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Research article

Existence and stability analysis for a coupled system of q-fractional implicit jerk Caputo derivatives with q-Erdélyi-Kober fractional integral conditions

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Abstract: This manuscript demonstrates the existence, uniqueness, and different kinds of Ulam stability for a q-Caputo implicit fractional jerk coupled system involving q-fractional Erdélyi-Kober integral conditions. The existence and uniqueness results were investigated by employing the Leray-Schauder alternative and the Banach contraction mapping principle. We also derived various kinds of Ulam stability under specific conditions. We provided an example to verify our main results.

Keywords: coupled system; fractional jerk equation; Erdélyi-Kober q-fractional integral conditions; fixed point theorem; Ulam-Hyers stability

1. Introduction

Fractional differential equations ($\mathcal{FDE}s$) have gained more attention from researchers and mathematicians in recent times. These equations incorporate derivatives of non-integer order and are recently gaining importance due to their great accuracy and practical applications in various scientific fields. $\mathcal{FDE}s$ are special mathematical tools that help us to create models for different phenomena. $\mathcal{FDE}s$ have a variety of applications in various areas, including polymer rheology, electrodynamics of complex medium, biology, aerodynamics, and more. For more information, visit [1–5]. The study of coupled systems (CSs) of $\mathcal{FDE}s$ has drawn interest in various fields including physics, biology, and psychology. The reader may refer to [6, 7]. Due to the non-local nature of $\mathcal{FDE}s$ and their applicability in real-world problems such as disease models, synchronization of chaotic systems, anomalous diffusion, etc., efforts [8, 9] are made to the theoretical work of CSs of classical differential equations ($\mathcal{DE}s$) and $\mathcal{FDE}s$.

In 1910, Jackson [10] made significant contributions in quantum calculus ($\mathcal{QC}$). Later, Agarwal [11] and Salam [12] made more progress and presented important theoretical results in $\mathcal{QC}$. Many researchers have taken interest in this field, particularly solving problems involving q-$\mathcal{FDE}s$ [13, 14]. $\mathcal{QC}$ theory includes many practical aspects in areas of particle physics, discrete mathematics, the theory of relativity, combinatorics, and complex analysis. For more information on the primitive notion of $\mathcal{QC}$, see [15–17] and the references therein.

The basic requirement of qualitative theory is the particular behavior of solutions, for which we need the existence and uniqueness ($\mathscr{E}\mathscr{U}$) of solutions [3, 6, 18]. Various fixed point (FP) theorems [19] such as the Leray-Schauder alternative, the Banach contraction principle, and the Arzela-Ascoli theorem are utilized for the existing solutions of $\mathcal{FDE}s$. Moreover, stability analysis is an important aspect of qualitative theory of $\mathcal{DE}s$. Ulam stability ($\mathscr{U}\mathscr{S}$) was introduced in 1940 by Ulam [20] and had been further expanded by Hyers and Rassias [21]. Recent advancements on Ulam-Hyers stability (UHS) have relaxed the stability conditions. The idea of $\mathscr{U}\mathscr{S}$ is beneficial for various fields including biology, numerical analysis, economics, optimization theory, and so on [22]. For further details, see [14, 23–26].

In 1978, Schot [27] presented the idea of the jerk equation ($\mathcal{JE}$) which is described as the rate of change of acceleration. A well known, three dimensional dynamic system can be given by $\mathsf{m}'(\varrho) = a$, $a'(\varrho) = e$, $e'(\varrho) = \mathsf{f}(\mathsf{m}, a, e)$, which can be written in the form of $\mathsf{m}''' = \mathsf{f}(\mathsf{m}, \mathsf{m}', \mathsf{m}'')$. The $\mathcal{JE}$ is a third-order $\mathcal{DE}$ that has been applied in various fields, such as secure communication, signal processing, control systems, electrical engineering, economic systems, bio-mechanics, acoustics, electrical circuits, mechanics, and dynamical processes [28, 29]. The reader may refer to [29–32] for details on applications of $\mathcal{JE}s$.

Alam et al. [33] demonstrated different kinds of $\mathscr{U}\mathscr{S}$ and the existence uniqueness ($\mathscr{E}\mathscr{U}$) for the implicit q-fractional CS involving Erdélyi-Kober and non-local Reimann-Liouville ($\mathcal{RL}$) q-fractional integral conditions (FICs):

$$\begin{cases} \begin{cases} {}^{\mathcal{RL}}D^{\alpha}_{\mathsf{q},0}\mathsf{m}(\varrho) + \mathsf{f}_1(\varrho, \mathsf{m}(\varrho), {}^{\mathcal{RL}}D^{\beta}_{\mathsf{q},0}\mathsf{u}(\varrho)) = 0, \varrho \in \mathcal{J} = (0, T), \\ \varrho^{2-\alpha}\mathsf{m}(\varrho)|_{\varrho=0} = \mathcal{I}^{\alpha}_{\mathsf{q},0}\mathsf{m}(\zeta_1), \alpha \in (1, 2], \\ c_1\mathsf{m}(T) = \sum_{i=1}^{n} \lambda_i \mathcal{I}^{\eta_i,\mu_i,\beta_i}_{\mathsf{q},0}\mathsf{m}(\Upsilon_i) + \int_0^T \mathsf{m}(\varrho)d_{\mathsf{q}}\varrho, \end{cases} \\ \begin{cases} {}^{\mathcal{RL}}D^{\beta}_{\mathsf{q},0}\mathsf{u}(\varrho) + \mathsf{f}_2(\varrho, \mathsf{u}(\varrho), {}^{\mathcal{RL}}D^{\alpha}_{\mathsf{q},0}\mathsf{m}(\varrho)) = 0, \varrho \in \mathcal{J} = (0, T), \\ \varrho^{2-\beta}\mathsf{u}(\varrho)|_{\varrho=0} = \mathcal{I}^{\alpha}_{\mathsf{q},0}\mathsf{u}(\zeta_2), \beta \in (1, 2], \\ c_2\mathsf{u}(T) = \sum_{i=1}^{n} \lambda_i \mathcal{I}^{\eta_i,\mu_i,\beta_i}_{\mathsf{q},0}\mathsf{u}(\Upsilon_i) + \int_0^T \mathsf{u}(\varrho)d_{\mathsf{q}}\varrho, \end{cases} \end{cases}$$

where $\mathsf{f}_1, \mathsf{f}_2$ are appropriate functions, ${}^{\mathcal{RL}}D^{\alpha}_{\mathsf{q},0}$ and ${}^{\mathcal{RL}}D^{\beta}_{\mathsf{q},0}$ are the $\mathcal{RL}$ fractional q-derivatives having orders α and β, on $\mathcal{J}$ with 0 as the lower limit, $\mathcal{I}^{\eta_i,\mu_i,\beta_i}_{\mathsf{q},0}$ represents the q- Erdélyi-Kober fractional integral ($\mathcal{FI}$) and c_1, c_2, λ_i $(i = 1, 2, \ldots, n)$ are the particular constants.

In [34], the authors investigated the existence of coupled fractional jerk differential equations ($\mathscr{J}\mathscr{D}\mathscr{E}$) with multi-point boundary conditions (BCs):

$$\begin{cases} \begin{cases} D^{\alpha}_{0+}\mathsf{m}_1(\varrho) = \mathsf{m}_2(\varrho), \\ D^{\beta}_{0+}\mathsf{m}_2(\varrho) = \mathsf{m}_3(\varrho), \\ D^{\gamma}_{0+}\mathsf{m}_3(\varrho) = \mathsf{f}(\mathsf{u}_1(\varrho), \mathsf{u}_2(\varrho), \mathsf{u}_3(\varrho)), \\ \quad \mathsf{m}_2(0) = \mathsf{m}_3(0) = 0, \\ \quad \mathsf{m}_1(0) = \sum_{i=1}^{n-1} a_i \mathsf{m}_1(\chi_i), \end{cases} \\ \begin{cases} D^{\theta}_{0+}\mathsf{u}_1(\varrho) = \mathsf{u}_2(\varrho), \\ D^{\zeta}_{0+}\mathsf{u}_2(\varrho) = \mathsf{u}_3(\varrho), \\ D^{\eta}_{0+}\mathsf{u}_3(\varrho) = g(\mathsf{m}_1(\varrho), \mathsf{m}_2(\varrho), \mathsf{m}_3(\varrho)), \\ \quad \mathsf{u}_2(0) = \mathsf{u}_3(0) = 0, \\ \quad \mathsf{u}_1(0) = \sum_{i=1}^{n-1} b_i \mathsf{u}_1(\xi_i), \end{cases} \end{cases}$$

where $\sigma \in (0, 1)$, $\sigma = \{\alpha, \beta, \gamma, \theta, \zeta, \eta\}$, D^{σ}_{0+} is the Caputo fractional derivative (CFD). f, g are some suitable functions. $\varrho \in (0, 1)$, $0 \leqslant \chi_1 \leqslant \chi_2 \leqslant ... \leqslant \chi_{n-1} \leqslant 1$, $0 \leqslant \xi_1 \leqslant \xi_2 \leqslant ... \leqslant \xi_{n-1} \leqslant 1$, $\sum_{i=1}^{n-1} a_i = \sum_{i=1}^{n-1} b_i = 1$, $\sum_{i=1}^{n-1} a_i(\chi_i) = \sum_{i=1}^{n-1} b_i(\xi_i) \neq 0$.

Majeed et al. [35] studied the implicit Langevin equations using Stieltjes integral BCs and mixed derivatives:

$$\begin{cases} \begin{cases} {}^{C}D^{\alpha}_{0,\varrho}(D + \lambda_1)\mathsf{m}(\varrho) = \mathsf{f}_1(\varrho, \mathsf{m}(\varrho)), \quad \varrho \in (0, 1], \\ \qquad \mathsf{m}(0) = 0, \\ {}^{C}D^{\beta_0}_{0,\varrho}\mathsf{m}(1) = \sum_{i=1}^{p} \int_0^1 {}^{C}D^{\beta_i}_{0,\varrho}\mathsf{m}(\varrho)d_{\mu_i}\varrho, \end{cases} \\ \begin{cases} {}^{C}D^{\gamma}_{0,\varrho}(D + \lambda_2)\mathsf{u}(\varrho) = \mathsf{f}_2(\varrho, \mathsf{u}(\varrho)), \quad \varrho \in (0, 1], \\ \qquad \mathsf{u}(0) = 0, \\ {}^{C}D^{\zeta_0}_{0,\varrho}\mathsf{u}(1) = \sum_{i=1}^{p} \int_0^1 {}^{C}D^{\zeta_i}_{0,\varrho}\mathsf{u}(\varrho)d_{\mu_i}\varrho, \end{cases} \end{cases}$$

where $\mathsf{f}_1, \mathsf{f}_2$ are continuous functions, ${}^{C}D^{\alpha}_{0,\varrho}$ and ${}^{C}D^{\gamma}_{0,\varrho}$ represent the Caputo derivatives ($\mathcal{CD}$) of orders α and γ with 0 as a lower bound, $\alpha, \gamma \in (0, 1)$, λ_1, λ_2 are real constants except zero. $p, j \in N$, $\beta_i \in \mathscr{R}$ ($\forall\ i = 0, \ldots, p$), $\beta_0 \in [0, \alpha)$, $\zeta_i \in \mathscr{R}$ ($\forall\ i = 0, \ldots, j$), $\zeta_0 \in [0, \gamma)$, and the integrals resulting from BCs are Reimann-Stieltjes integrals with μ_i ($i = 1, \ldots, p/j$) being functions of bounded variation.

In [23], Wang et al. studied the various forms of $\mathscr{U}\mathscr{S}$ and $\mathscr{E}\mathscr{U}$ for the following nonlinear implicit fractional integro-$\mathcal{DE}s$ involving $\mathcal{CD}s$ of fractional order:

$$\begin{cases} \begin{cases} {}^CD^{\alpha}\mathsf{m}(\varrho) = \chi(\varrho, \mathsf{u}(\varrho), {}^CD^{\alpha}\mathsf{m}(\varrho)) + \displaystyle\int_0^{\varrho} \frac{(\varrho - s)^{\nu-1}}{\Gamma(\sigma)} \mathsf{f}_1(s, \mathsf{u}(s), \\ \qquad\qquad {}^CD^{\alpha}\mathsf{m}(s))ds, \varrho \in \mathcal{J} = [0, T], \\ \mathsf{m}(\varrho)|_{\varrho=0} = -\mathsf{m}(\varrho)|_{\varrho=T}, \\ {}^CD^{\beta}\mathsf{m}(\varrho)|_{\varrho=0} = -{}^CD^{\beta}\mathsf{m}(\varrho)|_{\varrho=T}, \end{cases} \\ \begin{cases} {}^CD^{\zeta}\mathsf{u}(\varrho) = \xi(\varrho, \mathsf{m}(\varrho), {}^CD^{\zeta}\mathsf{u}(\varrho)) + \displaystyle\int_0^{\varrho} \frac{(\varrho - s)^{\nu-1}}{\Gamma(\sigma)} \mathsf{f}_2(s, \mathsf{m}(s), \\ \qquad\qquad {}^CD^{\zeta}\mathsf{u}(s))ds,\ \varrho \in \mathcal{J} = [0, T], \\ \mathsf{u}(\varrho)|_{\varrho=0} = -\mathsf{u}(\varrho)|_{\varrho=T}, \\ {}^CD^{\eta}\mathsf{u}(\varrho)|_{\varrho=0} = -{}^CD^{\eta}\mathsf{u}(\varrho)|_{\varrho=T}, \end{cases} \end{cases}$$

where $\mathcal{J} = [0, T]$ with $T > 0, \nu, \sigma > 0$, $1 < \alpha, \zeta \leqslant 2$, $0 \leqslant \beta, \eta \leqslant 2$, and $\chi, \xi, \mathsf{f}_1, \mathsf{f}_2$ are given real-valued continuous functions.

In [36], the authors investigated the following q-Caputo fractional implicit ($\mathscr{J}\mathscr{D}\mathscr{E}$) having Erdélyi-Kober q-$\mathcal{FI}$ conditions:

$$\begin{cases} {}^CD_{\mathsf{q}}^{\alpha}({}^CD_{\mathsf{q}}^{\omega}({}^CD_{\mathsf{q}}^{\theta} + \Omega))\mathsf{m}(\varrho) = \mathsf{f}(\varrho) - \phi(\varrho, \mathsf{m}(\varrho)) \\ -\Pi\xi(\varrho, \mathsf{m}(\varrho), {}^CD_{\mathsf{q}}^{\nu}\mathsf{m}(\varrho)) - \chi\rho(\varrho, \mathsf{m}(\varrho), {}^CD_{\mathsf{q}}^{\tau}\mathsf{m}(\varrho)) \\ -\displaystyle\int_0^{\varrho} \frac{(\varrho - \mathsf{q}s)^{\vartheta-1}}{\Gamma_{\mathsf{q}}(\varphi)} h(s, \mathsf{m}(s), {}^CD_{\mathsf{q}}^{\nu}\mathsf{m}(s))d_{\mathsf{q}}s, \varrho \in \mathcal{J} = (0, T), \\ \mathsf{m}(0) = 0, \\ ({}^CD_{\mathsf{q}}^{\omega}({}^CD_{\mathsf{q}}^{\theta} + \Omega))\mathsf{m}(\delta) = 0, \\ a\mathsf{m}(T) = \displaystyle\sum_{i=1}^{n} \lambda_i \mathcal{I}_{\mathsf{q}}^{\eta_i, \mu_i, \beta_i}\mathsf{m}(\zeta_i), \end{cases}$$

where $\Omega, \Pi, \chi \in \mathscr{R}^+$, $0 < \delta, \zeta_i < \mathrm{T}$, $0 < \alpha, \omega, \theta \leqslant 1$, $\nu < \theta$, $\tau < \theta$, and the CFD is $D_{\mathsf{q}}^{\varsigma}$, $\varsigma \in \{\alpha, \omega, \theta, \nu, \tau\}$ of order ς. $\mathsf{f} : \mathcal{J} \to \mathscr{R}$, $\phi : \mathcal{J} \times \mathscr{R} \to \mathscr{R}$, and $\xi, \rho, h : \mathcal{J} \times \mathscr{R}^2 \to \mathscr{R}$ are given continuous functions. $\mathcal{I}_{\mathsf{q}}^{\eta_i, \mu_i, \beta_i}$ denotes the Erdélyi-Kober q-$\mathcal{FI}$ having order μ_i on $\mathcal{J}$ where μ_i, β_i are greater than zero and η_i is a real number. $a, \vartheta, \varphi > 0, \lambda_i$ ($i = 1, 2, \ldots, n$) are appropriate constants.

Influenced by the aforementioned works, the objective of this dissertation is to examine the $\mathscr{E}\mathscr{U}$ and different forms of $\mathscr{U}\mathscr{S}$ for the following q-Caputo implicit fractional jerk coupled system ($\mathcal{FJCS}$) including q-Erdélyi-Kober FICs:

$$\begin{cases} \begin{cases} {}^CD_{\mathsf{q}}^{\alpha_1}({}^CD_{\mathsf{q}}^{\omega_1}({}^CD_{\mathsf{q}}^{\theta_1} + \Omega))\mathsf{m}(\varrho) = \mathsf{f}_1(\varrho) - \phi_1(\varrho, \mathsf{m}(\varrho)) \\ -\Pi_1\xi_1(\varrho, \mathsf{m}(\varrho), {}^CD_{\mathsf{q}}^{\nu_2}\mathsf{u}(\varrho)) - \chi_1\rho_1(\varrho, \mathsf{m}(\varrho), {}^CD_{\mathsf{q}}^{\tau_2}\mathsf{u}(\varrho)) \\ -\displaystyle\int_0^{\varrho} \frac{(\varrho - \mathsf{q}s)^{\vartheta_1-1}}{\Gamma_{\mathsf{q}}(\varphi_1)} h_1(s, \mathsf{m}(s), {}^CD_{\mathsf{q}}^{\nu_2}\mathsf{u}(s))d_{\mathsf{q}}s, \\ \qquad\qquad\qquad \varrho \in \mathcal{J} = (0, T), \\ \mathsf{m}(0) = 0, \\ ({}^CD_{\mathsf{q}}^{\omega_1}({}^CD_{\mathsf{q}}^{\theta_1} + \Omega))\mathsf{m}(\delta_1) = 0, \\ a_1\mathsf{m}(T) = \displaystyle\sum_{i=1}^{n} \lambda_i \mathcal{I}_{\mathsf{q}}^{\eta_i, \mu_i, \beta_i}\mathsf{m}(\zeta_i), \end{cases} \\ \begin{cases} {}^CD_{\mathsf{q}}^{\alpha_2}({}^CD_{\mathsf{q}}^{\omega_2}({}^CD_{\mathsf{q}}^{\theta_2} + \Upsilon))\mathsf{u}(\varrho) = \mathsf{f}_2(\varrho) - \phi_2(\varrho, \mathsf{u}(\varrho)) \\ -\Pi_2\xi_2(\varrho, \mathsf{u}(\varrho), {}^CD_{\mathsf{q}}^{\nu_1}\mathsf{m}(\varrho)) - \chi_2\rho_2(\varrho, \mathsf{u}(\varrho), {}^CD_{\mathsf{q}}^{\tau_1}\mathsf{m}(\varrho)) \\ -\displaystyle\int_0^{\varrho} \frac{(\varrho - \mathsf{q}s)^{\vartheta_2-1}}{\Gamma_{\mathsf{q}}(\varphi_2)} h_2(s, \mathsf{u}(s), {}^CD_{\mathsf{q}}^{\nu_1}\mathsf{m}(s))d_{\mathsf{q}}s, \\ \qquad\qquad\qquad \varrho \in \mathcal{J} = (0, T), \\ \mathsf{u}(0) = 0, \\ ({}^CD_{\mathsf{q}}^{\omega_2}({}^CD_{\mathsf{q}}^{\theta_2} + \Upsilon))\mathsf{u}(\delta_2) = 0, \\ a_2\mathsf{u}(T) = \displaystyle\sum_{i=1}^{n} \lambda_i \mathcal{I}_{\mathsf{q}}^{\eta_i, \mu_i, \beta_i}\mathsf{u}(\zeta_i), \end{cases} \end{cases} \tag{1.1}$$

where $\Omega, \Upsilon, \Pi_1, \Pi_2, \chi_1, \chi_2 \in \mathscr{R}^+$, $0 < \delta_1, \delta_2, \zeta_i < T$. $\alpha_r, \omega_r, \theta_r \in (0, 1]$, $\nu_1, \nu_2, \tau_1, \tau_2 < \theta_r$ with $\{r = 1, 2\}$, and the q-CFDs are $D_{\mathsf{q}}^{\varsigma}$, $\varsigma \in \{\alpha_1, \omega_1, \theta_1, \nu_2, \tau_2\}$ and $D_{\mathsf{q}}^{\varkappa}$, $\varkappa \in \{\alpha_2, \omega_2, \theta_2, \nu_1, \tau_1\}$ of orders ς and $\varkappa$, respectively. $\mathsf{f}_1, \mathsf{f}_2 : \mathcal{J} \to \mathscr{R}$, $\phi_1, \phi_2 : \mathcal{J} \times \mathscr{R} \to \mathscr{R}$ and $\xi_1, \xi_2, \rho_1, \rho_2, h_1, h_2 : \mathcal{J} \times \mathscr{R}^2 \to \mathscr{R}$ are given continuous functions. $\mathcal{I}_{\mathsf{q}}^{\eta_i, \mu_i, \beta_i}$ denotes the Erdélyi-Kober q-$\mathcal{FI}$ having order μ_i on $\mathcal{J}$ where μ_i, β_i are greater than zero and η_i is a real number. $a_1, a_2, \vartheta_1, \vartheta_2, \varphi_1, \varphi_2 > 0, \lambda_i$ ($i = 1, 2, \ldots, n$) are constants.

The main aspects of this typescript are as follows:

(1) We implement Erdélyi-Kober q-FICs motivated by previous work [33] in order to describe the q-Caputo implicit $\mathcal{FJCS}$ in the literature.

(2) In this dissertation, we establish the necessary conditions of $\mathscr{E}\mathscr{U}$ and various types of $\mathscr{U}\mathscr{S}$ for the suggested q-$\mathcal{FJCS}$ (1.1).

(3) Different from prior studies that used multi-point BCs [34], anti-periodic BCs [23], and implicit Langevin

equations with mixed derivatives [35], we achieve better results by employing Erdélyi-Kober q-FICs for q-$\mathcal{FJCS}$s.

This manuscript is organized in the following manner: In Section 2, we define some basic lemmas and theorems for the suggested q-$\mathcal{FJCS}$ (1.1). In Section 3, we establish sufficient criteria for the $\mathscr{EU}$ of the considered q-$\mathcal{FJCS}$ (1.1) by using different FP theorems such as the Leray-Schauder alternative and Banach contraction mapping principle. Various kinds of $\mathscr{US}$ are derived in Section 4 under specific conditions. In Section 5, an example is presented to verify our theoretical outcomes.

2. Basic results

Here we recall a few basic notions which will be utilized for our objectives.

Definition 2.1. *[37] The operator* ${}^{C}\mathrm{D}_{\mathrm{q}}^{\varsigma}$ *is* q-CFD *of order* ς *and is given by:*

$$ {}^{C}\mathrm{D}_{\mathrm{q}}^{\varsigma}\mathrm{m}(\varrho) = \mathcal{I}_{\mathrm{q}}^{\lceil\varsigma\rceil-\varsigma}\mathrm{D}_{\mathrm{q}}^{\lceil\varsigma\rceil}\mathrm{m}(\varrho), \quad \varsigma > 0, $$

and ${}^{C}\mathrm{D}_{\mathrm{q}}^{0}\mathrm{m}(\varrho) = \mathrm{m}(\varrho)$*, where* $\lceil\varsigma\rceil$ *denotes the greatest integer function.*

Definition 2.2. *[38] The fractional* $\mathcal{RL}$ *type* q*-integral of the function* m *is given by*

$$ \mathcal{I}_{\mathrm{q}}^{\varsigma}\mathrm{m}(\varrho) = \frac{1}{\Gamma_{\mathrm{q}}(\varsigma)}\int_0^{\varrho}(\varrho - \mathrm{q}s)^{\varsigma-1}\mathrm{m}(s)d_{\mathrm{q}}s, \quad \varrho > 0, \quad \varsigma > 0, $$

and $\mathcal{I}_{\mathrm{q}}^{0}\mathrm{m}(\varrho) = \mathrm{m}(\varrho)$.

Definition 2.3. *[39] For* $0 < \mathrm{q} < 1$*, the Erdélyi-Kober* q*-*$\mathcal{FI}$ *of order* $\mu > 0$ *with* $\beta > 0$ *and* $\eta \in \mathscr{R}$ *of a continuous function* $\mathrm{f} : \mathcal{J} \to \mathscr{R}$ *is defined as*

$$ \mathcal{I}_{\mathrm{q}}^{\eta,\mu,\beta}\mathrm{f}(\varrho) = \frac{\beta\varrho^{-\beta(\mu+\eta)}}{\Gamma_{\mathrm{q}}(\mu)}\int_0^{\varrho}(\varrho^{\beta} - \mathrm{q}s^{\beta})^{(\mu-1)}s^{\eta}\mathrm{f}(s)d_{\mathrm{q}}s. $$

Lemma 2.1. *[38] Let* $\varsigma \in \mathscr{R}^{+}\backslash\mathscr{N}$*. Then the following equality is fulfilled:*

$$ \mathcal{I}_{\mathrm{q}}^{\varsigma}{}^{C}\mathrm{D}_{\mathrm{q}}^{\varsigma}\mathrm{m}(\varrho) = \mathrm{m}(\varrho) - \sum_{j=0}^{n-1}\frac{\varrho^{j}}{\Gamma_{\mathrm{q}}(j+1)}{}^{C}\mathrm{D}_{\mathrm{q}}^{j}\mathrm{m}(0), $$

where n represents the greatest integer function.

Theorem 2.1. ***(Banach fixed point theorem, [19])*** *Let* X *be a Banach space. Then each contraction mapping* $B : \mathrm{X} \to \mathrm{X}$ *has a precisely unique* FP.

Lemma 2.2. *Let* $\mathrm{X} = \{\mathrm{m}(\varrho) : \mathrm{m} \in C(\mathcal{J}, \mathscr{R})\}$ *be a Banach space endowed with the norm given by* $\|\mathrm{m}\|_{\mathrm{X}} = \sup\{|\mathrm{m}(\varrho)| : \varrho \in \mathcal{J}\}$*. Consequently, the norm established on the product space is* $\|(\mathrm{m}, \mathrm{u})\|_{\mathrm{X}\times\mathrm{X}} = \|\mathrm{m}\|_{\mathrm{X}} + \|\mathrm{u}\|_{\mathrm{X}}$*. It is obvious that the Banach space is obtained by* $(\mathrm{X}\times\mathrm{X}, \|(\mathrm{m}, \mathrm{u})\|_{\mathrm{X}\times\mathrm{X}})$.

Theorem 2.2. ***(Leray-Schauder alternative, [19])*** *Let* $B : \mathrm{X} \to \mathrm{X}$ *be a completely continuous operator. Assume that the set* $\mathscr{H}(B) = \{\mathrm{m} \in \mathscr{H} : \mathrm{m} = \gamma B(\mathrm{m}),\ for\ some\ 0 < \gamma < 1\}$*. Then:*

(1) The set $\mathscr{H}(B)$ *is unbounded, or*

(2) B *has at least one* FP.

3. Existence and uniqueness

We present the sufficient criteria for the $\mathscr{EU}$ of the solution for the suggested system (1.1) in this section.

Lemma 3.1. *Assume that* $g_1 \in C(\mathcal{J}, \mathscr{R})$*. Then the* q*-fractional jerk problem* ($\mathcal{FJP}$) *is given as:*

$$ \begin{cases} {}^{C}\mathrm{D}_{\mathrm{q}}^{\alpha_1}({}^{C}\mathrm{D}_{\mathrm{q}}^{\omega_1}({}^{C}\mathrm{D}_{\mathrm{q}}^{\theta_1} + \Omega))\mathrm{m}(\varrho) = g_1(\varrho), \varrho \in \mathcal{J} = (0, T), \\ \mathrm{m}(0) = 0, \\ ({}^{C}\mathrm{D}_{\mathrm{q}}^{\omega_1}({}^{C}\mathrm{D}_{\mathrm{q}}^{\theta_1} + \Omega))\mathrm{m}(\delta_1) = 0, \\ a_1\mathrm{m}(T) = \sum_{i=1}^{n}\lambda_i\mathcal{I}_{\mathrm{q}}^{\eta_i,\mu_i,\beta_i}\mathrm{m}(\zeta_i), \end{cases} \tag{3.1} $$

where $0 < \alpha_1, \omega_1, \theta_1 \leqslant 1$, $\Omega > 0$, $0 < \delta_1, \zeta_i < T$. (3.1) *admits the following solution:*

$$ \begin{aligned} \mathrm{m}(\varrho) &= \int_0^{\varrho}\frac{(\varrho - \mathrm{q}s)^{\alpha_1+\omega_1+\theta_1-1}}{\Gamma_{\mathrm{q}}(\alpha_1 + \omega_1 + \theta_1)}g_1(s)d_{\mathrm{q}}s \\ &+ \frac{\varrho^{\theta_1}}{\Delta_1\Gamma_{\mathrm{q}}(\theta_1 + 1)}\Big(\sum_{i=1}^{n}\lambda_i\frac{\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathrm{q}}(\mu_i)}\int_0^{\zeta_i}(\zeta_i^{\beta_i} - \mathrm{q}s^{\beta_i})^{(\mu_i-1)}s^{\eta_i} \\ &\times\Big(\int_0^{\zeta_i}\frac{(\zeta_i - \mathrm{q}s)^{\alpha_1+\omega_1+\theta-1}}{\Gamma_{\mathrm{q}}(\alpha_1 + \omega_1 + \theta_1)}g_1(s)d_{\mathrm{q}}s\Big)d_{\mathrm{q}}s \\ &- a_1\int_0^{T}\frac{(T - \mathrm{q}s)^{\alpha_1+\omega_1+\theta_1-1}}{\Gamma_{\mathrm{q}}(\alpha_1 + \omega_1 + \theta_1)}g_1(s)d_{\mathrm{q}}s\Big) + \Omega\Big(\frac{\varrho^{\theta_1}}{\Delta_1\Gamma_{\mathrm{q}}(\theta_1 + 1)} \\ &\times\Big(a_1\int_0^{T}\frac{(T - \mathrm{q}s)^{\theta_1-1}}{\Gamma_{\mathrm{q}}(\theta_1)}\mathrm{m}(s)d_{\mathrm{q}}s \\ &- \sum_{i=1}^{n}\lambda_i\frac{\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathrm{q}}(\mu_i)}\int_0^{\zeta_i}(\zeta_i^{\beta_i} - \mathrm{q}s^{\beta_i})^{(\mu_i-1)}s^{\eta_i} \end{aligned} $$

$$\times\Big(\int_0^{\zeta_i}\frac{(\zeta_i-\mathsf{q}s)^{\theta_1-1}}{\Gamma_{\mathsf{q}}(\theta_1)}\mathsf{m}(s)d_{\mathsf{q}}s\Big)d_{\mathsf{q}}s\Big)$$
$$-\int_0^{\varrho}\frac{(\varrho-\mathsf{q}s)^{\theta_1-1}}{\Gamma_{\mathsf{q}}(\theta_1)}\mathsf{m}(s)d_{\mathsf{q}}s\Big)$$
$$+\int_0^{\delta_1}\frac{(\delta_1-\mathsf{q}s)^{\alpha_1-1}}{\Gamma_{\mathsf{q}}(\alpha_1)}\Big(\frac{\varrho^{\theta_1}}{\Delta_1\Gamma_{\mathsf{q}}(\theta_1+1)}\Big(a_1\frac{T^{\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\omega_1+\theta_1+1)}$$
$$-\sum_{i=1}^{n}\lambda_i\beta_i[\frac{1}{\beta_i}]_{\mathsf{q}}\frac{\Gamma_{\mathsf{q}}(\eta_i+1+\frac{\omega_1+\theta_1}{\beta_i})\zeta_i^{\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\mu_i+\eta i+1+\frac{\omega_1+\theta_1}{\beta_i})\Gamma_{\mathsf{q}}(\omega_1+\theta_1+1)}\Big)$$
$$-\frac{\varrho^{\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\omega_1+\theta_1+1)}\Big)g_1(s)d_{\mathsf{q}}s,$$

where Δ_1 is given by

$$\Delta_1=\frac{a_1T^{\theta_1}}{\Gamma_{\mathsf{q}}(\theta_1+1)}-\sum_{i=1}^{n}\lambda_i\mathcal{I}_{\mathsf{q}}^{\eta_i,\mu_i,\beta_i}\frac{\zeta_i^{\theta_1}}{\Gamma_{\mathsf{q}}(\theta_1+1)}.$$

From [36], it is easy to prove Lemma 3.1, so we omit its proof.

Consequently, the solution of the suggested system (1.1) is similar to the q-$\mathcal{FJCS}$ given by Lemma 3.1.

For $\varrho\in\mathcal{J}$, we have

$$\mathsf{u}(\varrho)=\int_0^{\varrho}\frac{(\varrho-\mathsf{q}s)^{\alpha_2+\omega_2+\theta_2-1}}{\Gamma_{\mathsf{q}}(\alpha_2+\omega_2+\theta_2)}g_2(s)d_{\mathsf{q}}s+\frac{\varrho^{\theta_2}}{\Delta_2\Gamma_{\mathsf{q}}(\theta_2+1)}$$
$$\times\Big(\sum_{i=1}^{n}\lambda_i\frac{\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)}\int_0^{\zeta_i}(\zeta_i^{\beta_i}-\mathsf{q}s^{\beta_i})^{(\mu_i-1)}s^{\eta_i}$$
$$\times\Big(\int_0^{\zeta_i}\frac{(\zeta_i-\mathsf{q}s)^{\alpha_2+\omega_2+\theta-2}}{\Gamma_{\mathsf{q}}(\alpha_2+\omega_2+\theta_2)}g_2(s)d_{\mathsf{q}}s\Big)d_{\mathsf{q}}s$$
$$-a_2\int_0^{T}\frac{(T-\mathsf{q}s)^{\alpha_2+\omega_2+\theta_2-1}}{\Gamma_{\mathsf{q}}(\alpha_2+\omega_2+\theta_2)}g_2(s)d_{\mathsf{q}}s\Big)$$
$$+\Upsilon\Big(\frac{\varrho^{\theta_2}}{\Delta_2\Gamma_{\mathsf{q}}(\theta_2+1)}\Big(a_2\int_0^{T}\frac{(T-\mathsf{q}s)^{\theta_2-1}}{\Gamma_{\mathsf{q}}(\theta_2)}\mathsf{u}(s)d_{\mathsf{q}}s$$
$$-\sum_{i=1}^{n}\lambda_i\frac{\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)}\int_0^{\zeta_i}(\zeta_i^{\beta_i}-\mathsf{q}s^{\beta_i})^{(\mu_i-1)}s^{\eta_i}$$
$$\times\Big(\int_0^{\zeta_i}\frac{(\zeta_i-\mathsf{q}s)^{\theta_2-1}}{\Gamma_{\mathsf{q}}(\theta_2)}\mathsf{u}(s)d_{\mathsf{q}}s\Big)d_{\mathsf{q}}s\Big)$$
$$-\int_0^{\varrho}\frac{(\varrho-\mathsf{q}s)^{\theta_2-1}}{\Gamma_{\mathsf{q}}(\theta_1)}\mathsf{m}(s)d_{\mathsf{q}}s\Big)$$
$$+\int_0^{\delta_2}\frac{(\delta_2-\mathsf{q}s)^{\alpha_2-1}}{\Gamma_{\mathsf{q}}(\alpha_2)}\Big(\frac{\varrho^{\theta_2}}{\Delta_2\Gamma_{\mathsf{q}}(\theta_2+1)}\Big(a_2\frac{T^{\omega_2+\theta_2}}{\Gamma_{\mathsf{q}}(\omega_2+\theta_2+1)}$$
$$-\sum_{i=1}^{n}\lambda_i\beta_i[\frac{1}{\beta_i}]_{\mathsf{q}}\frac{\Gamma_{\mathsf{q}}(\eta_i+1+\frac{\omega_2+\theta_2}{\beta_i})\zeta_i^{\omega_2+\theta_2}}{\Gamma_{\mathsf{q}}(\mu_i+\eta i+1+\frac{\omega_2+\theta_2}{\beta_i})\Gamma_{\mathsf{q}}(\omega_2+\theta_2+1)}\Big)$$
$$-\frac{\varrho^{\omega_2+\theta_2}}{\Gamma_{\mathsf{q}}(\omega_2+\theta_2+1)}\Big)g_2(s)d_{\mathsf{q}}s,$$

where Δ_2 is given by

$$\Delta_2=\frac{a_2T^{\theta_2}}{\Gamma_{\mathsf{q}}(\theta_2+1)}-\sum_{i=1}^{n}\lambda_i\mathcal{I}_{\mathsf{q}}^{\eta_i,\mu_i,\beta_i}\frac{\zeta_i^{\theta_2}}{\Gamma_{\mathsf{q}}(\theta_2+1)}.$$

If m,u are the solutions of the proposed system (1.1) and $\varrho\in\mathcal{J}$, then we have

$$\mathsf{m}(\varrho)=\int_0^{\varrho}\frac{(\varrho-\mathsf{q}s)^{\alpha_1+\omega_1+\theta_1-1}}{\Gamma_{\mathsf{q}}(\alpha_1+\omega_1+\theta_1)}g_1(s)d_{\mathsf{q}}s$$
$$+\frac{\varrho^{\theta_1}}{\Delta_1\Gamma_{\mathsf{q}}(\theta_1+1)}\Big(\sum_{i=1}^{n}\lambda_i\frac{\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)}\int_0^{\zeta_i}(\zeta_i^{\beta_i}-\mathsf{q}s^{\beta_i})^{(\mu_i-1)}s^{\eta_i}$$
$$\times\Big(\int_0^{\zeta_i}\frac{(\zeta_i-\mathsf{q}s)^{\alpha_1+\omega_1+\theta-1}}{\Gamma_{\mathsf{q}}(\alpha_1+\omega_1+\theta_1)}g_1(s)d_{\mathsf{q}}s\Big)d_{\mathsf{q}}s$$
$$-a_1\int_0^{T}\frac{(T-\mathsf{q}s)^{\alpha_1+\omega_1+\theta_1-1}}{\Gamma_{\mathsf{q}}(\alpha_1+\omega_1+\theta_1)}g_1(s)d_{\mathsf{q}}s\Big)$$
$$+\Omega\Big(\frac{\varrho^{\theta_1}}{\Delta_1\Gamma_{\mathsf{q}}(\theta_1+1)}\Big(a_1\int_0^{T}\frac{(T-\mathsf{q}s)^{\theta_1-1}}{\Gamma_{\mathsf{q}}(\theta_1)}\mathsf{m}(s)d_{\mathsf{q}}s$$
$$-\sum_{i=1}^{n}\lambda_i\frac{\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)}\int_0^{\zeta_i}(\zeta_i^{\beta_i}-\mathsf{q}s^{\beta_i})^{(\mu_i-1)}s^{\eta_i}$$
$$\times\Big(\int_0^{\zeta_i}\frac{(\zeta_i-\mathsf{q}s)^{\theta_1-1}}{\Gamma_{\mathsf{q}}(\theta_1)}\mathsf{m}(s)d_{\mathsf{q}}s\Big)d_{\mathsf{q}}s\Big)-\int_0^{\varrho}\frac{(\varrho-\mathsf{q}s)^{\theta_1-1}}{\Gamma_{\mathsf{q}}(\theta_1)}\mathsf{m}(s)d_{\mathsf{q}}s\Big)$$
$$+\int_0^{\delta_1}\frac{(\delta_1-\mathsf{q}s)^{\alpha_1-1}}{\Gamma_{\mathsf{q}}(\alpha_1)}\Big(\frac{\varrho^{\theta_1}}{\Delta_1\Gamma_{\mathsf{q}}(\theta_1+1)}\Big(a_1\frac{T^{\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\omega_1+\theta_1+1)}$$
$$-\sum_{i=1}^{n}\lambda_i\beta_i[\frac{1}{\beta_i}]_{\mathsf{q}}\frac{\Gamma_{\mathsf{q}}(\eta_i+1+\frac{\omega_1+\theta_1}{\beta_i})}{\Gamma_{\mathsf{q}}(\mu_i+\eta i+1+\frac{\omega_1+\theta_1}{\beta_i})\Gamma_{\mathsf{q}}(\omega_1+\theta_1+1)}\zeta_i^{\omega_1+\theta_1}\Big)$$
$$-\frac{\varrho^{\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\omega_1+\theta_1+1)}\Big)g_1(s)d_{\mathsf{q}}s,$$

and

$$\mathsf{u}(\varrho)=\int_0^{\varrho}\frac{(\varrho-\mathsf{q}s)^{\alpha_2+\omega_2+\theta_2-1}}{\Gamma_{\mathsf{q}}(\alpha_2+\omega_2+\theta_2)}g_2(s)d_{\mathsf{q}}s$$
$$+\frac{\varrho^{\theta_2}}{\Delta_2\Gamma_{\mathsf{q}}(\theta_2+1)}\Big(\sum_{i=1}^{n}\lambda_i\frac{\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)}\int_0^{\zeta_i}(\zeta_i^{\beta_i}-\mathsf{q}s^{\beta_i})^{(\mu_i-1)}s^{\eta_i}$$
$$\times\Big(\int_0^{\zeta_i}\frac{(\zeta_i-\mathsf{q}s)^{\alpha_2+\omega_2+\theta-2}}{\Gamma_{\mathsf{q}}(\alpha_2+\omega_2+\theta_2)}g_2(s)d_{\mathsf{q}}s\Big)d_{\mathsf{q}}s$$
$$-a_2\int_0^{T}\frac{(T-\mathsf{q}s)^{\alpha_2+\omega_2+\theta_2-1}}{\Gamma_{\mathsf{q}}(\alpha_2+\omega_2+\theta_2)}g_2(s)d_{\mathsf{q}}s\Big)$$
$$+\Upsilon\Big(\frac{\varrho^{\theta_2}}{\Delta_2\Gamma_{\mathsf{q}}(\theta_2+1)}\Big(a_2\int_0^{T}\frac{(T-\mathsf{q}s)^{\theta_2-1}}{\Gamma_{\mathsf{q}}(\theta_2)}\mathsf{u}(s)d_{\mathsf{q}}s$$
$$-\sum_{i=1}^{n}\lambda_i\frac{\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)}\int_0^{\zeta_i}(\zeta_i^{\beta_i}-\mathsf{q}s^{\beta_i})^{(\mu_i-1)}s^{\eta_i}$$
$$\times\Big(\int_0^{\zeta_i}\frac{(\zeta_i-\mathsf{q}s)^{\theta_2-1}}{\Gamma_{\mathsf{q}}(\theta_2)}\mathsf{u}(s)d_{\mathsf{q}}s\Big)d_{\mathsf{q}}s\Big)-\int_0^{\varrho}\frac{(\varrho-\mathsf{q}s)^{\theta_2-1}}{\Gamma_{\mathsf{q}}(\theta_1)}\mathsf{m}(s)d_{\mathsf{q}}s\Big)$$
$$+\int_0^{\delta_2}\frac{(\delta_2-\mathsf{q}s)^{\alpha_2-1}}{\Gamma_{\mathsf{q}}(\alpha_2)}\Big(\frac{\varrho^{\theta_2}}{\Delta_2\Gamma_{\mathsf{q}}(\theta_2+1)}\Big(a_2\frac{T^{\omega_2+\theta_2}}{\Gamma_{\mathsf{q}}(\omega_2+\theta_2+1)}$$

$$- \sum_{i=1}^{n} \lambda_i \beta_i [\frac{1}{\beta_i}]_{\mathsf{q}} \frac{\Gamma_{\mathsf{q}}(\eta_i + 1 + \frac{\omega_2+\theta_2}{\beta_i})}{\Gamma_{\mathsf{q}}(\mu_i + \eta i + 1 + \frac{\omega_2+\theta_2}{\beta_i})\Gamma_{\mathsf{q}}(\omega_2 + \theta_2 + 1)} \zeta_i^{\omega_2+\theta_2} \Big)$$
$$- \frac{\varrho^{\omega_2+\theta_2}}{\Gamma_{\mathsf{q}}(\omega_2 + \theta_2 + 1)} \Big) g_2(s) d_{\mathsf{q}} s,$$

where $g_1(\varrho)$ and $g_2(\varrho)$ are given by

$$\begin{aligned} g_1(\varrho) &= \mathsf{f}_1(\varrho) - \phi_1(\varrho, \mathsf{m}(\varrho)) - \Pi_1 \xi_1(\varrho, \mathsf{m}(\varrho), {}^C D_{\mathsf{q}}^{\nu_2} \mathsf{u}(\varrho)) \\ &\quad - \chi_1 \rho_1(\varrho, \mathsf{m}(\varrho), {}^C D_{\mathsf{q}}^{\tau_2} \mathsf{u}(\varrho)) \\ &\quad - \int_0^{\varrho} \frac{(\varrho - \mathsf{q}s)^{\vartheta_1 - 1}}{\Gamma_{\mathsf{q}}(\varphi_1)} h_1(s, \mathsf{m}(s), {}^C D_{\mathsf{q}}^{\nu_2} \mathsf{u}(s)) d_{\mathsf{q}} s, \\ g_2(\varrho) &= \mathsf{f}_2(\varrho) - \phi_2(\varrho, \mathsf{u}(\varrho)) - \Pi_2 \xi_2(\varrho, \mathsf{u}(\varrho), {}^C D_{\mathsf{q}}^{\nu_1} \mathsf{m}(\varrho)) \\ &\quad - \chi_2 \rho_2(\varrho, \mathsf{u}(\varrho), {}^C D_{\mathsf{q}}^{\tau_1} \mathsf{m}(\varrho)) \\ &\quad - \int_0^{\varrho} \frac{(\varrho - \mathsf{q}s)^{\vartheta_2 - 1}}{\Gamma_{\mathsf{q}}(\varphi_2)} h_1(s, \mathsf{u}(s), {}^C D_{\mathsf{q}}^{\nu_1} \mathsf{m}(s)) d_{\mathsf{q}} s. \end{aligned}$$

The next stage is to convert the proposed system q-$\mathcal{FJCS}$ (1.1) into an FP problem. Let the operator $B : \mathsf{X} \to \mathsf{X}$ be defined as follows:

$$B(\mathsf{m}, \mathsf{u})(\varrho) = \begin{pmatrix} B_1(\mathsf{m})(\varrho) \\ B_2(\mathsf{u})(\varrho) \end{pmatrix}. \tag{3.2}$$

Hence, the FP of ϱ and the solution of the system (1.1) are congruent, where

$$B_1(\mathsf{m})(\varrho) = \int_0^{\varrho} \frac{(\varrho - \mathsf{q}s)^{\alpha_1+\omega_1+\theta_1-1}}{\Gamma_{\mathsf{q}}(\alpha_1 + \omega_1 + \theta_1)} \Big(\mathsf{f}_1(s) - \phi_{1\mathsf{m}}^*(s)$$
$$- \Pi_1 \xi_{1\mathsf{m}}^*(s) - \chi_1 \rho_{1\mathsf{m}}^*(s) - h_{1\mathsf{m}}^*(s) \Big) d_{\mathsf{q}} s + \frac{\varrho^{\theta_1}}{\Delta_1 \Gamma_{\mathsf{q}}(\theta_1 + 1)}$$
$$\times \Big(\sum_{i=1}^{n} \lambda_i \frac{\beta_i \zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)} \int_0^{\zeta_i} (\zeta_i^{\beta_i} - \mathsf{q}s^{\beta_i})^{(\mu_i - 1)} s^{\eta_i}$$
$$\times \Big(\int_0^{\zeta_i} \frac{(\zeta_i - \mathsf{q}s)^{\alpha_1+\omega_1+\theta_1-1}}{\Gamma_{\mathsf{q}}(\alpha_1 + \omega_1 + \theta_1)} \Big(\mathsf{f}_1(s) - \phi_{1\mathsf{m}}^*(s)$$
$$- \Pi_1 \xi_{1\mathsf{m}}^*(s) - \chi_1 \rho_{1\mathsf{m}}^*(s) - h_{1\mathsf{m}}^*(s) \Big) d_{\mathsf{q}} s \Big) d_{\mathsf{q}} s$$
$$- a_1 \int_0^{T} \frac{(T - \mathsf{q}s)^{\alpha_1+\omega_1+\theta_1-1}}{\Gamma_{\mathsf{q}}(\alpha_1 + \omega_1 + \theta_1)} \Big(\mathsf{f}_1(s) - \phi_{1\mathsf{m}}^*(s)$$
$$- \Pi_1 \xi_{1\mathsf{m}}^*(s) - \chi_1 \rho_{1\mathsf{m}}^*(s) - h_{1\mathsf{m}}^*(s) \Big) d_{\mathsf{q}} s \Big)$$
$$+ \Omega \Big(\frac{\varrho^{\theta_1}}{\Delta_1 \Gamma_{\mathsf{q}}(\theta_1 + 1)} \Big(a_1 \int_0^{T} \frac{(T - \mathsf{q}s)^{\theta_1 - 1}}{\Gamma_{\mathsf{q}}(\theta_1)} \mathsf{m}(s) d_{\mathsf{q}} s$$
$$- \sum_{i=1}^{n} \lambda_i \frac{\beta_i \zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)} \int_0^{\zeta_i} (\zeta_i^{\beta_i} - \mathsf{q}s^{\beta_i})^{(\mu_i - 1)} s^{\eta_i}$$
$$\times \Big(\int_0^{\zeta_i} \frac{(\zeta_i - \mathsf{q}s)^{\theta_1 - 1}}{\Gamma_{\mathsf{q}}(\theta_1)} \mathsf{m}(s) d_{\mathsf{q}} s \Big) d_{\mathsf{q}} s \Big)$$
$$- \int_0^{\varrho} \frac{(\varrho - \mathsf{q}s)^{\theta_1 - 1}}{\Gamma_{\mathsf{q}}(\theta_1)} \mathsf{m}(s) d_{\mathsf{q}} s \Big) + \int_0^{\delta_1} \frac{(\delta_1 - \mathsf{q}s)^{\alpha_1 - 1}}{\Gamma_{\mathsf{q}}(\alpha_1)}$$
$$\times \Big(\frac{\varrho^{\theta_1}}{\Delta_1 \Gamma_{\mathsf{q}}(\theta_1 + 1)} \Big(a_1 \frac{T^{\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\omega_1 + \theta_1 + 1)} - \sum_{i=1}^{n} \lambda_i \beta_i [\frac{1}{\beta_i}]_{\mathsf{q}}$$
$$\times \frac{\Gamma_{\mathsf{q}}(\eta_i + 1 + \frac{\omega_1+\theta_1}{\beta_i}) \zeta_i^{\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\mu_i + \eta_i + 1 + \frac{\omega_1+\theta_1}{\beta_i})\Gamma_{\mathsf{q}}(\omega_1 + \theta_1 + 1)} \Big) - \frac{\varrho^{\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\omega_1 + \theta_1 + 1)} \Big)$$
$$\times \Big(\mathsf{f}_1(s) - \phi_{1\mathsf{m}}^*(s) - \Pi_1 \xi_{1\mathsf{m}}^*(s) - \chi_1 \rho_{1\mathsf{m}}^*(s) - h_{1\mathsf{m}}^*(s) \Big) d_{\mathsf{q}} s,$$

and

$$B_2(\mathsf{u})(\varrho) = \int_0^{\varrho} \frac{(\varrho - \mathsf{q}s)^{\alpha_2+\omega_2+\theta_2-1}}{\Gamma_{\mathsf{q}}(\alpha_2 + \omega_2 + \theta_2)} \Big(\mathsf{f}_2(s) - \phi_{2\mathsf{u}}^*(s)$$
$$- \Pi_2 \xi_{2\mathsf{u}}^*(s) - \chi_2 \rho_{2\mathsf{u}}^*(s) - h_{2\mathsf{u}}^*(s) \Big) d_{\mathsf{q}} s + \frac{\varrho^{\theta_2}}{\Delta_2 \Gamma_{\mathsf{q}}(\theta_2 + 1)}$$
$$\times \Big(\sum_{i=1}^{n} \lambda_i \frac{\beta_i \zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)} \int_0^{\zeta_i} (\zeta_i^{\beta_i} - \mathsf{q}s^{\beta_i})^{(\mu_i - 1)} s^{\eta_i}$$
$$\times \Big(\int_0^{\zeta_i} \frac{(\zeta_i - \mathsf{q}s)^{\alpha_2+\omega_2+\theta_2-1}}{\Gamma_{\mathsf{q}}(\alpha_2 + \omega_2 + \theta_2)} \Big(\mathsf{f}_2(s) - \phi_{2\mathsf{u}}^*(s)$$
$$- \Pi_2 \xi_{2\mathsf{u}}^*(s) - \chi_2 \rho_{2\mathsf{u}}^*(s) - h_{2\mathsf{u}}^*(s) \Big) d_{\mathsf{q}} s \Big) d_{\mathsf{q}} s$$
$$- a_2 \int_0^{T} \frac{(T - \mathsf{q}s)^{\alpha_2+\omega_2+\theta_2-1}}{\Gamma_{\mathsf{q}}(\alpha_2 + \omega_2 + \theta_2)} \Big(\mathsf{f}_2(s) - \phi_{2\mathsf{u}}^*(s) - \Pi_2 \xi_{2\mathsf{u}}^*(s)$$
$$- \chi_2 \rho_{2\mathsf{u}}^*(s) - h_{2\mathsf{u}}^*(s) \Big) d_{\mathsf{q}} s \Big)$$
$$+ \Upsilon \Big(\frac{\varrho^{\theta_2}}{\Delta_2 \Gamma_{\mathsf{q}}(\theta_2 + 1)} \Big(a_2 \int_0^{T} \frac{(T - \mathsf{q}s)^{\theta_2 - 1}}{\Gamma_{\mathsf{q}}(\theta_2)}$$
$$\times \mathsf{u}(s) d_{\mathsf{q}} s - \sum_{i=1}^{n} \lambda_i \frac{\beta_i \zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)} \int_0^{\zeta_i} (\zeta_i^{\beta_i} - \mathsf{q}s^{\beta_i})^{(\mu_i - 1)} s^{\eta_i}$$
$$\times \Big(\int_0^{\zeta_i} \frac{(\zeta_i - \mathsf{q}s)^{\theta_2 - 1}}{\Gamma_{\mathsf{q}}(\theta_2)} \mathsf{u}(s) d_{\mathsf{q}} s \Big) d_{\mathsf{q}} s \Big) - \int_0^{\varrho} \frac{(\varrho - \mathsf{q}s)^{\theta_2 - 1}}{\Gamma_{\mathsf{q}}(\theta_1)} \mathsf{u}(s) d_{\mathsf{q}} s \Big)$$
$$+ \int_0^{\delta_1} \frac{(\delta_2 - \mathsf{q}s)^{\alpha_2 - 1}}{\Gamma_{\mathsf{q}}(\alpha_2)} \Big(\frac{\varrho^{\theta_2}}{\Delta_2 \Gamma_{\mathsf{q}}(\theta_2 + 1)} \Big(a_2 \frac{T^{\omega_2+\theta_2}}{\Gamma_{\mathsf{q}}(\omega_2 + \theta_2 + 1)}$$
$$- \sum_{i=1}^{n} \lambda_i \beta_i [\frac{1}{\beta_i}]_{\mathsf{q}} \frac{\Gamma_{\mathsf{q}}(\eta_i + 1 + \frac{\omega_2+\theta_2}{\beta_i}) \zeta_i^{\omega_2+\theta_2}}{\Gamma_{\mathsf{q}}(\mu_i + \eta_i + 1 + \frac{\omega_2+\theta_2}{\beta_i})\Gamma_{\mathsf{q}}(\omega_2 + \theta_2 + 1)} \Big)$$
$$- \frac{\varrho^{\omega_2+\theta_2}}{\Gamma_{\mathsf{q}}(\omega_2 + \theta_2 + 1)} \Big) \Big(\mathsf{f}_2(s) - \phi_{2\mathsf{u}}^*(s) - \Pi_2 \xi_{2\mathsf{u}}^*(s)$$
$$- \chi_2 \rho_{2\mathsf{u}}^*(s) - h_{2\mathsf{u}}^*(s) \Big) d_{\mathsf{q}} s.$$

Also, we have

$$D_{\mathsf{q}}^{\wp_1}(B_1 \mathsf{m})(\varrho) = \int_0^{\varrho} \frac{(\varrho - \mathsf{q}s)^{(-\wp_1)}}{\Gamma_{\mathsf{q}}(1 - \wp_1)} (D_{\mathsf{q}} B_1 \mathsf{u})(s) d_{\mathsf{q}} s, \wp_1 = \nu_2, \tau_2, \tag{3.3}$$

and

$$D_{\mathsf{q}}^{\wp_2}(B_2\mathsf{u})(\varrho) = \int_0^{\varrho} \frac{(\varrho - \mathsf{q}s)^{(-\wp_2)}}{\Gamma_{\mathsf{q}}(1-\wp_2)}(D_{\mathsf{q}}B_2\mathsf{m})(s)d_{\mathsf{q}}s, \wp_2 = \nu_1, \tau_1, \tag{3.4}$$

where

$$D_{\mathsf{q}}(B_1\mathsf{m})(\varrho) = \int_0^{\varrho} \frac{(\varrho-\mathsf{q}s)^{\alpha_1+\omega_1+\theta_1-2}}{\Gamma_{\mathsf{q}}(\alpha_1+\omega_1+\theta_1-1)}\Big(\mathsf{f}_1(s) - \phi_{1\mathsf{m}}^*(s)$$
$$- \Pi_1\xi_{1\mathsf{m}}^*(s) - \chi_1\rho_{1\mathsf{m}}^*(s) - h_{1\mathsf{m}}^*(s)\Big)d_{\mathsf{q}}s$$
$$+ \frac{\lceil\theta_1\rceil_{\mathsf{q}}\varrho^{\theta_1-1}}{\Delta_1\Gamma_{\mathsf{q}}(\theta_1+1)}\Big(\sum_{i=1}^{n}\lambda_i\frac{\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)}\int_0^{\zeta_i}(\zeta_i^{\beta_i} - \mathsf{q}s^{\beta_i})^{(\mu_i-1)}s^{\eta_i}$$
$$\times\Big(\int_0^{\zeta_i}\frac{(\zeta_i-\mathsf{q}s)^{\alpha_1+\omega_1+\theta_1-1}}{\Gamma_{\mathsf{q}}(\alpha_1+\omega_1+\theta_1)}\Big(\mathsf{f}_1(s)-\phi_{1\mathsf{m}}^*(s)$$
$$- \Pi_1\xi_{1\mathsf{m}}^*(s) - \chi_1\rho_{1\mathsf{m}}^*(s) - h_{1\mathsf{m}}^*(s)\Big)d_{\mathsf{q}}s\Big)d_{\mathsf{q}}s$$
$$- a_1\int_0^{T}\frac{(T-\mathsf{q}s)^{\alpha_1+\omega_1+\theta_1-1}}{\Gamma_{\mathsf{q}}(\alpha_1+\omega_1+\theta_1)}\Big(\mathsf{f}_1(s)-\phi_{1\mathsf{m}}^*(s)$$
$$- \Pi_1\xi_{1\mathsf{m}}^*(s) - \chi_1\rho_{1\mathsf{m}}^*(s) - h_{1\mathsf{m}}^*(s)\Big)d_{\mathsf{q}}s\Big)$$
$$+ \Omega\Big(\frac{\lceil\theta_1\rceil_{\mathsf{q}}\varrho^{\theta_1-1}}{\Delta_1\Gamma_{\mathsf{q}}(\theta_1+1)}\Big(a_1\int_0^{T}\frac{(T-\mathsf{q}s)^{\theta_1-1}}{\Gamma_{\mathsf{q}}(\theta_1)}\mathsf{m}(s)d_{\mathsf{q}}s$$
$$- \sum_{i=1}^{n}\lambda_i\frac{\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)}\int_0^{\zeta_i}(\zeta_i^{\beta_i}-\mathsf{q}s^{\beta_i})^{(\mu_i-1)}s^{\eta_i}$$
$$\times\Big(\int_0^{\zeta_i}\frac{(\zeta_i-\mathsf{q}s)^{\theta_1-1}}{\Gamma_{\mathsf{q}}(\theta_1)}\mathsf{m}(s)d_{\mathsf{q}}s\Big)d_{\mathsf{q}}s\Big)$$
$$- \int_0^{\varrho}\frac{(\varrho-\mathsf{q}s)^{\theta_1-2}}{\Gamma_{\mathsf{q}}(\theta_1-1)}\mathsf{m}(s)d_{\mathsf{q}}s\Big) + \int_0^{\delta_1}\frac{(\delta_1-\mathsf{q}s)^{\alpha_1-1}}{\Gamma_{\mathsf{q}}(\alpha_1)}$$
$$\times\Big(\frac{\lceil\theta_1\rceil_{\mathsf{q}}\varrho^{\theta_1-1}}{\Delta_1\Gamma_{\mathsf{q}}(\theta_1+1)}\Big(a_1\frac{T^{\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\omega_1+\theta_1+1)}$$
$$- \sum_{i=1}^{n}\lambda_i\beta_i[\frac{1}{\beta_i}]_{\mathsf{q}}\frac{\Gamma_{\mathsf{q}}(\eta_i+1+\frac{\omega_1+\theta_1}{\beta_i})}{\Gamma_{\mathsf{q}}(\mu_i+\eta_i+1+\frac{\omega_1+\theta_1}{\beta_i})\Gamma_{\mathsf{q}}(\omega_1+\theta_1+1)}$$
$$\times\zeta_i^{\omega_1+\theta_1}\Big) - \frac{\lceil\omega_1+\theta_1\rceil_{\mathsf{q}}\varrho^{\omega_1+\theta_1-1}}{\Gamma_{\mathsf{q}}(\omega_1+\theta_1+1)}\Big)\Big(\mathsf{f}_1(s)-\phi_{1\mathsf{m}}^*(s)$$
$$- \Pi_1\xi_{1\mathsf{m}}^*(s) - \chi_1\rho_{1\mathsf{m}}^*(s) - h_{1\mathsf{m}}^*(s)\Big)d_{\mathsf{q}}s,$$

and

$$D_{\mathsf{q}}(B_2\mathsf{u})(\varrho) = \int_0^{\varrho}\frac{(\varrho-\mathsf{q}s)^{\alpha_2+\omega_2+\theta_2-2}}{\Gamma_{\mathsf{q}}(\alpha_2+\omega_2+\theta_2-1)}\Big(\mathsf{f}_2(s)-\phi_{2\mathsf{u}}^*(s)$$
$$\times p - \Pi_2\xi_{2\mathsf{u}}^*(s) - \chi_2\rho_{2\mathsf{u}}^*(s) - h_{2\mathsf{u}}^*(s)\Big)d_{\mathsf{q}}s + \frac{\lceil\theta_2\rceil_{\mathsf{q}}\varrho^{\theta_2-1}}{\Delta_2\Gamma_{\mathsf{q}}(\theta_2+1)}$$
$$\times\Big(\sum_{i=1}^{n}\lambda_i\frac{\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)}\int_0^{\zeta_i}(\zeta_i^{\beta_i}-\mathsf{q}s^{\beta_i})^{(\mu_i-1)}s^{\eta_i}$$
$$\times\Big(\int_0^{\zeta_i}\frac{(\zeta_i-\mathsf{q}s)^{\alpha_2+\omega_2+\theta_2-1}}{\Gamma_{\mathsf{q}}(\alpha_2+\omega_2+\theta_2)}\Big(\mathsf{f}_2(s)-\phi_{2\mathsf{u}}^*(s)$$
$$- \Pi_2\xi_{2\mathsf{u}}^*(s) - \chi_2\rho_{2\mathsf{u}}^*(s) - h_{2\mathsf{u}}^*(s)\Big)d_{\mathsf{q}}s\Big)d_{\mathsf{q}}s$$
$$- a_2\int_0^{T}\frac{(T-\mathsf{q}s)^{\alpha_2+\omega_2+\theta_2-1}}{\Gamma_{\mathsf{q}}(\alpha_2+\omega_2+\theta_2)}\Big(\mathsf{f}_2(s)-\phi_{2\mathsf{u}}^*(s)$$
$$- \Pi_2\xi_{2\mathsf{u}}^*(s) - \chi_2\rho_{2\mathsf{u}}^*(s) - h_{2\mathsf{u}}^*(s)\Big)d_{\mathsf{q}}s\Big)$$
$$+ \Upsilon\Big(\frac{\lceil\theta_2\rceil_{\mathsf{q}}\varrho^{\theta_2-1}}{\Delta_2\Gamma_{\mathsf{q}}(\theta_2+1)}\Big(a_2\int_0^{T}\frac{(T-\mathsf{q}s)^{\theta_2-1}}{\Gamma_{\mathsf{q}}(\theta_1)}\mathsf{u}(s)d_{\mathsf{q}}s$$
$$- \sum_{i=1}^{n}\lambda_i\frac{\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)}\int_0^{\zeta_i}(\zeta_i^{\beta_i}-\mathsf{q}s^{\beta_i})^{(\mu_i-1)}s^{\eta_i}$$
$$\times\Big(\int_0^{\zeta_i}\frac{(\zeta_i-\mathsf{q}s)^{\theta_2-1}}{\Gamma_{\mathsf{q}}(\theta_2)}\mathsf{u}(s)d_{\mathsf{q}}s\Big)d_{\mathsf{q}}s\Big)$$
$$- \int_0^{\varrho}\frac{(\varrho-\mathsf{q}s)^{\theta_2-2}}{\Gamma_{\mathsf{q}}(\theta_2-1)}\mathsf{u}(s)d_{\mathsf{q}}s\Big) + \int_0^{\delta_2}\frac{(\delta_2-\mathsf{q}s)^{\alpha_2-1}}{\Gamma_{\mathsf{q}}(\alpha_2)}$$
$$\times\Big(\frac{\lceil\theta_2\rceil_{\mathsf{q}}\varrho^{\theta_2-1}}{\Delta_2\Gamma_{\mathsf{q}}(\theta_2+1)}\Big(a_2\frac{T^{\omega_2+\theta_2}}{\Gamma_{\mathsf{q}}(\omega_2+\theta_2+1)} - \sum_{i=1}^{n}\lambda_i\beta_i[\frac{1}{\beta_i}]_{\mathsf{q}}$$
$$\times\frac{\Gamma_{\mathsf{q}}(\eta_i+1+\frac{\omega_2+\theta_2}{\beta_i})\zeta_i^{\omega_2+\theta_2}}{\Gamma_{\mathsf{q}}(\mu_i+\eta_i+1+\frac{\omega_2+\theta_2}{\beta_i})\Gamma_{\mathsf{q}}(\omega_2+\theta_2+1)}\Big)$$
$$- \frac{\lceil\omega_2+\theta_2\rceil_{\mathsf{q}}\varrho^{\omega_2+\theta_2-1}}{\Gamma_{\mathsf{q}}(\omega_2+\theta_2+1)}\Big)$$
$$\times\Big(\mathsf{f}_2(s)-\phi_{2\mathsf{u}}^*(s) - \Pi_2\xi_{2\mathsf{u}}^*(s) - \chi_2\rho_{2\mathsf{u}}^*(s) - h_{2\mathsf{u}}^*(s)\Big)d_{\mathsf{q}}s.$$

The following hypothesis will be used for further analysis.

(F$_1$) $\Delta_1 = \frac{a_1T^{\theta_1}}{\Gamma_{\mathsf{q}}(\theta_1+1)} - \sum_{i=1}^{n}\lambda_i\mathcal{I}_{\mathsf{q}}^{\eta_i,\mu_i,\beta_i}\frac{\zeta_i^{\theta_1}}{\Gamma_{\mathsf{q}}(\theta_1+1)} \neq 0,$

$\Delta_2 = \frac{a_2T^{\theta_2}}{\Gamma_{\mathsf{q}}(\theta_2+1)} - \sum_{i=1}^{n}\lambda_i\mathcal{I}_{\mathsf{q}}^{\eta_i,\mu_i,\beta_i}\frac{\zeta_i^{\theta_2}}{\Gamma_{\mathsf{q}}(\theta_2+1)} \neq 0.$

(F$_2$) The functions $\xi_1, \xi_2, \rho_1, \rho_2, h_1, h_2 \in C(\mathcal{J}, \mathscr{R}^2)$, $\phi_1, \phi_2 \in C(\mathcal{J}, \mathscr{R})$, and $\mathsf{f}_1, \mathsf{f}_2 \in C(\mathcal{J})$.

(F$_3$) $\exists$ constant $b_l > 0$, $l = 1, 2, 3, 4$ in such way that $\forall\ \varrho \in \mathcal{J}$ and $\mathsf{m}_l, w_l \in \mathscr{R}$ $(l = 1, 2)$, we have

$$|\xi_1(\varrho, \mathsf{m}_1, \mathsf{m}_2) - \xi_1(\varrho, w_1, w_2)| \leqslant b_1(|\mathsf{m}_1 - w_1| + |\mathsf{m}_2 - w_2|),$$

$$|\rho_1(\varrho, \mathsf{m}_1, \mathsf{m}_2) - \rho_1(\varrho, w_1, w_2)| \leqslant b_2(|\mathsf{m}_1 - w_1| + |\mathsf{m}_2 - w_2|),$$

$$|h_1(\varrho, \mathsf{m}_1, \mathsf{m}_2) - h_1(\varrho, w_1, w_2)| \leqslant b_3(|\mathsf{m}_1 - w_1| + |\mathsf{m}_2 - w_2|),$$

and

$$|\phi_1(\varrho, \mathsf{m}_1) - \phi_1(\varrho, w_1)| \leqslant b_4(|\mathsf{m}_1 - w_1|).$$

(F$_4$) $\exists$ constant $p_k > 0$, $k = 1, 2, 3, 4$ in such way that $\forall\ \varrho \in \mathcal{J}$ and $\mathsf{u}_k, x_k \in \mathscr{R}$ $(k = 1, 2)$, we have

$$|\xi_2(\varrho, \mathsf{u}_1, \mathsf{u}_2) - \xi_2(\varrho, x_1, x_2)| \leqslant p_1(|\mathsf{u}_1 - x_1| + |\mathsf{u}_2 - x_2|),$$

$$|\rho_2(\varrho,\mathsf{u}_1,\mathsf{u}_2)-\rho_2(\varrho,x_1,x_2)|\leqslant p_2(|\mathsf{u}_1-x_1|+|\mathsf{u}_2-x_2|),$$

$$|h_2(\varrho,\mathsf{u}_1,\mathsf{u}_2)-h_2(\varrho,x_1,x_2)|\leqslant p_3(|\mathsf{u}_1-x_1|+|\mathsf{u}_2-x_2|),$$

and

$$|\phi_2(\varrho,\mathsf{u}_1)-\phi_2(\varrho,x_2)|\leqslant p_4(|\mathsf{u}_1-x_1|).$$

($\mathbf{F_5}$) $\exists$ constants $\nabla_l>0,\ l=1,2,3,4,5$ in such way that $\forall$ $\varrho\in\mathcal{J}$ and $\mathsf{m},w\in\mathscr{R}$, we have

$$|\xi_1(\varrho,\mathsf{m},w)|\leqslant\nabla_1,\quad |\rho_1(\varrho,\mathsf{m},w)|\leqslant\nabla_2,$$
$$|h_1(\varrho,\mathsf{m},w)|\leqslant\nabla_3,\quad |\phi_1(\varrho,\mathsf{m})|\leqslant\nabla_4,\quad |\mathsf{f}_1(\varrho)|\leqslant\nabla_5.$$

($\mathbf{F_6}$) $\exists$ constants $\Lambda_k>0,\ k=1,2,3,4,5$ in such way that $\forall$ $\varrho\in\mathcal{J}$ and $\mathsf{u},x\in\mathscr{R}$, we have

$$|\xi_2(\varrho,\mathsf{u},x)|\leqslant\Lambda_1,\quad |\rho_2(\varrho,\mathsf{u},x)|\leqslant\Lambda_2,$$
$$|h_2(\varrho,\mathsf{u},x)|\leqslant\Lambda_3,\quad |\phi_2(\varrho,\mathsf{u})|\leqslant\Lambda_4,\quad |\mathsf{f}_2(\varrho)|\leqslant\Lambda_5.$$

($\mathbf{F_7}$) $\exists\, y,z\in C(\mathcal{J},\mathscr{R}_+)$ is increasing and $\mathcal{M}_y,\mathcal{M}_z>0$, such that

$$\mathcal{I}_{\mathsf{q}}^{\alpha_1+\omega_1+\theta_1}[y(\varrho)]\leqslant\mathcal{M}_y y(\varrho),$$
$$\mathcal{I}_{\mathsf{q}}^{\alpha_2+\omega_2+\theta_2}[z(\varrho)]\leqslant\mathcal{M}_z z(\varrho).$$

The following notations will be used in the upcoming results.

$$\mathcal{U}_1=\Big(\Pi_1 b_1+\chi_1 b_2+\frac{\varrho^{\vartheta_1}}{\lceil\vartheta_1\rceil_{\mathsf{q}}\Gamma_{\mathsf{q}}(\varphi_1)}b_3+b_4\Big),$$
$$\mathcal{U}_3=(1-\Psi_1)(\mathcal{A}_1^{-1}),$$

where

$$\mathcal{A}_1=\Big(\varTheta_1+\frac{\pi_1}{\Gamma_{\mathsf{q}}(2-\nu_2)}+\frac{\pi_1}{\Gamma_{\mathsf{q}}(2-\tau_2)}\Big),$$
$$\Psi_1=\varPhi_1+\frac{\varpi_1}{\Gamma_{\mathsf{q}}(2-\nu_2)}+\frac{\varpi_1}{\Gamma_{\mathsf{q}}(2-\tau_2)},$$

with

$$\begin{aligned}\varTheta_1=&\Bigg[\frac{T^{\alpha_1+\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\alpha_1+\omega_1+\theta_1+1)}\\&+\frac{T^{\theta_1}}{|\Delta_1|\Gamma_{\mathsf{q}}(\theta_1+1)}\Big(\sum_{i=1}^{n}\frac{\lambda_i\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)}\\&\times\int_0^{\zeta_i}\frac{(\zeta_i^{\beta_i}-\mathsf{q}s^{\beta_i})^{(\mu_i-1)}s^{\eta_i}\zeta_i^{\alpha_1+\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\alpha_1+\omega_1+\theta_1+1)}d_{\mathsf{q}}s\\&+\frac{a_1T^{\alpha_1+\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\alpha_1+\omega_1+\theta_1+1)}\Big)\\&+\frac{\delta_1^{\alpha_1}}{\Gamma_{\mathsf{q}}(\alpha_1+1)}\Big(\frac{T^{\theta_1}}{|\Delta_1|\Gamma_{\mathsf{q}}(\theta_1+1)}\Big(\frac{a_1T^{\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\omega_1+\theta_1+1)}\\&+\sum_{i=1}^{n}\lambda_i\beta_i[\frac{1}{\beta_i}]_{\mathsf{q}}\frac{\Gamma_{\mathsf{q}}(\eta_i+1+\frac{\omega_1+\theta_1}{\beta_i})\zeta_i^{\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\mu_i+\eta_i+1+\frac{\omega_1+\theta_1}{\beta_i})\Gamma_{\mathsf{q}}(\omega_1+\theta_1+1)}\Big)\\&+\frac{T^{\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\omega_1+\theta_1+1)}\Big)\Bigg],\end{aligned}$$

$$\begin{aligned}\varPhi_1=&\,\Omega\Big(\frac{T^{\theta_1}}{|\Delta_1|\Gamma_{\mathsf{q}}(\theta_1+1)}\Big(\frac{a_1T^{\theta_1}}{\Gamma_{\mathsf{q}}(\theta_1+1)}+\sum_{i=1}^{n}\frac{\lambda_i\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)}\\&\times\int_0^{\zeta_i}\frac{(\zeta_i^{\beta_i}-\mathsf{q}s^{\beta_i})^{(\mu_i-1)}s^{\eta_i}\zeta_i^{\theta}}{\Gamma_{\mathsf{q}}(\theta_1+1)}d_{\mathsf{q}}s\Big)+\frac{T^{\theta_1}}{\Gamma_{\mathsf{q}}(\theta_1+1)}\Big),\end{aligned}$$

and

$$\begin{aligned}\pi_1=&\Bigg[\frac{T^{\alpha_1+\omega_1+\theta_1-1}}{\Gamma_{\mathsf{q}}(\alpha_1+\omega_1+\theta_1)}+\frac{\lceil\theta_1\rceil_{\mathsf{q}}T^{\theta_1-1}}{|\Delta_1|\Gamma_{\mathsf{q}}(\theta_1+1)}\\&\times\Big(\sum_{i=1}^{n}\lambda_i\frac{\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)}\int_0^{\zeta_i}\frac{(\zeta_i^{\beta_i}-\mathsf{q}s^{\beta_i})^{(\mu_i-1)}s^{\eta_i}\zeta_i^{\alpha_1+\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\alpha_1+\omega_1+\theta_1+1)}d_{\mathsf{q}}s\\&+\frac{a_1T^{\alpha_1+\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\alpha_1+\omega_1+\theta_1+1)}\Big)\\&+\frac{\delta_1^{\alpha_1}}{\Gamma_{\mathsf{q}}(\alpha_1+1)}\Big(\frac{\lceil\theta_1\rceil_{\mathsf{q}}T^{\theta_1-1}}{|\Delta_1|\Gamma_{\mathsf{q}}(\theta_1+1)}\Big(\frac{a_1T^{\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\omega_1+\theta_1+1)}\\&+\sum_{i=1}^{n}\lambda_i\beta_i[\frac{1}{\beta_i}]_{\mathsf{q}}\frac{\Gamma_{\mathsf{q}}(\eta_i+1+\frac{\omega_1+\theta_1}{\beta_i})\zeta_i^{\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\mu_i+\eta_i+1+\frac{\omega_1+\theta_1}{\beta_i})\Gamma_{\mathsf{q}}(\omega_1+\theta_1+1)}\Big)\\&+\frac{\lceil\omega_1+\theta_1\rceil_{\mathsf{q}}T^{\omega_1+\theta_1-1}}{\Gamma_{\mathsf{q}}(\omega_1+\theta_1+1)}\Big)\Bigg],\end{aligned}$$

$$\begin{aligned}\varpi_1=&\,\Omega\Big(\frac{\lceil\theta_1\rceil_{\mathsf{q}}T^{\theta_1-1}}{|\Delta_1|\Gamma_{\mathsf{q}}(\theta_1+1)}\Big(\frac{a_1T^{\theta_1}}{\Gamma_{\mathsf{q}}(\theta_1+1)}+\sum_{i=1}^{n}\frac{\lambda_i\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)}\\&\times\int_0^{\zeta_i}\frac{(\zeta_i^{\beta_i}-\mathsf{q}s^{\beta_i})^{(\mu_i-1)}s^{\eta_i}\zeta_i^{\theta_1}}{\Gamma_{\mathsf{q}}(\theta_1+1)}d_{\mathsf{q}}s\Big)+\frac{T^{\theta_1-1}}{\Gamma_{\mathsf{q}}(\theta_1)}\Big).\end{aligned}$$

Similarly,

$$\mathcal{U}_2=\Big(\Pi_2 p_1+\chi_2 p_2+\frac{\varrho^{\vartheta_2}}{\lceil\vartheta_2\rceil_{\mathsf{q}}\Gamma_{\mathsf{q}}(\varphi_2)}p_3+p_4\Big),$$
$$\mathcal{U}_4=(1-\Psi_2)(\mathcal{A}_2^{-1}),$$

where

$$\mathcal{A}_2=\Big(\varTheta_2+\frac{\pi_2}{\Gamma_{\mathsf{q}}(2-\nu_1)}+\frac{\pi_2}{\Gamma_{\mathsf{q}}(2-\tau_1)}\Big),$$
$$\Psi_2=\varPhi_2+\frac{\varpi_2}{\Gamma_{\mathsf{q}}(2-\nu_1)}+\frac{\varpi_2}{\Gamma_{\mathsf{q}}(2-\tau_1)},$$

with

$$\varTheta_2=\Bigg[\frac{T^{\alpha_2+\omega_2+\theta_2}}{\Gamma_{\mathsf{q}}(\alpha_2+\omega_2+\theta_2+1)}+\frac{T^{\theta_2}}{|\Delta_2|\Gamma_{\mathsf{q}}(\theta_2+1)}$$

$$\times\Big(\sum_{i=1}^{n}\frac{\lambda_i\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)}\int_0^{\zeta_i}\frac{(\zeta_i^{\beta_i}-\mathsf{q}s^{\beta_i})^{(\mu_i-1)}s^{\eta_i}\zeta_i^{\alpha_1+\omega_2+\theta_2}}{\Gamma_{\mathsf{q}}(\alpha_2+\omega_2+\theta_2+1)}d_{\mathsf{q}}s$$
$$+\frac{a_2T^{\alpha_2+\omega_2+\theta_2}}{\Gamma_{\mathsf{q}}(\alpha_2+\omega_2+\theta_2+1)}\Big)$$
$$+\frac{\delta_2^{\alpha_2}}{\Gamma_{\mathsf{q}}(\alpha_2+1)}\Big(\frac{T^{\theta_2}}{|\Delta_2|\Gamma_{\mathsf{q}}(\theta_2+1)}\Big(\frac{a_2T^{\omega_2+\theta_2}}{\Gamma_{\mathsf{q}}(\omega_2+\theta_2+1)}$$
$$+\sum_{i=1}^{n}\lambda_i\beta_i[\frac{1}{\beta_i}]_{\mathsf{q}}\frac{\Gamma_{\mathsf{q}}(\eta_i+1+\frac{\omega_1+\theta_1}{\beta_i})\zeta_i^{\omega_2+\theta_2}}{\Gamma_{\mathsf{q}}(\mu_i+\eta_i+1+\frac{\omega_2+\theta_2}{\beta_i})\Gamma_{\mathsf{q}}(\omega_2+\theta_2+1)}\Big)$$
$$+\frac{T^{\omega_2+\theta_2}}{\Gamma_{\mathsf{q}}(\omega_2+\theta_2+1)}\Big)\Big],$$
$$\varPhi_2=\Upsilon\Big(\frac{T^{\theta_2}}{|\Delta_2|\Gamma_{\mathsf{q}}(\theta_2+1)}\Big(\frac{a_2T^{\theta_2}}{\Gamma_{\mathsf{q}}(\theta_2+1)}+\sum_{i=1}^{n}\frac{\lambda_i\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)}$$
$$\times\int_0^{\zeta_i}\frac{(\zeta_i^{\beta_i}-\mathsf{q}s^{\beta_i})^{(\mu_i-1)}s^{\eta_i}\zeta_i^{\theta}}{\Gamma_{\mathsf{q}}(\theta_2+1)}d_{\mathsf{q}}s\Big)+\frac{T^{\theta_2}}{\Gamma_{\mathsf{q}}(\theta_2+1)}\Big),$$

and

$$\pi_2=\Big[\frac{T^{\alpha_2+\omega_2+\theta_2-1}}{\Gamma_{\mathsf{q}}(\alpha_2+\omega_2+\theta_2)}+\frac{\lceil\theta_2\rceil_{\mathsf{q}}T^{\theta_2-1}}{|\Delta_2|\Gamma_{\mathsf{q}}(\theta_2+1)}$$
$$\times\Big(\sum_{i=1}^{n}\frac{\lambda_i\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)}\int_0^{\zeta_i}\frac{(\zeta_i^{\beta_i}-\mathsf{q}s^{\beta_i})^{(\mu_i-1)}s^{\eta_i}\zeta_i^{\alpha_2+\omega_2+\theta_2}}{\Gamma_{\mathsf{q}}(\alpha_2+\omega_2+\theta_2+1)}d_{\mathsf{q}}s$$
$$+\frac{a_2T^{\alpha_2+\omega_2+\theta_2}}{\Gamma_{\mathsf{q}}(\alpha_2+\omega_2+\theta_2+1)}\Big)$$
$$+\frac{\delta_2^{\alpha_2}}{\Gamma_{\mathsf{q}}(\alpha_2+1)}\Big(\frac{\lceil\theta_2\rceil_{\mathsf{q}}T^{\theta_2-1}}{|\Delta_2|\Gamma_{\mathsf{q}}(\theta_2+1)}\Big(\frac{a_2T^{\omega_2+\theta_2}}{\Gamma_{\mathsf{q}}(\omega_2+\theta_2+1)}$$
$$+\sum_{i=1}^{n}\lambda_i\beta_i[\frac{1}{\beta_i}]_{\mathsf{q}}\frac{\Gamma_{\mathsf{q}}(\eta_i+1+\frac{\omega_2+\theta_2}{\beta_i})\zeta_i^{\omega_1+\theta_2}}{\Gamma_{\mathsf{q}}(\mu_i+\eta_i+1+\frac{\omega_2+\theta_2}{\beta_i})\Gamma_{\mathsf{q}}(\omega_2+\theta_2+1)}\Big)$$
$$+\frac{\lceil\omega_2+\theta_2\rceil_{\mathsf{q}}T^{\omega_2+\theta_2-1}}{\Gamma_{\mathsf{q}}(\omega_2+\theta_2+1)}\Big)\Big],$$
$$\varpi_2=\Upsilon\Big(\frac{\lceil\theta_2\rceil_{\mathsf{q}}T^{\theta_2-1}}{|\Delta_2|\Gamma_{\mathsf{q}}(\theta_2+1)}\Big(\frac{a_2T^{\theta_2}}{\Gamma_{\mathsf{q}}(\theta_2+1)}+\sum_{i=1}^{n}\frac{\lambda_i\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)}$$
$$\times\int_0^{\zeta_i}\frac{(\zeta_i^{\beta_i}-\mathsf{q}s^{\beta_i})^{(\mu_i-1)}s^{\eta_i}\zeta_i^{\theta_2}}{\Gamma_{\mathsf{q}}(\theta_2+1)}d_{\mathsf{q}}s\Big)+\frac{T^{\theta_2-1}}{\Gamma_{\mathsf{q}}(\theta_2)}\Big).$$

Theorem 3.1. *Assume that* $(\mathbf{F_2})$–$(\mathbf{F_4})$ *and*

$$\begin{cases}\mathcal{U}_1<\mathcal{U}_3,\\ \mathcal{U}_2<\mathcal{U}_4\end{cases}\tag{3.5}$$

hold. Then q-$\mathcal{FJCS}$ (1.1) *has a unique solution.*

Proof. Consider

$$v_1\geq\frac{\mathcal{A}_1\Big(\Pi_1+\chi_1+\frac{\varrho^{\vartheta_1}}{\lceil\vartheta_1\rceil_{\mathsf{q}}\Gamma_{\mathsf{q}}(\varphi_1)}+2\Big)\mathcal{V}}{1-\Big(\mathcal{A}_1(\mathcal{U}_1)+\Psi_1\Big)},\tag{3.6}$$

and $\mathcal{V}=\max\{\mathcal{V}_o,\ o=1,2,3,4,5\}$. Here $\mathcal{V}_o$ are given by

$$\mathcal{V}_1=sup_{\varrho\in\mathcal{J}}|\xi_{1\mathsf{m}}(\varrho,0,0)|,\ \mathcal{V}_2=sup_{\varrho\in\mathcal{J}}|\rho_{1\mathsf{m}}(\varrho,0,0)|,$$
$$\mathcal{V}_3=sup_{\varrho\in\mathcal{J}}|h_{1\mathsf{m}}(\varrho,0,0)|,\mathcal{V}_4=sup_{\varrho\in\mathcal{J}}|\phi_{1\mathsf{m}}(\varrho,0)|,$$

and $\mathcal{V}_5=sup_{\varrho\in\mathcal{J}}|\mathsf{f}_1(\varrho)|$. Then, we show that $B\mathcal{W}_{v_1}\subset\mathcal{W}_{v_1}$, where

$$\mathcal{W}_{v_1}=\{\mathsf{m}\in\mathsf{X}:\|\mathsf{m}\|_{\mathsf{X}}\leqslant v_1\}.$$

Using $(\mathbf{F_3})$, we get

$$\begin{aligned}|\xi_{1\mathsf{m}}^{*,v_2}(\varrho)|&=|\xi_1(\varrho,\mathsf{m}(\varrho),{}^CD_{\mathsf{q}}^{v_2}\mathsf{m}(\varrho))|\\&\leqslant|\xi_1(\varrho,\mathsf{m}(\varrho),{}^CD_{\mathsf{q}}^{v_2}\mathsf{m}(\varrho))-\xi_1(\varrho,0,0)|+|\xi_1(\varrho,0,0)|\\&\leqslant b_1(\|\mathsf{m}\|+\|{}^CD_{\mathsf{q}}^{v_2}\mathsf{m}\|)+\mathcal{V}_1\\&\leqslant b_1\|\mathsf{m}\|_{\mathsf{X}}+\mathcal{V}_1\leqslant b_1v_1+\mathcal{V}_1,\end{aligned}\tag{3.7}$$

$$\begin{aligned}|\rho_{1\mathsf{m}}^{*,\tau_2}(\varrho)|&=|\mathsf{m}(\varrho,\mathsf{m}(\varrho),{}^CD_{\mathsf{q}}^{\tau_2}\mathsf{m}(\varrho))|\\&\leqslant|\rho_1(\varrho,\mathsf{m}(\varrho),{}^CD_{\mathsf{q}}^{\tau_2}\mathsf{m}(\varrho))-\rho_1(\varrho,0,0)|+|\rho_1(\varrho,0,0)|\\&\leqslant b_2(\|\mathsf{m}\|+\|{}^CD_{\mathsf{q}}^{\tau_2}\mathsf{m}\|)+\mathcal{V}_2\\&\leqslant b_2\|\mathsf{m}\|_{\mathsf{X}}+\mathcal{V}_2\leqslant b_2v_1+\mathcal{V}_2,\end{aligned}\tag{3.8}$$

$$\begin{aligned}|h_{1\mathsf{m}}^{*,v_2}(\varrho)|&=\int_0^{\varrho}\frac{(\varrho-\mathsf{q}s)^{\vartheta_1-1}}{\Gamma_{\mathsf{q}}(\varphi_1)}|h_1(s,\mathsf{m}(s),{}^CD_{\mathsf{q}}^{v_2}\mathsf{m}(s))|d_{\mathsf{q}}(s)\\&\leqslant\frac{\varrho^{\vartheta_1}}{\lceil\vartheta_1\rceil_{\mathsf{q}}\Gamma_{\mathsf{q}}(\varphi_1)}|h_1(s,\mathsf{m}(s),{}^CD_{\mathsf{q}}^{v_2}\mathsf{m}(s))-h_1(s,0,0)|\\&\quad+|h_1(s,0,0)|\\&\leqslant\frac{\varrho^{\vartheta_1}}{\lceil\vartheta_1\rceil_{\mathsf{q}}\Gamma_{\mathsf{q}}(\varphi_1)}b_3(\|\mathsf{m}\|+\|{}^CD_{\mathsf{q}}^{v_2}\mathsf{m}\|)+\mathcal{V}_3\\&\leqslant\frac{\varrho^{\vartheta_1}}{\lceil\vartheta_1\rceil_{\mathsf{q}}\Gamma_{\mathsf{q}}(\varphi_1)}b_3\|\mathsf{m}\|_{\mathsf{X}}+\mathcal{V}_3\\&\leqslant\frac{\varrho^{\vartheta_1}}{\lceil\vartheta_1\rceil_{\mathsf{q}}\Gamma_{\mathsf{q}}(\varphi_1)}b_3v_1+\mathcal{V}_3,\end{aligned}\tag{3.9}$$

and

$$\begin{aligned}|\phi_{1\mathsf{m}}^{*}(\varrho)|&=|\phi_1(\varrho,\mathsf{m}(\varrho))|\\&\leqslant|\phi_1(\varrho,\mathsf{m}(\varrho))-\phi_1(\varrho,0)|+|\phi_1(\varrho,0)|\\&\leqslant b_4(\|\mathsf{m}\|)+\mathcal{V}_4\\&\leqslant b_4\|\mathsf{m}\|_{\mathsf{X}}+\mathcal{V}_4\\&\leqslant b_4v_1+\mathcal{V}_4.\end{aligned}\tag{3.10}$$

From (3.7)–(3.10), we can write

$$
\begin{aligned}
\|B_1(\mathsf{m})\| \leqslant& \Big[\Big(\Pi_1 b_1 + \chi_1 b_2 + \frac{\varrho^{\vartheta_1}}{\lceil\vartheta_1\rceil_{\mathsf{q}}\Gamma_{\mathsf{q}}(\varphi_1)} b_3 + b_4\Big)v_1 \\
&+ \Big(\Pi_1 + \chi_1 + \frac{\varrho^{\vartheta_1}}{\lceil\vartheta_1\rceil_{\mathsf{q}}\Gamma_{\mathsf{q}}(\varphi_1)} + 2\Big)\mathcal{V}\Big] \\
&\times \Big[\frac{T^{\alpha_1+\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\alpha_1+\omega_1+\theta_1+1)} + \frac{T^{\theta_1}}{|\Delta_1|\Gamma_{\mathsf{q}}(\theta_1+1)} \\
&\times \Big(\sum_{i=1}^{n}\lambda_i \frac{\beta_i \zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)} \int_0^{\zeta_i} (\zeta_i^{\beta_i} - \mathsf{q}s^{\beta_i})^{(\mu_i-1)} s^{\eta_i} \\
&\times \Big(\frac{\zeta_i^{\alpha_1+\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\alpha_1+\omega_1+\theta+1)}\Big) d_{\mathsf{q}}s \\
&+ \frac{a_1 T^{\alpha_1+\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\alpha_1+\omega_1+\theta_1+1)}\Big) + \frac{\delta_1^{\alpha_1}}{\Gamma_{\mathsf{q}}(\alpha_1+1)}\Big(\frac{T^{\theta_1}}{|\Delta_1|\Gamma_{\mathsf{q}}(\theta_1+1)} \\
&\times \Big(\frac{a_1 T^{\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\omega_1+\theta_1+1)} + \sum_{i=1}^{n}\lambda_i\beta_i[\frac{1}{\beta_i}]_{\mathsf{q}} \\
&\times \frac{\Gamma_{\mathsf{q}}(\eta_i + 1 + \frac{\omega_1+\theta_1}{\beta_i})}{\Gamma_{\mathsf{q}}(\mu_i+\eta_i+1+\frac{\omega_1+\theta_1}{\beta_i})\Gamma_{\mathsf{q}}(\omega_1+\theta_1+1)} \zeta_i^{\omega_1+\theta_1}\Big) \\
&+ \frac{T^{\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\omega_1+\theta_1+1)}\Big)\Big] + \Omega\Big(\frac{T^{\theta_1}}{|\Delta_1|\Gamma_{\mathsf{q}}(\theta_1+1)}\Big(\frac{a_1 T^{\theta_1}}{\Gamma_{\mathsf{q}}(\theta_1+1)} \\
&+ \sum_{i=1}^{n}\lambda_i \frac{\beta_i \zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)} \int_0^{\zeta_i} (\zeta_i^{\beta_i} - \mathsf{q}s^{\beta_i})^{(\mu_i-1)} s^{\eta_i} \\
&\times \Big(\frac{\zeta_i^{\theta}}{\Gamma_{\mathsf{q}}(\theta_1+1)}\Big) d_{\mathsf{q}}s\Big) + \frac{T^{\theta_1}}{\Gamma_{\mathsf{q}}(\theta_1+1)}\Big) \\
=& \Big(\Theta_1(\mathcal{U}_1) + \Phi_1\Big)v_1 + \Big(\Pi_1 + \chi_1 + \frac{\varrho^{\vartheta_1}}{\lceil\vartheta_1\rceil_{\mathsf{q}}\Gamma_{\mathsf{q}}(\varphi_1)} + 2\Big)\mathcal{V}.
\end{aligned}
$$

Also, we can observe that

$$
\begin{aligned}
\|{}^{C}D_{\mathsf{q}} B_1(\mathsf{m})\| \leqslant& \Big[\Big(\Pi_1 b_1 + \chi_1 b_2 + \frac{\varrho^{\vartheta_1}}{\lceil\vartheta_1\rceil_{\mathsf{q}}\Gamma_{\mathsf{q}}(\varphi_1)} b_3 + b_4\Big)v_1 \\
&+ \Big(\Pi_1 + \chi_1 + \frac{\varrho^{\vartheta_1}}{\lceil\vartheta_1\rceil_{\mathsf{q}}\Gamma_{\mathsf{q}}(\varphi_1)} + 2\Big)\mathcal{V}\Big] \\
&\times \Big[\frac{T^{\alpha_1+\omega_1+\theta_1-1}}{\Gamma_{\mathsf{q}}(\alpha_1+\omega_1+\theta_1)} + \frac{\lceil\theta_1\rceil_{\mathsf{q}} T^{\theta_1-1}}{|\Delta_1|\Gamma_{\mathsf{q}}(\theta_1+1)}\Big(\sum_{i=1}^{n}\lambda_i \frac{\beta_i \zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)} \\
&\times \int_0^{\zeta_i} (\zeta_i^{\beta_i} - \mathsf{q}s^{\beta_i})^{(\mu_i-1)} s^{\eta_i} \times \Big(\frac{\zeta_i^{\alpha_1+\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\alpha_1+\omega_1+\theta_1+1)}\Big) d_{\mathsf{q}}s \\
&+ \frac{a_1 T^{\alpha_1+\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\alpha_1+\omega_1+\theta_1+1)}\Big) + \frac{\delta_1^{\alpha_1}}{\Gamma_{\mathsf{q}}(\alpha_1+1)}\Big(\frac{\lceil\theta_1\rceil_{\mathsf{q}} T^{\theta_1-1}}{|\Delta_1|\Gamma_{\mathsf{q}}(\theta_1+1)} \\
&\times \Big(\frac{a_1 T^{\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\omega_1+\theta_1+1)} + \sum_{i=1}^{n}\lambda_i\beta_i[\frac{1}{\beta_i}]_{\mathsf{q}} \\
&\times \frac{\Gamma_{\mathsf{q}}(\eta_i + 1 + \frac{\omega_1+\theta_1}{\beta_i})}{\Gamma_{\mathsf{q}}(\mu_i+\eta_i+1+\frac{\omega_1+\theta_1}{\beta_i})\Gamma_{\mathsf{q}}(\omega_1+\theta_1+1)} \zeta_i^{\omega_1+\theta_1}\Big) \\
&+ \frac{\lceil\omega_1+\theta_1\rceil_{\mathsf{q}} T^{\omega_1+\theta_1-1}}{\Gamma_{\mathsf{q}}(\omega_1+\theta_1+1)}\Big)\Big] + \Omega\Big(\frac{\lceil\theta_1\rceil_{\mathsf{q}} T^{\theta_1-1}}{|\Delta_1|\Gamma_{\mathsf{q}}(\theta_1+1)}\Big(\frac{a_1 T^{\theta_1}}{\Gamma_{\mathsf{q}}(\theta+1)} \\
&+ \sum_{i=1}^{n}\lambda_i \frac{\beta_i \zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)} \int_0^{\zeta_i} (\zeta_i^{\beta_i} - \mathsf{q}s^{\beta_i})^{(\mu_i-1)} s^{\eta_i} \\
&\times \Big(\frac{\zeta_i^{\theta}}{\Gamma_{\mathsf{q}}(\theta+1)}\Big) d_{\mathsf{q}}s\Big) + \frac{T^{\theta_1-1}}{\Gamma_{\mathsf{q}}(\theta_1)}\Big) \\
=& \Big(\pi_1(\mathcal{U}_1) + \varpi_1\Big)v_1 + \Big(\Pi_1 + \chi_1 + \frac{\varrho^{\vartheta_1}}{\lceil\vartheta_1\rceil_{\mathsf{q}}\Gamma_{\mathsf{q}}(\varphi_1)} + 2\Big)\mathcal{V}.
\end{aligned}
$$

Using (3.3), we get

$$
\begin{aligned}
\|{}^{C}D_{\mathsf{q}}^{\nu_2} B_1(\mathsf{m})\| \leqslant& \frac{T^{1-\nu_2}}{\Gamma_{\mathsf{q}}(2-\nu_2)}\Big(\Big[\pi_1(\mathcal{U}_1) + \varpi_1\Big]v_1 \\
&+ \Big(\Pi_1 + \chi_1 + \frac{\varrho^{\vartheta_1}}{\lceil\vartheta_1\rceil_{\mathsf{q}}\Gamma_{\mathsf{q}}(\varphi_1)} + 2\Big)\mathcal{V}\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
\|{}^{C}D_{\mathsf{q}}^{\tau_2} B_1(\mathsf{m})\| \leqslant& \frac{T^{1-\tau_2}}{\Gamma_{\mathsf{q}}(2-\tau_2)}\Big(\Big[\pi_1(\mathcal{U}_1) + \varpi_1\Big]v_1 \\
&+ \Big(\Pi_1 + \chi_1 + \frac{\varrho^{\vartheta_1}}{\lceil\vartheta_1\rceil_{\mathsf{q}}\Gamma_{\mathsf{q}}(\varphi_1)} + 2\Big)\mathcal{V}\Big).
\end{aligned}
$$

In consequence, we obtain

$$
\begin{aligned}
&\|B_1(\mathsf{m})\|_{B} \\
&= \|B_1(\mathsf{m})\| + \|{}^{C}D_{\mathsf{q}}^{\nu_2} B_1(\mathsf{m})\| + \|{}^{C}D_{\mathsf{q}}^{\tau_2} B_1(\mathsf{m})\| \\
&\leqslant \Big[\Big(\Theta_1 + \frac{\pi_1}{\Gamma_{\mathsf{q}}(2-\nu_2)} + \frac{\pi_1}{\Gamma_{\mathsf{q}}(2-\tau_2)}\Big)(\mathcal{U}_1) + \Psi_1\Big]v_1 \\
&\quad + \Big(\Pi_1 + \chi_1 + \frac{\varrho^{\vartheta_1}}{\lceil\vartheta_1\rceil_{\mathsf{q}}\Gamma_{\mathsf{q}}(\varphi_1)} + 2\Big)\mathcal{V} \\
&= \Big[\mathcal{A}_1(\mathcal{U}_1) + \Psi_1\Big]v_1 + \mathcal{A}_1\Big(\Pi_1 + \chi_1 + \frac{\varrho^{\vartheta_1}}{\lceil\vartheta_1\rceil_{\mathsf{q}}\Gamma_{\mathsf{q}}(\varphi_1)} + 2\Big)\mathcal{V} \\
&\leqslant v_1,
\end{aligned}
$$

which means that $B\mathcal{W}_{v_1} \subset \mathcal{W}_{v_1}$. For $\mathsf{m}, w \in \mathcal{W}_{v_1}$ and by (**F$_3$**), we have

$$
\begin{aligned}
&|B_1\mathsf{m}(\varrho) - B_1 w(\varrho)| \\
&= \int_0^{\varrho} \frac{(\varrho - \mathsf{q}s)^{\alpha_1+\omega_1+\theta_1-1}}{\Gamma_{\mathsf{q}}(\alpha_1+\omega_1+\theta_1)}\Big(|\phi_{1\mathsf{m}}^{*}(s) - \phi_{1w}^{*}(s)| + \Pi_1|\xi_{1\mathsf{m}}^{*}(s) \\
&\quad - \xi_{1w}^{*}(s)| + \chi_1|\rho_{1\mathsf{m}}^{*}(s) - \rho_{1w}^{*}(s)| + |h_{1\mathsf{m}}^{*}(s) - h_{1w}^{*}(s)|\Big) d_{\mathsf{q}}s \\
&\quad + \frac{\varrho^{\theta_1}}{\Delta_1\Gamma_{\mathsf{q}}(\theta_1+1)}\Big(\sum_{i=1}^{n}\lambda_i \frac{\beta_i \zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)} \int_0^{\zeta_i} (\zeta_i^{\beta_i} - \mathsf{q}s^{\beta_i})^{(\mu_i-1)} s^{\eta_i} \\
&\quad \times \Big(\int_0^{\zeta_i} \frac{(\zeta_i - \mathsf{q}s)^{\alpha_1+\omega_1+\theta_1-1}}{\Gamma_{\mathsf{q}}(\alpha_1+\omega_1+\theta_1)}\Big(|\phi_{1\mathsf{m}}^{*}(s) - \phi_{1w}^{*}(s)| + \Pi_1|\xi_{1\mathsf{m}}^{*}(s) \\
&\quad - \xi_{1w}^{*}(s)| + \chi_1|\rho_{1\mathsf{m}}^{*}(s) - \rho_{1w}^{*}(s)| + |h_{1\mathsf{m}}^{*}(s) - h_{1w}^{*}(s)|\Big) d_{\mathsf{q}}s\Big) d_{\mathsf{q}}s \\
&\quad + a_1 \int_0^{T} \frac{(T - \mathsf{q}s)^{\alpha_1+\omega_1+\theta_1-1}}{\Gamma_{\mathsf{q}}(\alpha_1+\omega_1+\theta_1)}\Big(|\phi_{1\mathsf{m}}^{*}(s) - \phi_{1w}^{*}(s)| + \Pi_1|\xi_{1\mathsf{m}}^{*}(s)
\end{aligned}
$$

$$- \xi^*_{1w}(s)| + \chi_1|\rho^*_{1m}(s) - \rho^*_{1w}(s)| + |h^*_{1m}(s) - h^*_{1w}(s)|\Big)d_q s\Big)$$
$$+ \int_0^{\delta_1} \frac{(\delta_1 - qs)^{\alpha_1 - 1}}{\Gamma_q(\alpha_1)}\Big(\frac{\varrho^{\theta_1}}{\Delta_1\Gamma_q(\theta_1+1)}\Big(a_1\frac{T^{\omega_1+\theta_1}}{\Gamma_q(\omega_1+\theta_1+1)}$$
$$+ \sum_{i=1}^{n} \lambda_i\beta_i[\frac{1}{\beta_i}]_q \frac{\Gamma_q(\eta_i + 1 + \frac{\omega_1+\theta_1}{\beta_i})\zeta_i^{\omega_1+\theta_1}}{\Gamma_q(\mu_i+\eta_i+1+\frac{\omega_1+\theta_1}{\beta_i})\Gamma_q(\omega_1+\theta_1+1)}\Big)$$
$$+ \frac{\varrho^{\omega_1+\theta_1}}{\Gamma_q(\omega_1+\theta_1+1)}\Big)\Big(|\phi^*_{1m}(s) - \phi^*_{1w}(s)| + \Pi_1|\xi^*_{1m}(s) - \xi^*_{1w}(s)|$$
$$+ \chi|\rho^*_{1m}(s) - \rho^*_{1w}(s)| + |h^*_{1m}(s) - h^*_{1w}(s)|\Big)d_q s.$$

So, we obtain

$$\|B_1(m) - B_1(w)\| \leqslant \Big(\Theta_1(\mathcal{U}_1) + \Phi_1\Big)\|m - w\|_X.$$

On the other hand, we have

$$\|{}^C D_q B_1(m) - {}^C D_q B_1(w)\| \leqslant \Big(\pi_1(\mathcal{U}_1) + \varpi_1\Big)\|m - w\|_X.$$

By (3.3), we have

$$\|{}^C D_q^{\nu_2} B_1(m) - {}^C D_q^{\nu_2} B_1(w)\| \leqslant \frac{T^{1-\nu_2}}{\Gamma_q(2-\nu_2)}\Big(\pi_1(\mathcal{U}_1) + \varpi_1\Big)\|m - w\|_X,$$

$$\|{}^C D_q^{\tau_2} B_1(m) - {}^C D_q^{\tau_2} B_1(w)\| \leqslant \frac{T^{1-\tau_2}}{\Gamma_q(2-\tau_2)}\Big(\pi_1(\mathcal{U}_1) + \varpi_1\Big)\|m - w\|_X.$$

Consequently, we obtain

$$\|B_1(m) - B_1(w)\|_X \leqslant \Big(\mathcal{A}_1(\mathcal{U}_1) + \Psi_1\Big)\|m - w\|_X.$$

By applying the same procedure and ($\mathbf{F_4}$), we get

$$\|B_2(u) - B_2(x)\|_X \leqslant \Big(\mathcal{A}_2(\mathcal{U}_2) + \Psi_2\Big)\|u - x\|_X.$$

The solution of q-$\mathcal{FJCS}$ (1.1) has a unique FP. □

We show the existence of solutions for q-$\mathcal{FJCS}$ (1.1) by employing Lemma 2.2. The following notations will be used in the next theorem.

$$\Psi_1 = \Phi_1 + \frac{\varpi_1}{\Gamma_q(2-\nu_2)} + \frac{\varpi_1}{\Gamma_q(2-\tau_2)},$$
$$\Psi_2 = \Phi_2 + \frac{\varpi_2}{\Gamma_q(2-\nu_1)} + \frac{\varpi_2}{\Gamma_q(2-\tau_1)},$$

$$\Psi = max\{\Psi_1, \Psi_2\}. \tag{3.11}$$

Theorem 3.2. *Let* $f_1, f_2, \phi_1, \phi_2, \xi_1, \xi_2, \rho_1, \rho_2, h_1, h_2$ *be continuous and satisfy* ($\mathbf{F_2}$), ($\mathbf{F_5}$), *and* ($\mathbf{F_6}$). *If* $\Psi < 1$, *where* Ψ *is given by* (3.11), *then the* q-$\mathcal{FJCS}$ *has at least one solution.*

Proof. From the continuity of f_1, f_2, ϕ_1, ϕ_2, ξ_1, ξ_2, ρ_1, ρ_2, h_1, h_2, and B are also continuous. Assume that the set $\mathcal{E}$ is bounded. Thus, for each $m, u \in \mathcal{E}$, we get

$$\mathcal{E}_\sigma = \{(m, u) \in X : \|m\|_X \leqslant \sigma,\ \|u\|_X \leqslant \sigma, \quad \sigma > 0\}.$$

Then, for all $m \in \mathcal{E}_\sigma$ and ($\mathbf{F_5}$), by using (3.2), we have

$$\|B_1(m)\| \leqslant \Big[\frac{T^{\alpha_1+\omega_1+\theta_1}}{\Gamma_q(\alpha_1+\omega_1+\theta_1+1)} + \frac{T^{\theta_1}}{|\Delta_1|\Gamma_q(\theta_1+1)}$$
$$\times\Big(\sum_{i=1}^{n}\lambda_i\frac{\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_q(\mu_i)}\int_0^{\zeta_i}(\zeta_i^{\beta_i} - qs^{\beta_i})^{(\mu_i-1)}s^{\eta_i}$$
$$\times\Big(\frac{\zeta_i^{\alpha_1+\omega_1+\theta_1}}{\Gamma_q(\alpha_1+\omega_1+\theta_1+1)}\Big)d_q s + \frac{a_1 T^{\alpha_1+\omega_1+\theta_1}}{\Gamma_q(\alpha_1+\omega_1+\theta_1+1)}\Big)$$
$$+ \frac{\delta_1^{\alpha_1}}{\Gamma_q(\alpha_1+1)}\Big(\frac{T^{\theta_1}}{|\Delta_1|\Gamma_q(\theta_1+1)}\Big(\frac{a_1 T^{\omega_1+\theta_1}}{\Gamma_q(\omega_1+\theta_1+1)} + \sum_{i=1}^{n}\lambda_i\beta_i[\frac{1}{\beta_i}]_q$$
$$\times\frac{\Gamma_q(\eta_i+1+\frac{\omega_1+\theta_1}{\beta_i})\zeta_i^{\omega_1+\theta_1}}{\Gamma_q(\mu_i+\eta_i+1+\frac{\omega_1+\theta_1}{\beta_i})\Gamma_q(\omega_1+\theta_1+1)}\Big) + \frac{T^{\omega_1+\theta_1}}{\Gamma_q(\omega_1+\theta_1+1)}\Big)\Big]$$
$$\times\Big(\Pi_1\nabla_1 + \chi_1\nabla_2 + \frac{\varrho^{\vartheta_1}}{\lceil\vartheta_1\rceil_q\Gamma_q(\Phi_1)}\nabla_3 + \sum_{l=4}^{5}\nabla_l\Big)$$
$$+ \sigma\Omega\Big(\frac{T^{\theta_1}}{|\Delta_1|\Gamma_q(\theta_1+1)}\Big(\frac{a_1T^{\theta_1}}{\Gamma_q(\theta_1+1)} + \sum_{i=1}^{n}\lambda_i\frac{\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_q(\mu_i)}$$
$$\times\int_0^{\zeta_i}(\zeta_i^{\beta_i} - qs^{\beta_i})^{(\mu_i-1)}s^{\eta_i}\times\Big(\frac{\zeta_i^{\theta_1}}{\Gamma_q(\theta_1+1)}\Big)d_q s\Big) + \frac{T^{\theta_1}}{\Gamma_q(\theta+1)}\Big)$$
$$= \Theta_1\Big(\Pi_1\nabla_1 + \chi_1\nabla_2 + \frac{\varrho^{\vartheta_1}}{\lceil\vartheta_1\rceil_q\Gamma_q(\Phi_1)}\nabla_3 + \sum_{l=4}^{5}\nabla_l\Big) + \Phi_1\sigma.$$

In a similar way, for all $u \in \mathcal{E}_\sigma$ and ($\mathbf{F_6}$), we get

$$\|B_1(u)\| \leqslant \Theta_2\Big(\Pi_2\Lambda_1 + \chi_2\Lambda_2 + \frac{\varrho^{\vartheta_2}}{\lceil\vartheta_2\rceil_q\Gamma_q(\Phi_2)}\Lambda_3 + \sum_{k=4}^{5}\Lambda_k\Big) + \Phi_2\sigma.$$

On the other hand, for any $m, u \in \mathcal{E}_\sigma$, we get

$$\|{}^C D_q B_1(m)\| \leqslant \pi_1\Big(\Pi_1\nabla_1 + \chi_1\nabla_2 + \frac{\varrho^{\vartheta_1}}{\lceil\vartheta_1\rceil_q\Gamma_q(\Phi_1)}\nabla_3 + \sum_{l=4}^{5}\nabla_l\Big) + \varpi_1\sigma,$$

and

$$\|{}^{C}D_{\mathsf{q}}B_2(\mathsf{u})\| \leqslant \pi_2\Big(\Pi_2\Lambda_1 + \chi_2\Lambda_2 + \frac{\varrho^{\vartheta_2}}{\lceil\vartheta_2\rceil_{\mathsf{q}}\Gamma_{\mathsf{q}}(\Phi_2)}\Lambda_3 + \sum_{k=4}^{5}\Lambda_k\Big) + \varpi_2\sigma.$$

By using (3.3), we get

$$\|{}^{C}D_{\mathsf{q}}^{\nu_2}B_1(\mathsf{m})\| \leqslant \frac{T^{1-\nu_2}}{\Gamma_{\mathsf{q}}(2-\nu_2)}\Big[\pi_1\Big(\Pi_1\nabla_1 + \chi_1\nabla_2 + \frac{\varrho^{\vartheta_1}}{\lceil\vartheta_1\rceil_{\mathsf{q}}\Gamma_{\mathsf{q}}(\Phi_1)}\nabla_3 + \sum_{l=4}^{5}\nabla_l\Big) + \varpi_1\sigma\Big],$$

and

$$\|{}^{C}D_{\mathsf{q}}^{\tau_2}B_1(\mathsf{m})\| \leqslant \frac{T^{1-\tau_2}}{\Gamma_{\mathsf{q}}(2-\tau_2)}\Big[\pi_1\Big(\Pi_1\nabla_1 + \chi_1\nabla_2 + \frac{\varrho^{\vartheta_1}}{\lceil\vartheta_1\rceil_{\mathsf{q}}\Gamma_{\mathsf{q}}(\Phi_1)}\nabla_3 + \sum_{l=4}^{5}\nabla_l\Big) + \varpi_1\sigma\Big].$$

In a similar way, by using (3.4), we get

$$\|{}^{C}D_{\mathsf{q}}^{\nu_1}B_2(\mathsf{u})\| \leqslant \frac{T^{1-\nu_1}}{\Gamma_{\mathsf{q}}(2-\nu_1)}\Big[\pi_2\Big(\Pi_2\Lambda_1 + \chi_2\Lambda_2 + \frac{\varrho^{\vartheta_2}}{\lceil\vartheta_2\rceil_{\mathsf{q}}\Gamma_{\mathsf{q}}(\Phi_2)}\nabla_3 + \sum_{k=4}^{5}\Lambda_k\Big) + \varpi_2\sigma\Big],$$

and

$$\|{}^{C}D_{\mathsf{q}}^{\tau_1}B_2(\mathsf{u})\| \leqslant \frac{T^{1-\tau_1}}{\Gamma_{\mathsf{q}}(2-\tau_1)}\Big[\pi_2\Big(\Pi_2\Lambda_1 + \chi_2\Lambda_2 + \frac{\varrho^{\vartheta_2}}{\lceil\vartheta_2\rceil_{\mathsf{q}}\Gamma_{\mathsf{q}}(\Phi_2)}\Lambda_3 + \sum_{k=4}^{5}\Lambda_k\Big) + \varpi_2\sigma\Big].$$

The above mentioned inequalities imply that an operator B exhibits uniform boundedness. Next, we confirm that B has equi-continuous form. Let $\mathsf{m}, \mathsf{u} \in \mathcal{E}_\sigma$ and $\varrho_1, \varrho_2 \in \mathcal{J}$ with $\varrho_2 < \varrho_1$. Then by $(\mathbf{F_5})$, we obtain

$$\begin{aligned}
&|B_1\mathsf{m}(\varrho_1) - B_1\mathsf{m}(\varrho_2)| \\
&\leqslant \Bigg[\frac{|\varrho_1^{\alpha_1+\omega_1+\theta_1} - \varrho_2^{\alpha_1+\omega_1+\theta_1}|}{\Gamma_{\mathsf{q}}(\alpha_1+\omega_1+\theta_1+1)} + \frac{|\varrho_1^{\theta_1} - \varrho_2^{\theta_1}|}{|\Delta_1|\Gamma_{\mathsf{q}}(\theta_1+1)} \\
&\times\Big(\sum_{i=1}^{n}\lambda_i\frac{\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)}\int_0^{\zeta_i}(\zeta_i^{\beta_i} - \mathsf{q}s^{\beta_i})^{(\mu_i-1)}s^{\eta_i} \\
&\times\Big(\frac{\zeta_i^{\alpha_1+\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\alpha_1+\omega_1+\theta_1+1)}\Big)d_{\mathsf{q}}s + \frac{a_1T^{\alpha_1+\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\alpha_1+\omega_1+\theta_1+1)}\Big) \\
&+ \frac{\delta_1^{\alpha_1}}{\Gamma_{\mathsf{q}}(\alpha_1+1)}\Big(\frac{|\varrho_1^{\theta_1} - \varrho_2^{\theta_1}|}{|\Delta_1|\Gamma_{\mathsf{q}}(\theta_1+1)}\Big(\frac{a_1T^{\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\omega_1+\theta_1+1)} \\
&+ \sum_{i=1}^{n}\lambda_i\beta_i[\frac{1}{\beta_i}]_{\mathsf{q}}\frac{\Gamma_{\mathsf{q}}(\eta_i+1+\frac{\omega_1+\theta_1}{\beta_i})}{\Gamma_{\mathsf{q}}(\mu_i+\eta_i+1+\frac{\omega_1+\theta_1}{\beta_i})\Gamma_{\mathsf{q}}(\omega_1+\theta_1+1)}\zeta_i^{\omega_1+\theta_1}\Big) \\
&+ \frac{|\varrho_2^{\omega_1+\theta_1} - \varrho_1^{\omega_1+\theta_1}|}{\Gamma_{\mathsf{q}}(\omega_1+\theta_1+1)}\Big)\Bigg]\times\Big(\Pi_1\nabla_1 + \chi_1\nabla_2 + \frac{\varrho^{\vartheta_1}}{\lceil\vartheta_1\rceil_{\mathsf{q}}\Gamma_{\mathsf{q}}(\Phi_1)}\nabla_3 \\
&+ \sum_{l=4}^{5}\nabla_l\Big) + \sigma\Omega\Big(\frac{|\varrho_1^{\theta_1} - \varrho_2^{\theta_1}|}{|\Delta_1|\Gamma_{\mathsf{q}}(\theta_1+1)}\Big(\frac{a_1T^{\theta_1}}{\Gamma_{\mathsf{q}}(\theta_1+1)} + \sum_{i=1}^{n}\lambda_i \\
&\times\frac{\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)}\int_0^{\zeta_i}(\zeta_i^{\beta_i} - \mathsf{q}s^{\beta_i})^{(\mu_i-1)}s^{\eta_i} \\
&\times\Big(\frac{\zeta_i^{\theta_1}}{\Gamma_{\mathsf{q}}(\theta_1+1)}\Big)d_{\mathsf{q}}s\Big) + \frac{|\varrho_2^{\theta_1} - \varrho_1^{\theta_1}|}{\Gamma_{\mathsf{q}}(\theta_1+1)}\Big).
\end{aligned}$$

In a similar way, using $(\mathbf{F_6})$ we have

$$\begin{aligned}
&|B_2\mathsf{u}(\varrho_1) - B_2\mathsf{u}(\varrho_2)| \\
&\leqslant \Bigg[\frac{|\varrho_1^{\alpha_2+\omega_2+\theta_2} - \varrho_2^{\alpha_2+\omega_2+\theta_2}|}{\Gamma_{\mathsf{q}}(\alpha_2+\omega_2+\theta_2+1)} + \frac{|\varrho_1^{\theta_2} - \varrho_2^{\theta_2}|}{|\Delta_2|\Gamma_{\mathsf{q}}(\theta_2+1)} \\
&\times\Big(\sum_{i=1}^{n}\lambda_i\frac{\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)}\int_0^{\zeta_i}(\zeta_i^{\beta_i} - \mathsf{q}s^{\beta_i})^{(\mu_i-1)}s^{\eta_i} \\
&\times\Big(\frac{\zeta_i^{\alpha_2+\omega_2+\theta_2}}{\Gamma_{\mathsf{q}}(\alpha_2+\omega_2+\theta_2+1)}\Big)d_{\mathsf{q}}s + \frac{a_2T^{\alpha_2+\omega_2+\theta_2}}{\Gamma_{\mathsf{q}}(\alpha_2+\omega_2+\theta_2+1)}\Big) \\
&+ \frac{\delta_2^{\alpha_2}}{\Gamma_{\mathsf{q}}(\alpha_2+1)}\Big(\frac{|\varrho_1^{\theta_2} - \varrho_2^{\theta_2}|}{|\Delta_2|\Gamma_{\mathsf{q}}(\theta_2+1)}\Big(\frac{a_2T^{\omega_2+\theta_2}}{\Gamma_{\mathsf{q}}(\omega_2+\theta_2+1)} \\
&+ \sum_{i=1}^{n}\lambda_i\beta_i[\frac{1}{\beta_i}]_{\mathsf{q}}\frac{\Gamma_{\mathsf{q}}(\eta_i+1+\frac{\omega_2+\theta_2}{\beta_i})}{\Gamma_{\mathsf{q}}(\mu_i+\eta_i+1+\frac{\omega_2+\theta_2}{\beta_i})\Gamma_{\mathsf{q}}(\omega_2+\theta_2+1)}\zeta_i^{\omega_2+\theta_2}\Big) \\
&+ \frac{|\varrho_2^{\omega_2+\theta_2} - \varrho_1^{\omega_2+\theta_2}|}{\Gamma_{\mathsf{q}}(\omega_2+\theta_2+1)}\Big)\Bigg]\times\Big(\Pi_2\Lambda_1 + \chi_2\Lambda_2 + \frac{\varrho^{\vartheta_2}}{\lceil\vartheta_2\rceil_{\mathsf{q}}\Gamma_{\mathsf{q}}(\Phi_2)}\Lambda_3 \\
&+ \sum_{k=4}^{5}\Lambda_k\Big) + \sigma\Upsilon\Big(\frac{|\varrho_1^{\theta_2} - \varrho_2^{\theta_2}|}{|\Delta_2|\Gamma_{\mathsf{q}}(\theta_2+1)}\Big(\frac{a_2T^{\theta_2}}{\Gamma_{\mathsf{q}}(\theta_2+1)} + \sum_{i=1}^{n}\lambda_i \\
&\times\frac{\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)}\int_0^{\zeta_i}(\zeta_i^{\beta_i} - \mathsf{q}s^{\beta_i})^{(\mu_i-1)}s^{\eta_i} \\
&\times\Big(\frac{\zeta_i^{\theta_2}}{\Gamma_{\mathsf{q}}(\theta_2+1)}\Big)d_{\mathsf{q}}s\Big) + \frac{|\varrho_2^{\theta_2} - \varrho_1^{\theta_2}|}{\Gamma_{\mathsf{q}}(\theta_2+1)}\Big).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&|{}^{C}D_{\mathsf{q}}B_1\mathsf{m}(\varrho_1) - {}^{C}D_{\mathsf{q}}B_1\mathsf{m}(\varrho_2)| \leqslant \Bigg[\frac{|\varrho_1^{\alpha_1+\omega_1+\theta_1-1} - \varrho_2^{\alpha_1+\omega_1+\theta_1-1}|}{\Gamma_{\mathsf{q}}(\alpha_1+\omega_1+\theta_1)} \\
&+ \frac{\lceil\theta_1\rceil_{\mathsf{q}}|\varrho_1^{\theta_1-1} - \varrho_2^{\theta_1-1}|}{|\Delta_1|\Gamma_{\mathsf{q}}(\theta_1+1)}\Big(\sum_{i=1}^{n}\lambda_i\frac{\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)}\int_0^{\zeta_i}(\zeta_i^{\beta_i} - \mathsf{q}s^{\beta_i})^{(\mu_i-1)}s^{\eta_i} \\
&\times\Big(\frac{\zeta_i^{\alpha_1+\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\alpha_1+\omega_1+\theta_1+1)}\Big)d_{\mathsf{q}}s + \frac{a_1T^{\alpha_1+\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\alpha_1+\omega_1+\theta_1+1)}\Big) \\
&+ \frac{\delta_1^{\alpha_1}}{\Gamma_{\mathsf{q}}(\alpha_1+1)}\Big(\frac{\lceil\theta_1\rceil_{\mathsf{q}}|\varrho_1^{\theta_1-1} - \varrho_2^{\theta_1-1}|}{|\Delta_1|\Gamma_{\mathsf{q}}(\theta_1+1)}\Big(\frac{a_1T^{\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\omega_1+\theta_1+1)}
\end{aligned}$$

$$+\sum_{i=1}^{n}\lambda_i\beta_i[\frac{1}{\beta_i}]_{\mathsf{q}}\frac{\Gamma_{\mathsf{q}}(\eta_i+1+\frac{\omega_1+\theta_1}{\beta_i})}{\Gamma_{\mathsf{q}}(\mu_i+\eta_i+1+\frac{\omega_1+\theta_1}{\beta_i})\Gamma_{\mathsf{q}}(\omega_1+\theta_1+1)}\zeta_i^{\omega_1+\theta_1}\Big)$$
$$+\frac{[\omega_1+\theta_1]_{\mathsf{q}}|\varrho_2^{\omega_1+\theta_1-1}-\varrho_1^{\omega_1+\theta_1}|}{\Gamma_{\mathsf{q}}(\omega_1+\theta_1+1)}\Big)\Big]$$
$$\times\Big(\Pi_1\nabla_1+\chi_1\nabla_2+\frac{\varrho^{\vartheta_1}}{[\vartheta_1]_{\mathsf{q}}\Gamma_{\mathsf{q}}(\Phi_1)}\nabla_3+\sum_{l=4}^{5}\nabla_l\Big)$$
$$+\sigma\Omega\Big(\frac{[\theta_1]_{\mathsf{q}}|\varrho_1^{\theta_1-1}-\varrho_2^{\theta_1-1}|}{|\Delta_1|\Gamma_{\mathsf{q}}(\theta_1+1)}\Big(\frac{a_1T^{\theta_1}}{\Gamma_{\mathsf{q}}(\theta_1+1)}$$
$$+\sum_{i=1}^{n}\lambda_i\frac{\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)}\int_0^{\zeta_i}(\zeta_i^{\beta_i}-\mathsf{q}s^{\beta_i})^{(\mu_i-1)}s^{\eta_i}$$
$$\times\Big(\frac{\zeta_i^{\theta_1}}{\Gamma_{\mathsf{q}}(\theta_1+1)}\Big)d_{\mathsf{q}}s\Big)+\frac{|\varrho_2^{\theta_1-1}-\varrho_1^{\theta_1-1}|}{\Gamma_{\mathsf{q}}(\theta_1)}\Big).$$

Similarly,

$$|{}^CD_{\mathsf{q}}B_2\mathsf{u}(\varrho_1)-{}^CD_{\mathsf{q}}B_2\mathsf{u}(\varrho_2)|$$
$$\leqslant\Big[\frac{|\varrho_1^{\alpha_2+\omega_2+\theta_2-1}-\varrho_2^{\alpha_2+\omega_2+\theta_2-1}|}{\Gamma_{\mathsf{q}}(\alpha_2+\omega_2+\theta_2)}$$
$$+\frac{[\theta_1]_{\mathsf{q}}|\varrho_1^{\theta_2-1}-\varrho_2^{\theta_2-1}|}{|\Delta_2|\Gamma_{\mathsf{q}}(\theta_2+1)}\Big(\sum_{i=1}^{n}\lambda_i\frac{\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)}$$
$$\times\int_0^{\zeta_i}(\zeta_i^{\beta_i}-\mathsf{q}s^{\beta_i})^{(\mu_i-1)}s^{\eta_i}$$
$$\times\Big(\frac{\zeta_i^{\alpha_2+\omega_2+\theta_2}}{\Gamma_{\mathsf{q}}(\alpha_2+\omega_2+\theta_2+1)}\Big)d_{\mathsf{q}}s+\frac{a_2T^{\alpha_2+\omega_2+\theta_2}}{\Gamma_{\mathsf{q}}(\alpha_2+\omega_2+\theta_2+1)}\Big)$$
$$+\frac{\delta_2^{\alpha_2}}{\Gamma_{\mathsf{q}}(\alpha_2+1)}\Big(\frac{[\theta_2]_{\mathsf{q}}|\varrho_1^{\theta_2-1}-\varrho_2^{\theta_2-1}|}{|\Delta_2|\Gamma_{\mathsf{q}}(\theta_2+1)}\Big(\frac{a_2T^{\omega_2+\theta_2}}{\Gamma_{\mathsf{q}}(\omega_2+\theta_2+1)}$$
$$+\sum_{i=1}^{n}\lambda_i\beta_i[\frac{1}{\beta_i}]_{\mathsf{q}}\frac{\Gamma_{\mathsf{q}}(\eta_i+1+\frac{\omega_2+\theta_2}{\beta_i})}{\Gamma_{\mathsf{q}}(\mu_i+\eta_i+1+\frac{\omega_2+\theta_2}{\beta_i})\Gamma_{\mathsf{q}}(\omega_2+\theta_2+1)}\zeta_i^{\omega_2+\theta_2}\Big)$$
$$+\frac{[\omega_2+\theta_2]_{\mathsf{q}}|\varrho_2^{\omega_2+\theta_2-1}-\varrho_1^{\omega_2+\theta_2}|}{\Gamma_{\mathsf{q}}(\omega_2+\theta_2+1)}\Big)\Big]\times\Big(\Pi_2\Lambda_1+\chi_2\Lambda_2$$
$$+\frac{\varrho^{\vartheta_2}}{[\vartheta_2]_{\mathsf{q}}\Gamma_{\mathsf{q}}(\Phi_2)}\Lambda_3+\sum_{k=4}^{5}\Lambda_k\Big)+\sigma\Upsilon\Big(\frac{[\theta_2]_{\mathsf{q}}|\varrho_1^{\theta_2-1}-\varrho_2^{\theta_2-1}|}{|\Delta_2|\Gamma_{\mathsf{q}}(\theta_2+1)}$$
$$\times\Big(\frac{a_2T^{\theta_2}}{\Gamma_{\mathsf{q}}(\theta_2+1)}+\sum_{i=1}^{n}\lambda_i\frac{\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)}\int_0^{\zeta_i}(\zeta_i^{\beta_i}-\mathsf{q}s^{\beta_i})^{(\mu_i-1)}s^{\eta_i}$$
$$\times\Big(\frac{\zeta_i^{\theta_2}}{\Gamma_{\mathsf{q}}(\theta_1+1)}\Big)d_{\mathsf{q}}s\Big)+\frac{|\varrho_2^{\theta_2-1}-\varrho_1^{\theta_2-1}|}{\Gamma_{\mathsf{q}}(\theta_2)}\Big).$$

Now by (3.3), we have

$$|{}^CD_{\mathsf{q}}^{\nu_2}B_1\mathsf{m}(\varrho_1)-{}^CD_{\mathsf{q}}^{\nu_2}B_1\mathsf{m}(\varrho_2)|$$
$$\leqslant\frac{|\varrho_1^{1-\nu_2}-\varrho_2^{1-\nu_2}|}{\Gamma_{\mathsf{q}}(2-\nu_2)}|{}^CD_{\mathsf{q}}B_1\mathsf{m}(\varrho_1)-{}^CD_{\mathsf{q}}B_1\mathsf{m}(\varrho_2)|,$$

and

$$|{}^CD_{\mathsf{q}}^{\tau_2}B_1\mathsf{m}(\varrho_1)-{}^CD_{\mathsf{q}}^{\tau_2}B_1\mathsf{m}(\varrho_2)|$$
$$\leqslant\frac{|\varrho_1^{1-\tau_2}-\varrho_2^{1-\tau_2}|}{\Gamma_{\mathsf{q}}(2-\tau_2)}|{}^CD_{\mathsf{q}}B_1\mathsf{m}(\varrho_1)-{}^CD_{\mathsf{q}}B_1\mathsf{m}(\varrho_2)|.$$

Similarly, by (3.4), we get

$$|{}^CD_{\mathsf{q}}^{\nu_1}B_2\mathsf{u}(\varrho_1)-{}^CD_{\mathsf{q}}^{\nu_1}B_2\mathsf{u}(\varrho_2)|$$
$$\leqslant\frac{|\varrho_1^{1-\nu_1}-\varrho_2^{1-\nu_1}|}{\Gamma_{\mathsf{q}}(2-\nu_1)}|{}^CD_{\mathsf{q}}B_2\mathsf{u}(\varrho_1)-{}^CD_{\mathsf{q}}B_1\mathsf{u}(\varrho_2)|,$$

and

$$|{}^CD_{\mathsf{q}}^{\tau_1}B_2\mathsf{u}(\varrho_1)-{}^CD_{\mathsf{q}}^{\tau_1}B_2\mathsf{u}(\varrho_2)|$$
$$\leqslant\frac{|\varrho_1^{1-\tau_1}-\varrho_2^{1-\tau_1}|}{\Gamma_{\mathsf{q}}(2-\tau_1)}|{}^CD_{\mathsf{q}}B_2\mathsf{u}(\varrho_1)-{}^CD_{\mathsf{q}}B_1\mathsf{u}(\varrho_2)|.$$

From the above inequalities, we can state that $\|B_1\mathsf{m}(\varrho_1)-B_1\mathsf{m}(\varrho_2)\|\to 0$ and $\|B_2\mathsf{u}(\varrho_1)-B_2\mathsf{u}(\varrho_2)\|\to 0$ as $\varrho_1\to\varrho_2$. It is inferred that operator B is completely continuous by utilizing the Arzela-Ascoli theorem. Lastly, the validity of a set $\mathscr{H}$ is provided as follows:

$$\mathscr{H}=\{(\mathsf{m},\mathsf{u})\in\mathsf{X}\to\mathsf{X}:(\mathsf{m},\mathsf{u})=\gamma B(\mathsf{m},\mathsf{u}),0<\gamma<1\}$$

is bounded. Let $(\mathsf{m},\mathsf{u})\in\mathscr{H}$, and then $(\mathsf{m},\mathsf{u})=\gamma B(\mathsf{m},\mathsf{u})$ for some $0<\gamma<1$. Thus for any $\varrho\in\mathcal{J}$, we can write $\mathsf{m}(\varrho)=\gamma B_1(\mathsf{m})(\varrho)$, $\mathsf{u}(\varrho)=\gamma B_2(\mathsf{u})(\varrho)$. Then by ($\mathbf{F_5}$), we get

$$\|\mathsf{m}\|\leqslant\gamma\Theta_1\Big(\Pi_1\nabla_1+\chi_1\nabla_2+\frac{\varrho^{\vartheta_1}}{[\vartheta_1]_{\mathsf{q}}\Gamma_{\mathsf{q}}(\Phi_1)}\nabla_3+\sum_{l=4}^{5}\nabla_l\Big)+\Phi_1\|\mathsf{m}\|_{\mathsf{X}}.$$

In a similar way, using ($\mathbf{F_6}$), we obtain

$$\|\mathsf{u}\|\leqslant\gamma\Theta_2\Big(\Pi_2\Lambda_1+\chi_2\Lambda_2+\frac{\varrho^{\vartheta_2}}{[\vartheta_2]_{\mathsf{q}}\Gamma_{\mathsf{q}}(\Phi_2)}\Lambda_3+\sum_{k=4}^{5}\Lambda_k\Big)+\Phi_1\|\mathsf{u}\|_{\mathsf{X}}.$$

On the other hand, we get

$$\|{}^CD_{\mathsf{q}}(\mathsf{m})\|$$
$$\leqslant\gamma\pi_1\Big(\Pi_1\nabla_1+\chi_1\nabla_2+\frac{\varrho^{\vartheta_1}}{[\vartheta_1]_{\mathsf{q}}\Gamma_{\mathsf{q}}(\Phi_1)}\nabla_3+\sum_{l=4}^{5}\nabla_l\Big)+\varpi_1\|\mathsf{m}\|_{\mathsf{X}}.$$

Similarly,

$$\|{}^{C}D_{\mathsf{q}}(\mathsf{u})\|$$
$$\leqslant \gamma\pi_2\Big(\Pi_2\Lambda_1+\chi_2\Lambda_2+\frac{\varrho^{\vartheta_2}}{\lceil\vartheta_2\rceil_{\mathsf{q}}\Gamma_{\mathsf{q}}(\Phi_2)}\Lambda_3+\sum_{k=4}^{5}\Lambda_k\Big)+\varpi_2\|\mathsf{u}\|_{\mathsf{X}}.$$

Using (3.3), we have

$$\|{}^{C}D_{\mathsf{q}}^{\nu_2}(\mathsf{m})\|\leqslant\frac{T^{1-\nu_2}}{\Gamma_{\mathsf{q}}(2-\nu_2)}\Big[\gamma\pi_1\Big(\Pi_1\nabla_1+\chi_1\nabla_2+\frac{\varrho^{\vartheta_1}}{\lceil\vartheta_1\rceil_{\mathsf{q}}\Gamma_{\mathsf{q}}(\Phi_1)}\nabla_3+\sum_{l=4}^{5}\nabla_l\Big)+\varpi_1\|\mathsf{m}\|_{\mathsf{X}}\Big],$$

and

$$\|{}^{C}D_{\mathsf{q}}^{\tau_2}(\mathsf{m})\|\leqslant\frac{T^{1-\tau_2}}{\Gamma_{\mathsf{q}}(2-\tau_2)}\Big[\gamma\pi_1\Big(\Pi_1\nabla_1+\chi_1\nabla_2+\frac{\varrho^{\vartheta_1}}{\lceil\vartheta_1\rceil_{\mathsf{q}}\Gamma_{\mathsf{q}}(\Phi_1)}\nabla_3+\sum_{l=4}^{5}\nabla_l\Big)+\varpi_1\|\mathsf{m}\|_{\mathsf{X}}\Big].$$

Similarly, using (3.4), we get

$$\|{}^{C}D_{\mathsf{q}}^{\nu_1}(\mathsf{u})\|\leqslant\frac{T^{1-\nu_1}}{\Gamma_{\mathsf{q}}(2-\nu_1)}\Big[\gamma\pi_2\Big(\Pi_2\Lambda_1+\chi_2\Lambda_2+\frac{\varrho^{\vartheta_2}}{\lceil\vartheta_2\rceil_{\mathsf{q}}\Gamma_{\mathsf{q}}(\Phi_2)}\Lambda_3+\sum_{k=4}^{5}\Lambda_k\Big)+\varpi_2\|\mathsf{u}\|_{\mathsf{X}}\Big],$$

and

$$\|{}^{C}D_{\mathsf{q}}^{\tau_1}(\mathsf{u})\|\leqslant\frac{T^{1-\tau_1}}{\Gamma_{\mathsf{q}}(2-\tau_1)}\Big[\gamma\pi_2\Big(\Pi_2\Lambda_1+\chi_2\Lambda_2+\frac{\varrho^{\vartheta_2}}{\lceil\vartheta_2\rceil_{\mathsf{q}}\Gamma_{\mathsf{q}}(\Phi_2)}\Lambda_3+\sum_{k=4}^{5}\Lambda_k\Big)+\varpi_2\|\mathsf{u}\|_{\mathsf{X}}\Big].$$

It follows from the above inequalities that

$$\|\mathsf{m}\|_{\mathsf{X}}\leqslant\frac{\mathcal{A}_1\Big(\Pi_1\nabla_1+\chi_1\nabla_2+\frac{\varrho^{\vartheta_1}}{\lceil\vartheta_1\rceil_{\mathsf{q}}\Gamma_{\mathsf{q}}(\Phi_1)}\nabla_3+\sum_{l=4}^{5}\nabla_l\Big)}{1-\Psi_1}=\frac{\aleph_1}{1-\Psi_1},\tag{3.12}$$

and similarly

$$\|\mathsf{u}\|_{\mathsf{X}}\leqslant\frac{\mathcal{A}_2\Big(\Pi_2\Lambda_1+\chi_2\Lambda_2+\frac{\varrho^{\vartheta_2}}{\lceil\vartheta_2\rceil_{\mathsf{q}}\Gamma_{\mathsf{q}}(\Phi_2)}\Lambda_3+\sum_{k=4}^{5}\Lambda_k\Big)}{1-\Psi_2}=\frac{\aleph_2}{1-\Psi_2}.\tag{3.13}$$

Let $\Psi=\max\{\Psi_1,\Psi_2\}$, and then (3.12) and (3.13) become

$$\|(\mathsf{m},\mathsf{u})\|_{\mathsf{X}\times\mathsf{X}}\leqslant\frac{\aleph_1+\aleph_2}{1-\Psi},$$

where $\aleph_1,\aleph_2$ are given by

$$\aleph_1=\mathcal{A}_2\Big(\Pi_2\Lambda_1+\chi_2\Lambda_2+\frac{\varrho^{\vartheta_2}}{\lceil\vartheta_2\rceil_{\mathsf{q}}\Gamma_{\mathsf{q}}(\Phi_2)}\Lambda_3+\sum_{k=4}^{5}\Lambda_k\Big),$$

with

$$\mathcal{A}_1=\Big(\Theta_1+\frac{\pi_1}{\Gamma_{\mathsf{q}}(2-\nu_2)}+\frac{\pi_1}{\Gamma_{\mathsf{q}}(2-\tau_2)}\Big).$$

Similarly,

$$\aleph_2=\mathcal{A}_2\Big(\Pi_2\Lambda_1+\chi_2\Lambda_2+\frac{\varrho^{\vartheta_2}}{\lceil\vartheta_2\rceil_{\mathsf{q}}\Gamma_{\mathsf{q}}(\Phi_2)}\Lambda_3+\sum_{k=4}^{5}\Lambda_k\Big),$$

with

$$\mathcal{A}_2=\Big(\Theta_2+\frac{\pi_2}{\Gamma_{\mathsf{q}}(2-\nu_1)}+\frac{\pi_2}{\Gamma_{\mathsf{q}}(2-\tau_1)}\Big),$$

and Ψ is given by (3.11). This proves that $\mathscr{H}$ is a bounded set. Thus, by Theorem 2.2, the operator B has at least one FP. Therefore, q$-\mathcal{FJCS}$ (1.1) has at least one solution. □

4. Stability results

We will discuss types of $\mathscr{U}\mathscr{S}$ in the q-$\mathcal{FJCS}$ (1.1) described in [40] in this section.

Definition 4.1. *The* q-$\mathcal{FJCS}$ (1.1) *is said to be Ulam-Hyers stable* (UHS) *if there exists a real number* $\mathrm{D}_{\phi^*,\xi^*,\rho^*,h^*,\mathsf{f}}=\max(\mathrm{D}_{\phi^*_{\mathsf{m}},\xi^*_{\mathsf{m}},\rho^*_{\mathsf{m}},h^*_{\mathsf{m}},\mathsf{f}_1},\mathrm{D}_{\phi^*_{\mathsf{u}},\xi^*_{\mathsf{u}},\rho^*_{\mathsf{u}},h^*_{\mathsf{u}},\mathsf{f}_2})>0$ *in such a way that for each* $\epsilon=\max(\epsilon_{\mathsf{m}},\epsilon_{\mathsf{u}})>0$ *and* $(\mathsf{m},\mathsf{u})\in\mathsf{X}\times\mathsf{X}$ *satisfying*

$$\begin{cases}|{}^{C}\mathrm{D}_{\mathsf{q}}^{\alpha_1}({}^{C}\mathrm{D}_{\mathsf{q}}^{\omega_1}({}^{C}\mathrm{D}_{\mathsf{q}}^{\theta_1}+\Omega))\mathsf{m}(\varrho)-(\mathsf{f}_1(\varrho)-\phi_{1\mathsf{m}}^{*}(\varrho)\\ \quad-\Pi_1\xi_{1\mathsf{u}}^{*,\nu_2}(\varrho)-\chi_1\rho_{1\mathsf{u}}^{*,\tau_2}(\varrho)-h_{1\mathsf{u}}^{*,\nu_2}(\varrho))|\leqslant\epsilon_{\mathsf{m}},\quad\varrho\in\mathcal{J},\\ |{}^{C}\mathrm{D}_{\mathsf{q}}^{\alpha_2}({}^{C}\mathrm{D}_{\mathsf{q}}^{\omega_2}({}^{C}\mathrm{D}_{\mathsf{q}}^{\theta_2}+\Upsilon))\mathsf{u}(\varrho)-(\mathsf{f}_2(\varrho)-\phi_{2\mathsf{u}}^{*}(\varrho)\\ \quad-\Pi_2\xi_{2\mathsf{m}}^{*,\nu_1}(\varrho)-\chi_2\rho_{2\mathsf{m}}^{*,\tau_1}(\varrho)-h_{2\mathsf{m}}^{*,\nu_1}(\varrho))|\leqslant\epsilon_{\mathsf{u}},\quad\varrho\in\mathcal{J},\end{cases}\tag{4.1}$$

$\exists$ *a solution* $(\hat{\mathsf{m}},\hat{\mathsf{u}})\in\mathsf{X}\times\mathsf{X}$ *of* (1.1) *with*

$$|(\hat{\mathsf{m}},\hat{\mathsf{u}})(\varrho)-(\mathsf{m},\mathsf{u})(\varrho)|\leqslant\mathrm{D}_{\phi^*,\xi^*,\rho^*,h^*,\mathsf{f}}\epsilon_{\mathsf{m},\mathsf{u}},\quad\varrho\in\mathcal{J}.$$

Definition 4.2. *The* q-$\mathcal{FJCS}$ (1.1) *is said to be Ulam-Hyers Rassias stable (*UHRS*) corresponding to* $\mathcal{M}_{y,z} = (\mathcal{M}_y, \mathcal{M}_z)$ *with* $\mathcal{M}_{y,z} \in C(\mathsf{X}, \mathscr{R}_+)$*, if* $\exists\ \mathrm{D}_{\phi^*,\xi^*,\rho^*,h^*,\mathsf{f}} = \max(\mathrm{D}_{\phi^*_{\mathsf{m}},\xi^*_{\mathsf{m}},\rho^*_{\mathsf{m}},h^*_{\mathsf{m}},\mathsf{f}_1}, \mathrm{D}_{\phi^*_{\mathsf{u}},\xi^*_{\mathsf{u}},\rho^*_{\mathsf{u}},h^*_{\mathsf{u}},\mathsf{f}_2}) > 0$ *in such a way that for each* $\epsilon = \max(\epsilon_{\mathsf{m}}, \epsilon_{\mathsf{u}}) > 0$ *and* $(\mathsf{m}, \mathsf{u}) \in \mathsf{X} \times \mathsf{X}$*, satisfying*

$$\begin{cases} |{}^C\mathrm{D}_{\mathsf{q}}^{\alpha_1}({}^C\mathrm{D}_{\mathsf{q}}^{\omega_1}({}^C\mathrm{D}_{\mathsf{q}}^{\theta_1} + \Omega))\mathsf{m}(\varrho) - (\mathsf{f}_1(\varrho) - \phi_{1\mathsf{m}}^*(\varrho) \\ - \Pi_1 \xi_{1\mathsf{u}}^{*,\nu_2}(\varrho) - \chi_1 \rho_{1\mathsf{u}}^{*,\tau_2}(\varrho) - h_{1\mathsf{u}}^{*,\nu_2}(\varrho))| \leqslant \mathcal{M}_y(\varrho)\epsilon_{\mathsf{m}}, \\ \qquad \varrho \in \mathcal{J}, \\ |{}^C\mathrm{D}_{\mathsf{q}}^{\alpha_2}({}^C\mathrm{D}_{\mathsf{q}}^{\omega_2}({}^C\mathrm{D}_{\mathsf{q}}^{\theta_2} + \Upsilon))\mathsf{u}(\varrho) - (\mathsf{f}_2(\varrho) - \phi_{2\mathsf{u}}^*(\varrho) \\ - \Pi_2 \xi_{2\mathsf{m}}^{*,\nu_1}(\varrho) - \chi_2 \rho_{2\mathsf{m}}^{*,\tau_1}(\varrho) - h_{2\mathsf{m}}^{*,\nu_1}(\varrho))| \leqslant \mathcal{M}_z(\varrho)\epsilon_{\mathsf{u}}, \\ \qquad \varrho \in \mathcal{J}, \end{cases} \tag{4.2}$$

$\exists$ *a solution* $(\hat{\mathsf{m}}, \hat{\mathsf{u}}) \in \mathsf{X} \times \mathsf{X}$ *of* (1.1) *with*

$$|(\hat{\mathsf{m}}, \hat{\mathsf{u}})(\varrho) - (\mathsf{m}, \mathsf{u})(\varrho)| \leqslant \mathrm{D}_{\phi^*,\xi^*,\rho^*,h^*,\mathsf{f}} \mathcal{M}_{y,z}(\varrho)\epsilon_{\mathsf{m},\mathsf{u}}, \quad \varrho \in \mathcal{J}.$$

Remark 4.1. *Consider* $(\mathsf{m}, \mathsf{u}) \in \mathsf{X} \times \mathsf{X}$ *is a solution of the inequality* (4.1)*, if there exist functions* $\mathcal{L}_{\mathsf{m}}, \mathcal{L}_{\mathsf{u}} \in C(\mathcal{J}, \mathscr{R})$ *depending on (*m, u*), respectively, such that*

(1) $|\mathcal{L}_{\mathsf{m}}(\varrho)| \leqslant \epsilon_{\mathsf{m}}, \quad |\mathcal{L}_{\mathsf{u}}(\varrho)| \leqslant \epsilon_{\mathsf{u}}, \qquad \varrho \in \mathcal{J}$,

(2)

$$\begin{cases} {}^C\mathrm{D}_{\mathsf{q}}^{\alpha_1}({}^C\mathrm{D}_{\mathsf{q}}^{\omega_1}({}^C\mathrm{D}_{\mathsf{q}}^{\theta_1} + \Omega))\mathsf{m}(\varrho) - (\mathsf{f}_1(\varrho) - \phi_{1\mathsf{m}}^*(\varrho) \\ - \Pi_1 \xi_{1\mathsf{u}}^{*,\nu_2}(\varrho) - \chi_1 \rho_{1\mathsf{u}}^{*,\tau_2}(\varrho) - h_{1\mathsf{u}}^{*,\nu_2}(\varrho)) + \mathcal{L}_{\mathsf{m}}(\varrho), \\ \qquad \varrho \in \mathcal{J}, \\ {}^C\mathrm{D}_{\mathsf{q}}^{\alpha_2}({}^C\mathrm{D}_{\mathsf{q}}^{\omega_2}({}^C\mathrm{D}_{\mathsf{q}}^{\theta_2} + \Upsilon))\mathsf{u}(\varrho) - (\mathsf{f}_2(\varrho) - \phi_{2\mathsf{u}}^*(\varrho) \\ - \Pi_2 \xi_{2\mathsf{m}}^{*,\nu_1}(\varrho) - \chi_2 \rho_{2\mathsf{m}}^{*,\tau_1}(\varrho) - h_{2\mathsf{m}}^{*,\nu_1}(\varrho)) + \mathcal{L}_{\mathsf{u}}(\varrho), \\ \qquad \varrho \in \mathcal{J}. \end{cases}$$

Remark 4.2. $(\mathsf{m}, \mathsf{u}) \in \mathsf{X}$ *is a solution of* (4.2)*, if and only if* $\exists\ \mathcal{M}_{y,z} \in \mathsf{X}$ *depending on (*m, u*), respectively, such that*

(1) $|\mathcal{M}_y(\varrho)| \leqslant \epsilon_{\mathsf{m}}\mathcal{M}_y(\varrho), \ \ |\mathcal{M}_z(\varrho)| \leqslant \epsilon_{\mathsf{u}}\mathcal{M}_z(\varrho) \ \ \forall \ \ \varrho \in \mathcal{J}$;

(2)

$$\begin{cases} |{}^C\mathrm{D}_{\mathsf{q}}^{\alpha_1}({}^C\mathrm{D}_{\mathsf{q}}^{\omega_1}({}^C\mathrm{D}_{\mathsf{q}}^{\theta_1} + \Omega))\mathsf{m}(\varrho) - (\mathsf{f}_1(\varrho) - \phi_{1\mathsf{m}}^*(\varrho) \\ - \Pi_1 \xi_{1\mathsf{u}}^{*,\nu_2}(\varrho) - \chi_1 \rho_{1\mathsf{u}}^{*,\tau_2}(\varrho) - h_{1\mathsf{u}}^{*,\nu_2}(\varrho))| \\ \leqslant \mathcal{M}_y(\varrho), \ \varrho \in \mathcal{J}, \\ |{}^C\mathrm{D}_{\mathsf{q}}^{\alpha_2}({}^C\mathrm{D}_{\mathsf{q}}^{\omega_2}({}^C\mathrm{D}_{\mathsf{q}}^{\theta_2} + \Upsilon))\mathsf{u}(\varrho) - (\mathsf{f}_2(\varrho) - \phi_{2\mathsf{u}}^*(\varrho) \\ - \Pi_2 \xi_{2\mathsf{m}}^{*,\nu_1}(\varrho) - \chi_2 \rho_{2\mathsf{m}}^{*,\tau_1}(\varrho) - h_{2\mathsf{m}}^{*,\nu_1}(\varrho))| \\ \leqslant \mathcal{M}_z(\varrho), \ \varrho \in \mathcal{J}. \end{cases}$$

Theorem 4.1. *If the assumptions* ($\mathbf{F_2}$)–($\mathbf{F_4}$) *and* (3.5) *hold, then the* q-$\mathcal{FJCS}$ (1.1) *is* UHS.

Proof. Consider $\mathsf{m} \in \mathsf{X}$ is the unique solution of the given problem:

$$\begin{cases} {}^C D_{\mathsf{q}}^{\alpha_1}({}^C D_{\mathsf{q}}^{\omega_1}({}^C D_{\mathsf{q}}^{\theta_1} + \Omega))\mathsf{m}(\varrho) = \mathsf{f}_1(\varrho) - \phi_{1\mathsf{m}}^*(\varrho) - \Pi_1 \xi_{1\mathsf{m}}^{*,\nu_2}(\varrho) \\ - \chi_1 \rho_{1\mathsf{m}}^{*,\tau_2}(\varrho) - h_{\mathsf{m}}^{*,\nu_2}(\varrho), \quad \varrho \in \mathcal{J} = (0, T), \\ \hat{\mathsf{m}}(0) = \mathsf{m}(0), \\ ({}^C D_{\mathsf{q}}^{\omega_1}({}^C D_{\mathsf{q}}^{\theta_1} + \Omega))\hat{\mathsf{m}}(\delta_1) = ({}^C D_{\mathsf{q}}^{\omega_1}({}^C D_{\mathsf{q}}^{\theta_1} + \Omega))\mathsf{m}(\delta_1), \\ a_1 \hat{\mathsf{m}}(T) = a_1 \mathsf{m}(T), \\ \sum_{i=1}^{n} \lambda_i \mathcal{I}_{\mathsf{q}}^{\eta_i,\mu_i,\beta_i} \hat{\mathsf{m}}(\zeta_i) = \sum_{i=1}^{n} \lambda_i \mathcal{I}_{\mathsf{q}}^{\eta_i,\mu_i,\beta_i} \mathsf{m}(\zeta_i), \end{cases} \tag{4.3}$$

such that inequality (4.1) can be solved for $\hat{\mathsf{m}} \in \mathsf{X}$. By Remark 4.1, we have

$$\mathsf{m}(\varrho) = \mathcal{I}_{\mathsf{q}}^{\alpha_1+\omega_1+\theta_1}[g_{1\hat{\mathsf{m}}}(\varrho)] - \Omega \mathcal{I}_{\mathsf{q}}^{\theta_1}[\hat{\mathsf{m}}(\varrho)] + \hat{c}_0 \frac{\varrho^{\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\omega_1 + \theta_1 + 1)} + \hat{c}_1 \frac{\varrho^{\theta_1}}{\Gamma_{\mathsf{q}}(\theta_1 + 1)} + \hat{c}_2 + \mathcal{I}_{\mathsf{q}}^{\alpha_1+\omega_1+\theta_1}[\mathcal{L}_{\mathsf{m}}(\varrho)],$$

where

$$g_{1\hat{\mathsf{m}}}(\varrho) = \mathsf{f}_1(\varrho) - \phi_{1\hat{\mathsf{m}}}^*(\varrho) - \Pi_1 \xi_{1\hat{\mathsf{m}}}^{*,\nu_2}(\varrho) - \chi_1 \rho_{\hat{\mathsf{m}}}^{*,\tau_2}(\varrho) - h_{\hat{\mathsf{m}}}^{*,\nu_2}(\varrho),$$

and $|\mathcal{L}_{\mathsf{m}}(\varrho)| \leqslant \epsilon_{\mathsf{m}}$, $\varrho \in \mathcal{J}$. By Lemma 3.1 and Remark 4.1, we have

$$|\hat{\mathsf{m}}(\varrho) - B\hat{\mathsf{m}}(\varrho)| = |\mathcal{I}_{\mathsf{q}}^{\alpha_1+\omega_1+\theta_1}[\mathcal{L}_{\mathsf{m}}(\varrho)]| \leqslant \epsilon_{\mathsf{m}} \mathcal{I}_{\mathsf{q}}^{\alpha_1+\omega_1+\theta_1}[T].$$

So,

$$\|\hat{\mathsf{m}} - B\hat{\mathsf{m}}\|_{\mathsf{X}} \leqslant \frac{\epsilon_{\mathsf{m}} T^{\alpha_1+\omega_1+\theta_1+1}}{\Gamma_{\mathsf{q}}(2 + \alpha_1 + \omega_1 + \theta_1)}. \tag{4.4}$$

On the other hand, we have

$$\begin{aligned} |\hat{\mathsf{m}}(\varrho) - \mathsf{m}(\varrho)| &= \Big|\hat{\mathsf{m}}(\varrho) - \mathcal{I}_{\mathsf{q}}^{\alpha_1+\omega_1+\theta_1}[g_{1\mathsf{m}}(\varrho)] + \Omega \mathcal{I}_{\mathsf{q}}^{\theta_1}[\mathsf{m}(\varrho)] \\ &\quad - c_0 \frac{\varrho^{\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\omega_1 + \theta_1 + 1)} - c_1 \frac{\varrho^{\theta_1}}{\Gamma_{\mathsf{q}}(\theta_1 + 1)} - c_2\Big| \\ &= |\hat{\mathsf{m}}(\varrho) - B\hat{\mathsf{m}}(\varrho) + B\hat{\mathsf{m}}(\varrho) - B\mathsf{m}(\varrho)| \\ &\leqslant |\hat{\mathsf{m}}(\varrho) - B\hat{\mathsf{m}}(\varrho)| + | + B\hat{\mathsf{m}}(\varrho) - B\mathsf{m}(\varrho)|. \end{aligned}$$

By (4.4) and ($\mathbf{F_3}$), we obtain

$$\|\hat{\mathsf{m}} - \mathsf{m}\|_{\mathsf{X}} \leqslant \frac{\epsilon_{\mathsf{m}} T^{\alpha_1+\omega_1+\theta_1+1}}{\Gamma_{\mathsf{q}}(2 + \alpha_1 + \omega_1 + \theta_1)} + (\mathcal{A}_1(\mathcal{U}_1) + \Psi_1)\|\hat{\mathsf{m}} - \mathsf{m}\|_{\mathsf{X}}.$$

Then

$$\|\hat{\mathsf{m}}-\mathsf{m}\|_{\mathsf{X}} \leqslant \frac{\epsilon_{\mathsf{m}} T^{\alpha_1+\omega_1+\theta_1+1}}{\Gamma_{\mathsf{q}}(2+\alpha_1+\omega_1+\theta_1)[1-(\mathcal{A}_1(\mathcal{U}_1)+\Psi_1)]}. \tag{4.5}$$

If we put

$$D_{\phi^*_{\mathsf{m}},\xi^*_{\mathsf{m}},\rho^*_{\mathsf{m}},h_{\mathsf{m}^*},\mathsf{f}_1} = \frac{T^{\alpha_1+\omega_1+\theta_1+1}}{\Gamma_{\mathsf{q}}(2+\alpha_1+\omega_1+\theta_1)[1-(\mathcal{A}_1(\mathcal{U}_1)+\Psi_1)]}.$$

Similarly, by using Remark 4.1 and (**F$_4$**), we obtain

$$\|\hat{\mathsf{u}}-\mathsf{u}\|_{\mathsf{X}} \leqslant \frac{\epsilon_{\mathsf{u}} T^{\alpha_2+\omega_2+\theta_2+1}}{\Gamma_{\mathsf{q}}(2+\alpha_2+\omega_2+\theta_2)[1-(\mathcal{A}_2(\mathcal{U}_2)+\Psi_2)]}. \tag{4.6}$$

If we take

$$D_{\phi^*_{\mathsf{u}},\xi^*_{\mathsf{u}},\rho^*_{\mathsf{u}},h_{\mathsf{u}^*},\mathsf{f}_2} = \frac{T^{\alpha_2+\omega_2+\theta_2+1}}{\Gamma_{\mathsf{q}}(2+\alpha_2+\omega_2+\theta_2)[1-(\mathcal{A}_2(\mathcal{U}_2)+\Psi_2)]}.$$

From (4.5) and (4.6), we get

$$\|(\hat{\mathsf{m}},\hat{\mathsf{u}})(\varrho)-(\hat{\mathsf{m}},\hat{\mathsf{u}})(\varrho)\|_{\mathsf{X}\times\mathsf{X}} \leqslant D_{\phi^*,\xi^*,\rho^*,h^*,\mathsf{f}}\epsilon_{\mathsf{m},\mathsf{u}},$$

where $D_{\phi^*,\xi^*,\rho^*,h^*,\mathsf{f}} = max(D_{\phi^*_{\mathsf{m}},\xi^*_{\mathsf{m}},\rho^*_{\mathsf{m}},h^*_{\mathsf{m}},\mathsf{f}_1}, D_{\phi^*_{\mathsf{u}},\xi^*_{\mathsf{u}},\rho^*_{\mathsf{u}},h^*_{\mathsf{u}},\mathsf{f}_2})$ and $\epsilon_{\mathsf{m},\mathsf{u}} = max(\epsilon_{\mathsf{m}},\epsilon_{\mathsf{u}})$. Therefore, we have derived that q-$\mathcal{FJCS}$ given by (1.1) is UHS. □

Theorem 4.2. *If the assumptions* (**F$_2$**)–(**F$_4$**)*,* (**F$_7$**)*, and* (3.5) *hold, then the* q-$\mathcal{FJCS}$ (1.1) *is* UHRS.

Proof. From Remark 4.2, we get

$$\begin{aligned}\mathsf{m}(\varrho) &= \mathcal{I}_{\mathsf{q}}^{\alpha_1+\omega_1+\theta_1}[g_{1\hat{\mathsf{m}}}(\varrho)] - \Omega\mathcal{I}_{\mathsf{q}}^{\theta_1}[\hat{\mathsf{m}}(\varrho)] + \hat{c}_0\frac{\varrho^{\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\omega_1+\theta_1+1)}\\ &+ \hat{c}_1\frac{\varrho^{\theta_1}}{\Gamma_{\mathsf{q}}(\theta_1+1)} + \hat{c}_2 + \mathcal{I}_{\mathsf{q}}^{\alpha_1+\omega_1+\theta_1}[\mathcal{M}_y(\varrho)],\end{aligned}$$

where $|\mathcal{M}_y(\varrho)| \leqslant \epsilon_{\mathsf{m}}\mathcal{M}_y(\varrho)$, $\varrho\in\mathcal{J}$. Thus, the inequality (4.2) has a solution $\hat{\mathsf{m}}\in\mathsf{X}$. The unique solution of (4.3) is given by $\mathsf{m}\in\mathsf{X}$. By applying Lemma 3.1 and (**F$_7$**), we get

$$\begin{aligned}|(\mathsf{m}) - B\hat{\mathsf{m}}(\varrho)| &= |\mathcal{I}_{\mathsf{q}}^{\alpha_1+\omega_1+\theta_1}[\mathcal{M}_y(\varrho)]|\\ &\leqslant \epsilon_{\mathsf{m}}\mathcal{I}_{\mathsf{q}}^{\alpha_1+\omega_1+\theta_1}[y(\varrho)] \qquad (4.7)\\ &\leqslant \epsilon_{\mathsf{m}}\mathcal{M}_y y(\varrho).\end{aligned}$$

From these relations, we have

$$\begin{aligned}|\hat{\mathsf{m}}(\varrho)-\mathsf{m}(\varrho)| &= \Big|\hat{\mathsf{m}}(\varrho) - \mathcal{I}_{\mathsf{q}}^{\alpha_1+\omega_1+\theta_1}[g_{1\mathsf{m}}(\varrho)] + \Omega\mathcal{I}_{\mathsf{q}}^{\theta_1}[\mathsf{m}(\varrho)]\\ &- c_0\frac{\varrho^{\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\omega_1+\theta_1+1)} - c_1\frac{\varrho^{\theta_1}}{\Gamma_{\mathsf{q}}(\theta_1+1)} - c_2\Big|\\ &= |\hat{\mathsf{m}}(\varrho) - B\hat{\mathsf{m}}(\varrho) + B\hat{\mathsf{m}}(\varrho) - B\mathsf{m}(\varrho)|\\ &\leqslant |\hat{\mathsf{m}}(\varrho) - B\hat{\mathsf{m}}(\varrho)| + | + B\hat{\mathsf{m}}(\varrho) - B\mathsf{m}(\varrho)|.\end{aligned}$$

By (4.7), we have

$$\|\hat{\mathsf{m}}-\mathsf{m}\|_{\mathsf{X}} \leqslant \epsilon_{\mathsf{m}}\mathcal{M}_y y(\varrho) + [\mathcal{A}_1(\mathcal{U}_1)+\Psi_1]\|\hat{\mathsf{m}}-\mathsf{m}\|_{\mathsf{X}}.$$

Then, we get

$$\|\hat{\mathsf{m}}-\mathsf{m}\|_{\mathsf{X}} \leqslant \frac{\mathcal{M}_y}{1-[\mathcal{A}_1(\mathcal{U}_1)+\Psi_1]}\epsilon_{\mathsf{m}} y(\varrho), \quad \varrho\in\mathcal{J}. \tag{4.8}$$

If we take

$$D_{\phi^*_{\mathsf{m}},\xi^*_{\mathsf{m}},\rho^*_{\mathsf{m}},h^*_{\mathsf{m}},\mathsf{f}_1} = \frac{\mathcal{M}_y}{1-[\mathcal{A}_1(\mathcal{U}_1)+\Psi_1]},$$

similarly, using Remark 4.2, we have

$$\|\hat{\mathsf{u}}-\mathsf{u}\|_{\mathsf{X}} \leqslant \frac{\mathcal{M}_z}{1-[\mathcal{A}_2(\mathcal{U}_2)+\Psi_2]}\epsilon_{\mathsf{u}} z(\varrho), \quad \varrho\in\mathcal{J}. \tag{4.9}$$

If we take

$$D_{\phi^*_{\mathsf{u}},\xi^*_{\mathsf{u}},\rho^*_{\mathsf{u}},h^*_{\mathsf{u}},\mathsf{f}_2} = \frac{\mathcal{M}_z}{1-[\mathcal{A}_2(\mathcal{U}_2)+\Psi_2]},$$

from (4.8) and (4.9), we obtain

$$\|(\hat{\mathsf{m}},\hat{\mathsf{u}})(\varrho)-(\mathsf{m},\mathsf{u})(\varrho)\|_{\mathsf{X}\times\mathsf{X}} \leqslant D_{\phi^*,\xi^*,\rho^*,h^*,\mathsf{f}}\mathcal{M}_{y,z}\epsilon_{\mathsf{m},\mathsf{u}},$$

where $D_{\phi^*,\xi^*,\rho^*,h^*,\mathsf{f}} = \max(D_{\phi^*_{\mathsf{m}},\xi^*_{\mathsf{m}},\rho^*_{\mathsf{m}},h^*_{\mathsf{m}},\mathsf{f}_1}, D_{\phi^*_{\mathsf{u}},\xi^*_{\mathsf{u}},\rho^*_{\mathsf{u}},h^*_{\mathsf{u}},\mathsf{f}_2})$ and $\epsilon_{\mathsf{m},\mathsf{u}} = \max(\epsilon_{\mathsf{m}},\epsilon_{\mathsf{u}})$. Thus, the q-$\mathcal{FJCS}$ (1.1) is UHRS. □

5. Example

Example 5.1. *Consider the* q-$\mathcal{FJCS}$ *as follows:*

$$
\begin{cases}
{}^{C}\mathrm{D}^{\frac{5}{7}}_{\frac{1}{3}}({}^{C}\mathrm{D}^{\frac{4}{5}}_{\frac{1}{3}}({}^{C}\mathrm{D}^{\frac{1}{6}}_{\frac{1}{3}}+\frac{1}{10}))\mathrm{m}(\varrho) \\
=\frac{sinh(\varrho)+3}{1+e^{\varrho}}-\frac{cos(\mathrm{m}(\varrho))}{24(e^{\varrho}+4)}-\frac{\sqrt{13}}{7\sqrt{51\pi}}\Bigg(\frac{sin(e^{\varrho}+2)}{5} \\
\quad+\frac{|\mathrm{m}(\varrho)|}{450\Big((\frac{\sqrt{e^{11}}}{45}+ln(\varrho))+21\Big)(|\frac{1}{4}+\mathrm{m}(\varrho)|)} \\
\quad+\frac{tan^{-1}({}^{C}\mathrm{D}^{\frac{3}{13}}_{\frac{1}{3}}\mathrm{m}(\varrho))}{450\Big((\frac{\sqrt{e^{11}}}{45}+ln(\varrho))+21\Big)}\Bigg) \\
\quad-\frac{\sqrt{31}}{201e^{17}}\Bigg(\frac{ln(e^{\varrho}+1)+4}{5\pi}+\frac{arccos(\mathrm{m}(\varrho))}{234\Big(e^{\varrho+17}+\frac{2\pi}{13}\Big)} \\
\quad+\frac{|{}^{C}\mathrm{D}^{\frac{10}{13}}_{\frac{1}{3}}\mathrm{m}(\varrho)|}{234\Big(e^{\varrho+17}+\frac{2\pi}{13}\Big)\Big(|{}^{C}\mathrm{D}^{\frac{3}{13}}_{\frac{1}{3}}\mathrm{m}(\varrho)|+2\Big)}\Bigg) \\
\quad-\int_0^{\varrho}\frac{(\varrho-\frac{1}{3}s)^{\frac{7}{8}-1}}{\Gamma_{\frac{1}{3}}(\frac{7}{8})}\frac{sin(e)}{20\sqrt{11}}\Bigg(\frac{tan(e^{s}+5)}{35} \\
\quad+\frac{|\mathrm{m}(s)|}{150\Big((\frac{\sqrt{24}}{135}+ln(s))+435\Big)(|\frac{1}{3}+\mathrm{m}(s)|)} \\
\quad+\frac{cos^{-1}({}^{C}\mathrm{D}^{\frac{3}{13}}_{\frac{1}{3}}\mathrm{m}(s))}{150\Big((\frac{\sqrt{24}}{135}+ln(s))+435\Big)}\Bigg)d_{\mathrm{q}}s, \quad \varrho\in\mathcal{J}=(0,1), \\
\mathrm{m}(0)=0, \\
\Big({}^{C}\mathrm{D}^{\frac{4}{5}}_{\frac{1}{3}}\Big({}^{C}\mathrm{D}^{\frac{1}{6}}_{\frac{1}{3}}+\frac{1}{10}\Big)\Big)\mathrm{m}\Big(\frac{1}{4}\Big)=0, \\
2\mathrm{m}(1)=\frac{1}{15}\mathcal{I}^{\frac{3}{17},\frac{2}{19},\frac{1}{14}}_{\frac{1}{3}}\mathrm{m}\Big(\frac{1}{17}\Big)+\frac{4}{15}\mathcal{I}^{\frac{4}{17},\frac{3}{19},\frac{3}{14}}_{\frac{1}{3}}\mathrm{m}\Big(\frac{2}{17}\Big),
\end{cases}
\tag{5.1}
$$

and

$$
\begin{cases}
{}^{C}\mathrm{D}^{\frac{4}{7}}_{\frac{1}{3}}({}^{C}\mathrm{D}^{\frac{5}{9}}_{\frac{1}{3}}({}^{C}\mathrm{D}^{\frac{2}{5}}_{\frac{1}{3}}+\frac{1}{11}))\mathrm{u}(\varrho) \\
=\frac{cosh(\varrho)+7}{34+e^{\varrho}}-\frac{cos(\mathrm{u}(\varrho))}{45(\sqrt{\varrho}+4)}-\frac{log(23)}{19e^{6}}\Bigg(\frac{cos(e^{\varrho}+5)}{7} \\
\quad+\frac{|\mathrm{u}(\varrho)|}{559\Big((\frac{\sqrt{e^{12}}}{45}+ln(\varrho))+41\Big)(|\frac{1}{4}+\mathrm{u}(\varrho)|)} \\
\quad+\frac{sin^{-1}({}^{C}\mathrm{D}^{\frac{3}{11}}_{\frac{1}{3}}\mathrm{u}(\varrho))}{559\Big((\frac{\sqrt{e^{12}}}{45}+ln(\varrho))+41\Big)}\Bigg)-\frac{log(43)}{345\sqrt{19}}\Bigg(\frac{ln(e^{\varrho}+4)+5}{7\pi} \\
\quad+\frac{arctan(\mathrm{u}(\varrho))}{167\Big(e^{\varrho+19}+\frac{7\pi}{33}\Big)}+\frac{|{}^{C}\mathrm{D}^{\frac{5}{11}}_{\frac{1}{3}}\mathrm{u}(\varrho)|}{167\Big(e^{\varrho+19}+\frac{7\pi}{33}\Big)}\Bigg) \\
\quad-\int_0^{\varrho}\frac{(\varrho-\frac{1}{3}s)^{\frac{5}{6}-1}}{\Gamma_{\frac{1}{3}}(\frac{5}{6})}\frac{cos(\sqrt{e})}{49\sqrt{11}}\Bigg(\frac{tanh(e^{s}+4)}{19} \\
\quad+\frac{|\mathrm{u}(s)|}{257\Big((\frac{\sqrt{43}}{567}+ln(s))+348\Big)(|\frac{8}{5}+\mathrm{u}(s)|)} \\
\quad+\frac{sin^{-1}({}^{C}\mathrm{D}^{\frac{3}{11}}_{\frac{1}{3}}\mathrm{u}(s))}{234\Big((\frac{\sqrt{43}}{567}+ln(s))+348\Big)}\Bigg)d_{\mathrm{q}}s, \quad \varrho\in\mathcal{J}=(0,1), \\
\mathrm{u}(0)=0, \\
\Big({}^{C}\mathrm{D}^{\frac{5}{9}}_{\frac{1}{3}}\Big({}^{C}\mathrm{D}^{\frac{2}{5}}_{\frac{1}{3}}+\frac{1}{11}\Big)\Big)\mathrm{u}\Big(\frac{8}{13}\Big)=0, \\
4\mathrm{u}(1)=\frac{4}{11}\mathcal{I}^{\frac{7}{10},\frac{1}{12},\frac{6}{13}}_{\frac{1}{3}}\mathrm{u}\Big(\frac{9}{14}\Big)+\frac{7}{11}\mathcal{I}^{\frac{3}{10},\frac{5}{12},\frac{2}{13}}_{\frac{1}{3}}\mathrm{u}\Big(\frac{3}{14}\Big),
\end{cases}
\tag{5.2}
$$

where $\mathrm{q}=\frac{1}{3}$, $\alpha_1=\frac{5}{7}$, $\alpha_2=\frac{4}{7}$, $\omega_1=\frac{4}{5}$, $\omega_2=\frac{5}{9}$, $\theta_1=\frac{1}{6}$, $\theta_2=\frac{2}{5}$, $\nu_1=\frac{3}{11}$, $\nu_2=\frac{3}{13}$, $\tau_1=\frac{5}{11}$, $\tau_2=\frac{10}{13}$, $\Omega=\frac{1}{10}$, $\Upsilon=\frac{1}{11}$, $\delta_1=\frac{1}{4}$, $\delta_2=\frac{8}{13}$, $\vartheta_1=\varphi_1=\frac{7}{8}$, $\vartheta_2=\varphi_2=\frac{5}{6}$, $\beta_1=\frac{1}{14}$, $\beta_2=\frac{3}{14}$, $\beta_3=\frac{6}{13}$, $\beta_4=\frac{2}{13}$, $\lambda_1=\frac{1}{15}$, $\lambda_2=\frac{4}{15}$, $\lambda_3=\frac{4}{11}$, $\lambda_4=\frac{7}{11}$, $\eta_1=\frac{3}{17}$, $\eta_2=\frac{4}{17}$, $\eta_3=\frac{7}{10}$, $\eta_4=\frac{3}{10}$, $\zeta_1=\frac{1}{17}$, $\zeta_2=\frac{2}{17}$, $\zeta_3=\frac{9}{14}$, $\zeta_4=\frac{1}{14}$, $\mu_1=\frac{2}{19}$, $\mu_2=\frac{3}{19}$, $\mu_3=\frac{1}{12}$, $\mu_4=\frac{5}{12}$, $a_1=2$, $a_2=4$, $n=2$, *and* $T=1$.

$$
\begin{aligned}
&\Bigg|{}^{C}\mathrm{D}^{\frac{5}{7}}_{\frac{1}{3}}\Big({}^{C}\mathrm{D}^{\frac{4}{5}}_{\frac{1}{3}}\Big({}^{C}\mathrm{D}^{\frac{1}{6}}_{\frac{1}{3}}+\frac{1}{10}\Big)\Big)\mathrm{m}(\varrho)-\Big(\frac{sinh\varrho+3}{1+e^{\varrho}}-\phi^{*}_{1\mathrm{m}} \\
&\quad-\frac{\sqrt{13}}{7\sqrt{51\pi}}\xi^{*,\nu_2}_{1\mathrm{m}}-\frac{\sqrt{31}}{201e^{17}}\rho^{*,\tau_2}_{1\mathrm{m}}-\frac{sin(e)}{20\sqrt{11}}h^{*,\nu_2}_{1\mathrm{m}}\Big)\Bigg|\leqslant\epsilon_{\mathrm{m}}, \\
&\Bigg|{}^{C}\mathrm{D}^{\frac{5}{7}}_{\frac{1}{3}}\Big({}^{C}\mathrm{D}^{\frac{4}{5}}_{\frac{1}{3}}\Big({}^{C}\mathrm{D}^{\frac{1}{6}}_{\frac{1}{3}}+\frac{1}{10}\Big)\Big)\mathrm{m}(\varrho)-\Big(\frac{sinh\varrho+3}{1+e^{\varrho}}-\phi^{*}_{1\mathrm{m}} \\
&\quad-\frac{\sqrt{13}}{7\sqrt{51\pi}}\xi^{*,\nu_2}_{1\mathrm{m}}-\frac{\sqrt{31}}{201e^{17}}\rho^{*,\tau_2}_{1\mathrm{m}}-\frac{sin(e)}{20\sqrt{11}}h^{*,\nu_2}_{1\mathrm{m}}\Big)\Bigg|\leqslant\mathcal{M}_y(\varrho)\epsilon_{\mathrm{m}},
\end{aligned}
$$

where

$$
\begin{aligned}
\phi^{*}_{1\mathrm{m}}(\varrho)&=\frac{cos(\mathrm{m}(\varrho))}{24(e^{\varrho}+4)}, \\
\xi^{*,\nu_2}_{1\mathrm{m}}(\varrho)&=\frac{sin(e^{\varrho}+2)}{5}+\frac{|\mathrm{m}(\varrho)|}{450\Big((\frac{\sqrt{e^{11}}}{45}+ln(\varrho))+21\Big)(|\frac{1}{4}+\mathrm{m}(\varrho)|)} \\
&\quad+\frac{tan^{-1}({}^{C}\mathrm{D}^{\frac{3}{13}}_{\frac{1}{3}}\mathrm{m}(\varrho))}{450\Big((\frac{\sqrt{e^{11}}}{45}+ln(\varrho))+21\Big)}, \\
\rho^{*,\tau_2}_{\mathrm{m}}(\varrho)&=\frac{ln(e^{\varrho}+1)+4}{5\pi}+\frac{arccos(\mathrm{m}(\varrho))}{234\Big(e^{\varrho+17}+\frac{2\pi}{13}\Big)} \\
&\quad+\frac{|{}^{C}\mathrm{D}^{\frac{10}{13}}_{\frac{1}{3}}\mathrm{m}(\varrho)|}{234\Big(e^{\varrho+17}+\frac{2\pi}{13}\Big)\Big(|{}^{C}\mathrm{D}^{\frac{3}{13}}_{\frac{1}{3}}\mathrm{m}(\varrho)|+2\Big)}, \\
h^{*,\nu_2}_{\mathrm{m}}(\varrho)&=\int_0^{\varrho}\frac{(\varrho-\frac{1}{3}s)^{\frac{7}{8}-1}}{\Gamma_{\frac{1}{3}}(\frac{7}{8})}\Bigg(\frac{tan(e^{s}+5)}{35} \\
&\quad+\frac{|\mathrm{m}(s)|}{150\Big((\frac{\sqrt{24}}{135}+ln(s))+435\Big)(|\frac{1}{3}+\mathrm{m}(s)|)} \\
&\quad+\frac{cos^{-1}({}^{C}\mathrm{D}^{\frac{3}{13}}_{\frac{1}{3}}\mathrm{m}(s))}{150\Big((\frac{\sqrt{24}}{135}+ln(s))+435\Big)}\Bigg)d_{\mathrm{q}}s.
\end{aligned}
$$

Here, we have

$$
\Pi_1=\frac{\sqrt{13}}{7\sqrt{51\pi}},\ \chi_1=\frac{\sqrt{31}}{201e^{17}},\ \Omega=\frac{1}{10},\ b_1=\frac{1}{450(\frac{\sqrt{e^{11}}}{45}+21)},
$$

$$
b_2=\frac{1}{234(e^{17}+\frac{2\pi}{13})},\ b_3=\frac{1}{150(\frac{\sqrt{24}}{135}+435)},\ b_4=\frac{1}{e^{1}+4}.
$$

$$\mathcal{U}_1 = \Pi_1 b_1 + \chi_1 b_2 + \frac{\varrho^{\vartheta_1}}{\lceil \vartheta_1 \rceil_{\mathsf{q}} \Gamma_{\mathsf{q}}(\varphi_1)} b_3 + b_4 \approx 0.00648873.$$

Similarly,

$$\begin{aligned}
&\Pi_2 = \frac{log(23)}{19e^6}, \chi_2 = \frac{log(43)}{345\sqrt{19}},\\
&\Upsilon = \frac{1}{11},\ p_1 = \frac{1}{559(\frac{\sqrt{e^{12}}}{45} + 41)},\\
&p_2 = \frac{1}{167(e^{19} + \frac{7\pi}{33})}, p_3 = \frac{1}{257(\frac{\sqrt{43}}{567} + 348)}, p_4 = \frac{1}{225}.\\
&\mathcal{U}_2 = \Pi_2 p_1 + \chi_2 p_2 + \frac{\varrho^{\vartheta_2}}{\lceil \vartheta_2 \rceil_{\mathsf{q}} \Gamma_{\mathsf{q}}(\varphi_2)} p_3 + p_4 \approx 0.00445599.\\
&\lceil \theta_1 \rceil_{\mathsf{q}} = 0.250975,\ \lceil \theta_2 \rceil_{\mathsf{q}} = 0.533409,\\
&\lceil \omega_1 + \theta_1 \rceil_{\mathsf{q}} = 0.98135,\ \lceil \omega_2 + \theta_2 \rceil_{\mathsf{q}} = 0.974981.
\end{aligned}$$

We can also find

$$\begin{aligned}
\mathit{\Theta}_1 = \Bigg[&\frac{T^{\alpha_1+\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\alpha_1+\omega_1+\theta_1+1)} + \frac{T^{\theta_1}}{|\Delta_1|\Gamma_{\mathsf{q}}(\theta_1+1)}\\
&\times\Bigg(\sum_{i=1}^{n} \frac{\lambda_i\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)} \int_0^{\zeta_i} \frac{(\zeta_i^{\beta_i} - \mathsf{q}s^{\beta_i})^{(\mu_i-1)} s^{\eta_i} \zeta_i^{\alpha_1+\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\alpha_1+\omega_1+\theta_1+1)} d_{\mathsf{q}}s\\
&+ \frac{a_1 T^{\alpha_1+\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\alpha_1+\omega_1+\theta_1+1)}\Bigg) + \frac{\delta_1^{\alpha_1}}{\Gamma_{\mathsf{q}}(\alpha_1+1)}\Bigg(\frac{T^{\theta_1}}{|\Delta_1|\Gamma_{\mathsf{q}}(\theta_1+1)}\\
&\times\Bigg(\frac{a_1 T^{\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\omega_1+\theta_1+1)} + \sum_{i=1}^{n} \lambda_i\beta_i[\frac{1}{\beta_i}]_{\mathsf{q}}\\
&\times \frac{\Gamma_{\mathsf{q}}(\eta_i+1+\frac{\omega_1+\theta_1}{\beta_i})\zeta_i^{\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\mu_i+\eta_i+1+\frac{\omega_1+\theta_1}{\beta_i})\Gamma_{\mathsf{q}}(\omega_1+\theta_1+1)}\Bigg)\\
&+ \frac{T^{\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\omega_1+\theta_1+1)}\Bigg)\Bigg] \approx 2.41643,\\
\mathit{\Theta}_2 = \Bigg[&\frac{T^{\alpha_2+\omega_2+\theta_2}}{\Gamma_{\mathsf{q}}(\alpha_2+\omega_2+\theta_2+1)} + \frac{T^{\theta_2}}{|\Delta_2|\Gamma_{\mathsf{q}}(\theta_2+1)}\\
&\times\Bigg(\sum_{i=1}^{n} \frac{\lambda_i\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)} \int_0^{\zeta_i} \frac{(\zeta_i^{\beta_i} - \mathsf{q}s^{\beta_i})^{(\mu_i-1)} s^{\eta_i} \zeta_i^{\alpha_1+\omega_2+\theta_2}}{\Gamma_{\mathsf{q}}(\alpha_2+\omega_2+\theta_2+1)} d_{\mathsf{q}}s\\
&+ \frac{a_2 T^{\alpha_2+\omega_2+\theta_2}}{\Gamma_{\mathsf{q}}(\alpha_2+\omega_2+\theta_2+1)}\Bigg)\\
&+ \frac{\delta_2^{\alpha_2}}{\Gamma_{\mathsf{q}}(\alpha_2+1)}\Bigg(\frac{T^{\theta_2}}{|\Delta_2|\Gamma_{\mathsf{q}}(\theta_2+1)}\Bigg(\frac{a_2 T^{\omega_2+\theta_2}}{\Gamma_{\mathsf{q}}(\omega_2+\theta_2+1)}\\
&+ \sum_{i=1}^{n} \lambda_i\beta_i[\frac{1}{\beta_i}]_{\mathsf{q}} \frac{\Gamma_{\mathsf{q}}(\eta_i+1+\frac{\omega_1+\theta_1}{\beta_i})\zeta_i^{\omega_2+\theta_2}}{\Gamma_{\mathsf{q}}(\mu_i+\eta_i+1+\frac{\omega_2+\theta_2}{\beta_i})\Gamma_{\mathsf{q}}(\omega_2+\theta_2+1)}\Bigg)\\
&+ \frac{T^{\omega_2+\theta_2}}{\Gamma_{\mathsf{q}}(\omega_2+\theta_2+1)}\Bigg)\Bigg] \approx 3.31115,\\
\mathit{\Phi}_1 = \Omega\Bigg(&\frac{T^{\theta_1}}{|\Delta_1|\Gamma_{\mathsf{q}}(\theta_1+1)}\Bigg(\frac{a_1 T^{\theta_1}}{\Gamma_{\mathsf{q}}(\theta_1+1)} + \sum_{i=1}^{n} \frac{\lambda_i\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)}\\
&\times \int_0^{\zeta_i} \frac{(\zeta_i^{\beta_i} - \mathsf{q}s^{\beta_i})^{(\mu_i-1)} s^{\eta_i} \zeta_i^{\theta}}{\Gamma_{\mathsf{q}}(\theta_1+1)} d_{\mathsf{q}}s\Bigg) + \frac{T^{\theta_1}}{\Gamma_{\mathsf{q}}(\theta_1+1)}\Bigg)\\
&\approx 0.205737,\\
\mathit{\Phi}_2 = \Upsilon\Bigg(&\frac{T^{\theta_2}}{|\Delta_2|\Gamma_{\mathsf{q}}(\theta_2+1)}\Bigg(\frac{a_2 T^{\theta_2}}{\Gamma_{\mathsf{q}}(\theta_2+1)} + \sum_{i=1}^{n} \frac{\lambda_i\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)}\\
&\times \int_0^{\zeta_i} \frac{(\zeta_i^{\beta_i} - \mathsf{q}s^{\beta_i})^{(\mu_i-1)} s^{\eta_i} \zeta_i^{\theta}}{\Gamma_{\mathsf{q}}(\theta_2+1)} d_{\mathsf{q}}s\Bigg) + \frac{T^{\theta_2}}{\Gamma_{\mathsf{q}}(\theta_2+1)}\Bigg)\\
&\approx 0.18897,
\end{aligned}$$

and

$$\begin{aligned}
\pi_1 = \Bigg[&\frac{T^{\alpha_1+\omega_1+\theta_1-1}}{\Gamma_{\mathsf{q}}(\alpha_1+\omega_1+\theta_1)} + \frac{\lceil\theta_1\rceil_{\mathsf{q}} T^{\theta_1-1}}{|\Delta_1|\Gamma_{\mathsf{q}}(\theta_1+1)}\Bigg(\sum_{i=1}^{n} \frac{\lambda_i\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)}\\
&\times \int_0^{\zeta_i} \frac{(\zeta_i^{\beta_i} - \mathsf{q}s^{\beta_i})^{(\mu_i-1)} s^{\eta_i} \zeta_i^{\alpha_1+\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\alpha_1+\omega_1+\theta_1+1)} d_{\mathsf{q}}s\\
&+ \frac{a_1 T^{\alpha_1+\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\alpha_1+\omega_1+\theta_1+1)}\Bigg)\\
&+ \frac{\delta_1^{\alpha_1}}{\Gamma_{\mathsf{q}}(\alpha_1+1)}\Bigg(\frac{\lceil\theta_1\rceil_{\mathsf{q}} T^{\theta_1-1}}{|\Delta_1|\Gamma_{\mathsf{q}}(\theta_1+1)}\Bigg(\frac{a_1 T^{\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\omega_1+\theta_1+1)} + \sum_{i=1}^{n} \lambda_i\beta_i[\frac{1}{\beta_i}]_{\mathsf{q}}\\
&\times \frac{\Gamma_{\mathsf{q}}(\eta_i+1+\frac{\omega_1+\theta_1}{\beta_i})\zeta_i^{\omega_1+\theta_1}}{\Gamma_{\mathsf{q}}(\mu_i+\eta_i+1+\frac{\omega_1+\theta_1}{\beta_i})\Gamma_{\mathsf{q}}(\omega_1+\theta_1+1)}\Bigg)\\
&+ \frac{\lceil\omega_1+\theta_1\rceil_{\mathsf{q}} T^{\omega_1+\theta_1-1}}{\Gamma_{\mathsf{q}}(\omega_1+\theta_1+1)}\Bigg)\Bigg] \approx 0.694472,\\
\pi_2 = \Bigg[&\frac{T^{\alpha_2+\omega_2+\theta_2-1}}{\Gamma_{\mathsf{q}}(\alpha_2+\omega_2+\theta_2)} + \frac{\lceil\theta_2\rceil_{\mathsf{q}} T^{\theta_2-1}}{|\Delta_2|\Gamma_{\mathsf{q}}(\theta_2+1)}\Bigg(\sum_{i=1}^{n} \frac{\lambda_i\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)}\\
&\times \int_0^{\zeta_i} \frac{(\zeta_i^{\beta_i} - \mathsf{q}s^{\beta_i})^{(\mu_i-1)} s^{\eta_i} \zeta_i^{\alpha_2+\omega_2+\theta_2}}{\Gamma_{\mathsf{q}}(\alpha_2+\omega_2+\theta_2+1)} d_{\mathsf{q}}s + \frac{a_2 T^{\alpha_2+\omega_2+\theta_2}}{\Gamma_{\mathsf{q}}(\alpha_2+\omega_2+\theta_2+1)}\Bigg)\\
&+ \frac{\delta_2^{\alpha_2}}{\Gamma_{\mathsf{q}}(\alpha_2+1)}\Bigg(\frac{\lceil\theta_2\rceil_{\mathsf{q}} T^{\theta_2-1}}{|\Delta_2|\Gamma_{\mathsf{q}}(\theta_2+1)}\Bigg(\frac{a_2 T^{\omega_2+\theta_2}}{\Gamma_{\mathsf{q}}(\omega_2+\theta_2+1)} + \sum_{i=1}^{n} \lambda_i\beta_i[\frac{1}{\beta_i}]_{\mathsf{q}}\\
&\times \frac{\Gamma_{\mathsf{q}}(\eta_i+1+\frac{\omega_2+\theta_2}{\beta_i})\zeta_i^{\omega_1+\theta_2}}{\Gamma_{\mathsf{q}}(\mu_i+\eta_i+1+\frac{\omega_2+\theta_2}{\beta_i})\Gamma_{\mathsf{q}}(\omega_2+\theta_2+1)}\Bigg)\\
&+ \frac{\lceil\omega_2+\theta_2\rceil_{\mathsf{q}} T^{\omega_2+\theta_2-1}}{\Gamma_{\mathsf{q}}(\omega_2+\theta_2+1)}\Bigg)\Bigg] \approx 1.68793,\\
\varpi_1 = \Omega\Bigg(&\frac{\lceil\theta_1\rceil_{\mathsf{q}} T^{\theta_1-1}}{|\Delta_1|\Gamma_{\mathsf{q}}(\theta_1+1)}\Bigg(\frac{a_1 T^{\theta_1}}{\Gamma_{\mathsf{q}}(\theta_1+1)} + \sum_{i=1}^{n} \frac{\lambda_i\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)}\\
&\times \int_0^{\zeta_i} \frac{(\zeta_i^{\beta_i} - \mathsf{q}s^{\beta_i})^{(\mu_i-1)} s^{\eta_i} \zeta_i^{\theta_1}}{\Gamma_{\mathsf{q}}(\theta_1+1)} d_{\mathsf{q}}s\Bigg) + \frac{T^{\theta_1-1}}{\Gamma_{\mathsf{q}}(\theta_1)}\Bigg) \approx 0.0516349,\\
\varpi_2 = \Upsilon\Bigg(&\frac{\lceil\theta_2\rceil_{\mathsf{q}} T^{\theta_2-1}}{|\Delta_2|\Gamma_{\mathsf{q}}(\theta_2+1)}\Bigg(\frac{a_2 T^{\theta_2}}{\Gamma_{\mathsf{q}}(\theta_2+1)} + \sum_{i=1}^{n} \frac{\lambda_i\beta_i\zeta_i^{-\beta_i(\mu_i+\eta_i)}}{\Gamma_{\mathsf{q}}(\mu_i)}\\
&\times \int_0^{\zeta_i} \frac{(\zeta_i^{\beta_i} - \mathsf{q}s^{\beta_i})^{(\mu_i-1)} s^{\eta_i} \zeta_i^{\theta_2}}{\Gamma_{\mathsf{q}}(\theta_2+1)} d_{\mathsf{q}}s\Bigg) + \frac{T^{\theta_2-1}}{\Gamma_{\mathsf{q}}(\theta_2)}\Bigg) \approx 0.100798.
\end{aligned}$$

$$\mathcal{A}_1 = \left(\Theta_1 + \frac{\pi_1}{\Gamma_{\mathsf{q}}(2-\nu_2)} + \frac{\pi_1}{\Gamma_{\mathsf{q}}(2-\tau_2)}\right) \approx 3.87166,$$
$$\Psi_1 = \Phi_1 + \frac{\varpi_1}{\Gamma_{\mathsf{q}}(2-\nu_2)} + \frac{\varpi_1}{\Gamma_{\mathsf{q}}(2-\tau_2)} \approx 0.313935,$$
$$\mathcal{U}_3 = (1-\Psi_1)(\mathcal{A}_1^{-1}) \approx 0.177202,$$
$$\mathcal{U}_1 \approx 0.00648873 \leqslant 0.177202 \approx \mathcal{U}_3.$$

Similarly,

$$\mathcal{A}_2 = \left(\Theta_2 + \frac{\pi_2}{\Gamma_{\mathsf{q}}(2-\nu_1)} + \frac{\pi_2}{\Gamma_{\mathsf{q}}(2-\tau_1)}\right) \approx 6.87511,$$
$$\Psi_2 = \Phi_2 + \frac{\varpi_2}{\Gamma_{\mathsf{q}}(2-\nu_1)} + \frac{\varpi_2}{\Gamma_{\mathsf{q}}(2-\tau_1)} \approx 0.401798,$$
$$\mathcal{U}_4 = (1-\Psi_2)(\mathcal{A}_2^{-1}) \approx 0.0870098,$$
$$\mathcal{U}_2 \approx 0.00445599 \leqslant 0.0870098 \approx \mathcal{U}_4.$$

Hence, the inequality (3.5) *in Theorem 3.1 holds. On the other hand, we see that*

$$\begin{aligned}
\mathcal{V}_1 &= sup_{\varrho\in\mathcal{J}}|\xi_{1\mathsf{m}}(\varrho,0,0)| \\
&= sup_{\varrho\in\mathcal{J}}\left|\frac{sin(e^{\varrho}+2)}{5}\right| = 0.199997, \\
\mathcal{V}_2 &= sup_{\varrho\in\mathcal{J}}|\rho_{1\mathsf{m}}(\varrho,0,0)| \\
&= sup_{\varrho\in\mathcal{J}}\left|\frac{ln(e^{\varrho}+1)+4}{5\pi}\right| = 0.130099, \\
\mathcal{V}_3 &= sup_{\varrho\in\mathcal{J}}|h_{1\mathsf{m}}(\varrho,0,0)| \\
&= sup_{\varrho\in\mathcal{J}}\left|\frac{tan(e^{\varrho}+5)}{35}\right| = 0.209255, \\
\mathcal{V}_4 &= sup_{\varrho\in\mathcal{J}}|\phi_{1\mathsf{m}}(\varrho,0)| \\
&= sup_{\varrho\in\mathcal{J}}\left|\frac{cos(\mathsf{m}(\varrho))}{24(e^{\varrho}+4)}\right| = 0.00620198, \\
\mathcal{V}_5 &= sup_{\varrho\in\mathcal{J}}|\mathsf{f}_1(\varrho)| \\
&= sup_{\varrho\in\mathcal{J}}\left|\frac{sinh(\varrho)+3}{1+e^{\varrho}}\right| = 1.12288.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathcal{K}_1 &= sup_{\varrho\in\mathcal{J}}|\xi_{2\mathsf{u}}(\varrho,0,0)| \\
&= sup_{\varrho\in\mathcal{J}}\left|\frac{cos(e^{\varrho}+5)}{7}\right| = 0.0193262, \\
\mathcal{K}_2 &= sup_{\varrho\in\mathcal{J}}|\rho_{2\mathsf{u}}(\varrho,0,0)| \\
&= sup_{\varrho\in\mathcal{J}}\left|\frac{ln(e^{\varrho}+5)+4}{7\pi}\right| = 0.111915, \\
\mathcal{K}_3 &= sup_{\varrho\in\mathcal{J}}|h_{2\mathsf{u}}(\varrho,0,0)| \\
&= sup_{\varrho\in\mathcal{J}}\left|\frac{tanh(e^{\varrho}+4)}{19}\right| = 0.0526314, \\
\mathcal{K}_4 &= sup_{\varrho\in\mathcal{J}}|\phi_{2\mathsf{u}}(\varrho,0)| \\
&= sup_{\varrho\in\mathcal{J}}\left|\frac{cos(\mathsf{u}(\varrho))}{45(\sqrt{\varrho}+4)}\right| = 0.00444444, \\
\mathcal{K}_5 &= sup_{\varrho\in\mathcal{J}}|\mathsf{f}_2(\varrho)| \\
&= sup_{\varrho\in\mathcal{J}}\left|\frac{cosh(\varrho)+7}{34+e^{\varrho}}\right| = 0.232666.
\end{aligned}$$

Hence, $\mathcal{V} = \max\mathcal{V}_o \approx 1.12288$ *and* $\mathcal{K} = \max\mathcal{K}_o \approx 0.232666$. *Therefore, according to Theorem 3.1, the* q-$\mathcal{FJCS}$ *defined by* (5.1) *possesses a unique solution within the interval* $[0,1]$. *Now,*

$$\Psi = \max\{\Psi_1,\Psi_2\} = \max\{0.313935, 0.401798\} < 1,$$

which shows that (3.11) *in Theorem 3.2 holds. Therefore,* (5.1) *has at least one solution in the interval* $[0,1]$. *Furthermore, in accordance with Theorem 4.1, the* UHS *is given by*

$$\begin{aligned}
\|\hat{\mathsf{m}}-\mathsf{m}\|_{\mathsf{X}} &\leqslant \frac{\epsilon_{\mathsf{m}} T^{\alpha_1+\omega_1+\theta_1+1}}{\Gamma_{\mathsf{q}}(2+\alpha_1+\omega_1+\theta_1)[1-(\mathcal{A}_1(\mathcal{U}_1)+\Psi_1)]} \\
&\leqslant 0.454923\,\epsilon_{\mathsf{m}}, \quad \epsilon_{\mathsf{m}} > 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|\hat{\mathsf{u}}-\mathsf{u}\|_{\mathsf{X}} &\leqslant \frac{\epsilon_{\mathsf{u}} T^{\alpha_2+\omega_2+\theta_2+1}}{\Gamma_{\mathsf{q}}(2+\alpha_2+\omega_2+\theta_2)[1-(\mathcal{A}_2(\mathcal{U}_2)+\Psi_2)]} \\
&\leqslant 0.452914\,\epsilon_{\mathsf{u}}, \quad \epsilon_{\mathsf{u}} > 0.
\end{aligned}$$

Consider $y(\varrho) = \varrho^{\frac{e^2}{13}}$, *and then we obtain*

$$\mathcal{I}_{\frac{1}{3}}^{\frac{5}{7}+\frac{4}{5}+\frac{1}{6}}[y(\varrho)] \leqslant 0.646431\varrho^{\frac{e^2}{13}} = \mathcal{M}_y y(\varrho).$$

Similarly, $z(\varrho) = \varrho^{\frac{\sqrt{3}}{14}}$, *and then we get*

$$\mathcal{I}_{\frac{1}{3}}^{\frac{4}{7}+\frac{5}{9}+\frac{2}{5}}[z(\varrho)] \leqslant 0.812674\varrho^{\frac{\sqrt{3}}{14}} = \mathcal{M}_z z(\varrho).$$

Consequently, $y(\varrho) = \varrho^{\frac{e^2}{13}}$, $z(\varrho) = \varrho^{\frac{\sqrt{3}}{14}}$, $\mathcal{M}_y = 0.646431$, *and* $\mathcal{M}_z = 0.812674$ *satisfy hypothesis* ($\mathbf{F}_7$). *Theorem 4.2 indicates that* q-$\mathcal{FJCS}$ (5.1) *is* UHR *stable, having*

$$\begin{aligned}
\|\hat{\mathsf{m}}-\mathsf{m}\|_{\mathsf{X}} &\leqslant \frac{\mathcal{M}_y}{1-[\mathcal{A}_1(\mathcal{U}_1)+\Psi_1]}\epsilon_{\mathsf{m}} y(\varrho) \\
&\leqslant 0.978044\,\epsilon_{\mathsf{m}}\,\varrho^{\frac{e^2}{13}}, \quad \epsilon_{\mathsf{m}} > 0,\ \varrho\in[0,1],
\end{aligned}$$

and similarly we have

$$\begin{aligned}
\|\hat{\mathsf{u}}-\mathsf{u}\|_{\mathsf{X}} &\leqslant \frac{\mathcal{M}_z}{1-[\mathcal{A}_2(\mathcal{U}_2)+\Psi_2]}\epsilon_{\mathsf{u}} z(\varrho) \\
&\leqslant 1.43186\,\epsilon_{\mathsf{u}}\,\varrho^{\frac{\sqrt{3}}{14}}, \quad \epsilon_{\mathsf{u}} > 0,\ \varrho\in[0,1].
\end{aligned}$$

6. Conclusions

In this article, the sufficient conditions for obtaining the $\mathscr{E}\mathscr{U}$ of the solution of our considered system (1.1) have been derived with the help of the Leray-Schauder alternative and the Banach FP theorem. Moreover, the different kinds of $\mathscr{U}\mathscr{S}$ have been derived for q-($\mathcal{FJCS}$) (1.1) under specific assumptions. An example is presented at the end to verify our main theoretical results. For the notion of Hyers-Ulam stability, there is a gap between the exact and approximate solutions of the problem under consideration. That is why this theory has great importance in approximation theory and numerical analysis. That is why our results may have many applications in system theories of the mentioned areas.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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