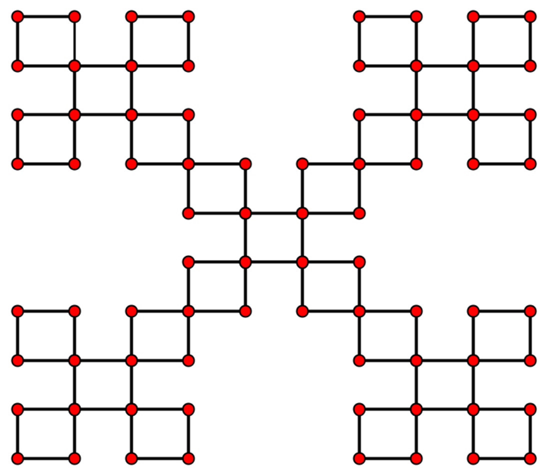
The structure of square fractal network (SFN) is designed in Figure 3. The first iteration of the SFN is a simple square at the beginning of the set. Using vertex-type connectivity, a new square is formed from all vertices of the initial square, which eventually forms the second iteration. The construction of the third iteration of the fractal in Figure 3 is accomplished by connecting four similar copies of the second iteration with the end vertices of a copy of the second iteration, which is placed in a center part of the new iteration. Similarly, the k^{th} iteration of the SFN (SFN$_k$) is designed using the pre-fractal iteration SFN$_{k-1}$. A SFN is made by creating an infinite number of repeated iterations. This is also known as Vicsek's fractal. These iterations represent a variable scale based on the variable size of the study area, with the first three iterations representing the growing pre-fractal process. The different potential sizes of the study area may vary depending on the iteration being examined. The first iteration is unique, and the results are outward. These growing fractals have often been used to act as a simple fractal model of urban development. Twenty years ago, a growing fractal was constructed and is now used in urban studies to model or simulate urban growth by introducing chance factors. There is an analogy between the development of a fractal and the growth of a city. This represents a ceaseless cumulative process, which represents a step-wise expansion of urban populations [23–25].



(a) SFN$_1$



(b) SFN$_2$



(c) SFN$_3$

Figure 3. Three iterations of the SFN.

The similarity dimension of the SFN is estimated as shown below:

$$D = \frac{log\, 5}{log\, 3} = 1.46497.$$

As illustrated in Figure 4, the third iteration of the SFN is split into two partitions A and B. Among them, A has four sub-partitions G_1, G_2, G_3, and G_4, while B has one sub-partition G_5. In the same way, every iteration is split into two partitions, A and B, which are further split into five sub-partitions, namely G_1, G_2, G_3, G_4, and G_5.

Consequently, the GFD for SFN_k ($k \geq 3$) is

$$D_q = \lim_{\varepsilon \to 0} \frac{1}{q-1} \frac{ln\left((\frac{G_1}{G})^q + (\frac{G_2}{G})^q + (\frac{G_3}{G})^q + (\frac{G_4}{G})^q + (\frac{G_5}{G})^q \right)}{ln\, \varepsilon}. \tag{2.5}$$

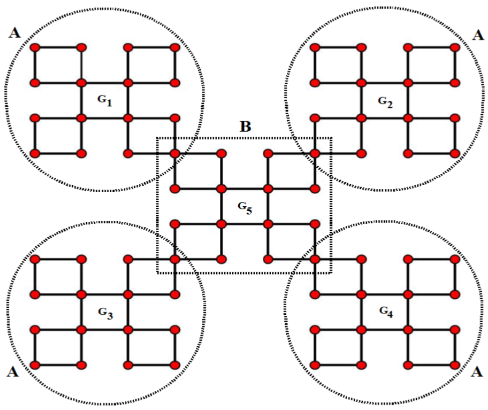


Figure 4. Partitions and sub-partitions in the SFN_3 structure.

3. Special topological indices for self-similar networks

3.1. Neighborhood degree-based topological indices

A topological index is a numerical quantity associated with a network or graph that characterizes the overall structural properties. In fact, in many studies, the terms commonly used for these numerical quantities are called topological indices. In addition, a topological index is a form of a molecular descriptor. Molecular descriptors have been produced by assigning numerical values to molecular graphs and to characterize the molecule using those numerical values. Topological indices are used in QSPR and QSAR analyzes [26].

Let $G = (V, E, \phi)$ be a graph, where V is the vertex set, E is the edge set, and $\phi : E \times E \to V$ is the incidence function. Here, e is denoted as the edge connecting two vertices m and n. The number of edges that are incident on a vertex m is denoted by d_m and is referred to as the degree of the vertex [27].

A few neighborhood degree-based topological indices are discussed as follows.

The third ND_e index is denoted as ND_3(G) [28] and is defined as

$$ND_3(\text{G}) = \sum_{mn \in E} (\delta_m \times \delta_n)(\delta_m + \delta_n).$$

The neighborhood version of the hyper-Zegreb index is denoted as NHM(G) [29] and is defined as

$$NHM(\text{G}) = \sum_{mn \in E} (\delta_m + \delta_n)^2.$$

The neighborhood forgotten topological index is denoted as NF^*(G) [30] and is defined as

$$NF^*(\text{G}) = \sum_{mn \in E} \left(\delta_m^2 + \delta_n^2\right).$$

The Sanskruti index is denoted as S(G) [10, 28, 31] and is defined as

$$S(\text{G}) = \sum_{mn \in E} \left(\frac{\delta_m \times \delta_n}{\delta_m + \delta_n - 2} \right)^3.$$

The neighborhood inverse sum index is denoted as NI(G) [9] and is defined as

$$NI(\text{G}) = \sum_{mn \in E} \left(\frac{\delta_m \times \delta_n}{\delta_m + \delta_n} \right).$$

Here, δ_m is the sum of degrees of all neighboring vertices of $m \in V(G)$, i.e., $\delta_m = \sum_{n \in N_m} d_n$, $N_m = \{n : mn \in E\}$, and δ_n can be defined similarly.

3.2. Derivation of neighborhood degree-based topological indices for DFN_k

In this section, the general results for total structure and partitions of DFN_k are derived by using the topological indices, discussed in Section 3.1.

Main theorem for DFN_k

Theorem 3.1. *Let* DFN_k *be the total structure of the* k^{th} *iteration of the DFN. In this case,*

(1) $ND_3(\text{DFN}_k) = 1557.6 \times 6^{k-1} - 160.8 \times 6^k - 1996.8$,

(2) $NHM(\text{DFN}_k) = 370.8 \times 6^{k-1} - 32.4 \times 6^k - 446.4$,

(3) $NF^*(\text{DFN}_k) = 189.6 \times 6^{k-1} - 16.8 \times 6^k - 220.8$,

(4) $S(\text{DFN}_k) = 134.13335 \times 6^{k-1} - 12.49676 \times 6^k - 175.51844$,

(5) $NI(\text{DFN}_k) = 4.95865 \times 6^{k-1} - 0.27981 \times 6^k - 5.01976$.

Proof. Consider a graph with the total structure of vertices and edges of the k^{th} iteration of the DFN, represented by DFN_k.

The cardinality of the vertex set is $\frac{2}{5}\left(6^k - 4\times 6^{k-1} + 3\right)$, and the edge set is 6^{k-1} for the graph of DFN_k. According to the degrees of the vertices, the vertex set is divided into five groups. The collection of vertices of the degree d is designated by the symbol V_d. For DFN_k ($k \geq 2$), we have $|V_3| = 2$, $|V_5| = 2$, $|V_6| = \frac{1}{5}\left(3\times 6^k - 16\times 6^{k-1} + 4\times 6^{k-2} - 6\right)$, $|V_7| = \frac{1}{5}\left(7\times 6^{k-1} - 6^k - 6\right)$, and $|V_9| = \frac{1}{5}\left(6^{k-1} - 4\times 6^{k-2} - 2\right)$. The set of edges partitions the graph of DFN_k into several portions that correspond to the sum of degrees of the neighborhood, which are

$$E_{(3,5)} = \{mn \in E(\mathrm{DFN}_k) \,|\, \delta_m = 3 \text{ and } \delta_n = 5\},$$
$$E_{(5,6)} = \{mn \in E(\mathrm{DFN}_k) \,|\, \delta_m = 5 \text{ and } \delta_n = 6\},$$
$$E_{(6,6)} = \{mn \in E(\mathrm{DFN}_k) \,|\, \delta_m = 6 \text{ and } \delta_n = 6\},$$
$$E_{(6,7)} = \{mn \in E(\mathrm{DFN}_k) \,|\, \delta_m = 6 \text{ and } \delta_n = 7\},$$
$$E_{(7,9)} = \{mn \in E(\mathrm{DFN}_k) \,|\, \delta_m = 7 \text{ and } \delta_n = 9\}.$$

In DFN_k ($k \geq 2$), we can see that $|E_{(3,5)}| = 2$, $|E_{(5,6)}| = 4$, $|E_{(6,6)}| = \frac{1}{5}\left(3\times 6^k - 16\times 6^{k-1} - 12\right)$, $|E_{(6,7)}| = \frac{2}{5}\left(7\times 6^{k-1} - 6^k - 6\right)$, and $|E_{(7,9)}| = \frac{1}{5}\left(7\times 6^{k-1} - 6^k - 6\right)$.

By definition, the $ND_3(\mathrm{DFN}_k)$ index is computed by using the following edge partition:

$$\begin{aligned}
ND_3(\mathrm{DFN}_k) = &\sum_{mn\in E} (\delta_m \times \delta_n)(\delta_m + \delta_n) \\
=&(2)[(3\times 5)(3+5)] + (4)[(5\times 6)(5+6)] \\
&+ \frac{1}{5}\left(3\times 6^k - 16\times 6^{k-1} - 12\right)[(6\times 6)(6+6)] \\
&+ \frac{2}{5}\left(7\times 6^{k-1} - 6^k - 6\right)[(6\times 7)(6+7)] \\
&+ \frac{1}{5}\left(7\times 6^{k-1} - 6^k - 6\right)[(7\times 9)(7+9)].
\end{aligned}$$

By simplifying the previous expression, the following required result is reached:

$$ND_3(\mathrm{DFN}_k) = 1557.6\times 6^{k-1} - 160.8\times 6^k - 1996.8.$$

By definition, the $NHM(\mathrm{DFN}_k)$ index is computed as follows by utilizing the edge partition:

$$\begin{aligned}
NHM(\mathrm{DFN}_k) = &\sum_{mn\in E} (\delta_m + \delta_n)^2 \\
=&(2)(3+5)^2 + (4)(5+6)^2 \\
&+ \frac{1}{5}\left(3\times 6^k - 16\times 6^{k-1} - 12\right)(6+6)^2 \\
&+ \frac{2}{5}\left(7\times 6^{k-1} - 6^k - 6\right)(6+7)^2 \\
&+ \frac{1}{5}\left(7\times 6^{k-1} - 6^k - 6\right)(7+9)^2.
\end{aligned}$$

After reducing the form above to its essence, we get following the result:

$$NHM(\mathrm{DFN}_k) = 370.8\times 6^{k-1} - 32.4\times 6^k - 446.4.$$

The following $NF^*(\mathrm{DFN}_k)$ index is computed according to the definition, utilizing the edge partition:

$$\begin{aligned}
NF^*(\mathrm{DFN}_k) = &\sum_{mn\in E} \left(\delta_m^2 + \delta_n^2\right) \\
=&(2)(3^2+5^2) + (4)(5^2+6^2) \\
&+ \frac{1}{5}\left(3\times 6^k - 16\times 6^{k-1} - 12\right)(6^2+6^2) \\
&+ \frac{2}{5}\left(7\times 6^{k-1} - 6^k - 6\right)(6^2+7^2) \\
&+ \frac{1}{5}\left(7\times 6^{k-1} - 6^k - 6\right)(7^2+9^2).
\end{aligned}$$

After changing the form above, we get the result

$$NF^*(\mathrm{DFN}_k) = 189.6\times 6^{k-1} - 16.8\times 6^k - 220.8.$$

By definition, the $S(\mathrm{DFN}_k)$ index is computed as follows by utilizing the edge partition:

$$\begin{aligned}
S(\mathrm{DFN}_k) = &\sum_{mn\in E} \left(\frac{\delta_m \times \delta_n}{\delta_m + \delta_n - 2}\right)^3 \\
=&(2)\left(\frac{3\times 5}{3+5-2}\right)^3 + (4)\left(\frac{5\times 6}{5+6-2}\right)^3 \\
&+ \frac{1}{5}\left(3\times 6^k - 16\times 6^{k-1} - 12\right)\left(\frac{6\times 6}{6+6-2}\right)^3 \\
&+ \frac{2}{5}\left(7\times 6^{k-1} - 6^k - 6\right)\left(\frac{6\times 7}{6+7-2}\right)^3 \\
&+ \frac{1}{5}\left(7\times 6^{k-1} - 6^k - 6\right)\left(\frac{7\times 9}{7+9-2}\right)^3.
\end{aligned}$$

By simplifying the previous calculation, we reach the required result

$$S(\mathrm{DFN}_k) = 134.13335\times 6^{k-1} - 12.49676\times 6^k - 175.51844.$$

The $NI(\mathrm{DFN}_k)$ index is computed according to the definition, utilizing the edge partition as follows:

$$
\begin{aligned}
NI(\mathrm{DFN}_k) &= \sum_{mn\in E}\left(\frac{\delta_m\times\delta_n}{\delta_m+\delta_n}\right)\\
&=(2)\left(\frac{3\times 5}{3+5}\right)+(4)\left(\frac{5\times 6}{5+6}\right)\\
&\quad+\frac{1}{5}\left(3\times 6^{k}-16\times 6^{k-1}-12\right)\left(\frac{6\times 6}{6+6}\right)\\
&\quad+\frac{2}{5}\left(7\times 6^{k-1}-6^{k}-6\right)\left(\frac{6\times 7}{6+7}\right)\\
&\quad+\frac{1}{5}\left(7\times 6^{k-1}-6^{k}-6\right)\left(\frac{7\times 9}{7+9}\right).
\end{aligned}
$$

After reducing the form above to its essence, we get the result

$$NI(\mathrm{DFN}_k) = 4.95865\times 6^{k-1} - 0.27981\times 6^{k} - 5.01976.$$

□

Theorem 3.2. *Let* DFN_k *be the* k^{th} *iteration of the DFN. Suppose that the partition* A *of* DFN_k *consists of the sub-partitions* G_1 *and* G_2 *(*G_1 *and* G_2 *are isomorphisms), as described in Figure 2. In this case*

(1) $ND_3(\mathrm{DFN}_k) = 138\times 6^{k-1} - 235.2\times 6^{k-2} - 676.8$,

(2) $NHM(\mathrm{DFN}_k) = 27.4\times 6^{k-1} + 12\times 6^{k-2} - 158.4$,

(3) $NF^{*}(\mathrm{DFN}_k) = 14.2\times 6^{k-1} + 3.6\times 6^{k-2} - 76.8$,

(4) $S(\mathrm{DFN}_k) = 10.69528\times 6^{k-1} - 5.01893\times 6^{k-2} - 62.76569$,

(5) $NI(\mathrm{DFN}_k) = 0.23365\times 6^{k-1} + 1.87788\times 6^{k-2} - 1.95026$.

Proof. As shown in Figure 2, consider the edge sub-partition graph G_1 in the partition A of the k^{th} iteration of the DFN, denoted DFN_k . The cardinality of the edge set for each sub-partition graph in DFN_k is 6^{k-2}. The set of edges partitions G_1 into several portions that correspond to the sum of the degrees of the neighborhood, which are

$$
\begin{aligned}
E_{(3,5)} &= \{mn\in E(\mathrm{DFN}_k)\,|\,\delta_m = 3 \text{ and } \delta_n = 5\},\\
E_{(5,6)} &= \{mn\in E(\mathrm{DFN}_k)\,|\,\delta_m = 5 \text{ and } \delta_n = 6\},\\
E_{(6,6)} &= \{mn\in E(\mathrm{DFN}_k)\,|\,\delta_m = 6 \text{ and } \delta_n = 6\},\\
E_{(6,7)} &= \{mn\in E(\mathrm{DFN}_k)\,|\,\delta_m = 6 \text{ and } \delta_n = 7\},\\
E_{(7,9)} &= \{mn\in E(\mathrm{DFN}_k)\,|\,\delta_m = 7 \text{ and } \delta_n = 9\}.
\end{aligned}
$$

From the sub-partition G_1 of the graph of DFN_k ($k \geq 3$), we can see that $|E_{(3,5)}| = 1$, $|E_{(5,6)}| = 2$, $|E_{(6,6)}| = \frac{2}{5}\left(7\times 6^{k-2}-6^{k-1}-6\right)$, $|E_{(6,7)}| = \frac{1}{5}\left(6^{k-1}-4\times 6^{k-2}-2\right)$, and $|E_{(7,9)}| = \frac{1}{5}\left(6^{k-1}-5\times 6^{k-2}-1\right)$.

Then, the $ND_3(\mathrm{DFN}_k)$ index is computed as follows, according to the definition and utilizing the edge partition:

$$
\begin{aligned}
ND_3(\mathrm{DFN}_k) &= \sum_{mn\in E}(\delta_m\times\delta_n)(\delta_m+\delta_n)\\
&=(1)[(3\times 5)(3+5)]+(2)[(5\times 6)(5+6)]\\
&\quad+\frac{2}{5}\left(7\times 6^{k-2}-6^{k-1}-6\right)[(6\times 6)(6+6)]\\
&\quad+\frac{1}{5}\left(6^{k-1}-4\times 6^{k-2}-2\right)[(6\times 7)(6+7)]\\
&\quad+\frac{1}{5}\left(6^{k-1}-5\times 6^{k-2}-1\right)[(7\times 9)(7+9)].
\end{aligned}
$$

By simplifying the previous calculation, we reach the required result as follows:

$$ND_3(\mathrm{DFN}_k) = 138\times 6^{k-1} - 235.2\times 6^{k-2} - 676.8.$$

By definition, the $NHM(\mathrm{DFN}_k)$ index is computed as follows by utilizing the edge partition.

$$
\begin{aligned}
NHM(\mathrm{DFN}_k) &= \sum_{mn\in E}(\delta_m+\delta_n)^2\\
&=(1)(3+5)^2+(2)(5+6)^2\\
&\quad+\frac{2}{5}\left(7\times 6^{k-2}-6^{k-1}-6\right)(6+6)^2\\
&\quad+\frac{1}{5}\left(6^{k-1}-4\times 6^{k-2}-2\right)(6+7)^2\\
&\quad+\frac{1}{5}\left(6^{k-1}-5\times 6^{k-2}-1\right)(7+9)^2.
\end{aligned}
$$

After reducing the form above to its essence, we get the result

$$NHM(\mathrm{DFN}_k) = 27.4\times 6^{k-1} + 12\times 6^{k-2} - 158.4.$$

The $NF^{*}(\mathrm{DFN}_k)$ index is computed according to the definition, utilizing the edge partition and defined as

$$
\begin{aligned}
NF^{*}(\mathrm{DFN}_k) &= \sum_{mn\in E}\left(\delta_m^2+\delta_n^2\right)\\
&=(1)(3^2+5^2)+(2)(5^2+6^2)\\
&\quad+\frac{2}{5}\left(7\times 6^{k-2}-6^{k-1}-6\right)(6^2+6^2)\\
&\quad+\frac{1}{5}\left(6^{k-1}-4\times 6^{k-2}-2\right)(6^2+7^2)\\
&\quad+\frac{1}{5}\left(6^{k-1}-5\times 6^{k-2}-1\right)(7^2+9^2).
\end{aligned}
$$

After changing the form above, we get the result

$$NF^{*}(\text{DFN}_k) = 14.2 \times 6^{k-1} + 3.6 \times 6^{k-2} - 76.8.$$

By definition, the $S(\text{DFN}_k)$ index is computed as follows by utilizing the edge partition:

$$\begin{aligned} S(\text{DFN}_k) = &\sum_{mn\in E}\left(\frac{\delta_m \times \delta_n}{\delta_m + \delta_n - 2}\right)^3 \\ =&(1)\left(\frac{3\times 5}{3+5-2}\right)^3 + (2)\left(\frac{5\times 6}{5+6-2}\right)^3 \\ &+ \frac{2}{5}\left(7\times 6^{k-2} - 6^{k-1} - 6\right)\left(\frac{6\times 6}{6+6-2}\right)^3 \\ &+ \frac{1}{5}\left(6^{k-1} - 4\times 6^{k-2} - 2\right)\left(\frac{6\times 7}{6+7-2}\right)^3 \\ &+ \frac{1}{5}\left(6^{k-1} - 5\times 6^{k-2} - 1\right)\left(\frac{7\times 9}{7+9-2}\right)^3. \end{aligned}$$

By simplifying the previous calculation, we reach the required result as follows:

$$S(\text{DFN}_k) = 10.69528 \times 6^{k-1} - 5.01893 \times 6^{k-2} - 62.76569.$$

The $NI(\text{DFN}_k)$ index is computed according to the definition and utilizing the edge partition as follows:

$$\begin{aligned} NI(\text{DFN}_k) = &\sum_{mn\in E}\left(\frac{\delta_m \times \delta_n}{\delta_m + \delta_n}\right) \\ =&(1)\left(\frac{3\times 5}{3+5}\right) + (2)\left(\frac{5\times 6}{5+6}\right) \\ &+ \frac{2}{5}\left(7\times 6^{k-2} - 6^{k-1} - 6\right)\left(\frac{6\times 6}{6+6}\right) \\ &+ \frac{1}{5}\left(6^{k-1} - 4\times 6^{k-2} - 2\right)\left(\frac{6\times 7}{6+7}\right) \\ &+ \frac{1}{5}\left(6^{k-1} - 5\times 6^{k-2} - 1\right)\left(\frac{7\times 9}{7+9}\right). \end{aligned}$$

After reducing the form above to its essence, we get the result

$$NI(\text{DFN}_k) = 0.23365 \times 6^{k-1} + 1.87788 \times 6^{k-2} - 1.95026.$$

Similarly, consider the edge sub-partition graph G_2 found in partition A of the k^{th} iteration of the DFN as shown in Figure 2. G_2 has the same structure as G_1, since G_1 and G_2 are isomorphic graphs. Therefore, the derived general forms of all neighborhood degree-based topological indices for G_1 are applicable to G_2. □

Theorem 3.3. *Let* DFN_k *be the* k^{th} *iteration of the DFN. Suppose that the partition* B *of* DFN_k *consists of the sub-partitions* G_3, G_4, G_5, *and* G_6*; and* G_3, G_4, G_5, *and* G_6 *are isomorphic graphs, as shown in Figure 2. In this case*

(1) $ND_3(\text{DFN}_k) = 138 \times 6^{k-1} - 235.2 \times 6^{k-2} - 160.8$,
(2) $NHM(\text{DFN}_k) = 27.4 \times 6^{k-1} + 12 \times 6^{k-2} - 32.4$,
(3) $NF^{*}(\text{DFN}_k) = 14.2 \times 6^{k-1} + 3.6 \times 6^{k-2} - 16.8$,
(4) $S(\text{DFN}_k) = 10.69528 \times 6^{k-1} - 5.01893 \times 6^{k-2} - 12.49676$,
(5) $NI(\text{DFN}_k) = 0.23365 \times 6^{k-1} + 1.87788 \times 6^{k-2} - 0.27981$.

Proof. In Figure 2, consider the edge sub-partition graph G_3 in the partition B of the k^{th} iteration of the DFN, denoted DFN_k. The cardinality of the edge set for each sub-partition graph in DFN_k is 6^{k-2}. The set of edges partitions G_3 into several portions that correspond to the sum of the degrees of the neighborhood, which are

$$\begin{aligned} E_{(6,6)} &= \{mn \in E(\text{DFN}_k) \,|\, \delta_m = 6 \text{ and } \delta_n = 6\}, \\ E_{(6,7)} &= \{mn \in E(\text{DFN}_k) \,|\, \delta_m = 6 \text{ and } \delta_n = 7\}, \\ E_{(7,9)} &= \{mn \in E(\text{DFN}_k) \,|\, \delta_m = 7 \text{ and } \delta_n = 9\}. \end{aligned}$$

From the sub-partition G_3 of the graph of DFN_k ($k \geq 3$), we can see that $|E_{(6,6)}| = \frac{1}{5}\left(14.6^{k-2} - 2.6^{k-1} + 3\right)$, $|E_{(6,7)}| = \frac{1}{5}\left(6^{k-1} - 4.6^{k-2} - 2\right)$, and $|E_{(7,9)}| = \frac{1}{5}\left(6^{k-1} - 5.6^{k-2} - 1\right)$.

Then, the $ND_3(\text{DFN}_k)$ index is computed as follows, according to the definition and utilizing the edge partition technique:

$$\begin{aligned} ND_3(\text{DFN}_k) = &\sum_{mn\in E} (\delta_m \times \delta_n)(\delta_m + \delta_n) \\ =&\frac{1}{5}\left(14\times 6^{k-2} - 2\times 6^{k-1} + 3\right)[(6\times 6)(6+6)] \\ &+ \frac{1}{5}\left(6^{k-1} - 4\times 6^{k-2} - 2\right)[(6\times 7)(6+7)] \\ &+ \frac{1}{5}\left(6^{k-1} - 5\times 6^{k-2} - 1\right)[(7\times 9)(7+9)]. \end{aligned}$$

By simplifying the previous expression, we reach the required result

$$ND_3(\text{DFN}_k) = 138 \times 6^{k-1} - 235.2 \times 6^{k-2} - 160.8.$$

By definition, the $NHM(\text{DFN}_k)$ index is computed as follows by utilizing the edge partition:

$$\begin{aligned} NHM(\mathrm{DFN}_k) &= \sum_{mn\in E} (\delta_m + \delta_n)^2 \\ &= \frac{1}{5}\left(14\times 6^{k-2} - 2\times 6^{k-1} + 3\right)(6+6)^2 \\ &+ \frac{1}{5}\left(6^{k-1} - 4\times 6^{k-2} - 2\right)(6+7)^2 \\ &+ \frac{1}{5}\left(6^{k-1} - 5\times 6^{k-2} - 1\right)(7+9)^2. \end{aligned}$$

After reducing the form above to its essence, we get the result $NHM(\mathrm{DFN}_k) = 27.4\times 6^{k-1} + 12\times 6^{k-2} - 32.4$.

The $NF^*(\mathrm{DFN}_k)$ index is computed according to the definition, utilizing the edge partition and defined as

$$\begin{aligned} NF^*(\mathrm{DFN}_k) &= \sum_{mn\in E} \left(\delta_m^2 + \delta_n^2\right) \\ &= \frac{1}{5}\left(14\times 6^{k-2} - 2\times 6^{k-1} + 3\right)(6^2+6^2) \\ &+ \frac{1}{5}\left(6^{k-1} - 4\times 6^{k-2} - 2\right)(6^2+7^2) \\ &+ \frac{1}{5}\left(6^{k-1} - 5\times 6^{k-2} - 1\right)(7^2+9^2). \end{aligned}$$

After changing the form above, we get the result

$$NF^*(\mathrm{DFN}_k) = 14.2\times 6^{k-1} + 3.6\times 6^{k-2} - 16.8.$$

By definition, the $S(\mathrm{DFN}_k)$ index is computed as follows by utilizing the edge partition:

$$\begin{aligned} S(\mathrm{DFN}_k) &= \sum_{mn\in E} \left(\frac{\delta_m\times\delta_n}{\delta_m+\delta_n-2}\right)^3 \\ &= \frac{1}{5}\left(14\times 6^{k-2} - 2\times 6^{k-1} + 3\right)\left(\frac{6\times 6}{6+6-2}\right)^3 \\ &+ \frac{1}{5}\left(6^{k-1} - 4\times 6^{k-2} - 2\right)\left(\frac{6\times 7}{6+7-2}\right)^3 \\ &+ \frac{1}{5}\left(6^{k-1} - 5\times 6^{k-2} - 1\right)\left(\frac{7\times 9}{7+9-2}\right)^3. \end{aligned}$$

By simplifying the previous calculation, we reach the required result

$$S(\mathrm{DFN}_k) = 10.69528\times 6^{k-1} - 5.01893\times 6^{k-2} - 12.49676.$$

The $NI(\mathrm{DFN}_k)$ index is computed according to the definition and utilizing the edge partition idea and is derived as follows:

$$\begin{aligned} NI(\mathrm{DFN}_k) &= \sum_{mn\in E} \left(\frac{\delta_m\times\delta_n}{\delta_m+\delta_n}\right) \\ &= \frac{1}{5}\left(14\times 6^{k-2} - 2\times 6^{k-1} + 3\right)\left(\frac{6\times 6}{6+6}\right) \\ &+ \frac{1}{5}\left(6^{k-1} - 4\times 6^{k-2} - 2\right)\left(\frac{6\times 7}{6+7}\right) \\ &+ \frac{1}{5}\left(6^{k-1} - 5\times 6^{k-2} - 1\right)\left(\frac{7\times 9}{7+9}\right). \end{aligned}$$

After reducing the form above to its essence, we get the result

$$NI(\mathrm{DFN}_k) = 0.23365\times 6^{k-1} + 1.87788\times 6^{k-2} - 0.27981.$$

Similarly, consider the edge sub-partition graphs G_4, G_5, and G_6 found in partition B of the k^{th} iteration of the DFN as shown in Figure 2. The structure of G_3 is replicated in G_4, G_5, and G_6 as G_3, G_4, G_5, and G_6 are isomorphic graphs. Therefore, the derived general forms of all neighborhood degree-based topological indices for G_3 are applicable to G_4, G_5, and G_6. □

3.3. *Derivation of neighborhood degree-based topological indices for* SFN_k

The topological indices described in Section 3.1 are derived for the total structure and its partitions of SFN_k in this section.

Main theorem for SFN_k

Theorem 3.4. *Let* SFN_k *be the total structure of the* k^{th} *iteration of the SFN. In this case*

(1) $ND_3(\mathrm{SFN}_k) = 1024\times 5^k + 896\times 5^{k-1} + 1664\times 5^{k-2} - 5632,$

(2) $NHM(\mathrm{SFN}_k) = 256\times 5^k - 24\times 5^{k-1} + 344\times 5^{k-2} - 928,$

(3) $NF^*(\mathrm{SFN}_k) = 128\times 5^k + 8\times 5^{k-1} + 184\times 5^{k-2} - 416,$

(4) $S(\mathrm{SFN}_k) = 95.53353\times 5^k + 16.41764\times 5^{k-1} + 112.72954\times 5^{k-2} - 516.44893,$

(5) $NI(\mathrm{SFN}_k) = 4\times 5^k - 3.6\times 5^{k-1} + 2.8\times 5^{k-2} - 9.6.$

Proof. Consider a graph with the total structure of vertices and edges of the k^{th} iteration of the SFN, represented by

SFN$_k$. The cardinality of the vertex set is $5^k - 2\times5^{k-1} + 1$ and the edge set is $5^k - 5^{k-1}$ for the graph of SFN$_k$. According to the degrees of the vertices, the vertices are divided into four groups. The collection of vertices of the degree d is designated by the symbol V_d. For SFN$_k$ ($k \geq 2$), we have $|V_4| = 2\times5^{k-2}+2$, $|V_6| = 5^{k-1}-5^{k-2}+4$, $|V_8| = 5^k-4\times5^{k-1}-5$, and $|V_{12}| = 5^{k-1} - 5^{k-2}$. According to the distribution of neighborhood degree, the edges are partitioned into the following groups:

$$E_{(4,6)} = \{mn \in E(\mathrm{SFN}_k) \,|\, \delta_m = 4 \text{ and } \delta_n = 6\},$$
$$E_{(6,12)} = \{mn \in E(\mathrm{SFN}_k) \,|\, \delta_m = 6 \text{ and } \delta_n = 12\},$$
$$E_{(8,8)} = \{mn \in E(\mathrm{SFN}_k) \,|\, \delta_m = 8 \text{ and } \delta_n = 8\},$$
$$E_{(8,12)} = \{mn \in E(\mathrm{SFN}_k) \,|\, \delta_m = 8 \text{ and } \delta_n = 12\},$$
$$E_{(12,12)} = \{mn \in E(\mathrm{SFN}_k) \,|\, \delta_m = 12 \text{ and } \delta_n = 12\}.$$

From the graph of SFN$_k$ ($k \geq 2$), we can see that $|E_{(4,6)}| = 5^{k-1}-5^{k-2}+4$, $|E_{(6,12)}| = 5^{k-1}-5^{k-2}+4$, $|E_{(8,8)}| = 5^k-4\times5^{k-1}-5^{k-2}-4$, $|E_{(8,12)}| = 4\times5^{k-2}-4$, and $|E_{(12,12)}| = 5^{k-1}-5^{k-2}$.

The $ND_3(\mathrm{SFN}_k)$ index is computed as follows, according to the definition and utilizing the edge partition:

$$\begin{aligned} ND_3(\mathrm{SFN}_k) = &\sum_{mn\in E} (\delta_m \times \delta_n)(\delta_m + \delta_n) \\ =&(5^{k-1} - 5^{k-2} + 4)[(4\times6)(4+6)] \\ &+ (5^{k-1} - 5^{k-2} + 4)[(6\times12)(6+12)] \\ &+ (5^k - 4\times5^{k-1} - 5^{k-2} - 4)[(8\times8)(8+8)] \\ &+ (4\times5^{k-2} - 4)[(8\times12)(8+12)] \\ &+ (5^{k-1} - 5^{k-2})[(12\times12)(12+12)]. \end{aligned}$$

By simplifying the previous calculation, we reach the required result

$$ND_3(\mathrm{SFN}_k) = 1024\times5^k + 896\times5^{k-1} + 1664\times5^{k-2} - 5632.$$

By definition, the $NHM(\mathrm{SFN}_k)$ index is computed as follows by utilizing the edge partition:

$$\begin{aligned} NHM(\mathrm{SFN}_k) = &\sum_{mn\in E} (\delta_m + \delta_n)^2 \\ =&(5^{k-1} - 5^{k-2} + 4)(4+6)^2 \\ &+ (5^{k-1} - 5^{k-2} + 4)(6+12)^2 \\ &+ (5^k - 4\times5^{k-1} - 5^{k-2} - 4)(8+8)^2 \\ &+ (4\times5^{k-2} - 4)(8+12)^2 \\ &+ (5^{k-1} - 5^{k-2})(12+12)^2. \end{aligned}$$

After reducing the form above to its essence, we get the result

$$NHM(\mathrm{SFN}_k) = 256\times5^k - 24\times5^{k-1} + 344\times5^{k-2} - 928.$$

The $NF^*(\mathrm{SFN}_k)$ index is computed according to the definition and utilizing the edge partition and is derived as follows:

$$\begin{aligned} NF^*(\mathrm{SFN}_k) = &\sum_{mn\in E} \left(\delta_m^2 + \delta_n^2\right) \\ =&(5^{k-1} - 5^{k-2} + 4)(4^2 + 6^2) \\ &+ (5^{k-1} - 5^{k-2} + 4)(6^2 + 12^2) \\ &+ (5^k - 4\times5^{k-1} - 5^{k-2} - 4)(8^2 + 8^2) \\ &+ (4\times5^{k-2} - 4)(8^2 + 12^2) \\ &+ (5^{k-1} - 5^{k-2})(12^2 + 12^2). \end{aligned}$$

After changing the form above, we get the result

$$NF^*(\mathrm{SFN}_k) = 128\times5^k + 8\times5^{k-1} + 184\times5^{k-2} - 416.$$

By definition, the $S(\mathrm{SFN}_k)$ index is computed as follows by utilizing the edge partition:

$$\begin{aligned} S(\mathrm{SFN}_k) = &\sum_{mn\in E} \left(\frac{\delta_m\times\delta_n}{\delta_m+\delta_n-2}\right)^3 \\ =&(5^{k-1} - 5^{k-2} + 4)\left(\frac{4\times6}{4+6-2}\right)^3 \\ &+ (5^{k-1} - 5^{k-2} + 4)\left(\frac{6\times12}{6+12-2}\right)^3 \\ &+ (5^k - 4\times5^{k-1} - 5^{k-2} - 4)\left(\frac{8\times8}{8+8-2}\right)^3 \\ &+ (4\times5^{k-2} - 4)\left(\frac{8\times12}{8+12-2}\right)^3 \\ &+ (5^{k-1} - 5^{k-2})\left(\frac{12\times12}{12+12-2}\right)^3. \end{aligned}$$

By simplifying the previous calculation, we reach the required result

$$\begin{aligned} S(\mathrm{SFN}_k) =&95.53353\times5^k + 16.41764\times5^{k-1} \\ &+ 112.72954\times5^{k-2} - 516.44893. \end{aligned}$$

The $NI(\mathrm{SFN}_k)$ index is computed according to the definition and utilizing the edge partition as follows:

$$
\begin{aligned}
NI(\mathrm{SFN}_k) = &\sum_{mn\in E}\left(\frac{\delta_m\times\delta_n}{\delta_m+\delta_n}\right)\\
=&(5^{k-1}-5^{k-2}+4)\left(\frac{4\times 6}{4+6}\right)\\
&+(5^{k-1}-5^{k-2}+4)\left(\frac{6\times 12}{6+12}\right)\\
&+(5^{k}-4\times 5^{k-1}-5^{k-2}-4)\left(\frac{8\times 8}{8+8}\right)\\
&+(4\times 5^{k-2}-4)\left(\frac{8\times 12}{8+12}\right)\\
&+(5^{k-1}-5^{k-2})\left(\frac{12\times 12}{12+12}\right).
\end{aligned}
$$

After reducing the form above to its essence, we get the result

$$NI(\mathrm{SFN}_k) = 4\times 5^{k} - 3.6\times 5^{k-1} + 2.8\times 5^{k-2} - 9.6.$$

□

Theorem 3.5. *Let* SFN_k *be the* k^{th} *iteration of the SFN. Suppose that the partition* A *of* SFN_k *consists of the sub-partitions* G_1, G_2, G_3, *and* G_4 *(*G_1, G_2, G_3, *and* G_4 *are isomorphisms), as described in Figure 4. In this case*

(1) $ND_3(\mathrm{SFN}_k) = 1920\times 5^{k-1} - 2960\times 5^{k-2} - 1456\times 5^{k-3} - 2816$,

(2) $NHM(\mathrm{SFN}_k) = 400\times 5^{k-1} - 668\times 5^{k-2} - 36\times 5^{k-3} - 464$,

(3) $NF^*(\mathrm{SFN}_k) = 208\times 5^{k-1} - 364\times 5^{k-2} + 44\times 5^{k-3} - 208$,

(4) $S(\mathrm{SFN}_k) = 151.70370\times 5^{k-1} - 203.85454\times 5^{k-2} - 190.16398\times 5^{k-3} - 258.22446$,

(5) $NI(\mathrm{SFN}_k) = 4.8\times 5^{k-1} - 6.8\times 5^{k-2} - 1.2\times 5^{k-3} - 4.8$.

Proof. In Figure 4, consider the edge sub-partition graph G_1 in the partition A of the k^{th} iteration of the SFN, denoted as SFN_k. The cardinality of the edge set for each sub-partition graph in SFN_k is $5^{k-1} - 5^{k-2}$. The set of edges partitions G_1 into several portions that correspond to the sum of the degrees of the neighborhood, which are

$$
\begin{aligned}
E_{(4,6)} &= \{mn\in E(\mathrm{SFN}_k)\,|\,\delta_m = 4 \text{ and } \delta_n = 6\},\\
E_{(6,12)} &= \{mn\in E(\mathrm{SFN}_k)\,|\,\delta_m = 6 \text{ and } \delta_n = 12\},\\
E_{(8,8)} &= \{mn\in E(\mathrm{SFN}_k)\,|\,\delta_m = 8 \text{ and } \delta_n = 8\},\\
E_{(8,12)} &= \{mn\in E(\mathrm{SFN}_k)\,|\,\delta_m = 8 \text{ and } \delta_n = 12\},\\
E_{(12,12)} &= \{mn\in E(\mathrm{SFN}_k)\,|\,\delta_m = 12 \text{ and } \delta_n = 12\}.
\end{aligned}
$$

From the sub-partition G_1 of the graph of SFN_k ($k \geq 3$), we can see that $|E_{(4,6)}| = 5^{k-2} - 5^{k-3} + 2$, $|E_{(6,12)}| = 4\times 5^{k-3} + 2$, $|E_{(8,8)}| = 5^{k-2} - 5^{k-3} - 2$, $|E_{(8,12)}| = 5^{k-1} - 4\times 5^{k-2} - 5^{k-3} - 2$, and $|E_{(12,12)}| = 5^{k-2} - 5^{k-3}$.

The $ND_3(\mathrm{SFN}_k)$ index is computed as follows, according to the definition and utilizing the edge partition:

$$
\begin{aligned}
ND_3(\mathrm{SFN}_k) = &\sum_{mn\in E}(\delta_m\times\delta_n)(\delta_m+\delta_n)\\
=&(5^{k-2}-5^{k-3}+2)[(4\times 6)(4+6)]\\
&+(4\times 5^{k-3}+2)[(6\times 12)(6+12)]\\
&+(5^{k-2}-5^{k-3}-2)[(8\times 8)(8+8)]\\
&+(5^{k-1}-4\times 5^{k-2}-5^{k-3}-2)[(8\times 12)(8+12)]\\
&+(5^{k-2}-5^{k-3})[(12\times 12)(12+12)].
\end{aligned}
$$

By simplifying the previous calculation, we reach the required result as follows:

$$ND_3(\mathrm{SFN}_k) = 1920\times 5^{k-1} - 2960\times 5^{k-2} - 1456\times 5^{k-3} - 2816.$$

By definition, $NHM(\mathrm{SFN}_k)$ index is computed as follows by utilizing the edge partition:

$$
\begin{aligned}
NHM(\mathrm{SFN}_k) = &\sum_{mn\in E}(\delta_m+\delta_n)^2\\
=&(5^{k-2}-5^{k-3}+2)(4+6)^2\\
&+(4\times 5^{k-3}+2)(6+12)^2\\
&+(5^{k-2}-5^{k-3}-2)(8+8)^2\\
&+(5^{k-1}-4\times 5^{k-2}-5^{k-3}-2)(8+12)^2\\
&+(5^{k-2}-5^{k-3})(12+12)^2.
\end{aligned}
$$

After reducing the form above to its essence, we get the result

$$NHM(\mathrm{SFN}_k) = 400\times 5^{k-1} - 668\times 5^{k-2} - 36\times 5^{k-3} - 464.$$

The $NF^*(\mathrm{SFN}_k)$ index is computed according to the definition and utilizing the edge partition as described below:

$$
\begin{aligned}
NF^*(\mathrm{SFN}_k) = &\sum_{mn\in E}\left(\delta_m^2+\delta_n^2\right)\\
=&(5^{k-2}-5^{k-3}+2)(4^2+6^2)\\
&+(4\times 5^{k-3}+2)(6^2+12^2)\\
&+(5^{k-2}-5^{k-3}-2)(8^2+8^2)\\
&+(5^{k-1}-4\times 5^{k-2}-5^{k-3}-2)(8^2+12^2)\\
&+(5^{k-2}-5^{k-3})(12^2+12^2).
\end{aligned}
$$

After changing the form above, we get the result

$$NF^{*}(\mathrm{SFN}_k) = 208 \times 5^{k-1} - 364 \times 5^{k-2} + 44 \times 5^{k-3} - 208.$$

By definition, the $S(\mathrm{SFN}_k)$ index is computed as follows by utilizing the edge partition:

$$\begin{aligned}S(\mathrm{SFN}_k) = &\sum_{mn\in E}\left(\frac{\delta_m \times \delta_n}{\delta_m + \delta_n - 2}\right)^3\\
&=(5^{k-2} - 5^{k-3} + 2)\left(\frac{4\times 6}{4+6-2}\right)^3\\
&+ (4\times 5^{k-3} + 2)\left(\frac{6\times 12}{6+12-2}\right)^3\\
&+ (5^{k-2} - 5^{k-3} - 2)\left(\frac{8\times 8}{8+8-2}\right)^3\\
&+ (5^{k-1} - 4\times 5^{k-2} - 5^{k-3} - 2)\left(\frac{8\times 12}{8+12-2}\right)^3\\
&+ (5^{k-2} - 5^{k-3})\left(\frac{12\times 12}{12+12-2}\right)^3.\end{aligned}$$

By simplifying the previous calculation, we reach the required result

$$\begin{aligned}S(\mathrm{SFN}_k) =&151.70370 \times 5^{k-1} - 203.85454 \times 5^{k-2}\\
&- 190.16398 \times 5^{k-3} - 258.22446.\end{aligned}$$

The $NI(\mathrm{SFN}_k)$ index is computed according to the definition and utilizing the edge partition as mentioned below:

$$\begin{aligned}NI(\mathrm{SFN}_k) = &\sum_{mn\in E}\left(\frac{\delta_m \times \delta_n}{\delta_m + \delta_n}\right)\\
&=(5^{k-2} - 5^{k-3} + 2)\left(\frac{4\times 6}{4+6}\right)\\
&+ (4\times 5^{k-3} + 2)\left(\frac{6\times 12}{6+12}\right)\\
&+ (5^{k-2} - 5^{k-3} - 2)\left(\frac{8\times 8}{8+8}\right)\\
&+ (5^{k-1} - 4\times 5^{k-2} - 5^{k-3} - 2)\left(\frac{8\times 12}{8+12}\right)\\
&+ (5^{k-2} - 5^{k-3})\left(\frac{12\times 12}{12+12}\right).\end{aligned}$$

After reducing the form above to its essence, we get the result

$$NI(\mathrm{SFN}_k) = 4.8 \times 5^{k-1} - 6.8 \times 5^{k-2} - 1.2 \times 5^{k-3} - 4.8.$$

Similarly, consider the edge sub-partition graphs G_2, G_3, and G_4 found in partition A of the k^{th} iteration of the SFN as in Figure 4. Since G_1, G_2, G_3, and G_4 are isomorphic graphs, the shapes of G_2, G_3, and G_4 are identical to G_1. Therefore, the general forms of all neighborhood degree-based topological indices are derived for G_1, which are applicable to G_2, G_3, and G_4. □

Theorem 3.6. *Let* SFN_k *be the* k^{th} *iteration of the SFN. Suppose that the partition* B *of* SFN_k *consists of the sub-partition* G_5*, as portrayed in Figure 4. In this case*

(1) $ND_3(\mathrm{SFN}_k) = 1296 \times 5^{k-1} + 1216 \times 5^{k-2} - 6736 \times 5^{k-3} + 5632,$

(2) $NHM(\mathrm{SFN}_k) = 324 \times 5^{k-1} - 64 \times 5^{k-2} - 1156 \times 5^{k-3} + 928,$

(3) $NF^{*}(\mathrm{SFN}_k) = 180 \times 5^{k-1} - 96 \times 5^{k-2} - 596 \times 5^{k-3} + 416,$

(4) $S(\mathrm{SFN}_k) = 91.125 \times 5^{k-1} + 163.16398 \times 5^{k-2} - 510.78898 \times 5^{k-3} + 516.44893,$

(5) $NI(\mathrm{SFN}_k) = 4 \times 5^{k-1} - 1.2 \times 5^{k-2} - 9.2 \times 5^{k-3} + 9.6.$

Proof. As in Figure 4, consider the edge sub-partition graph G_5 in the partition B of the k^{th} iteration of the SFN, denoted SFN_k. The cardinality of the edge set for the sub-partition graph G_5 in SFN_k is $5^{k-1} - 5^{k-2}$. The set of edges partitions G_5 into several portions that correspond to the sum of the degrees of the neighborhood, which are

$$\begin{aligned}E_{(4,6)} &= \{mn \in E(\mathrm{SFN}_k) \mid \delta_m = 4 \text{ and } \delta_n = 6\},\\
E_{(6,12)} &= \{mn \in E(\mathrm{SFN}_k) \mid \delta_m = 6 \text{ and } \delta_n = 12\},\\
E_{(8,8)} &= \{mn \in E(\mathrm{SFN}_k) \mid \delta_m = 8 \text{ and } \delta_n = 8\},\\
E_{(8,12)} &= \{mn \in E(\mathrm{SFN}_k) \mid \delta_m = 8 \text{ and } \delta_n = 12\},\\
E_{(12,12)} &= \{mn \in E(\mathrm{SFN}_k) \mid \delta_m = 12 \text{ and } \delta_n = 12\}.\end{aligned}$$

From the sub-partition G_5 of the graph of SFN_k ($k \geq 3$), we can see that $|E_{(4,6)}| = 4\times 5^{k-3} - 4$, $|E_{(6,12)}| = 5^{k-1} - 4\times 5^{k-2} - 5^{k-3} - 4$, $|E_{(8,8)}| = 5^{k-2} - 5^{k-3} + 4$, $|E_{(8,12)}| = 5^{k-2} - 5^{k-3} + 4$, and $|E_{(12,12)}| = 5^{k-2} - 5^{k-3}$.

The $ND_3(\mathrm{SFN}_k)$ index is computed as follows, according to the definition and utilizing the edge partition:

$$ND_3(\mathrm{SFN}_k) = \sum_{mn\in E} (\delta_m \times \delta_n)(\delta_m + \delta_n)$$
$$\begin{aligned}=&(4 \times 5^{k-3} - 4)[(4 \times 6)(4 + 6)]\\ &+ (5^{k-1} - 4 \times 5^{k-2} - 5^{k-3} - 4)[(6 \times 12)(6 + 12)]\\ &+ (5^{k-2} - 5^{k-3} + 4)[(8 \times 8)(8 + 8)]\\ &+ (5^{k-2} - 5^{k-3} + 4)[(8 \times 12)(8 + 12)]\\ &+ (5^{k-2} - 5^{k-3})[(12 \times 12)(12 + 12)].\end{aligned}$$

By simplifying the previous calculation, we reach the required result

$$ND_3(\mathrm{SFN}_k) = 1296\times5^{k-1}+1216\times5^{k-2}-6736\times5^{k-3}+5632.$$

By definition, $NHM(\mathrm{SFN}_k)$ index is computed as follows by utilizing the edge partition:

$$NHM(\mathrm{SFN}_k) = \sum_{mn\in E} (\delta_m + \delta_n)^2$$
$$\begin{aligned}=&(4 \times 5^{k-3} - 4)(4 + 6)^2\\ &+ (5^{k-1} - 4 \times 5^{k-2} - 5^{k-3} - 4)(6 + 12)^2\\ &+ (5^{k-2} - 5^{k-3} + 4)(8 + 8)^2\\ &+ (5^{k-2} - 5^{k-3} + 4)(8 + 12)^2\\ &+ (5^{k-2} - 5^{k-3})(12 + 12)^2.\end{aligned}$$

After reducing the form above to its essence, we get the result we seek

$$NHM(\mathrm{SFN}_k) = 324 \times 5^{k-1} - 64 \times 5^{k-2} - 1156 \times 5^{k-3} + 928.$$

The $NF^*(\mathrm{SFN}_k)$ index is computed according to the definition and utilizing the edge partition, and is defined as

$$NF^*(\mathrm{SFN}_k) = \sum_{mn\in E} \left(\delta_m^2 + \delta_n^2\right)$$
$$\begin{aligned}=&(4 \times 5^{k-3} - 4)(4^2 + 6^2)\\ &+ (5^{k-1} - 4 \times 5^{k-2} - 5^{k-3} - 4)(6^2 + 12^2)\\ &+ (5^{k-2} - 5^{k-3} + 4)(8^2 + 8^2)\\ &+ (5^{k-2} - 5^{k-3} + 4)(8^2 + 12^2)\\ &+ (5^{k-2} - 5^{k-3})(12^2 + 12^2).\end{aligned}$$

After changing the form above, we get the result we want, as follows:

$$NF^*(\mathrm{SFN}_k) = 180 \times 5^{k-1} - 96 \times 5^{k-2} - 596 \times 5^{k-3} + 416.$$

By definition, the $S(\mathrm{SFN}_k)$ index is computed as follows by utilizing the edge partition:

$$S(\mathrm{SFN}_k) = \sum_{mn\in E} \left(\frac{\delta_m \times \delta_n}{\delta_m + \delta_n - 2}\right)^3$$
$$\begin{aligned}=&(4 \times 5^{k-3} - 4)\left(\frac{4 \times 6}{4 + 6 - 2}\right)^3\\ &+ (5^{k-1} - 4 \times 5^{k-2} - 5^{k-3} - 4)\left(\frac{6 \times 12}{6 + 12 - 2}\right)^3\\ &+ (5^{k-2} - 5^{k-3} + 4)\left(\frac{8 \times 8}{8 + 8 - 2}\right)^3\\ &+ (5^{k-2} - 5^{k-3} + 4)\left(\frac{8 \times 12}{8 + 12 - 2}\right)^3\\ &+ (5^{k-2} - 5^{k-3})\left(\frac{12 \times 12}{12 + 12 - 2}\right)^3.\end{aligned}$$

By simplifying the previous calculation, we reach the required result

$$\begin{aligned}S(\mathrm{SFN}_k) =&91.125 \times 5^{k-1} + 163.16398 \times 5^{k-2}\\ &- 510.78898 \times 5^{k-3} + 516.44893.\end{aligned}$$

The $NI(\mathrm{SFN}_k)$ index is computed according to the definition and utilizing the edge partition.

$$NI(\mathrm{SFN}_k) = \sum_{mn\in E} \left(\frac{\delta_m \times \delta_n}{\delta_m + \delta_n}\right)$$
$$\begin{aligned}=&(4 \times 5^{k-3} - 4)\left(\frac{4 \times 6}{4 + 6}\right)\\ &+ (5^{k-1} - 4 \times 5^{k-2} - 5^{k-3} - 4)\left(\frac{6 \times 12}{6 + 12}\right)\\ &+ (5^{k-2} - 5^{k-3} + 4)\left(\frac{8 \times 8}{8 + 8}\right)\\ &+ (5^{k-2} - 5^{k-3} + 4)\left(\frac{8 \times 12}{8 + 12}\right)\\ &+ (5^{k-2} - 5^{k-3})\left(\frac{12 \times 12}{12 + 12}\right).\end{aligned}$$

After reducing the form above to its essence, we get the result we seek

$$NI(\mathrm{SFN}_k) = 4 \times 5^{k-1} - 1.2 \times 5^{k-2} - 9.2 \times 5^{k-3} + 9.6.$$

□

4. Results and discussion

4.1. Estimating the GFD for DFN_k using neighborhood degree-based topological indices

As proposed in Section 3.2, the GFD are computed for the DFN using a simplified version of the neighborhood degree-based topological indices. Additionally, the GFD obtained by using neighborhood degree-based topological indices for each iteration ($k \geq 3$) of DFN$_k$ are compared graphically.

Initially, the neighborhood degree-based topological indices for DFN$_k$ are compared iteratively and represented graphically in Figure 5. The derived neighborhood degree-based topological indices for DFN$_k$ are displayed on the vertical axis, while the horizontal axis displays the number of iterations (k). The given neighborhood degree-based topological indices can be examined individually according to their values, as depicted in Figure 5, in which all the curves are independent and not related to each other. In Figure 5, as the number of iterations (k) increases, the value of each index increases rapidly. The comparison using values obtained iteration-wise analyzes the similarity level of DFN$_k$ structures based on the neighborhood degree-based topological indices.

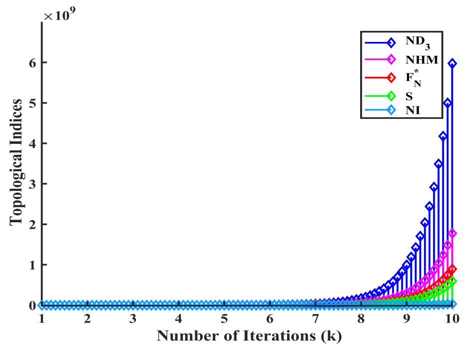
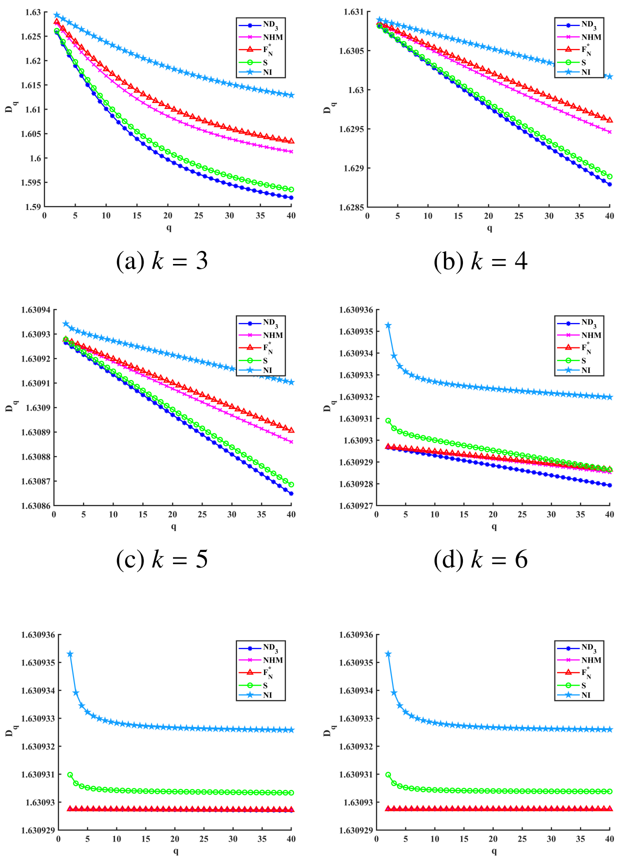


Figure 5. Iteration-based comparison of neighborhood degree-based topological indices for DFN$_k$.

The generalized fractal dimensional values are illustrated in Figure 6, which calculates the GFD using a simplified version of the neighborhood degree-based topological indices, obtained for iterations of the representative self-similar DFN to highlight the complexity of the self-similar scale at each iteration of the network. Moreover, the GFD spectra for DFN$_k$ are obtained by neighborhood degree-based topological indices such as ND_3, NHM, F_N^*, S, and NI. Furthermore, the GFD spectra for iterations from $K = 3$ to $k = 8$ are clearly displayed in Figure 6. It shows an apparent distinction in the GFD curves of the representative complex network. In addition, GFD spectra in Figure 6 demonstrates that, the value of GFD decreases, as q increases.



(a) $k = 3$ (b) $k = 4$

(c) $k = 5$ (d) $k = 6$

(e) $k = 7$ (f) $k = 8$

Figure 6. Comparison of GFD spectra for iterations from $k = 3$ to $k = 8$ of the DFN using different neighborhood degree-based topological indices.

In Figure 6a–c, the GFD spectral curves of ND_3 and S for the iterations from $k = 3$ to $k = 5$ are lower than the GFD curves of other indices. Among these, the GFD spectra of the indices ND_3 and S are extremely close and the GFD spectra of the indices NHM and F_N^* are significantly close. The GFD curves of these four indices are lower than the GFD curve of the index NI. In addition, the GFD values of index NI explains DFN_k better than other indices. Similarly, the GFD curves for other iterations are compared using neighborhood degree-based topological indices, as shown in Figure 6. For the iteration $k = 6$, the GFD curves of indices ND_3, NHM, F_N^* and S are significantly close to one another. Figure 6d demonstrates that the NI index-based GFD curve is higher than the curves of the other indices.

In addition, the iterations $k = 7$ and $k = 8$ in Figures 6e, 6f reveal that the GFD spectra have a similar nature. The GFD curves of the ND_3, NHM, and F_N^* indices are quite close, whereas the GFD curve of the S index is found to be greater than the curves of these three indices. Furthermore, the GFD curve of the NI index is higher than the GFD curve of the S index. Therefore, it is clear that the GFD values of the NI index are slightly higher than the GFD values of the other four indices for iterations of the complex self-similar DFN. Figure 6 shows that increasing the value of q decreases the value of GFD. Thus, the GFD values of the NI index provide a clear description of the complexity of the self-similarity level and the properties of the DFN when compared with the GFD values using other indices.

4.2. *Estimating GFD for SFN_k using the neighborhood degree-based topological indices*

As discussed in Section 3.3, the GFD are determined for the SFN using the general form of neighborhood degree-based topological indices. Furthermore, the computed GFD using neighborhood degree-based topological indices for each iteration ($k \geq 3$) of SFN_k are compared geometrically.

Figure 7 shows an iterative comparison and graphical representation of neighborhood degree-based topological indices for SFN_k. The vertical axis illustrates the estimated neighborhood degree-based topological indices for SFN_k and the horizontal axis represents the number of iterations (k). As observed in Figure 7, each curve is independent and unrelated to the others, and hence the neighborhood degree-based topological indices can be examined separately according to their values. The value of each index in Figure 7 increases quickly, as the number of iterations (k) a rises. This iteration-wise graphical comparison is used to analyze the similarity level of SFN_k structures based on neighborhood degree-based topological indices.

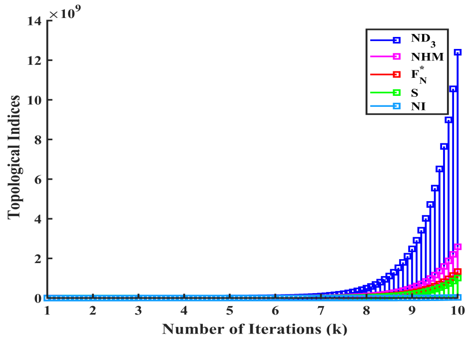
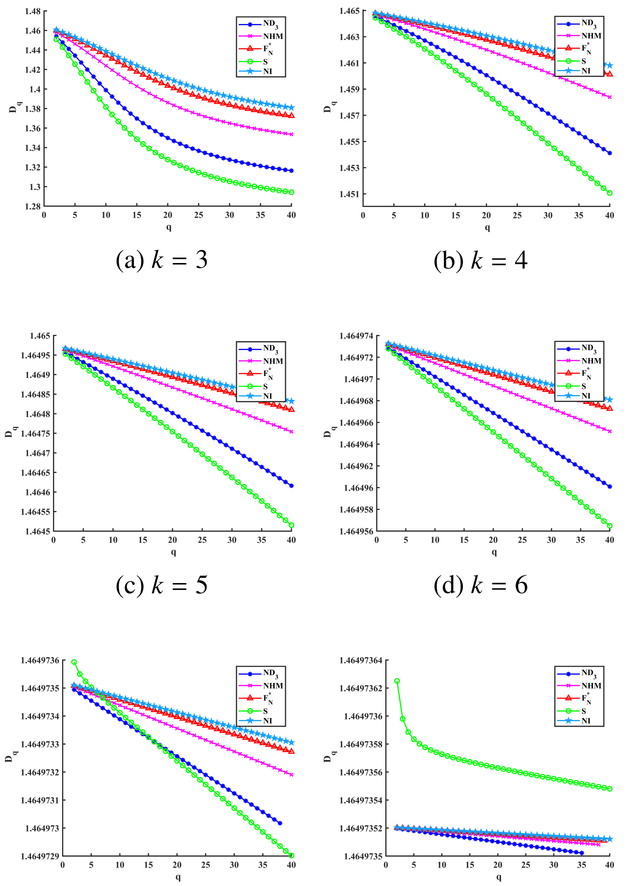


Figure 7. Iteration-based comparison of neighborhood degree-based topological indices for SFN_k.

Figure 8 shows the values of the GFD of iterations from $k = 3$ to $k = 8$ of the SFN. To emphasize the intricacy of the scale at each iteration of the network, the GFD is computed using a general form of the neighborhood degree-based topological indices acquired for the iterations of the representative self-similar SFN. Moreover, neighborhood degree-based topological indices like ND_3, NHM, F_N^*, S, and NI are used to calculate the GFD spectra for SFN_k. The apparent discrimination in the GFD curves of this representative complex network is depicted in Figure 8a–f. As shown in Figure 8a–d, the GFD curves of the two indices F_N^* and NI for iterations from $k = 3$ to $k = 6$ are extremely similar. Moreover, the GFD for the F_N^* and NI indices are higher than the GFD values for other three indices. Additionally, the GFD curve of the NI index is slightly higher than the GFD curve of the F_N^* index. When compared with the other indices, the GFD values of the index NI examines SFN_k in detail. Likewise, as illustrated in Figure 8, the GFD curves for the last two iterations are depicted using neighborhood degree-based topological indices.

For the iteration $k = 7$, the GFD curves of the ND_3 and S indices are slightly closer, whereas those of the NI and F_N^* indices are extremely close to each other. As in Figure 8e, the NI index's GFD curve is larger than the curves of the other indices. Furthermore, the GFD curves of the indices ND_3, NHM, F_N^* and NI are quite closer in the eighth iteration, as shown in Figure 8f. The GFD curve of the S index is higher than the curves of the other four indices. The NI index's GFD curve is greater than the curves of the other indices, as observed from iterations $k = 3$ to $k = 7$.



(a) $k = 3$ (b) $k = 4$

(c) $k = 5$ (d) $k = 6$

(e) $k = 7$ (f) $k = 8$

Figure 8. Comparison of GFD spectra for iterations from $k = 3$ to $k = 8$ of the SFN using different neighborhood degree-based topological indices.

In the iteration $k = 8$, the GFD curve of the index S is higher than the others. Therefore, the GFD curves of the given neighborhood degree-based topological indices vary depending on the iterations of the complex self-similar SFN. It can be observed from Figure 8 that as the value of q increases, the value of GFD decreases. Eventually, it can be seen that the GFD values of the NI index for SFN_k are slightly higher and gives a clear picture of the complex level of self-similarity and the characteristics of the SFN.

The structural characteristics of networks, including their association, node distribution, and overall geometry, can be quantitatively described using topological indices. To comprehend how the structure of a network affects its behavior, topological indices are crucial. The intricacy of a network's topology and its relationship to fractal features are better captured by including topological indices in the GFD computation. Additionally, deriving the GFD from topological indices also aids in comprehending the behavior of the network's features at various scales.

Finally, the multifractal dimension measures, calculated for specific representational networks, demonstrate self-similarity and scale invariance at various levels of these systems through GFD values derived from topological indices. Additionally, they offer an in-depth and detailed understanding of the complex systems within the given representational networks. This multifractal analysis proves especially valuable in systems where the scaling behavior varies with the spatial or temporal scale, and multifractal dimensions are used to gain insights into the behavior of various networks. By analyzing the behavior of a system at different scales, multifractal dimensionality measures uncover concealed patterns and allow for a more in-depth examination of both natural events and engineered systems.

5. Conclusions

In this context, the self-similarity properties are analyzed by computing multifractal dimensions based on neighborhood degree-based topological indices for complex self-similar networks. In this paper, the GFD are calculated for two representative networks, namely the DFN and SFN. A few notable neighborhood degree-based topological indices are calculated for DFN_k and SFN_k, and we compared them graphically on an iteration basis. Multifractal GFD spectra are constructed for representative networks using neighborhood degree-based topological indices such as ND_3, NHM, F_N^*, S, and NI. The GFD spectra are also compared by representative topological indices for the given networks. For DFN_k and SFN_k structures, the GFD curve of the NI index is higher than the curves of other indices, and hence the NI index provides clear information through its GFD values. The GFD curves obtained by these topological indices help us to understand the self-similar networks DFN_k and SFN_k by examining the complexity level of their self-similarity and its properties. The self-similar characteristics obtained for the iterations (DFN_k & SFN_k) via the GFD based on topological indices can provide an idea for studying the limiting case of DFNs and SFNs.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no conflicts of interest.

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