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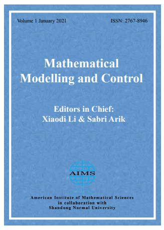

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Research article

Higher-order dynamics and optimal control of SEIR rumor propagation models in homogeneous and heterogeneous networks

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Abstract: Higher-order interactions play a critical role in driving the rapid propagation of rumors. We propose a susceptible–exposed–infected–removed (SEIR) model that incorporates group interactions as higher-order terms. Detailed analysis reveals the emergence of bistability in both homogeneous and heterogeneous networks and shows that the bistable region expands as group interactions intensify. By utilizing group interactions as the bifurcation parameter, we derive explicit bifurcation conditions for both network types. For heterogeneous networks, we establish global stability criteria and propose an effective control strategy. Numerical simulations demonstrate the significant influence of group interactions on the bistable region. This study advances the theoretical understanding of rumor propagation dynamics and provides a practical strategy for controlling rumor propagation.

Keywords: simplicial complexes; rumor propagation; higher-order dynamics; optimal control

1. Introduction

In recent decades, complex networks have emerged as an influential framework for understanding social contagion mechanisms [1–3]. Rumor propagation, a key area of social contagion, has garnered significant attention due to its impact on social stability and public health. By representing individuals as nodes and interactions as edges, network models offer an intuitive and effective framework for understanding the complex phenomenon of rumor propagation [4, 5]. Traditionally, rumor propagation has been modeled through pairwise interactions between individuals, which has been the focus of numerous previous studies [6–9]. However, in reality, rumor propagation often involves group discussions. For example, when three individuals engage in a group chat on a social platform such as QQ or WeChat, their interactions influence one another. Such group interactions are better represented by higher-order network models, which can capture complex connections and collective behaviors [10–16].

The propagation of rumors is similar to the transmission of contagious diseases, where individuals who are "infected" with a rumor spread it to others [17, 18]. Many researchers have used epidemic models to investigate rumor propagation. Studies have shown that epidemic models based on higher-order networks exhibit rich dynamic behaviors such as abrupt transitions, multistability, and bifurcations [19–21]. Iacopini et al. proposed a simplicial complex model that reveals discontinuous phase transitions and bistability. They also introduced an innovative approach to construct two-dimensional simplicial complexes with tunable average degree $\langle k \rangle$ and average 2-simplex degree $\langle k_{\triangle} \rangle$ [22]. Li et al. investigated the competition dynamics of SIS epidemic models on simplicial complexes. Their results show that the infection strength of 2-simplices greatly influences the system's phase diagram [23]. Matamalas et al. established a simplicial propagation model based on individual interactions. Their results show that an increase in the infection rate leads to a sudden phase transition [24]. Ferraz et al. proposed that the system's multistability or intermittency can be controlled by adjusting the number or strength of connections between communities [25]. Kiss et al. proposed that bistability and multistability emerge from higher-order contagion mechanisms without the need for complex contact patterns [26].

Real-world social networks are typically heterogeneous, which significantly impacts the dynamics of rumor propagation, such as propagation speed, thresholds, and stability [27]. Ding et al. investigated the impact of time delay and saturated conversion function on the dynamics of rumor propagation in heterogeneous networks, and their study revealed that the introduction of the saturated conversion function causes the system to exhibit backward bifurcation under certain conditions [28]. Chowdhary et al. proposed a framework that extends the simplicial contagion model to time-varying networks. Their results show that the dynamics of heterogeneous simplicial complexes are less constrained by temporality compared to homogeneous networks [29]. St-Onge et al. proposed an approximate master equation framework for contagion on heterogeneous structured hypergraphs. Their findings reveal that variations in group sizes and nonlinear contagion dynamics facilitate mesoscopic localization, thereby suppressing the emergence of bistability [30].

Although several extensions of classical epidemic models, such as susceptible–exposed–infected–removed (SEIR) formulations, have been employed to study rumor spreading, most of these studies primarily focus on pairwise interactions or simplified assumptions about group structures [31, 32]. Consequently, they often overlook the higher-order dynamics induced by multi-individual interactions and the crucial role of network heterogeneity. Likewise, existing rumor control approaches usually rely on heuristic strategies (e.g., reducing contact rates or increasing removal probabilities) without explicitly incorporating the structural complexity of social networks into the control framework [25,28]. Recent advances in control theory, such as resilient asynchronous estimation and event-triggered schemes in complex stochastic systems, provide promising methodological insights for developing more rigorous rumor suppression strategies [33, 34]. These limitations motivate the need for models that can simultaneously capture higher-order contagion processes, heterogeneous topologies, and control strategies grounded in rigorous mathematical analysis.

To address this gap and reveal the higher-order dynamics of rumor propagation, this study proposes an SEIR model with parameters designed to precisely describe the transition probabilities of susceptible individuals to infected, exposed, or removed states. We systematically examine the existence of equilibrium points, stability ranges, and bifurcation phenomena in both homogeneous and heterogeneous networks, using group interaction parameters as bifurcation parameters. Furthermore, by incorporating network heterogeneity, we investigate bistability in heterogeneous networks and propose optimal control strategies for rumor propagation. This extension not only enhances the applicability of the original model but also underscores the pivotal role of network heterogeneity in system dynamics, providing fresh insights into complex network research.

In Section 2, we introduce the spreading dynamics of the simplicial SEIR model for rumor propagation. In Sections 3 and 4, we establish the SEIR rumor propagation model in homogeneous and heterogeneous networks in which the existence of equilibrium points, multistability, and bifurcations is analyzed. In Section 5, we discuss the optimal control problem for rumor propagation on heterogeneous networks. In Section 6, we provide numerical simulation results. In Section 7, we conclude the paper.

2. Simplicial SEIR model for rumor propagation

In the simplicial SEIR (susceptible-exposed-infected-removed) rumor propagation model, the variables S, E, I, and R represent distinct categories of individuals, which can be described as follows.

Susceptible: This category represents individuals who are susceptible to the new rumor but have not yet been exposed to it.

Exposed: This category represents individuals who have encountered and believe the new rumor but do not propagate it.

Infected: This category represents individuals who have encountered and believe the new rumor and actively propagate it.

Removed: This category represents individuals who no longer believe the new rumor and no longer propagate it.

The propagation process of the SEIR model can be described as follows (Figure 1).

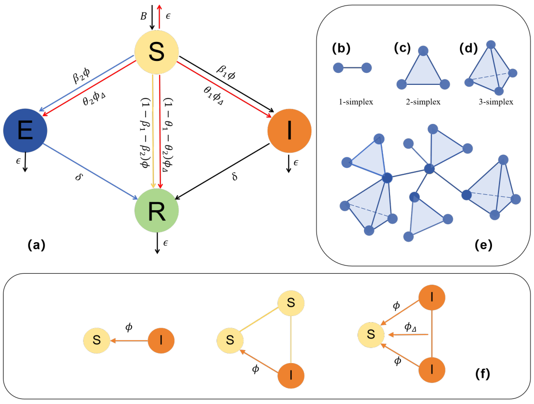


Figure 1. (a) Propagation rules of the SEIR model. (b) A 1-simplex (edge). (c) A 2-simplex (triangle). (d) A 3-simplex (tetrahedron). (e) A simplicial complex constructed from simplices, including nodes, edges, orange triangles, and blue tetrahedra. (f) Occurrence of pairwise and group interactions.

(1) During rumor propagation, B represents the rate of new individuals entering the system per unit time, and all these new individuals are in the susceptible state. Additionally, all individuals share the same emigration rate ϵ.

(2) When a susceptible individual encounters an infected individual, the susceptible individual transitions to a different state with a transmission rate denoted by ϕ. The probabilities of transitioning to the infected, exposed, or removed states are represented by β_1, β_2, and $1-\beta_1-\beta_2$, respectively. These parameters are constant proportions within the interval $[0,1]$.

(3) A group effect emerges in rumor propagation when a susceptible individual shares a 2-simplex with two infected individuals. The rate at which a susceptible individual changes to other states due to group interactions is denoted by $\phi_\triangle$. The probabilities of transitioning to the infected, exposed, or removed states under this group effect are represented by θ_1, θ_2, and $(1-\theta_1-\theta_2)$, respectively. These parameters are constant proportions within the interval $[0,1]$.

(4) Due to each individual's self-renewal and judgment abilities, an exposed or infected individual can transition to the removed state at a rate δ, which represents the self-recovery rate.

Figure 1(a) shows the dynamic transitions among different categories. Figure 1(b)–(d) shows the structures of a 1-simplex, 2-simplex, and 3-simplex, respectively. Figure 1(e) presents a simplicial complex constructed from simplices. Figure 1(f) shows the occurrence of pairwise interactions and group interactions.

The main symbols used in this study and their corresponding meanings are summarized in Table 1.

Table 1. List of main symbols used in the model.

Symbol	Description
$\rho^S(t)$	Density of susceptible individuals at time t.
$\rho^E(t)$	Density of exposed individuals at time t.
$\rho^I(t)$	Density of infected individuals at time t.
$\rho^R(t)$	Density of removed individuals at time t.
B	Immigration rate of individuals.
ϵ	Emigration rate of individuals.
ϕ	Transmission rate when a susceptible meets an infected individual.
$\phi_\triangle$	Transmission rate via group interactions.
β_1	Probability that a susceptible becomes infected after contact. ([0, 1])
β_2	Probability that a susceptible becomes exposed after contact. ([0, 1])
$1-\beta_1-\beta_2$	Probability that a susceptible becomes removed after contact.
θ_1	Probability that a susceptible becomes infected due to group effect. ([0, 1])
θ_2	Probability that a susceptible becomes exposed due to group effect. ([0, 1])
$1-\theta_1-\theta_2$	Probability that a susceptible becomes removed due to group effect.
δ	Self-recovery rate.
$\langle k\rangle$	Average degree of the network.
$\langle k_\triangle\rangle$	Average number of 2-simplices.

3. Simplicial SEIR rumor propagation model in homogeneous networks

The dynamic equations of the simplicial SEIR rumor propagation model in homogeneous networks are presented as follows:

$$\begin{cases} \dfrac{d\rho^S(t)}{dt} = B - \phi\langle k\rangle\rho^S(t)\rho^I(t) - \phi_\triangle\langle k_\triangle\rangle\rho^S(t)(\rho^I(t))^2 - \epsilon\rho^S(t), \\ \dfrac{d\rho^E(t)}{dt} = \beta_2\phi\langle k\rangle\rho^S(t)\rho^I(t) + \theta_2\phi_\triangle\langle k_\triangle\rangle\rho^S(t)(\rho^I(t))^2 - (\delta+\epsilon)\rho^E(t), \\ \dfrac{d\rho^I(t)}{dt} = \beta_1\phi\langle k\rangle\rho^S(t)\rho^I(t) + \theta_1\phi_\triangle\langle k_\triangle\rangle\rho^S(t)(\rho^I(t))^2 - (\delta+\epsilon)\rho^I(t), \\ \dfrac{d\rho^R(t)}{dt} = (1-\beta_1-\beta_2)\phi\langle k\rangle\rho^S(t)\rho^I(t) + \delta[\rho^E(t)+\rho^I(t)] \\ \qquad + (1-\theta_1-\theta_2)\phi_\triangle\langle k_\triangle\rangle\rho^S(t)(\rho^I(t))^2 - \epsilon\rho^R(t), \end{cases} \tag{3.1}$$

in which $\rho^S(t)$, $\rho^E(t)$, $\rho^I(t)$, and $\rho^R(t)$ represent the densities of susceptible, exposed, infected, and removed individuals at time t. Here, $\langle k\rangle$ is the average degree, and $\langle k_\triangle\rangle$ is the average number of 2-simplices.

In this model, the term $\phi_\triangle\langle k_\triangle\rangle\rho^S(t)(\rho^I(t))^2$ captures the core of the higher-order interactions. Specifically, it describes the scenario where a susceptible individual interacts simultaneously with two infected individuals through a 2-simplex.

The total number of individuals at time t is given by $\rho^N(t)$. It is evident that

$$\frac{d\rho^N(t)}{dt} = B - \epsilon\rho^N(t).$$

Subsequently, we have

$$\lim_{t\to+\infty}\rho^N(t) = \frac{B}{\epsilon}.$$

The initial conditions of system (3.1) are $\rho^S(0)\geq 0$, $\rho^E(0)\geq 0$, $\rho^I(0) > 0$, and $\rho^R(0)\geq 0$.

Obviously, we can find that $\rho^E(t)$ and $\rho^R(t)$ are independent of the differential equations for $\rho^S(t)$ and $\rho^I(t)$. Thus, we can solve the system of ordinary differential equations for $\rho^S(t)$ and $\rho^I(t)$ without considering $\rho^E(t)$ and $\rho^R(t)$. Then, the simplified equations can be obtained as follows:

$$\begin{cases} \dfrac{d\rho^S(t)}{dt} = B - \phi\langle k\rangle\rho^S(t)\rho^I(t) - \phi_\triangle\langle k_\triangle\rangle\rho^S(t)(\rho^I(t))^2 - \epsilon\rho^S(t), \\ \dfrac{d\rho^I(t)}{dt} = \beta_1\phi\langle k\rangle\rho^S(t)\rho^I(t) + \theta_1\phi_\triangle\langle k_\triangle\rangle\rho^S(t)(\rho^I(t))^2 \\ \qquad - (\delta+\epsilon)\rho^I(t), \end{cases} \tag{3.2}$$

where the initial conditions satisfy $\rho^S(0)\geq 0$ and $\rho^I(0) > 0$.

To derive the basic reproduction number R_0, we apply the next-generation matrix method [35]. We select the equation that represents the changes in the state of infected individuals and linearize it using the Jacobian matrix at the equilibrium point.

Then, we decompose the linearized matrix as F and V. It can be derived that

$$F = \beta_1\phi\langle k\rangle\rho^S(0) = \beta_1\phi\langle k\rangle\frac{B}{\epsilon}, \quad V = (\delta+\epsilon).$$

Because $\rho^I(0) = 0$ at the rumor-free equilibrium, V simplifies to $\delta+\epsilon$. Subsequently, we have

$$FV^{-1} = \beta_1\phi\langle k\rangle\frac{B}{\epsilon}\cdot\frac{1}{\delta+\epsilon}.$$

Thus, we get

$$R_0 = \rho(FV^{-1}) = \frac{B\beta_1\phi\langle k\rangle}{\epsilon(\delta+\epsilon)}.$$

3.1. Existence of equilibrium points

It is evident that $E_0 = (B/\epsilon, 0)$ is the rumor-free equilibrium point. Then, we solve Eq (3.3) to find the rumor-prevailing equilibrium point, $E_* = (\rho^S_*, \rho^I_*)$.

$$\begin{cases} 0 = B - \phi\langle k\rangle\rho^S_*\rho^I_* - \phi_\triangle\langle k_\triangle\rangle\rho^S_*(\rho^I_*)^2 - \epsilon\rho^S_*, \\ 0 = \beta_1\phi\langle k\rangle\rho^S_* + \theta_1\phi_\triangle\langle k_\triangle\rangle\rho^S_*\rho^I_* - (\delta+\epsilon). \end{cases} \tag{3.3}$$

We obtain

$$\rho^S_* = \frac{\delta+\epsilon}{\beta_1\phi\langle k\rangle + \theta_1\phi_\triangle\langle k_\triangle\rangle\rho^I_*}.$$

Substituting this into Eq (3.3) yields

$$B\left(\frac{\beta_1\phi\langle k\rangle + \theta_1\phi_\triangle\langle k_\triangle\rangle\rho^I_*}{\delta+\epsilon}\right) - \phi\langle k\rangle\rho^I_* - \phi_\triangle\langle k_\triangle\rangle(\rho^I_*)^2 - \epsilon = 0.$$

Let

$$h(\rho^I_*) = a(\rho^I_*)^2 + b\rho^I_* + c = 0, \tag{3.4}$$

where

$$\begin{cases} a = \phi_\triangle\langle k_\triangle\rangle, \\ b = \phi\langle k\rangle - \dfrac{B\theta_1\phi_\triangle\langle k_\triangle\rangle}{\delta+\epsilon}, \\ c = \epsilon(1-R_0), \end{cases}$$

where

$$R_0 = \frac{B/\beta_1\phi\langle k\rangle}{\delta+\epsilon}.$$

The following equivalence relations can be easily verified:

$$\begin{aligned} c<0 \Leftrightarrow \phi > \frac{\epsilon(\epsilon+\delta)}{B\beta_1\langle k\rangle}, b<0 \Leftrightarrow \phi_\triangle > \frac{(\epsilon+\delta)\langle k\rangle}{B\theta_1\langle k_\triangle\rangle}\phi, \\ c=0 \Leftrightarrow \phi = \frac{\epsilon(\epsilon+\delta)}{B\beta_1\langle k\rangle}, b=0 \Leftrightarrow \phi_\triangle = \frac{(\epsilon+\delta)\langle k\rangle}{B\theta_1\langle k_\triangle\rangle}\phi, \\ c>0 \Leftrightarrow \phi < \frac{\epsilon(\epsilon+\delta)}{B\beta_1\langle k\rangle}, b>0 \Leftrightarrow \phi_\triangle < \frac{(\epsilon+\delta)\langle k\rangle}{B\theta_1\langle k_\triangle\rangle}\phi. \end{aligned}$$

Let $\Delta = b^2 - 4ac$. The solutions of Eq (3.4) can be categorized into the following cases:

Case 1. If $c < 0$, there exists one positive solution and one negative solution.

Case 2. If $\Delta > 0$, $c > 0$, $b > 0$, there are two negative solutions.

Case 3. If $\Delta > 0$, $c > 0$, $b < 0$, there are two positive solutions.

Case 4. If $\Delta < 0$, there are no real solutions.

To facilitate the subsequent analysis, we introduce two scalar quantities:

$$\xi_1 = \frac{\epsilon(\epsilon+\delta)}{B\beta_1\langle k\rangle}, \quad \xi_2 = \frac{(\epsilon+\delta)\langle k\rangle}{B\theta_1\langle k_\triangle\rangle}.$$

The following equivalence relations can be easily obtained:

$$c < 0 \Leftrightarrow \phi > \xi_1, c = 0 \Leftrightarrow \phi = \xi_1, c > 0 \Leftrightarrow \phi < \xi_1;$$
$$b < 0 \Leftrightarrow \phi_\triangle > \xi_2\phi, b = 0 \Leftrightarrow \phi_\triangle = \xi_2\phi, b > 0 \Leftrightarrow \phi_\triangle < \xi_2\phi.$$

Theorem 3.1. *Consider system* (3.2). *The existence conditions for the rumor-prevailing equilibrium point(s) E_* are as follows:*
(i) If $\phi > \xi_1$, then a unique rumor-prevailing equilibrium E_* exists.
(ii) If $\Delta > 0$, $\phi < \xi_1$, and $\phi_\triangle < \xi_2\phi$, then no rumor-prevailing equilibrium exists.
(iii) If $\Delta > 0$, $\phi < \xi_1$, and $\phi_\triangle > \xi_2\phi$, then two distinct rumor-prevailing equilibria, E_{*1} and E_{*2}, exist.
(iv) If $\Delta < 0$, then no rumor-prevailing equilibrium exists.

3.2. Multistability analysis

Theorem 3.2. *For system* (3.2)*, the rumor-free equilibrium point $E_0 = (B/\epsilon, 0)$ is locally asymptotically stable when $R_0 < 1$.*

Proof. At E_0, the Jacobian matrix is given by

$$J(E_0) = \begin{pmatrix} -\epsilon & -\phi\langle k\rangle\frac{B}{\epsilon} \\ 0 & \beta_1\phi\langle k\rangle\frac{B}{\epsilon} - (\delta+\epsilon) \end{pmatrix}.$$

Then, the characteristic equation is

$$(\lambda+\epsilon)(\lambda - \beta_1\phi\langle k\rangle\frac{B}{\epsilon} + (\delta+\epsilon)) = 0. \tag{3.5}$$

If $R_0 < 1$, we have

$$\beta_1\phi\langle k\rangle\frac{B}{\epsilon} - (\delta+\epsilon) < 0.$$

According to the Routh–Hurwitz criterion, the real parts of all solutions to Eq (3.5) are negative if and only if $R_0 < 1$. Therefore, E_0 is locally asymptotically stable. □

Next, we analyze the local stability of system (3.2) at the rumor-prevailing equilibrium points.

When $\Delta > 0$, $\phi < \xi_1$, and $\phi_\triangle > \xi_2\phi$, Eq (3.4) admits two positive roots given by

$$\rho^I_{*1} = \frac{-b - \sqrt{\Delta}}{2a}, \quad \rho^I_{*2} = \frac{-b + \sqrt{\Delta}}{2a}.$$

Theorem 3.3. *Suppose that $\Delta > 0$, $\phi < \xi_1$, and $\phi_\triangle > \xi_2\phi$. Then, system* (3.2) *admits two rumor-prevailing equilibrium points, E_{*1} and E_{*2}, with the following stability properties:*
(i) The equilibrium $E_{*1} = (\rho^S_{*1}, \rho^I_{*1})$ is unstable.
(ii) The equilibrium $E_{*2} = (\rho^S_{*2}, \rho^I_{*2})$ is locally asymptotically stable if $\phi > (\delta+\epsilon)\xi_1/4\epsilon$.

Proof. For $E_{*1} = (\rho^S_{*1}, \rho^I_{*1})$, we compute the determinant of the Jacobian matrix

$$J(E_{*1}) = \begin{pmatrix} -\frac{B}{\rho^S_{*1}} & -\phi\langle k\rangle\rho^S_{*1} - 2\phi_\triangle\langle k_\triangle\rangle\rho^S_{*1}\rho^I_{*1} \\ \frac{(\delta+\epsilon)\rho^I_{*1}}{\rho^S_{*1}} & \theta_1\phi_\triangle\langle k_\triangle\rangle\rho^S_{*1}\rho^I_{*1} \end{pmatrix}.$$

The determinant is

$$\det(J(E_{*1})) = (\delta+\epsilon)\frac{\sqrt{\Delta}}{2a}(b + \sqrt{\Delta}).$$

We analyze its sign to determine the stability. Given that $a > 0$, $b < 0$, and $c > 0$, we have $b + \sqrt{\Delta} < 0$, implying $\det(J(E_{*1})) < 0$. This implies that $J(E_{*1})$ has a positive eigenvalue, making E_{*1} unstable.

Next, we analyze the stability of $E_{*2} = (\rho^S_{*2}, \rho^I_{*2})$:

$$J(E_{*2}) = \begin{pmatrix} -\frac{B}{\rho^S_{*2}} & -\phi\langle k\rangle\rho^S_{*2} - 2\phi_\triangle\langle k_\triangle\rangle\rho^S_{*2}\rho^I_{*2} \\ \frac{(\delta+\epsilon)\rho^I_{*2}}{\rho^S_{*2}} & \theta_1\phi_\triangle\langle k_\triangle\rangle\rho^S_{*2}\rho^I_{*2} \end{pmatrix}.$$

The characteristic equation is

$$\lambda^2 - tr(J(E_{*2}))\lambda + \det(J(E_{*2})) = 0. \tag{3.6}$$

Then,

$$\det(J(E_{*2})) = (\delta+\epsilon)\frac{\sqrt{\Delta}}{2a}(\sqrt{\Delta} - b).$$

From the given conditions, we have $\det(J(E_{*2})) > 0$,

$$\begin{aligned} tr(J(E_{*2})) = {} & \frac{-B\theta_1{}^2\phi_\triangle{}^2\langle k_\triangle\rangle^2}{(\delta+\epsilon)(\beta_1\phi\langle k\rangle + \theta_1\phi_\triangle\langle k_\triangle\rangle\rho^I_{*2})}(\rho^I_{*2})^2 \\ & + \frac{[(\delta+\epsilon)^2 - 2B\beta_1\phi\langle k\rangle]\theta_1\phi_\triangle\langle k_\triangle\rangle}{(\delta+\epsilon)(\beta_1\phi\langle k\rangle + \theta_1\phi_\triangle\langle k_\triangle\rangle\rho^I_{*2})}\rho^I_{*2} \\ & - \frac{B\beta_1{}^2\phi^2\langle k\rangle^2}{(\delta+\epsilon)(\beta_1\phi\langle k\rangle + \theta_1\phi_\triangle\langle k_\triangle\rangle\rho^I_{*2})}. \end{aligned}$$

We define

$$\begin{aligned} \Delta' &= [(\delta+\epsilon)^2 - 2B\beta_1\phi\langle k\rangle]^2\theta_1{}^2\phi_\triangle{}^2\langle k_\triangle\rangle^2 \\ &\quad - 4B^2\beta_1{}^2\phi^2\langle k\rangle^2\theta_1{}^2\phi_\triangle{}^2\langle k_\triangle\rangle^2 \\ &= \theta_1{}^2\phi_\triangle{}^2\langle k_\triangle\rangle^2(\delta+\epsilon)^3[\delta+\epsilon-4\epsilon R_0]. \end{aligned} \tag{3.7}$$

Then, we have

$$\Delta' < 0 \Leftrightarrow \phi > \frac{\delta+\epsilon}{4\epsilon}\xi_1, \Delta' = 0 \Leftrightarrow \phi = \frac{\delta+\epsilon}{4\epsilon}\xi_1,$$
$$\Delta' > 0 \Leftrightarrow \phi < \frac{\delta+\epsilon}{4\epsilon}\xi_1.$$

If $\Delta' < 0$, then $tr(J(E_{*2})) < 0$. Therefore, E_{*2} is locally asymptotically stable when $\phi > (\delta+\epsilon)\xi_1/4\epsilon$. □

Remark 3.1. When $\Delta > 0$, $\phi < \xi_1$, and $\phi_\triangle > \xi_2\phi$, both E_0 and a stable rumor-prevailing equilibrium coexist. This indicates the presence of a bistability, which demonstrates that the disappearance or outbreak of rumors is critically dependent on the initial conditions.

3.3. Bifurcation analysis

To further investigate the influence of group interactions on the dynamic behavior of system (3.2), we consider $\phi_\triangle$ (representing the strength of group interactions) as the bifurcation parameter.

Define the following auxiliary functions and critical values:

$$P(\rho^I_{*2}) = \frac{B}{\rho^S_{*2}} - \theta_1\phi_\triangle\langle k_\triangle\rangle\rho^S_{*2}\rho^I_{*2} = -\operatorname{tr}(J(E_{*2})),$$
$$Q(\rho^I_{*2}) = (\delta+\epsilon)(\phi\langle k\rangle + 2\phi_\triangle\langle k_\triangle\rangle\rho^I_{*2}) - B\theta_1\phi_\triangle\langle k_\triangle\rangle,$$
$$\phi_{\triangle_1} = \frac{B}{\theta_1\langle k_\triangle\rangle(\rho^S_{*2})^2\rho^I_{*2}}, \phi_{\triangle_2} = \frac{(\delta+\epsilon)\phi\langle k\rangle}{B\theta_1\langle k_\triangle\rangle - 2(\delta+\epsilon)\langle k_\triangle\rangle\rho^I_{*2}}.$$

Theorem 3.4. *Let* $\hat{\phi}_\triangle = \min\{\phi_{\triangle_1}, \phi_{\triangle_2}\}$. *Suppose that* $B\theta_1\langle k_\triangle\rangle - 2(\delta+\epsilon)\langle k_\triangle\rangle\rho^I_{*2} > 0$. *Then, the stability of the rumor-prevailing equilibrium* E_{*2} *is characterized as follows:*
(i) If $\phi_\triangle < \hat{\phi}_\triangle$, *the equilibrium* E_{*2} *is locally asymptotically stable.*
(ii) If $\phi_\triangle > \hat{\phi}_\triangle$, *the equilibrium* E_{*2} *becomes unstable.*

Furthermore, a Hopf bifurcation occurs at $\phi_\triangle = \hat{\phi}_\triangle$. *That is, system* (3.2) *undergoes a Hopf bifurcation at the equilibrium point* E_{*2} *when* $\phi_\triangle$ *crosses the critical value* $\hat{\phi}_\triangle$.

Proof. By solving $P(\rho^I_{*2}) = 0$, we obtain $\phi_\triangle = \phi_{\triangle_1}$ and the following equivalence relations:

$$P(\rho^I_{*2}) < 0 \Leftrightarrow \phi_\triangle > \phi_{\triangle_1}, P(\rho^I_{*2}) = 0 \Leftrightarrow \phi_\triangle = \phi_{\triangle_1},$$
$$P(\rho^I_{*2}) > 0 \Leftrightarrow \phi_\triangle < \phi_{\triangle_1}.$$

By solving $Q(\rho^I_{*2}) = 0$, we obtain $\phi_\triangle = \phi_{\triangle_2}$ and the following equivalence relations:

$$Q(\rho^I_{*2}) < 0 \Leftrightarrow \phi_\triangle > \phi_{\triangle_2}, Q(\rho^I_{*2}) = 0 \Leftrightarrow \phi_\triangle = \phi_{\triangle_2},$$
$$Q(\rho^I_{*2}) > 0 \Leftrightarrow \phi_\triangle < \phi_{\triangle_2}.$$

Assume that $\phi_{\triangle_1} > \phi_{\triangle_2}$. When $\phi_\triangle$ is in the neighborhood of $\phi_\triangle = \phi_{\triangle_2}$, there is only one positive root of the Eq (3.6), given by

$$\lambda_+ = \frac{-P(\rho^I_{*2}) + \sqrt{(P(\rho^I_{*2}))^2 - 4\rho^I_{*2}Q(\rho^I_{*2})}}{2}.$$

Then,

$$\frac{d\lambda_+}{d\phi_\triangle} = \frac{1}{2}\theta_1\langle k_\triangle\rangle\rho^S_{*2}\rho^I_{*2} + \frac{-P(\rho^I_{*2})\theta_1\langle k_\triangle\rangle\rho^S_{*2}\rho^I_{*2}}{2\sqrt{(P(\rho^I_{*2}))^2 - 4\rho^I_{*2}Q(\rho^I_{*2})}}$$
$$- \frac{\rho^I_{*2}[2(\delta+\epsilon)\langle k_\triangle\rangle\rho^I_{*2} - B\theta_1\langle k_\triangle\rangle]}{\sqrt{(P(\rho^I_{*2}))^2 - 4\rho^I_{*2}Q(\rho^I_{*2})}}.$$

When $\phi_\triangle = \phi_{\triangle_2}$, because $Q(\rho^I_{*2}) = 0$,

$$\left.\frac{d\lambda_+}{d\phi_\triangle}\right|_{\phi_\triangle=\phi_{\triangle_2}} = \frac{[B\theta_1\langle k_\triangle\rangle - 2(\delta+\epsilon)\langle k_\triangle\rangle\rho^I_{*2}]\rho^I_{*2}}{P(\rho^I_{*2})}.$$

Because $P(\rho^I_{*2})|_{\phi_\triangle=\phi_{\triangle 2}} > 0$, and $B\theta_1\langle k_\triangle\rangle - 2(\delta+\epsilon)\langle k_\triangle\rangle\rho^I_{*2} > 0$, it follows that

$$\left.\frac{d\lambda_+}{d\phi_\triangle}\right|_{\phi_\triangle=\phi_{\triangle_2}} > 0.$$

Hence, there exists a Hopf bifurcation at $\phi_\triangle = \phi_{\triangle_2}$.

Next, we assume that $\phi_{\triangle_2} > \phi_{\triangle_1}$. When $\phi_\triangle$ is in a neighborhood of $\phi_{\triangle_1}$, the characteristic Eq (3.6) has two positive roots. Let the positive roots be $\sigma_{1,2}$, given by

$$\sigma_{1,2} = \frac{-P(\rho^I_{*2}) \pm \sqrt{(P(\rho^I_{*2}))^2 - 4\rho^I_{*2}Q(\rho^I_{*2})}}{2}.$$

Then,

$$\frac{d\sigma_{1,2}}{d\phi_\triangle} = \frac{1}{2}\theta_1\langle k_\triangle\rangle\rho^S_{*2}\rho^I_{*2} \pm \frac{-P(\rho^I_{*2})\theta_1\langle k_\triangle\rangle\rho^S_{*2}\rho^I_{*2}}{2\sqrt{(P(\rho^I_{*2}))^2 - 4\rho^I_{*2}Q(\rho^I_{*2})}}$$
$$\pm \frac{-\rho^I_{*2}[2(\delta+\epsilon)\langle k_\triangle\rangle\rho^I_{*2} - B\theta_1\langle k_\triangle\rangle]}{\sqrt{(P(\rho^I_{*2}))^2 - 4\rho^I_{*2}Q(\rho^I_{*2})}}.$$

$$\left.\frac{d\sigma_{1,2}}{d\phi_\triangle}\right|_{\phi_\triangle=\phi_{\triangle_1}} = \frac{1}{2}\theta_1\langle k_\triangle\rangle\rho^S_{*2}\rho^I_{*2} \mp \frac{\rho^I_{*2}[2(\delta+\epsilon)\langle k_\triangle\rangle\rho^I_{*2} - B\theta_1\langle k_\triangle\rangle]}{2i\sqrt{\rho^I_{*2}Q(\rho^I_{*2})}}.$$

We can easily verify that

$$\Re\left(\left.\frac{d\sigma_{1,2}}{d\phi_\triangle}\right|_{\phi_\triangle=\phi_{\triangle_1}}\right) = \frac{1}{2}\theta_1\langle k_\triangle\rangle\rho^S_{*2}\rho^I_{*2} > 0.$$

Therefore, a Hopf bifurcation occurs at $\phi_\triangle = \phi_{\triangle_1}$. □

Remark 3.2. The coefficient $\phi_\triangle$, which characterizes the strength of high-order interactions, is chosen as the bifurcation parameter because it directly governs group effects beyond pairwise connections. As established in Theorem 3.4, $\phi_\triangle$ governs the stability of the equilibrium E_{*2}. Specifically, it controls the occurrence of a Hopf bifurcation, leading to a transition from stability to oscillatory dynamics. This underscores the critical influence of group dynamics in rumor propagation, emphasizing the need to account for higher-order effects in social contagion models.

4. Simplicial SEIR rumor propagation models in heterogeneous networks

4.1. Model formulation

To investigate the influence of heterogeneity and group interactions on rumor propagation, we propose the simplicial SEIR model on heterogeneous networks as follows:

$$\begin{cases} \frac{d\rho^{S_k}(t)}{dt} = B - \phi k\rho^{S_k}(t)\Xi(t) - \phi_\triangle\langle k_\triangle\rangle\rho^{S_k}(t)\Xi^2(t) - \epsilon\rho^{S_k}(t), \\ \frac{d\rho^{E_k}(t)}{dt} = \beta_2\phi k\rho^{S_k}(t)\Xi(t) + \theta_2\phi_\triangle\langle k_\triangle\rangle\rho^{S_k}(t)\Xi^2(t) - (\delta+\epsilon)\rho^{E_k}(t), \\ \frac{d\rho^{I_k}(t)}{dt} = \beta_1\phi k\rho^{S_k}(t)\Xi(t) + \theta_1\phi_\triangle\langle k_\triangle\rangle\rho^{S_k}(t)\Xi^2(t) - (\delta+\epsilon)\rho^{I_k}(t), \\ \frac{d\rho^{R_k}(t)}{dt} = (1-\beta_1-\beta_2)\phi k\rho^{S_k}(t)\Xi(t) + \delta[\rho^{E_k}(t)+\rho^{I_k}(t)] \\ \qquad + (1-\theta_1-\theta_2)\phi_\triangle\langle k_\triangle\rangle\rho^{S_k}(t)\Xi^2(t) - \epsilon\rho^{R_k}(t). \end{cases} \tag{4.1}$$

Here, $\rho^{S_k}(t)$, $\rho^{E_k}(t)$, $\rho^{I_k}(t)$, and $\rho^{R_k}(t)$ denote the densities of susceptible, exposed, infected, and removed individuals with degree k at time t, respectively. The average density of infected individuals is given by $\rho^I(t) = \sum_{k=1}^{n} P(k)\rho^{I_k}(t)$. $P(k)$ is the degree distribution of nodes, with n denoting the maximum degree. The rate at which an edge connects to an infected neighbor is given as

$$\Xi(t) = \frac{1}{\langle k\rangle}\sum_{k=1}^{n} kP(k)\rho^{I_k}(t). \tag{4.2}$$

Then, we obtain the simplified equations as follows:

$$\begin{cases} \frac{d\rho^{S_k}(t)}{dt} = B - \phi k\rho^{S_k}(t)\Xi(t) - \phi_\triangle\langle k_\triangle\rangle\rho^{S_k}(t)\Xi^2(t) - \epsilon\rho^{S_k}(t), \\ \frac{d\rho^{I_k}(t)}{dt} = \beta_1\phi k\rho^{S_k}(t)\Xi(t) + \theta_1\phi_\triangle\langle k_\triangle\rangle\rho^{S_k}(t)\Xi^2(t) - (\delta+\epsilon)\rho^{I_k}(t). \end{cases} \tag{4.3}$$

The initial conditions are

$$\rho^{S_k}(0) = \varphi_{S_k}, \rho^{I_k}(0) = \varphi_{I_k}, k = 1,2,3,\ldots,n,$$

where

$$0 \le \varphi_{S_k} \le \frac{B}{\epsilon}, 0 < \varphi_{I_k} \le \frac{B}{\epsilon}.$$

4.2. Existence of the equilibrium points

Theorem 4.1. *For system* (4.3)*, define the heterogeneous basic reproduction number as*

$$R_0^{\text{het}} = \frac{\langle k^2\rangle}{\langle k\rangle}\cdot\frac{\beta_1\phi}{\delta+\epsilon}\cdot\frac{B}{\epsilon}.$$

Then, the following assertions hold:
(i) $E_0 = (B/\epsilon, 0)_k$ is the rumor-free equilibrium point.
(ii) If $R_0^{\text{het}} > 1$, then the system admits a rumor-prevailing equilibrium $E^* = \left(\rho_*^{S_k}, \rho_*^{I_k}\right)$.

Proof. Part (i) is obvious. Then, we prove (ii). Let

$$\begin{cases} 0 = B - \phi k\rho^{S_k}\Xi - \phi_\triangle\langle k_\triangle\rangle\rho^{S_k}\Xi^2 - \epsilon\rho^{S_k}, \\ 0 = \beta_1\phi k\rho^{S_k}\Xi + \theta_1\phi_\triangle\langle k_\triangle\rangle\rho^{S_k}\Xi^2 - (\delta+\epsilon)\rho^{I_k}. \end{cases}$$

Then,

$$\begin{cases} \rho^{S_k} = \frac{B}{\phi k\Xi + \phi_\triangle\langle k_\triangle\rangle\Xi^2 + \epsilon}, \\ \rho^{I_k} = \frac{B}{\delta+\epsilon}\cdot\frac{\beta_1\phi k\Xi + \theta_1\phi_\triangle\langle k_\triangle\rangle\Xi^2}{\phi k\Xi + \phi_\triangle\langle k_\triangle\rangle\Xi^2 + \epsilon}. \end{cases}$$

Substituting ρ^{I_k} into (4.2), we obtain

$$\Xi = \frac{1}{\langle k\rangle}\sum_{k=1}^{n} kP(k)\frac{B}{\delta+\epsilon}\cdot\frac{\beta_1\phi k\Xi + \theta_1\phi_\triangle\langle k_\triangle\rangle\Xi^2}{\phi k\Xi + \phi_\triangle\langle k_\triangle\rangle\Xi^2 + \epsilon}.$$

Define

$$F(\Xi) = 1 - \frac{1}{\langle k\rangle}\sum_{k=1}^{n} kP(k)\frac{B}{\delta+\epsilon}\cdot\frac{\beta_1\phi k + \theta_1\phi_\triangle\langle k_\triangle\rangle\Xi}{\phi k\Xi + \phi_\triangle\langle k_\triangle\rangle\Xi^2 + \epsilon}.$$

Then, at $\Xi = 0$,

$$F(0) = 1 - \frac{1}{\langle k \rangle} \sum_{k=1}^{n} k^2 P(k) \frac{B\beta_1 \phi}{\epsilon(\delta+\epsilon)}.$$

At $\Xi = B/\epsilon$,

$$\begin{aligned} F\left(\frac{B}{\epsilon}\right) =& 1 - \frac{1}{\langle k \rangle} \sum_{k=1}^{n} kP(k) \frac{B}{\delta+\epsilon} \cdot \frac{\beta_1 \phi k + \theta_1 \phi_\triangle \langle k_\triangle \rangle \frac{B}{\epsilon}}{\phi k \frac{B}{\epsilon} + \phi_\triangle \langle k_\triangle \rangle (\frac{B}{\epsilon})^2 + \epsilon} \\ >& 1 - \frac{1}{\langle k \rangle} \sum_{k=1}^{n} kP(k) \frac{B}{\delta+\epsilon} \cdot \frac{\phi k + \phi_\triangle \langle k_\triangle \rangle \frac{B}{\epsilon}}{\phi k + \phi_\triangle \langle k_\triangle \rangle \frac{B}{\epsilon} + \frac{\epsilon^2}{B}} \cdot \frac{\epsilon}{B} \\ >& 1 - \frac{1}{\langle k \rangle} \sum_{k=1}^{n} kP(k) \frac{B}{\delta+\epsilon} \cdot \frac{\epsilon}{B} \\ >& 1 - \frac{\epsilon}{\delta+\epsilon} > 0. \end{aligned}$$

Next, noting that $\langle k^2 \rangle = \sum_{k=1}^{n} k^2 P(k)$, the basic reproduction number is defined as

$$R_0^{\text{het}} = \frac{\langle k^2 \rangle}{\langle k \rangle} \cdot \frac{B\beta_1 \phi}{\epsilon(\delta+\epsilon)}.$$

The function $F(\Xi)$ remains continuous on the interval $[0, B/\epsilon]$. When $R_0^{\text{het}} > 1$, we have $F(0) < 0$. Therefore, the equation $F(\Xi) = 0$ has at least one root in the interval $(0, B/\epsilon)$. Thus, system (4.3) admits a positive rumor-prevailing equilibrium $E^* = \left(\rho_*^{S_k}, \rho_*^{I_k}\right)$, explicitly given by

$$\rho_*^{S_k} = \frac{B}{\phi k \Xi_* + \phi_\triangle \langle k_\triangle \rangle \Xi_*^2 + \epsilon},$$

$$\rho_*^{I_k} = \frac{B[\beta_1 \phi k \Xi_* + \theta_1 \phi_\triangle \langle k_\triangle \rangle \Xi_*^2]}{(\delta+\epsilon)\phi k \Xi_* + \phi_\triangle \langle k_\triangle \rangle \Xi_*^2 + \epsilon}.$$

□

4.3. Analysis of stability

Theorem 4.2. *For system* (4.3)*, if* $R_0^{\text{het}} < 1$*, then the rumor-free equilibrium* $E_0 = (B/\epsilon, 0)_k$ *is locally asymptotically stable.*

Proof. At the rumor-free equilibrium $E_0 = (B/\epsilon, 0)_k$, the Jacobian matrix is given by

$$J(E_0) = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}_{2n\times 2n},$$

where

$$A_1 = \begin{bmatrix} -\epsilon & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & -\epsilon \end{bmatrix}_{n\times n}, A_3 = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix}_{n\times n},$$

$$A_2 = \begin{bmatrix} -\frac{B\phi}{\epsilon} \frac{1\cdot 1\cdot P(1)}{\langle k \rangle} & \dots & -\frac{B\phi}{\epsilon} \frac{1\cdot n\cdot P(n)}{\langle k \rangle} \\ \vdots & \ddots & \vdots \\ -\frac{B\phi}{\epsilon} \frac{n\cdot 1\cdot P(1)}{\langle k \rangle} & \dots & -\frac{B\phi}{\epsilon} \frac{n\cdot n\cdot P(n)}{\langle k \rangle} \end{bmatrix}_{n\times n},$$

$$A_4 = \begin{bmatrix} \frac{B\beta_1\phi}{\epsilon} \frac{1\cdot 1\cdot P(1)}{\langle k \rangle} - (\delta+\epsilon) & \dots & \frac{B\beta_1\phi}{\epsilon} \frac{1\cdot n\cdot P(n)}{\langle k \rangle} \\ \vdots & \ddots & \vdots \\ \frac{B\beta_1\phi}{\epsilon} \frac{n\cdot 1\cdot P(1)}{\langle k \rangle} & \dots & \frac{B\beta_1\phi}{\epsilon} \frac{n\cdot n\cdot P(n)}{\langle k \rangle} - (\delta+\epsilon) \end{bmatrix}_{n\times n}.$$

The characteristic equation is

$$|\lambda I - J(E_0)| = (\lambda+\epsilon)^n (\lambda+\delta+\epsilon)^{n-1} (\lambda+\delta+\epsilon - \frac{B\beta_1\phi}{\epsilon\langle k \rangle} \sum_{k=1}^{n} k^2 P(k)) = 0. \tag{4.4}$$

It is known that Eq (4.4) has an n-fold eigenvalue $\lambda_1 = -\epsilon$ and an $(n-1)$-fold eigenvalue $\lambda_2 = -(\delta+\epsilon)$. The $2n^{th}$ eigenvalue λ_3 is

$$\lambda_3 = \frac{B\beta_1\phi}{\epsilon} \frac{1}{\langle k \rangle} \sum_{k=1}^{n} k^2 P(k) - (\delta+\epsilon) = (\delta+\epsilon)(R_0^{het} - 1).$$

If $R_0^{het} < 1$, then $\lambda_3 < 0$, which implies that all eigenvalues of $J(E_0)$ are negative. Therefore, E_0 is locally asymptotically stable. □

Theorem 4.3. *For system* (4.3)*, define*

$$Q = R_0^{\text{het}} + \frac{B^2 \theta_1 \phi_\triangle \langle k_\triangle \rangle}{\epsilon^2(\delta+\epsilon)}.$$

If $Q < 1$*, then the rumor-free equilibrium point* $E_0 = (B/\epsilon, 0)_k$ *is globally asymptotically stable.*

Proof. Consider the Lyapunov function

$$V(t) = \sum_{k=1}^{n} \frac{kP(k)}{\langle k \rangle} \rho^{I_k}(t).$$

Then,

$$\begin{aligned} \frac{dV}{dt} =& \sum_{k=1}^{n} \frac{kP(k)}{\langle k \rangle} \Big([\beta_1 \phi k \Xi(t) + \theta_1 \phi_\triangle \langle k_\triangle \rangle \Xi^2(t)] \rho^{S_k}(t) - (\delta+\epsilon)\rho^{I_k}(t) \Big) \\ \le& \sum_{k=1}^{n} \frac{kP(k)}{\langle k \rangle} [\frac{B}{\epsilon}(\beta_1 \phi k \Xi(t) + \theta_1 \phi_\triangle \langle k_\triangle \rangle \Xi^2(t)) - (\delta+\epsilon)\rho^{I_k}(t)] \\ \le& \sum_{k=1}^{n} \frac{kP(k)}{\langle k \rangle} [\frac{B}{\epsilon}(\beta_1 \phi k \Xi(t) + \theta_1 \phi_\triangle \langle k_\triangle \rangle \frac{B}{\epsilon} \Xi(t)) - (\delta+\epsilon)\rho^{I_k}(t)] \\ =& [\frac{\langle k^2 \rangle}{\langle k \rangle} \frac{B\beta_1\phi}{\epsilon} + \frac{B^2 \theta_1 \phi_\triangle \langle k_\triangle \rangle}{\epsilon^2} - (\delta+\epsilon)]\Xi(t) \\ =& (\delta+\epsilon)[R_0^{\text{het}} + \frac{B^2 \theta_1 \phi_\triangle \langle k_\triangle \rangle}{\epsilon^2(\delta+\epsilon)} - 1]\Xi(t). \end{aligned}$$

Because $\Xi(t) > 0$, if $Q < 1$, then $dV/dt \leq 0$. Moreover, $dV/dt = 0$ if and only if $\rho^{I_k}(t) = 0$ for all k. Thus, E_0 is globally asymptotically stable. □

Theorem 4.4. *For system* (4.3)*, let*

$$\widetilde{\phi} = \frac{\phi(\delta+\epsilon)\langle k^3\rangle}{B\theta_1\langle k_\triangle\rangle\langle k^2\rangle},$$

where $\langle k^3\rangle = \sum_{k=1}^{n} k^3 P(k)$ *is the third moment of the degree distribution. Then, the bifurcation behavior of the rumor-prevailing equilibrium is as follows:*
(i) If $\phi_\triangle > \widetilde{\phi}$*, the system exhibits a backward bifurcation.*
(ii) If $\phi_\triangle < \widetilde{\phi}$*, the system exhibits a forward bifurcation.*

Proof. From the equilibrium condition $F(\Xi) = 0$, we have

$$\sum_{k=1}^{n} kP(k)\frac{BR_0^{\text{het}}(\beta_1\phi k + \theta_1\phi_\triangle\langle k_\triangle\rangle\Xi)}{(\delta+\epsilon)\langle k\rangle R_0^{\text{het}}(\phi k\Xi + \phi_\triangle\langle k_\triangle\rangle\Xi^2) + B\beta_1\phi\langle k^2\rangle} = 1, \tag{4.5}$$

where Ξ depends on R_0^{het}. The bifurcation type depends on the slope at $(R_0^{\text{het}}, \Xi) = (1, 0)$. This suggests that if

$$\frac{\partial\Xi}{\partial R_0^{\text{het}}}\Big|_{(R_0^{\text{het}},\Xi)=(1,0)} > 0,$$

a forward bifurcation will occur. If

$$\frac{\partial\Xi}{\partial R_0^{\text{het}}}\Big|_{(R_0^{\text{het}},\Xi)=(1,0)} < 0,$$

a backward bifurcation will occur. Differentiating Eq (4.5) with respect to R_0^{het}, we obtain

$$\sum_{k=1}^{n} kP(k)\frac{\hbar_1 - \hbar_2}{\left[(\delta+\epsilon)\langle k\rangle R_0^{\text{het}}(\phi k\Xi + \phi_\triangle\langle k_\triangle\rangle\Xi^2) + B\beta_1\phi\langle k^2\rangle\right]^2} = 0, \tag{4.6}$$

where

$$\begin{cases} \hbar_1 = \left[(\delta+\epsilon)\langle k\rangle R_0^{\text{het}}(\phi k\Xi + \phi_\triangle\langle k_\triangle\rangle\Xi^2) + B\beta_1\phi\left\langle k^2\right\rangle\right] \\ \quad \cdot B[(\beta_1\phi k + \theta_1\phi_\triangle\langle k_\triangle\rangle\Xi) + R_0^{\text{het}}\theta_1\phi_\triangle\langle k_\triangle\rangle\dfrac{\partial\Xi}{\partial R_0^{\text{het}}}], \\ \hbar_2 = BR_0^{\text{het}}(\beta_1\phi k + \theta_1\phi_\triangle\langle k_\triangle\rangle\Xi)\cdot(\delta+\epsilon)\langle k\rangle \\ \quad \cdot [(\phi k\Xi + \phi_\triangle\langle k_\triangle\rangle\Xi^2) + R_0^{\text{het}}(\phi k\dfrac{\partial\Xi}{\partial R_0^{\text{het}}} + 2\phi_\triangle\langle k_\triangle\rangle\Xi\dfrac{\partial\Xi}{\partial R_0^{\text{het}}})]. \end{cases}$$

Substituting $(R_0^{\text{het}}, \Xi) = (1, 0)$ into Eq (4.6), we obtain

$$\sum_{k=1}^{n} kP(k)\frac{\phi(\delta+\epsilon)k^2\langle k\rangle\frac{\partial\Xi}{\partial R_0^{\text{het}}} - B\theta_1\phi_\triangle\langle k_\triangle\rangle\left\langle k^2\right\rangle\frac{\partial\Xi}{\partial R_0^{\text{het}}}}{B\beta_1\phi\langle k^2\rangle^2} = 1.$$

Hence, we get

$$\frac{\partial\Xi}{\partial R_0^{\text{het}}}\Big|_{(R_0^{\text{het}},\Xi)=(1,0)} < 0 \Leftrightarrow \phi_\triangle > \frac{\phi(\delta+\epsilon)\left\langle k^3\right\rangle}{B\theta_1\langle k_\triangle\rangle\langle k^2\rangle},$$

$$\frac{\partial\Xi}{\partial R_0^{\text{het}}}\Big|_{(R_0^{\text{het}},\Xi)=(1,0)} > 0 \Leftrightarrow \phi_\triangle < \frac{\phi(\delta+\epsilon)\left\langle k^3\right\rangle}{B\theta_1\langle k_\triangle\rangle\langle k^2\rangle}.$$

This completes the proof. □

5. Optimal control model in heterogeneous networks

In this section, we introduce an optimal control model with a control variable which aims to mitigate rumor spread by regulating group influence. The control variable $u_k(t)$ modulates the strength of group interactions to minimize resource investment while effectively reducing the number of infected individuals.

The optimal control model is given by

$$\begin{cases} \dfrac{d\rho^{S_k}(t)}{dt} = B - \phi k\rho^{S_k}(t)\Xi(t) - u_k(t)\phi_\triangle\langle k_\triangle\rangle\rho^{S_k}(t)\Xi^2(t) - \epsilon\rho^{S_k}(t), \\ \dfrac{d\rho^{I_k}(t)}{dt} = \beta_1\phi k\rho^{S_k}(t)\Xi(t) + u_k(t)\theta_1\phi_\triangle\langle k_\triangle\rangle\rho^{S_k}(t)\Xi^2(t) \\ \quad -(\delta+\epsilon)\rho^{I_k}(t). \end{cases} \tag{5.1}$$

The initial conditions are $\rho^{S_k}(0) \geq 0, \rho^{I_k}(0) > 0$, and the control bounds are set as $0 \leq u_k(t) \leq 1$ with $0 \leq t \leq T$, $k = 1, 2, 3, \ldots, n$.

The objective function is defined as follows:

$$J(u) = \int_0^T \sum_{k=1}^{n}[\rho^{I_k}(t) + \frac{\alpha_k}{2}{u_k}^2(t)]dt,$$

where T is the final time, and $\alpha_k(\alpha_k > 0)$ is the weight for the control cost of degree k. The goal is to minimize $J(u)$ to reduce the number of infected individuals ρ^{I_k} while managing the control cost.

The Lagrangian function is

$$L(\rho^{I_k}, u_k) = \sum_{k=1}^{n}[\rho^{I_k}(t) + \frac{\alpha_k}{2}{u_k}^2(t)].$$

Then, there exists an optimal control $u_k^*(t) = (u_1^*(t), u_2^*(t), \ldots, u_n^*(t))$ that minimizes $J(u)$, that is, $J(u^*(t)) = \min J(u(t))$.

To find the optimal control that minimizes $J(u)$, we define the Hamiltonian function as

$$H=\sum_{k=1}^{n}[\rho^{I_k}(t)+\frac{\alpha_k}{2}u_k{}^2(t)]+\sum_{k=1}^{n}\lambda_1^k(t)\frac{d\rho^{S_k}(t)}{dt}+\sum_{k=1}^{n}\lambda_2^k(t)\frac{d\rho^{I_k}(t)}{dt} \tag{5.2}$$

with $\lambda_1^k(t),\lambda_2^k(t)$ variables associated with $\rho^{S_k}(t)$ and $\rho^{I_k}(t)$, respectively.

Theorem 5.1. *For the optimal control problem defined by the Hamiltonian function in Eq (5.2), let $\rho_*^{I_k}(t)=(\rho_*^{I_1}(t),\rho_*^{I_2}(t),\dots,\rho_*^{I_n}(t))$ denote the optimal state solution. The adjoint variables $\lambda_1^k(t)$ and $\lambda_2^k(t)$ satisfy the following system:*

$$\begin{cases}\frac{d\lambda_1^k(t)}{dt}=\sum_{k=1}^{n}\lambda_1^k(t)[\phi k\Xi_*(t)+u_k^*(t)\phi_\triangle\langle k_\triangle\rangle\Xi_*{}^2(t)+\epsilon]\\ \quad-\sum_{k=1}^{n}\lambda_2^k(t)[\beta_1\phi k\Xi_*(t)+u_k^*(t)\theta_1\phi_\triangle\langle k_\triangle\rangle\Xi_*{}^2(t)],\\ \frac{d\lambda_2^k(t)}{dt}=-1+\sum_{k=1}^{n}\lambda_1^k(t)[\phi k\rho_*^{S_k}(t)\frac{kP(k)}{\langle k\rangle}\\ \quad+2u_k^*(t)\phi_\triangle\langle k_\triangle\rangle\rho_*^{S_k}(t)\frac{kP(k)}{\langle k\rangle}\Xi_*(t)]\\ \quad-\sum_{k=1}^{n}\lambda_2^k(t)[\beta_1\phi k\rho_*^{S_k}(t)\frac{kP(k)}{\langle k\rangle}\\ \quad+2u_k^*(t)\theta_1\phi_\triangle\langle k_\triangle\rangle\rho_*^{S_k}(t)\frac{kP(k)}{\langle k\rangle}\Xi_*(t)+(\delta+\epsilon)],\end{cases} \tag{5.3}$$

and boundary condition $\lambda_1^k(T)=\lambda_2^k(T)=0,k=1,2,3,\dots,n$. The optimal control can be expressed as

$$u_k^*(t)=\begin{cases}0, & M<0,\\ \frac{M}{\alpha_k}\phi_\triangle\langle k_\triangle\rangle\rho_*^{S_k}(t)\Xi_*{}^2(t), & 0\le M\le\frac{\alpha_k}{\phi_\triangle\langle k_\triangle\rangle\rho_*^{S_k}(t)\Xi_*{}^2(t)},\\ 1, & M>\frac{\alpha_k}{\phi_\triangle\langle k_\triangle\rangle\rho_*^{S_k}(t)\Xi_*{}^2(t)},\end{cases}$$

where $M=\lambda_1^k(t)-\lambda_2^k(t)\theta_1$.

Proof. Assume the optimal states are $\rho^{S_k}(t)=\rho_*^{S_k}(t),\rho^{I_k}(t)=\rho_*^{I_k}(t)$, implying $\Xi(t)=\Xi_*(t)$. According to the necessary conditions for optimality, we compute the partial derivatives of the Hamiltonian H (defined in Eq (5.2)) with respect to $\rho^{S_k}(t)$, $\rho^{I_k}(t)$, and $u_k(t)$. The adjoint equations are

$$\begin{aligned}\frac{d\lambda_1^k(t)}{dt}&=-\frac{dH}{d\rho^{S_k}(t)}|_{\rho^{S_k}(t)=\rho_*^{S_k}(t),\rho^{I_k}(t)=\rho_*^{I_k}(t),u_k(t)=u_k^*(t)}\\&=\sum_{k=1}^{n}\lambda_1^k(t)[\phi k\Xi_*(t)+u_k^*(t)\phi_\triangle\langle k_\triangle\rangle\Xi_*{}^2(t)+\epsilon]\\&\quad-\sum_{k=1}^{n}\lambda_2^k(t)[\beta_1\phi k\Xi_*(t)+u_k^*(t)\theta_1\phi_\triangle\langle k_\triangle\rangle\Xi_*{}^2(t)],\end{aligned} \tag{5.4}$$

$$\begin{aligned}\frac{d\lambda_2^k(t)}{dt}&=-\frac{dH}{d\rho^{I_k}(t)}|_{\rho^{S_k}(t)=\rho_*^{S_k}(t),\rho^{I_k}(t)=\rho_*^{I_k}(t),u_k(t)=u_k^*(t)}\\&=-1+\sum_{k=1}^{n}\lambda_1^k(t)[\phi k\rho_*^{S_k}(t)\frac{kP(k)}{\langle k\rangle}\\&\quad+2u_k^*(t)\phi_\triangle\langle k_\triangle\rangle\rho_*^{S_k}(t)\frac{kP(k)}{\langle k\rangle}\Xi_*(t)]\\&\quad-\sum_{k=1}^{n}\lambda_2^k(t)[\beta_1\phi k\rho_*^{S_k}(t)\frac{kP(k)}{\langle k\rangle}\\&\quad+2u_k^*(t)\theta_1\phi_\triangle\langle k_\triangle\rangle\rho_*^{S_k}(t)\frac{kP(k)}{\langle k\rangle}\Xi_*(t)+(\delta+\epsilon)],\end{aligned} \tag{5.5}$$

and the optimality condition is

$$\begin{aligned}&\frac{\partial H}{\partial u_k(t)}|_{\rho^{S_k}(t)=\rho_*^{S_k}(t),\rho^{I_k}(t)=\rho_*^{I_k}(t),u_k(t)=u_k^*(t)}\\&=\alpha_k u_k^*(t)-(\lambda_1^k(t)-\lambda_2^k(t)\theta_1)\phi_\triangle\langle k_\triangle\rangle\rho_*^{S_k}(t)\Xi_*{}^2(t)\\&=0.\end{aligned} \tag{5.6}$$

Specifically, the adjoint equations are as follows:

Letting $M=\lambda_1^k(t)-\lambda_2^k(t)\theta_1$, we have the nonnegative root as

$$u_k^*(t)=\begin{cases}0, & M<0,\\ \frac{M}{\alpha_k}\phi_\triangle\langle k_\triangle\rangle\rho_*^{S_k}(t)\Xi_*{}^2(t), & 0\le M\le\frac{\alpha_k}{\phi_\triangle\langle k_\triangle\rangle\rho_*^{S_k}(t)\Xi_*{}^2(t)},\\ 1, & M>\frac{\alpha_k}{\phi_\triangle\langle k_\triangle\rangle\rho_*^{S_k}(t)\Xi_*{}^2(t)}.\end{cases}$$

This completes the proof. □

Remark 5.1. Theorem 5.1 shows that the optimal control $u_k^*(t)$ is influenced by both higher-order interaction dynamics and the heterogeneity of the network. Specifically, $u_k^*(t)$ depends not only on the higher-order interaction parameters $\phi_\triangle$ and $\langle k_\triangle\rangle$ but also on the $\Xi_*(t)$, which reflects the influence of the network's degree distribution. Consequently, the optimal control strategy integrates microscopic interaction mechanisms with macroscopic network topology, leading to a more precise and effective approach for rumor containment.

6. Numerical simulations

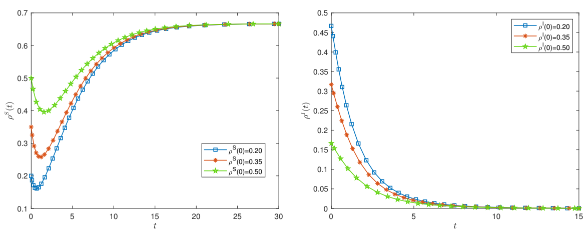
This section presents numerical simulations to validate the theoretical analysis and investigate the dynamic behavior of the rumor spreading model in networks that incorporate both pairwise and higher-order interactions. We generate networks with a vertex set V of 2000 nodes using the method described in [22], which constructs two-dimensional simplicial complexes with tunable average degree $\langle k \rangle$ and average number of 2-simplices $\langle k_\triangle \rangle$.

6.1. *Simulations in homogeneous networks*

Figure 2 illustrates that system (3.2) stabilizes at the equilibrium point $E_0 = (B/\epsilon, 0)$. The parameters are set as follows: $B = 0.2, \beta_1 = 0.3, \phi = 0.6, \phi_\triangle = 0.6, \theta_1 = 0.5, \delta = 0.5, \epsilon = 0.3, \langle k \rangle = 4$, and $\langle k_\triangle \rangle = 3$. Under these conditions, the basic reproduction number is calculated as $R_0 = 0.6 < 1$. In Figure 2(a), the densities of susceptible individuals converge to $B/\epsilon = 0.6667$. Figure 2(b) shows that the densities of infected individuals approach zero. These results confirm that the system stabilizes at the equilibrium point E_0, consistent with Theorem 3.2.

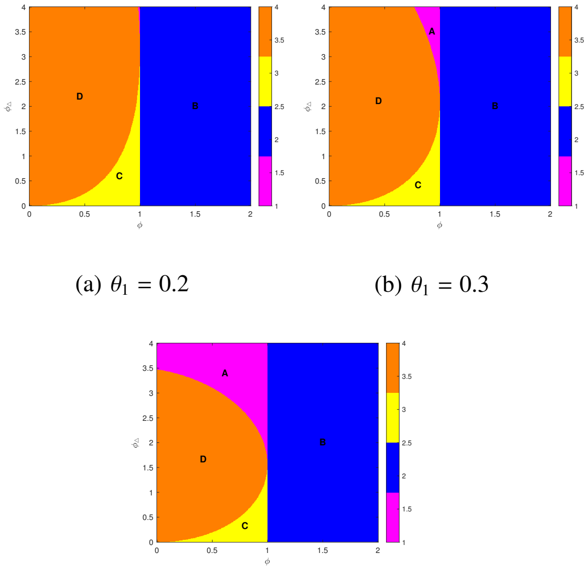
Figure 3 illustrates the bifurcation diagrams of system (3.2) for varying values of the parameters ϕ, $\phi_\triangle$, and θ_1. The other parameters are fixed as follows: $B = 1.2, \beta_1 = 2/9, \delta = 0.2, \epsilon = 0.8, \langle k \rangle = 3$, and $\langle k_\triangle \rangle = 4$. **In Domains C and D, E_0 is the only stable equilibrium.** Domain D corresponds to cases where the equation has no real roots, meaning that no positive equilibrium exists in the system. This suggests that, under these conditions, rumor propagation either fails to take hold or dies out quickly, preventing the emergence of a sustained rumor state. Domain C corresponds to cases where real roots exist but lie outside the valid interval $[0, B/\epsilon]$. In this domain, although a potential equilibrium could exist, it falls outside the range where it can be physically realized, implying that the system is not stable within the feasible limits of rumor propagation. **In Domain B, E_0 becomes unstable, and a stable positive equilibrium emerges.** This reflects a scenario where the rumor starts to spread more effectively, with the system transitioning from a state of no rumor to a state where the rumor actively propagates, driven by the parameters in this domain. **In Domain A, both E_0 and a stable positive equilibrium coexist, indicating bistability.** This suggests that the system can either stabilize at the rumor-free equilibrium E_0 or transition to the rumor-prevailing equilibrium, depending on initial conditions or small perturbations. The existence of bistability implies that the system is sensitive to initial conditions and external influences, such as control interventions. As θ_1 increases from 0.2 to 0.4, the region corresponding to Domain A expands, demonstrating that the parameter range for bistability broadens.

Figure 4 presents bifurcation diagrams illustrating the steady-state solutions E_* of system (3.2) for varying parameters ϕ, $\phi_\triangle$, and θ_1. The remaining parameters are fixed to the same values as in Figure 3: $B = 1.2$, $\beta_1 = 2/9$, $\delta = 0.2$, $\epsilon = 0.8$, $\langle k \rangle = 3$, and $\langle k_\triangle \rangle = 4$. In Figure 4(a), a simple transcritical bifurcation is observed. In Figure 4(b),4(c), two different bifurcation patterns emerge: a basic transcritical bifurcation and a fold bifurcation which leads to bistability. Notably, bistability occurs only when $\phi_\triangle$ reaches a certain value, indicating that the infection intensity of the group effect needs to reach a certain level to sustain the bistable region.



(a) Susceptible density converges to $B/\epsilon = 0.6667$ (b) Infected density converges to 0.

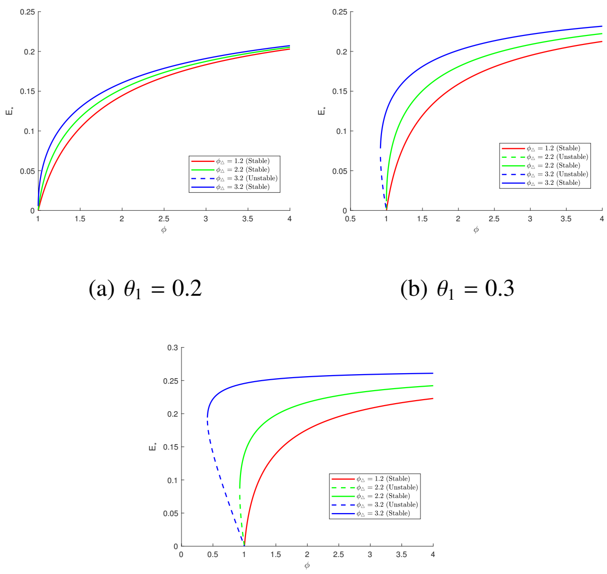
Figure 2. Evolution of system (3.2) at $E_0 = (B/\epsilon, 0)$.



(a) $\theta_1 = 0.2$ (b) $\theta_1 = 0.3$

(c) $\theta_1 = 0.4$

Figure 3. Bifurcation diagrams of system (3.2) with different ϕ, $\phi_\triangle$, and θ_1.

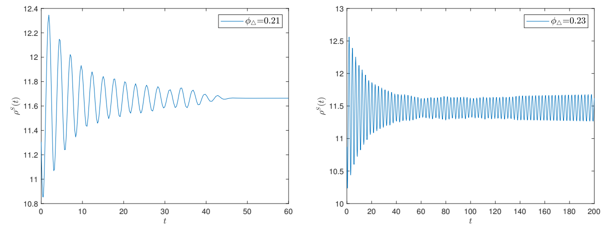


(a) $\theta_1 = 0.2$ (b) $\theta_1 = 0.3$

(c) $\theta_1 = 0.4$

Figure 4. Bifurcation diagrams illustrating the steady-state solutions E_* of system (3.2) for varying values of ϕ, $\phi_\triangle$, and θ_1.

Figure 5 illustrates the effect of the parameter $\phi_\triangle$ on the stability of system (3.2). The parameters are fixed as follows: $B = 15, \beta_1 = 0.2, \phi = 0.8, \theta_1 = 0.6, \delta = 8, \epsilon = 0.5,$ $\langle k \rangle = 4$, and $\langle k_\triangle \rangle = 3$. Under these conditions, we obtain the critical values $\phi_{\triangle_1} = 0.22$ and $\phi_{\triangle_2} = 1.36$. According to Theorem 3.4, system (3.2) undergoes a Hopf bifurcation at the equilibrium point E_{*2} when $\phi_\triangle = \hat{\phi}_\triangle = 0.22$. Figure 5(a) shows that system (3.2) is locally asymptotically stable at E_{*2} when $\phi_\triangle = 0.21$. Figure 5(b) shows that system (3.2) is unstable at E_{*2} when $\phi_\triangle = 0.23$.



(a) $\phi_\triangle = 0.21 < \hat{\phi}_\triangle = 0.22$ (b) $\phi_\triangle = 0.23 > \hat{\phi}_\triangle = 0.22$

Figure 5. Hopf bifurcation diagram of system (3.2).

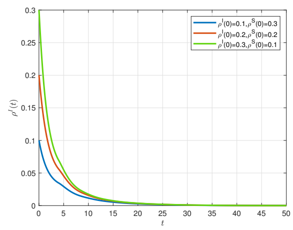
6.2. *Simulations in heterogeneous networks*

In this subsection, heterogeneous networks are generated using the methods described in [22], with parameters set as follows: $k_{min} = 1$, $k_{max} = 17$, $\langle k \rangle = 4.9981$, $\langle k^2 \rangle = 31.7322$, and $\langle k_\triangle \rangle = 1.0440$.

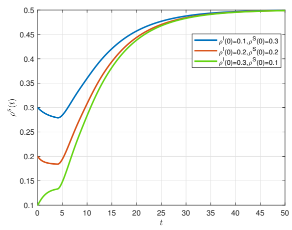
Figure 6 illustrates that system (4.3) stabilizes at the equilibrium point E_0. The parameters are fixed as follows: $B = 0.1, \beta_1 = 0.3, \beta_2 = 0.4, \phi = 0.3, \theta_1 = 0.2, \theta_2 = 0.3,$ $\delta = 0.4$, and $\epsilon = 0.2$. Under these conditions, the basic reproduction number is calculated as $R_0^{\text{het}} = 0.4761 < 1$. Simulation results show that the densities of susceptible individuals converge to $B/\epsilon = 0.5$, whereas the densities of infected individuals approach zero. These findings confirm that the system stabilizes at E_0, consistent with Theorem 4.2.

Figure 7 illustrates the global asymptotic stability of the equilibrium point E_0 for system (4.3). The parameters are fixed to the same values as in Figure 6: $B = 0.1, \beta_1 = 0.3,$ $\beta_2 = 0.4, \phi = 0.3, \theta_1 = 0.2, \theta_2 = 0.3, \delta = 0.4$, and $\epsilon = 0.2$, yielding $Q = 0.48067 < 1$. Simulations are conducted with multiple initial conditions: $(\rho^S(0), \rho^I(0))$ are set as (0.01, 0.1), (0.02, 0.15), (0.03, 0.2), (0.04, 0.25), (0.05, 0.3). The results show that all solution trajectories converge to E_0, consistent with Theorem 4.3.

Figure 8 illustrates the bistable phenomenon of system (4.3). The parameters are set as follows: $B = 0.3$, $\beta_1 = 0.2$, $\phi = 0.2$, $\theta_1 = 0.5$, $\delta = 0.3$, and $\epsilon = 0.3$. When $\phi_\triangle = 10$, all solution trajectories converge to E_0 regardless of the initial conditions. When $\phi_\triangle = 23$, the system exhibits bistability with two stable equilibrium points. Finally, when $\phi_\triangle = 36$, the system converges to the rumor-prevailing equilibrium point.



(a) Densities of infected individuals approach zero.



(b) Densities of susceptible individuals converge to $B/\epsilon = 0.5$

Figure 6. Evolution of system (4.3) with varying initial densities.

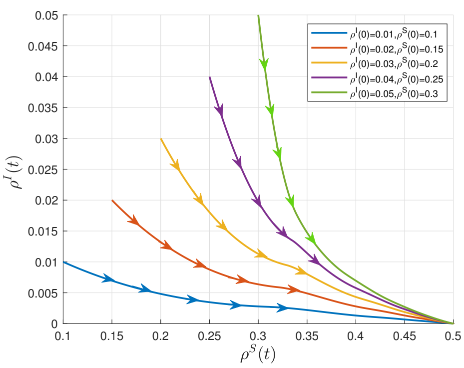
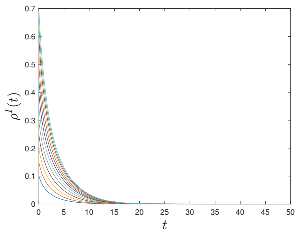
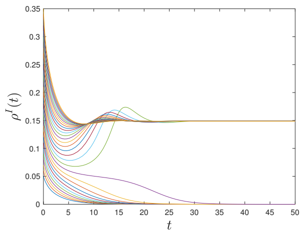


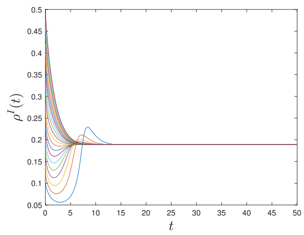
Figure 7. Global asymptotic stability at E_0 for system (4.3).



(a) $\phi_\triangle = 10$



(b) $\phi_\triangle = 23$



(c) $\phi_\triangle = 36$

Figure 8. Evolution of system (4.3) with varying initial conditions.

7. Conclusions

In this paper, we investigate a simplicial SEIR rumor propagation model on both homogeneous and heterogeneous networks. We analyze the existence of equilibrium points, explore multistability phenomena, examine bifurcation behavior, and propose an optimal control strategy to suppress rumor propagation. The theoretical analysis of system (3.2) reveals the influence of ϕ and $\phi_\triangle$ on rumor propagation. When $\Delta > 0$, $\phi < \xi_1$, and $\phi_\triangle > \xi_2\phi$, system (3.2) exhibits bistable behavior between the rumor-free equilibrium state and the rumor-prevailing equilibrium state. When $\phi > \xi_1$, only a stable positive equilibrium state exists. When $\Delta > 0$, $\phi < \xi_1$, and $\phi_\triangle < \xi_2\phi$ are satisfied or if $\Delta < 0$, the system (3.2) exclusively maintains stability at the rumor-free equilibrium. Bifurcation analysis identifies the critical threshold $\hat{\phi}_\triangle = min\{\phi_{\triangle_1}, \phi_{\triangle_2}\}$, beyond which the equilibrium point E_{*2} transitions from local asymptotic stability to instability. Simulations confirm that enhancing group interaction effects $\phi_\triangle$ and individual susceptibility θ_1 significantly expand the bistable region, demonstrating that intensified collective social dynamics amplify the system's multistable complexity. Extending to system (4.3) on heterogeneous networks, three key findings are established: (1) Global stability conditions for the rumor-free equilibrium are obtained, and reducing the group interactions' strength ($\phi_{\triangle}$) and infection transmission rate (θ_1) effectively promotes the system's global stability. (2) The parameter $\widetilde{\phi}$ serves as the bifurcation threshold governing equilibrium transitions: the endemic equilibrium curve undergoes a backward bifurcation when $\phi_{\triangle} > \widetilde{\phi}$, whereas a forward bifurcation emerges for $\phi_{\triangle} < \widetilde{\phi}$. (3) Control strategy is proposed for rumor propagation models incorporating group interactions in heterogeneous networks.

This study enhances our understanding of rumor propagation dynamics through the simplicial SEIR model on homogeneous and heterogeneous networks. The results not only strengthen the theoretical foundation of rumor dynamics but also provide practical insights for rumor control. Specifically, the proposed optimal control strategy can mitigate rumor spread by adjusting group interactions, which is crucial for applications like social media.

There are limitations that should be addressed in future work. First, the model assumes static networks with fixed topologies, whereas real-world networks are dynamic. Future studies could incorporate time-varying or high-dimensional networks to better reflect the evolving nature of social interactions. Additionally, the current model does not account for time delays in rumor propagation. In real scenarios, individuals often take time to react to or spread rumors. Including these delays would provide a more accurate representation of rumor dynamics in social networks. Finally, future research could explore the integration of real-world data to refine model parameters and validate the theoretical findings, further enhancing the model's applicability and accuracy in real-world scenarios.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest related to the publication of this research paper. The research was conducted impartially and without any financial or personal affiliations that might have influenced the outcome or interpretation of the findings. This work is solely based on scientific merit and the pursuit of knowledge in the respective field. We have no financial or nonfinancial relationships that could be perceived as potential conflicts of interest. We affirm that the submitted manuscript is an original work and has not been previously published in whole or in part. All authors have reviewed and approved the final version of the manuscript for submission.

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