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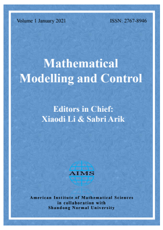

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Research article

Three weak solutions for double phase elliptic problem with indefinite weight

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Abstract: This paper investigates the multiplicity of weak solutions for a double-phase elliptic problem with indefinite interaction, where the nonlinearity involves a potential term $k(x)$ that may change sign and be singular within the domain. The problem is set up as a Dirichlet boundary value problem, where the differential operator has two different phases, involving two exponents p and q that meet a certain condition. Employing critical point theory, we demonstrate that at least one solution exists, and under suitable conditions, there are at least three solutions. These results come from applying abstract variational approaches, particularly the critical point theorems by Bonanno and Marano. The manuscript also presents a detailed variational framework and sets up the necessary preliminaries to support the main results.

Keywords: double phase; variational methods; critical point theorem; p-Laplacian

1. Introduction

The double-phase elliptic problem with indefinite interactions studied in this manuscript is closely related to a variety of physical phenomena, particularly in areas where material properties or media exhibit complex behaviors under different physical conditions. In mathematical physics, such problems arise in the modeling of heterogeneous media, such as composite materials, where different phases (for example, solid and fluid phases) interact with each other, exhibiting distinct mechanical and thermal properties. The double-phase operator, which includes two different exponents for the gradient term, is a natural extension of classical variational models and captures the interplay between different types of energy dissipation mechanisms (e.g., elastic and viscous), depending on the phase involved.

Moreover, the indefinite weight functions, such as $k(x)$, (see problem (P_λ)), represent spatially varying properties that are essential for describing phenomena like non-homogeneous material structures, fluid dynamics in porous media, or electrostatic fields in nonuniform media. These weights allow the model to account for situations where the physical properties of the system may vary unpredictably or even change sign across the domain, which is typical in many real-world applications, such as material design, geophysics, and the study of reaction-diffusion systems in biological and chemical processes.

The main point of this paper is that there are multiple weak solutions, which basically means a physical system can end up in different stable states or setups depending on things like boundary forces, external fields, or temperature. This multiplicity can be interpreted as the system's capacity to exhibit multiple stable or metastable states, a feature that is crucial in understanding how materials change phases, and how systems split into different behaviors in all sorts of physics areas. Thus, a rigorous mathematical analysis of these problems equips us to model, interpret, and anticipate the dynamics of intricate physical systems governed by heterogeneous and nonlinear interactions.

Double-phase problems arise in diverse applied contexts, such as electrorheological fluids [13], image restoration [1, 8, 9], elasticity theory [19], and the study of the Lavrentiev phenomenon [20]. Recent contributions can be found in [7, 16, 18]. For instance, in image processing, anisotropic operators with variable exponents model edge-preserving denoising [9], while in elasticity, they describe materials with hardening properties [19]. Our work extends these frameworks by incorporating indefinite weights, which model spatially heterogeneous media (e.g., composite materials with sign-changing conductivity). In [15], the authors investigate the existence of weak solutions for the following double-phase Dirichlet problem:

$$\begin{cases} -\operatorname{div}\left(|\nabla\xi|^{p-2}\nabla\xi + a(x)|\nabla\xi|^{q-2}\nabla\xi\right) = \lambda\beta(x)h(\xi) & \text{in } \mathcal{D}, \\ \xi = 0 & \text{on } \partial\mathcal{D}, \end{cases}$$

where $\mathcal{D} \subset \mathbb{R}^N$ is a bounded domain with a Lipschitz boundary, $N \geq 2$, and p is a constant such that $1 < p < N$; moreover, p and q satisfy the following structural conditions:

$$p^* = \frac{Np}{N-p} \quad \text{and} \quad p < q < p^*.$$

The coefficient $a \in L^\infty(\mathcal{D})$ is a nonnegative weight, λ is a positive real parameter, and $\beta \in \Lambda_+(\mathcal{D})$, where

$$\Lambda_+(\mathcal{D}) := \left\{\gamma \in L^\infty(\mathcal{D}) : \operatorname{ess}\inf_{x\in\mathcal{D}} \gamma(x) > 0\right\}.$$

Here, $h : [0, +\infty) \to \mathbb{R}$ is assumed to be a continuous functions that fulfills the given hypotheses:

$$(h_1) \quad \sup_{t>0} H(t) > 0, \quad \text{with} \quad H(t) := \int_0^t h(s)ds.$$

The paper [15] shows how the parameter λ controls the number of weak solutions. For $\lambda \in [0, \lambda^*)$, the unique solution is the trivial (zero) one, where λ^* is given by an explicit formula depending on the problem data. In contrast, when $\lambda > \lambda_*$, there are at least two distinct nonnegative weak solutions connected by an energy relation. The critical values λ^* and λ_* are derived independently, leaving the question of the existence of a solution in the interval $[\lambda^*, \lambda_*]$ open for study. These results clarify the transition from a single trivial solution to multiple nontrivial ones in double-phase problems. Additionally, in [2], the authors analyze the following Dirichelet double-phase problem involving variable exponents:

$$\begin{cases} -\operatorname{div}\left(|\nabla\xi|^{p(x)-2}\nabla\xi + b(x)|\nabla\xi|^{q(x)-2}\nabla\xi\right) = \lambda h(x,\xi) & \text{in } \mathcal{D}, \\ \xi = 0 & \text{on } \partial\mathcal{D}. \end{cases}$$

In this framework, $\mathcal{D} \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded domain with Lipschitz boundary $\partial\mathcal{D}$. The variable exponents $p, q \in C(\overline{\mathcal{D}})$ fulfill the conditions below:

Throughout the closure of the domain $\overline{\mathcal{D}}$, the variable exponent p satisfies $1 < p < N$, while q remains subcritical such that $p(x) < q(x) < \frac{Np(x)}{N-p(x)}$.

Additionally, $b(x)$ is a nonnegative function belonging to $L^\infty(\mathcal{D})$, and λ is a positive parameter. The nonlinearity $h : \mathcal{D} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with subcritical growth and prescribed behavior as $t \to \pm\infty$.

This work demonstrates the existence of weak solutions subject to standard hypotheses regarding superlinearity and subcritical growth rates. By leveraging a variational critical point theorem, the authors established the presence of two bounded weak solutions, characterized by having energy levels of opposite signs under specific additional assumptions, the non-negativity of these obtained solutions is further established. Contemporary literature regarding double-phase problems includes [3–5]. Inspired by these earlier studies, this paper investigates the presence of multiple weak solutions for a double-phase elliptic problem with an indefinite nonlinearity:

$$\begin{cases} -\operatorname{div}\left(|\nabla\xi|^{p-2}\nabla\xi + \mu(x)|\nabla\xi|^{q-2}\nabla\xi\right) = \lambda k(x)|\xi|^{s-2}\xi & \text{in } \mathcal{D}, \\ \xi = 0 & \text{on } \partial\mathcal{D}. \end{cases}$$

Here, $\mathcal{D} \subset \mathbb{R}^N$ with $N \geq 2$ denotes a bounded domain possessing a Lipschitz continuous boundary $\partial\mathcal{D}$. The parameters p, q, s, and γ are assumed to satisfy the following constraints:

$$\begin{aligned} &1 < s < p < N < \gamma, \\ &p < q < p^* := \frac{Np}{N-p}. \end{aligned} \tag{H}$$

The weight $\mu \in L^{\infty}(\mathcal{D})$ is nonnegative, while $k(x)$ may change sign and have singularities in $\mathcal{D}$, with $k \in L^{\gamma}(\mathcal{D})$. The parameter $\lambda > 0$ is positive.

We note that the double-phase operator is defined as follows:

$$-\operatorname{div}\left(|\nabla\xi|^{p-2}\nabla\xi + \mu(x)|\nabla\xi|^{q-2}\nabla\xi\right), \quad \xi \in W_0^{1,\mathcal{H}}(\mathcal{D}),$$

where $W_0^{1,\mathcal{H}}(\mathcal{D})$ denotes the Musielak–Orlicz–Sobolev space with zero boundary values and arises from a two-phase variational integral.

The results obtained in this paper establish the existence of one solution and the existence of three solutions, respectively, via to critical point theorems in [6, 14].

The paper is structured as follows: Section 2 presents the variational framework and preliminary results; Section 3 contains our main existence and multiplicity theorems; and finally, Section 4 provides concluding remarks and future research directions.

2. Variational setting and basic tools

In what follows, assume that $\mathcal{D} \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with Lipschitz boundary $\partial\mathcal{D}$. For $1 \leq r \leq \infty$, the Lebesgue space $L^r(\mathcal{D})$ is endowed with the norm $\|\cdot\|_r$. For $1 \leq r < \infty$, the spaces $W^{1,r}(\mathcal{D})$ and $W_0^{1,r}(\mathcal{D})$ are endowed with the norms

$$\|\xi\|_{1,r} \quad \text{and} \quad \|\xi\|_{1,r,0} := \|\nabla\xi\|_r,$$

respectively.

We introduce the function $\mathcal{H} : \mathcal{D} \times [0, +\infty) \to [0, +\infty)$, defined as

$$\mathcal{H}(x, t) := t^p + \mu(x)t^q,$$

which induces the corresponding modular:

$$\rho_{\mathcal{H}}(\xi) := \int_{\mathcal{D}} \mathcal{H}(x, |\xi|)\, dx = \int_{\mathcal{D}} (|\xi|^p + \mu(x)|\xi|^q)\, dx.$$

The Musielak–Orlicz space is then introduced as

$$L^{\mathcal{H}}(\mathcal{D}) := \{\xi : \mathcal{D} \to \mathbb{R} \text{ measurable} : \rho_{\mathcal{H}}(\xi) < +\infty\},$$

endowed with the so-called Luxemburg norm:

$$\|\xi\|_{\mathcal{H}} := \inf\left\{\eta > 0 : \rho_{\mathcal{H}}\left(\frac{\xi}{\eta}\right) \leq 1\right\}.$$

The next proposition summarizes the key properties of Musielak–Orlicz spaces; for a comprehensive discussion, we refer to Diening et al. [11].

Proposition 2.1. *Let p and q be such that condition (H) is satisfied, u belongs to $L^{\mathcal{H}}(\mathcal{D})$, and τ is a real number. The following statements can be made:*

(1) Given that $\xi \neq 0$, the equivalence

$$\|\xi\|_{\mathcal{H}} = \tau \Longleftrightarrow \rho_{\mathcal{H}}\left(\frac{\xi}{\tau}\right) = 1$$

holds.

(2) The condition $\|\xi\|_{\mathcal{H}} < 1$ (or $= 1, > 1$) is true if and only if $\rho_{\mathcal{H}}(\xi) < 1$ (or $= 1, > 1$).

(3) For $\|\xi\|_{\mathcal{H}} < 1$, the following inequalities are valid:

$$\|\xi\|_{\mathcal{H}}^q \leq \rho_{\mathcal{H}}(\xi) \leq \|\xi\|_{\mathcal{H}}^p.$$

(4) If $\|\xi\|_{\mathcal{H}} > 1$, then it follows that:

$$\|\xi\|_{\mathcal{H}}^p \leq \rho_{\mathcal{H}}(\xi) \leq \|\xi\|_{\mathcal{H}}^q.$$

(5) The limit $\|\xi\|_{\mathcal{H}} \to 0$ is equivalent to $\rho_{\mathcal{H}}(\xi) \to 0$.

(6) The limit $\|\xi\|_{\mathcal{H}} \to +\infty$ is equivalent to $\rho_{\mathcal{H}}(\xi) \to +\infty$.

(7) The limit $\|\xi\|_{\mathcal{H}} \to 1$ is equivalent to $\rho_{\mathcal{H}}(\xi) \to 1$.

(8) If ξ_n converges to ξ in $L^{\mathcal{H}}(\mathcal{D})$, then:

$$\rho_{\mathcal{H}}(\xi_n) \to \rho_{\mathcal{H}}(\xi).$$

The Musielak-Orlicz-Sobolev space $W^{1,\mathcal{H}}(\mathcal{D})$ is specified as

$$W^{1,\mathcal{H}}(\mathcal{D}) := \left\{\xi \in L^{\mathcal{H}}(\mathcal{D}) : |\nabla\xi| \in L^{\mathcal{H}}(\mathcal{D})\right\},$$

and is endowed with the norm

$$\|\xi\|_{1,\mathcal{H}} := \|\xi\|_{\mathcal{H}} + \|\nabla\xi\|_{\mathcal{H}},$$

where $\|\nabla\xi\|_{\mathcal{H}} = \||\nabla\xi|\|_{\mathcal{H}}$, defined as

$$\||\nabla\xi|\|_{\mathcal{H}} := \inf\left\{\tau > 0 : \int_{\mathcal{D}} \mathcal{H}\left(x, \frac{|\nabla\xi|}{\tau}\right) dx \leq 1\right\}.$$

We define $W_0^{1,\mathcal{H}}(\mathcal{D})$ as the closure of $C_0^{\infty}(\mathcal{D})$ in $W^{1,\mathcal{H}}(\mathcal{D})$. It is well-known that the Musielak-Orlicz and Musielak-Orlicz-Sobolev spaces possess a uniformly convex Banach structure, which ensures reflexivity; see Crespo-Blanco et al. [10]. Moreover, the following embeddings hold true:

Proposition 2.2 ([10] (Propositions 2.16, 2.18)). *Assuming that condition (H) is satisfied, we conclude:*

(1) The embedding $W_0^{1,\mathcal{H}}(\mathcal{D}) \hookrightarrow L^{\mathcal{H}}(\mathcal{D})$ is compact.
(2) The embedding $W_0^{1,\mathcal{H}}(\mathcal{D}) \hookrightarrow L^r(\mathcal{D})$ is compact for every $r \in [1, p^*)$.
(3) The embedding $W^{1,\mathcal{H}}(\mathcal{D}) \hookrightarrow W_0^{1,r}(\mathcal{D})$ remains continuous for all $r \in [1, p^*]$.
(4) The embedding $L^{\mathcal{H}}(\mathcal{D}) \hookrightarrow L^r(\mathcal{D})$ is continuous for all $r \in [1, p^*]$.

Additionally, a Poincaré-type inequality is applicable, which enables us to define the equivalent norm in $W_0^{1,\mathcal{H}}(\mathcal{D})$ as follows:

$$\|\xi\| := \|\nabla\xi\|_{\mathcal{H}}.$$

In what follows, let $r < p^*$ be chosen such that the embedding $W_0^{1,\mathcal{H}}(\mathcal{D}) \hookrightarrow L^r(\mathcal{D})$ is continuous. We define C_r as the optimal constant satisfying the corresponding inequality:

$$\|\xi\|_r \leq C_r\|\xi\| \quad \forall\xi \in W_0^{1,\mathcal{H}}(\mathcal{D}). \tag{2.1}$$

In other terms, i represents the operator norm for the embedding

$$i : \left(W_0^{1,\mathcal{H}}(\mathcal{D}), \|\cdot\|\right) \to (L^r(\mathcal{D}), \|\cdot\|_r).$$

The differential operator in (P_λ) is known as the double-phase operator:

$$-\operatorname{div}\left(|\nabla\xi|^{p-2}\nabla\xi + \mu(x)|\nabla\xi|^{q-2}\nabla\xi\right), \quad \xi \in W_0^{1,\mathcal{H}}(\mathcal{D}).$$

Definition 2.1. *We say that a function $\xi \in W_0^{1,\mathcal{H}}(\mathcal{D})$ is a* weak solution *of problem (P_λ) if for any function $\varphi \in W_0^{1,\mathcal{H}}(\mathcal{D})$, the following identity holds:*

$$\int_{\mathcal{D}} \left(|\nabla\xi|^{p-2}\nabla\xi + \mu(x)|\nabla\xi|^{q-2}\nabla\xi\right)\cdot\nabla\varphi\, dx = \lambda \int_{\mathcal{D}} k(x)|\xi|^{s-2}\xi\varphi\, dx.$$

For each $\lambda > 0$, we seek weak solutions in the sense of Definition 2.1.

Remark 2.2. *(1) From condition (H), we have:*

$$\frac{s\gamma}{\gamma-1} < \frac{s\gamma}{\gamma-s} < p^*,$$

where $p^* = \frac{Np}{N-p}$.

Proof. From condition (H), $sN < \gamma p$ implies $N\gamma - \gamma p < N\gamma - sN$. Since $\gamma > s$ and $N > p$, divide by $(\gamma - s)(N - p) > 0$ to get:

$$\frac{\gamma}{\gamma-s} < \frac{N}{N-p}.$$

Multiply by $s > 1$:

$$\frac{s\gamma}{\gamma-s} < \frac{sN}{N-p}.$$

Since $s < p$, we have $\frac{sN}{N-p} < \frac{pN}{N-p} = p^*$. Thus,

$$\frac{s\gamma}{\gamma-s} < p^*.$$

Since $1 < s < \gamma$, $\gamma - s < \gamma - 1$, so $\frac{s\gamma}{\gamma-s} > \frac{s\gamma}{\gamma-1}$. Hence,

$$\frac{s\gamma}{\gamma-1} < \frac{s\gamma}{\gamma-s} < p^*.$$

□

(2) For $\xi \in W_0^{1,\mathcal{H}}(\mathcal{D})$ and $\varphi \in W_0^{1,\mathcal{H}}(\mathcal{D})$, the term $k(x)|\xi|^{s-2}\xi\varphi$ belongs to $L^1(\mathcal{D})$.

Proof. Because:

- $k \in L^\gamma(\mathcal{D})$ by assumption,
- $|\xi|^{s-1} \in L^{\frac{s}{s-1}}(\mathcal{D})$,
- and $\varphi \in L^{\frac{\gamma s}{\gamma-s}}(\mathcal{D})$.

Hölder's inequality then yields:

$$\int_{\mathcal{D}} |k(x)|\xi|^{s-2}\xi\varphi|\, dx \leq \|k\|_\gamma\|\xi\|^{s-1}_{\frac{s}{s-1}}\|\varphi\|_{\frac{\gamma s}{\gamma-s}} < \infty,$$

via the embedding $W_0^{1,\mathcal{H}}(\mathcal{D}) \hookrightarrow L^{\frac{\gamma s}{\gamma-1}}(\mathcal{D})$. □

In the sequel, we define the functional $\mathcal{J}_\lambda \colon W_0^{1,\mathcal{H}}(\mathcal{D}) \to \mathbb{R}$ by

$$\mathcal{J}_\lambda(\xi) := \mathcal{A}(\xi) - \lambda\mathcal{B}(\xi),$$

where

$$\mathcal{A}(\xi) := \int_{\mathcal{D}} \left(\frac{|\nabla\xi|^p}{p} + \mu(x)\frac{|\nabla\xi|^q}{q}\right) \mathrm{d}x, \tag{2.2}$$

and

$$\mathcal{B}(\xi) := \int_{\mathcal{D}} \frac{1}{s}k(x)|\xi|^s dx. \tag{2.3}$$

It is well known that, under condition (H), that $\mathcal{A}$ is well defined and possesses continuous Gâteaux differentiability, defined by

$$\langle \mathcal{A}'(\xi), \varphi \rangle = \int_{\mathcal{D}} \Big(|\nabla \xi|^{p-2} \nabla \xi \cdot \nabla \varphi \; + \mu(x) |\nabla \xi|^{q-2} \nabla \xi \cdot \nabla \varphi \Big) dx;$$

moreover, by (H), we have $\frac{s\gamma}{\gamma-1} < p^*$. Consequently, $W_0^{1,\mathcal{H}}(\mathcal{D}) \hookrightarrow L^{\frac{s\gamma}{\gamma-1}}(\mathcal{D})$ is compact, so together with the Hölder inequality, we deduce that $\mathcal{B}$ is also well defined; in fact,

$$\int_{\mathcal{D}} \frac{k(x)}{s} |\xi|^s dx \leq \frac{1}{s} \|k\|_\gamma \| |\xi|^s \|_{\frac{\gamma}{\gamma-1}} \leq \frac{1}{s} \|k\|_\gamma \|\xi\|^s_{\frac{s\gamma}{\gamma-1}}.$$

Furthermore $\mathcal{B}$ is continuously differentiable in the Gâteaux sense, with

$$\langle \mathcal{B}'(\xi), \varphi \rangle = \int_{\mathcal{D}} k(x) |\xi|^{s-2} \xi \varphi \, dx.$$

Lemma 2.3. (*see [17], (Proposition 3.1)*)

- *The functional $\mathcal{A}'$ is Gâteaux differentiable and strictly monotone within $W_0^{1,\mathcal{H}}(\mathcal{D})$.*
- *The functional $\mathcal{A}'$ exhibits the properties of (S_+)-type map, that is, if $\xi_n \rightharpoonup \xi$ in $W_0^{1,\mathcal{H}}(\mathcal{D})$, and if $\overline{\lim}_{n\to\infty} \langle \mathcal{A}'(\xi_n) - \mathcal{A}'(\xi), \xi_n - \xi \rangle \leq 0$, then it follows that $\xi_n \to \xi$ in $W_0^{1,\mathcal{H}}(\mathcal{D})$.*
- *The functional $\mathcal{A}'$ is a homeomorphism.*

Lemma 2.4. *The functional $\mathcal{B}' : X := W_0^{1,\mathcal{H}}(\mathcal{D}) \to (W_0^{1,\mathcal{H}}(\mathcal{D}))^*$ is compact.*

Proof. Let $(\xi_k)_k \subset X$ be a sequence such that $\xi_k \rightharpoonup \xi$. Since $\frac{\gamma s}{\gamma-1} < p^*$, there exists a subsequence, also denoted by $(\xi_k)_k$, satisfying $\xi_k \to \xi$ strongly in $L^{\frac{\gamma s}{\gamma-1}}(\mathcal{D})$.

By the properties of the Nemytskii operator (see [12]), we have

$$N_s(\xi_k) = |\xi_k|^{s-2}\xi_k \to N_s(\xi) \quad \text{in } L^{\frac{s}{s-1}}(\mathcal{D}).$$

By using Hölder's inequality, one has

$$\begin{aligned} \Big| \int_{\mathcal{D}} k(x)|\xi_k|^{s-2}\xi_k \varphi dx - \int_{\mathcal{D}} k(x)|\xi|^{s-2}\xi\varphi dx \Big| \\ \leq 3\|k\|_\gamma \|N_s(\xi_k) - N_s(\xi)\|_{\frac{s}{s-1}} \|\varphi\|_{\frac{\gamma s}{\gamma - s}} \\ \leq 3c_{\gamma s} \|k\|_\gamma \|N_s(\xi_k) - N_s(\xi)\|_{\frac{s}{s-1}} \|\varphi\|. \end{aligned}$$

where $c_{\gamma s}$ is the embedding constant of $X \hookrightarrow L^{\frac{\gamma s}{\gamma-s}}(\mathcal{D})$. Thus,

$$\int_{\mathcal{D}} k(x)|\xi_k|^{s-2}\xi_k\varphi dx \to \int_{\mathcal{D}} k(x)|\xi|^{s-2}\xi\varphi dx, \quad \text{as } k \to +\infty. \tag{2.4}$$

Consequently, $\mathcal{B}'$ is compact. □

The following definition and critical point theorems constitute the principal tools used to obtain our result.

In order to formulate our existence result, we need these preliminary definitions and theorems.

Definition 2.5. *Consider $\mathcal{A}$ and $\mathcal{B}$ to be a pair of functionals acting on a real Banach space X that possess continuous Gâteaux derivatives, and let $d \in \mathbb{R}$ be fixed. $\mathcal{J} := \mathcal{A} - \mathcal{B}$ is said to satisfy the Palais-Smale requirement cut off at d (denoted as $(PS)^{[d]}$) if every sequence $\{\xi_n\}_{n\in\mathbb{N}} \subset X$ that meets the following criteria:*

- *$\mathcal{J}(\xi_n)$ remains bounded,*
- $\lim_{n\to+\infty} \|I'(\xi_n)\|_{X^*} = 0$,
- *and $\mathcal{A}(\xi_n) < d$ for all $n \in \mathbb{N}$,*

contains a subsequence that converges.

If $d = \infty$, then the functional $\mathcal{J} = \mathcal{A} - \mathcal{B}$ satisfies the Palais-Smale condition.

The primary existence result is due to the following theorem.

Theorem 2.6. *(Theorem 3.2 [6]) Let X be a real Banach space, and let $\mathcal{A}, \mathcal{B} : X \to \mathbb{R}$ be two functionals that are continuously Gâteaux differentiable such that*

$$(i) \qquad \inf_{x\in X} \mathcal{A} = \mathcal{A}(0) = \mathcal{B}(0) = 0.$$

Assume the existence of a positive constant $d \in \mathbb{R}$ and $\overline{x} \in X$ with $0 < \mathcal{A}(\overline{x}) < d$ satisfying

$$(ii) \qquad \frac{\sup_{x\in\mathcal{A}^{-1}(]-\infty,d[)} \mathcal{B}(x)}{d} < \frac{\mathcal{B}(\overline{x})}{\mathcal{A}(\overline{x})}$$

and

$$(iii) \quad \text{for every } \lambda \in \Lambda := \left] \frac{\mathcal{A}(\overline{x})}{\mathcal{B}(\overline{x})}, \frac{d}{\sup_{x\in\mathcal{A}^{-1}(]-\infty,d[)} \mathcal{B}(x)} \right[.$$

The functional $\mathcal{J}_\lambda = \mathcal{A} - \lambda\mathcal{B}$ satisfies the $(PS)^{[d]}$-condition.

Then, for every $\lambda \in \Lambda$, there exists $\xi_\lambda \in \mathcal{A}^{-1}(]0, d[)$ such that $\mathcal{J}_\lambda(\xi_\lambda) \leq I_\lambda(\xi)$ for all $\xi \in \mathcal{A}^{-1}(]0, d[)$ and $I'_\lambda(\xi_\lambda) = 0$.

Theorem 2.7 ([14, Theorem 3.6]). *Suppose X is a real reflexive Banach space, and the following assumptions are satisfied:*

- *$\mathcal{A} : X \to \mathbb{R}$ is a coercive functional, continuously Gâteaux differentiable, and weakly lower semicontinuous.*
- *The Gâteaux derivative of $\mathcal{A}$ admits a continuous inverse on the dual space X^*.*
- *$\mathcal{B} : X \to \mathbb{R}$ is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact.*

Additionally, assume that

$$(a_0) \quad \inf_X \mathcal{A} = \mathcal{A}(0) = 0 \quad \text{and} \quad \mathcal{B}(0) = 0.$$

There exists a positive constant d and an element $\overline{v} \in X$ such that $d < \mathcal{A}(\overline{v})$, with the following conditions:

$$(a_1) \quad \frac{\sup_{\mathcal{A}(x)<d} \mathcal{B}(x)}{d} < \frac{\mathcal{B}(\overline{v})}{\mathcal{A}(\overline{v})},$$

$$(a_2) \quad \text{For each } \lambda \in \Lambda_d := \left(\frac{\mathcal{A}(\overline{v})}{\mathcal{B}(\overline{v})}, \frac{d}{\sup_{\mathcal{A}(x)\leq d} \mathcal{B}(x)} \right),$$

the functional $I_\lambda := \mathcal{A} - \lambda\mathcal{B}$ is coercive.

Then, for every $\lambda \in \Lambda_d$, the functional $\mathcal{A} - \lambda\mathcal{B}$ possesses at least three distinct critical points in X.

3. Existence of weak solutions

This section is devoted to proving a multiplicity result, specifically demonstrating that problem (P_λ) admits at least three distinct weak solutions.

To begin, we note that, according to Proposition 2.1, we have

$$\begin{aligned}\mathcal{A}(\xi) &= \int_{\mathcal{D}} \left(\frac{|\nabla\xi|^p}{p} + \mu(x)\frac{|\nabla\xi|^q}{q} \right) \mathrm{d}x \\ &\geq \frac{1}{q}\rho_{\mathcal{H}}(\nabla\xi) \\ &\geq \frac{1}{q} \min\left(\|\xi\|^p, \|\xi\|^q\right),\end{aligned} \tag{3.1}$$

and thus, $\mathcal{A}$ is coercive and consequently bounded from below. Furthermore, we have the following lemma:

Lemma 3.1. *The functional $\mathcal{J}_\lambda$ satisfies the Palais-Smale condition for every $\lambda > 0$.*

Proof. Consider a Palais-Smale sequence $\{\xi_n\} \subseteq X$. This implies that

$$\sup_n \mathcal{J}_\lambda(\xi_n) < +\infty \quad \text{and} \quad \|\mathcal{J}'_\lambda(\xi_n)\|_{X^*} \to 0. \tag{3.2}$$

We will demonstrate that the sequence $\{\xi_n\}$ contains a convergent subsequence. By applying Hölder's inequality, we have

$$\begin{aligned}\langle \mathcal{B}'(\xi_n), \xi_n \rangle &= \int_{\mathcal{D}} k(x)|\xi_n|^s \, dx \\ &\leq \frac{1}{s}\|k\|_\gamma \||\xi_n|^s\|_{\gamma'} \\ &\leq \frac{1}{s}\|k\|_s c^s_{\gamma' s}\|\xi_n\|^s.\end{aligned}$$

Here, γ' represents the dual exponent corresponding to γ, and $c_{\gamma' s}$ is the constant from the continuous embedding of X into $W^{1,\gamma' s}(\mathcal{D})$.

For sufficiently large n, we obtain:

$$\begin{aligned}\langle \mathcal{J}'_\lambda(\xi_n), \xi_n \rangle &= \langle \mathcal{A}'_\lambda(\xi_n), \xi_n \rangle - \lambda\langle \mathcal{B}'_\lambda(\xi_n), \xi_n \rangle \\ &\geq \|\xi_n\|^p - \lambda\frac{1}{s}\|k\|_s c^s_{\gamma' s}\|\xi_n\|^s.\end{aligned}$$

Using (3.2), we find:

$$\|\xi_n\|^p \leq \lambda\frac{1}{s}\|k\|_s c^s_{\gamma' s}\|\xi_n\|^s.$$

Since $s < p$, it follows that $\{\xi_n\}$ is bounded. By passing to a subsequence if necessary, we can assume that $\xi_n \rightharpoonup \xi$. Consequently, due to the compactness of $\mathcal{B}'$, we have $\mathcal{B}'(\xi_n) \to \mathcal{B}'(\xi)$.

Combining this with the fact that $\mathcal{J}'_\lambda(\xi_n) = \mathcal{A}'(\xi_n) - \lambda\mathcal{B}'(\xi_n) \to 0$, we obtain:

$$\mathcal{A}'(\xi_n) \to \lambda\mathcal{B}'(\xi).$$

Since $\mathcal{A}'$ is a homeomorphism, it follows that $\xi_n \to \xi$. Therefore, we conclude that $\mathcal{J}_\lambda$ satisfies the Palais-Smale condition. □

We are now in a good position to state our main theorem. For this purpose, we introduce the following function:

$$\tilde{\delta}(x) := \sup\left\{\tilde{\delta} > 0 \mid B(x, \tilde{\delta}) \subseteq \mathcal{D}\right\}$$

for every $x \in \mathcal{D}$, where $B(x, \tilde{\delta})$ represents a ball centered at x with radius $\tilde{\delta}$. It is clear that there exists a point $x^0 \in \mathcal{D}$ such that $B(x^0, r) \subseteq \mathcal{D}$, where

$$r = \sup_{x\in\mathcal{D}} \tilde{\delta}(x).$$

In the following, assume that $k(x)$ satisfies the condition

$$k(x) := \begin{cases} \leq 0, & \text{for } x \in \mathcal{D} \setminus B(x^0, r), \\ \geq k_0, & \text{for } x \in B(x^0, \frac{r}{2}), \\ > 0, & \text{for } x \in B(x^0, r) \setminus B(x^0, \frac{r}{2}), \end{cases}$$

where k_0 is a positive constant. The symbol $\tilde{m}$ is defined as the constant

$$\tilde{m} = \frac{\pi^{\frac{N}{2}}}{\frac{N}{2}\Gamma\left(\frac{N}{2}\right)},$$

where Γ refers to the gamma function.

Theorem 3.2. *Consider the existence of two positive constants δ, d, such that*

$$\frac{1}{p}\left(\left(\frac{2\delta}{r}\right)^p + \left(\frac{2\delta}{r}\right)^q \|\mu\|_\infty\right)\tilde{m}\left(r^N - \left(\frac{r}{2}\right)^N\right) < d. \qquad (3.3)$$

Define

$$A_\delta := \frac{\frac{1}{p}\left(\left(\frac{2\delta}{r}\right)^p + \left(\frac{2\delta}{r}\right)^q \|\mu\|_\infty\right)\tilde{m}\left(r^N - \left(\frac{r}{2}\right)^N\right)}{\frac{1}{s}k_0\delta^s\tilde{m}\left(\frac{r}{2}\right)^N}$$

and

$$B_d := \frac{d}{\frac{1}{s}\|k\|_s c^s_{\gamma' s} \max\left\{(dq)^{\frac{s}{p}}, (dq)^{\frac{s}{q}}\right\}}.$$

Thus, for any λ within the interval (A_δ, B_d), the problem P_λ possesses at least one weak solution.

Proof. As showed earlier, the functionals $\mathcal{A}$ and $\mathcal{B}$ exhibit continuous Gâteaux differentiability, and moreover, condition (i) of Theorem 2.6 holds. Consider δ and d as in (3.3), and define $v_\delta \in X$ such that

$$v_\delta(x) := \begin{cases} 0 & x \in \mathcal{D}\setminus B\left(x^0, r\right), \\ \frac{2\delta}{r}\left(r - \left|x - x^0\right|\right) & x \in B\left(x^0, r\right)\setminus B\left(x^0, \frac{r}{2}\right), \\ \delta & x \in B\left(x^0, \frac{r}{2}\right). \end{cases}$$

Next, based on the definition of the functional $\mathcal{A}$, it follows that

$$\begin{aligned} &\frac{1}{p}\Big(\frac{2\delta}{r}\Big)^p\tilde{m}\left(r^N - \left(\frac{r}{2}\right)^N\right) \\ <&\mathcal{A}(v_\delta) \qquad (3.4) \\ \leq&\frac{1}{p}\left(\Big(\frac{2\delta}{r}\Big)^p + \Big(\frac{2\delta}{r}\Big)^q\|\mu\|_\infty\right)\tilde{m}\left(r^N - \left(\frac{r}{2}\right)^N\right). \end{aligned}$$

Thus, given that $0 < \mathcal{A}(v_\delta) < d$, combined with Lemma 3.1, we can conclude that $\mathcal{J}_\lambda$ satisfies $(PS)^{[d]}$-condition. Furthermore, based on the definition of $\mathcal{B}$, the form of v_δ, and the hypothesis on $k(x)$, it follows that

$$\mathcal{B}(v_\delta) \geq \int_{B\left(x_0, \frac{r}{2}\right)} \frac{k(x)}{s}|v_\delta|^\gamma\, dx \geq \frac{1}{s}k_0\delta^s\tilde{m}\left(\frac{r}{2}\right)^N. \qquad (3.5)$$

This leads to

$$\frac{\mathcal{B}(v_\delta)}{\mathcal{A}(v_\delta)} \geq \frac{\frac{1}{s}k_0\delta^s\tilde{m}\left(\frac{r}{2}\right)^N}{\frac{1}{p}\left(\left(\frac{2\delta}{r}\right)^p + \left(\frac{2\delta}{r}\right)^q\|\mu\|_\infty\right)\tilde{m}\left(r^N - \left(\frac{r}{2}\right)^N\right)}.$$

Moreover, for every $\xi \in \mathcal{A}^{-1}(]-\infty, d[)$, it follows that

$$\frac{1}{q}\min\left(\|\xi\|^p, \|\xi\|^q\right) \leq d. \qquad (3.6)$$

Consequently,

$$\|\xi\| \leq \max\left\{(dq)^{\frac{1}{p}}, (dq)^{\frac{1}{q}}\right\}.$$

To conclude, we obtain the following result:

$$\sup_{\mathcal{A}(\xi)<d}\mathcal{B}(\xi) \quad \leq \quad \frac{1}{s}\|k\|_s c^s_{\gamma' s}\max\left\{(dq)^{\frac{s}{p}}, (dq)^{\frac{s}{q}}\right\},$$

and

$$\frac{1}{d}\sup_{\mathcal{A}(\xi)<d}\mathcal{B}(\xi) < \frac{1}{\lambda}.$$

This concludes the proof. □

Theorem 3.3. *Assume there are two positive constants δ, d, such that*

$$\frac{1}{p}\left(\frac{2\delta}{r}\right)^p\tilde{m}\left(r^N - \left(\frac{r}{2}\right)^N\right) = d.$$

Then, for every λ in the interval (A_δ, B_d) (where A_δ and B_d are as defined in Theorem 3.2), the problem P_λ admits at least three weak solutions.

Proof. It is crucial to highlight that the functionals $\mathcal{A}$ and $\mathcal{B}$, linked to the problem (P_λ) and defined in (2.2) and (2.3), satisfy the regularity conditions outlined in Theorem 2.7. We now proceed to verify that conditions (a_1) and (a_2) hold. To this end, consider the equation

$$\frac{1}{p}\left(\frac{2\delta}{r}\right)^p \tilde{m}\left(r^N - \left(\frac{r}{2}\right)^N\right) = d.$$

Based on inequality (3.4), we deduce that $d < \mathcal{A}(v_\delta)$. Additionally, the coerciveness of $\mathcal{J}_\lambda$ for every $\lambda > 0$ can be established using inequality (3.5). For sufficiently large $\|\xi\|$, we have

$$\mathcal{A}(\xi) - \lambda\mathcal{B}(\xi) \geq \frac{1}{q}\|\xi\|^p - \lambda\frac{1}{s}\|k\|_s c^s_{\gamma' s}\|\xi\|^s.$$

Since $p > s$, this leads to the desired result. Finally, noting that

$$\bar{\Lambda}_d := (A_\delta, B_d) \subseteq \left(\frac{\mathcal{A}(v_\delta)}{\mathcal{B}(v_\delta)}, \frac{d}{\sup_{\mathcal{A}(\xi)<d}\mathcal{B}(\xi)}\right),$$

and given that all hypotheses of Theorem 2.7 are fulfilled, as a result for every λ that belongs into $\bar{\Lambda}_d$, $\mathcal{A} - \lambda\mathcal{B}$ admits at least three critical points in X. It follows that such critical points coincide with the weak solutions of problem (P_λ).

□

4. Conclusions and perspectives

In this work, we established the existence and multiplicity of weak solutions for a double-phase elliptic problem with an indefinite weight, employing variational methods and critical point theory. Our results extend previous studies by incorporating sign-changing and singular terms, demonstrating the applicability of these techniques to non-uniform media. Future research could explore variable exponents, stronger singularities, anisotropic settings, and numerical approximations, bridging theoretical advances with practical applications in physics and engineering.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare that there are no conflicts of interest regarding the publication of this paper.

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