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***Research article***

# Inverse problem for time dependent coefficients in the higher order pseudo-parabolic equation

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**Abstract:** In this paper, we considered an inverse problem of recovering the time dependent potential and force coefficients in the third order pseudoparabolic equation from nonlocal integral observations. Existence and uniqueness of the solution are proved by means of the contraction principle on a small time interval. The stability results for the inverse problem is presented. The unique solvability theorem for this inverse problem is proved. However, since the governing equation is yet ill-posed (very slight errors in the integral input may cause relatively significant errors in the output potential and heat source terms), we need to regularize the solution. Therefore, to get a stable solution, a regularized cost function is to be minimized for retrieval of the unknown terms. The third order pseudoparabolic problem is discretized using the cubic B-spline (CB-spline) collocation technique and reshaped as nonlinear least-squares optimization of the Tikhonov regularization function. This is numerically solved by means of the MATLAB subroutine *lsqnonlin* tool. Both analytical and perturbed data are inverted. Numerical outcomes for two benchmark test examples are reported and discussed. In addition, the von Neumann stability analysis for the proposed numerical approach has also been discussed.

**Keywords:** third-order pseudo-parabolic equation; inverse problem; stability analysis; Tikhonov regularization; nonlinear optimization

## 1. Introduction

An equation whose time derivative appears in the highest order term is called a pseudo-parabolic equation. The study of inverse boundary value problems involving pseudo-parabolic partial differential equations is a very interesting topic of research due to their wide range of applications in real life phenomena such as infiltration of homogeneous fluids in strata [1], nonsteady flows of second order fluids [2], the consolidation of clays [3], the theory of small oscillation of a rotating fluid [4], the diffusion of radiation [5], thermodynamics [6], waves in shallow water [7], nonlinear propagation of the quantum ion-acoustic waves [8], singular and dark soliton solutions [9], traveling waves [10], and the references therein [11–15]. However, relatively less amount of related work is found in open literature. The authors in [16–18] presented their work on detection of unknown source term in parabolic and hyperbolic equations subject to over specified data. Later on, Rundell [19] extended their work and proposed a technique based on semigroups of operators to compute an unknown forcing term in linear parabolic and pseudo-parabolic equations. Asanov and Atmanov [20] derived some results on existence of unique solutions of a class of inverse problems associated with the pseudo-parabolic operator equation. Elshafiey et al. [21] employed the Hopfield neural network to reconstruct the source term based on data obtained from electromagnetic measurements. Yaman and Gozukizil [22] examined the asymptotic nature of the solutions of the pseudo-parabolic inverse problem with the unknown nonhomogeneous term subject to the nonclassical over determination condition. Khonatbek [23] considered the three-dimensional pseudo-parabolic inverse problem representing Kelvin-Voight motion and determined the unknown terms describing pressure and velocity fields. The authors in [24, 25] examined the existence, regularity, and uniqueness conditions for inverse linear pseudo-parabolic problems representing the filtration process and determined the unknown coefficient of the leading term. Abylkairov and Khompysh [26] studied the existence of a unique solution for the inverse problem on the pseudo-parabolic equation modeling the motion of Kelvin-Voight fluids and determined the unknown space dependant coefficient. In recent years, Beshtokov [27] explored the numerical solution for a class of third order pseudo-parabolic inverse problems involving variable coefficients. Lyubanova and Velisevich [28] presented their work on detection of unknown coefficients in the lower term of pseudo-parabolic inverse problems involving diffusion and second order elliptic equations subject to Dirichlet type of boundary conditions. Baglan and Canel [29] investigated an inverse identification problem for quasi-linear pseudo-parabolic of heat conduction of poly from a heat moment additional condition.

The third order pseudo-parabolic problems arise in mathematical modeling of fluid filtration, moisture transfer and heat propagation [30–36]. The authors in [37] discussed the existence of a unique solution of a class of third order pseudo-parabolic inverse boundary value problems subject to no-classical conditions of first kind. Khompysh [38] investigated the reconstruction of unknown coefficient in pseudo-parabolic inverse problems with the integral over-determination condition and studied the uniqueness and existence of the solution by means of a method of successive approximations. Ramazanova et al. [39] theoretically studied the fourth-order inverse problem with the nonlocal integral condition. Serikbaev and Tokmagambetov [40] determined the space dependent forcing term for a class of pseudo-parabolic inverse problems. Huntul et al. [41] studied the reconstruction of unknown time and space-dependent coefficients in third order pseudo-parabolic inverse problems subject to nonlocal integral conditions. Further, Huntul et al. [42, 43] considered a fourth order pseudo-parabolic inverse problem with some additional information and identified the unknown time dependent potential coefficient. Mehraliyev and Shayeva [44] investigated an inverse problem theoretically for the third order pseudo-parabolic problem and proved the uniqueness of the solution. The existence and uniqueness of the solution of the inverse problem for the third order pseudo-parabolic equation with the integral over-determination condition is studied in [45]. Recently, Huntul et al. [46, 47] investigated the inverse problem of determining the time dependent coefficients from the time fractional third order and fourth order pseudo-parabolic equations, respectively.

In this work, we study the third order pseudo-parabolic problem to recover the time-dependent potential and heat source coefficients theoretically, i.e., existence and uniqueness, and numerically, for the first time, in the prescribed domain using the initial, Neumann boundary conditions, and the nonlocal integral data as an over-determination conditions. The stability results for the inverse problem are proved. The pre-eminent goal of the current work is to undertake the theory and numerical aspect of this problem.

This paper is presented as: The proposed inverse problem has been mathematically developed in Section 2. The unique solvability of the problem is proved in Section 3. Section 4 briefly explains the scheme for solving the direct problem by means of the CB-spline collocation technique. The unconditional stability has been proved in Section 5. The description of the numerical procedure to solve the minimization of the nonlinear Tikhonov regularization functional has been given in Section 6. The computational outcomes for some examples on the topic are discussed in the Section 7. Finally, concluding remarks are revealed in Section 8.

## 2. Mathematical formulation of the inverse problem

In the fixed domain $\Omega_T = \{(\kappa, \tau) : 0 \leq \kappa \leq 1, 0 \leq \tau \leq T\}$, we consider an inverse problem of recovering the time wise potential and heat source terms $a(\tau)$ and $b(\tau)$ in the third order pseudo-parabolic equation

$$z_\tau(\kappa,\tau) - \beta z_{\tau\kappa\kappa}(\kappa,\tau) - \alpha z_{\kappa\kappa}(\kappa,\tau) = F(\tau; a, b, z), (\kappa,\tau) \in \Omega_{T,} \tag{2.1}$$

subject to

$$z(\kappa, 0) = \varphi(\kappa), \quad \kappa \in [0, 1], \tag{2.2}$$

with the following Neumann type of boundary conditions

$$z_\kappa(0, \tau) = z_\kappa(1, \tau) = 0, \quad \tau \in [0, T], \tag{2.3}$$

and the integral observations

$$\int_0^1 z(\kappa, \tau) d\kappa = h_1(\tau), \quad \tau \in [0, T], \tag{2.4}$$

$$\int_0^1 \kappa z(\kappa, \tau) d\kappa = h_2(\tau), \quad \tau \in [0, T], \tag{2.5}$$

where $F(\tau; a, b, z) = a(\tau)z(\kappa, \tau) + b(\tau) f(\kappa, \tau)$, $\alpha$, $\beta > 0$ are constants. Equation (2.2) represents the mathematical model for the movement of moisture and salts in soils [48]. The function $h_1(\tau)$ in (2.4) corresponds the specification of energy or mass of the heat conducting system [49,50] whilst the function $h_2(\tau)$ in (2.5) represents the barycenter of the system [51].

## 3. Unique solvability of the problem

In this section, we will first define a Banach space corresponding to the eigenvalues and eigenfunctions of the auxiliary spectral problem of the problem (2.2)–(2.5). Then, we will set and prove the unique solvability theorem of the solution of the inverse initial-boundary value problem for the third order pseudo-parabolic equation by using the Banach fixed point theorem in this space.

**Definition 3.1.** *Let the triplet $\{a(\tau), b(\tau), z(\kappa, \tau)\}$ be from the class $C[0, T] \times C[0, T] \times C^{2,1}(\Omega_T)$ and satisfy the Eq (2.2) and conditions (2.2)–(2.5). Then, the triplet $\{a(\tau), b(\tau), z(\kappa, \tau)\}$ is noted as a classical solution of the problem (2.2)–(2.5).*

Since the boundary conditions are homogeneous, the Fourier method is suitable to obtain the solution of the problem (2.2)–(2.5). The auxiliary spectral problem (2.2)–(2.5) is

$$\begin{cases} W''(\kappa) + \lambda W(\kappa) = 0, & 0 \le \kappa \le 1, \\ W'(0) = W'(1) = 0. \end{cases} \tag{3.1}$$

The eigenvalues of the spectral problem (3.1) are $\lambda_n^2 = (n\pi)^2$ and the eigenfunctions corresponding to these eigenvalues are $W_0(\kappa) = 1$, $W_n(\kappa) = \sqrt{2}\cos(\lambda_n \kappa)$, $n = 1, 2, \cdots$. The system of eigenfunctions $W_n(\kappa)$ is bi-orthonormal on $[0, 1]$ and forms a Riesz basis in $L_2[0, 1]$.

In the rest of the paper, we will consider the following spaces to investigate unique solvability and conditional stability of the inverse problem (2.2)–(2.5):

**I:**

$$B_T = \Big\{ z(\kappa, \tau) = \textstyle\sum_{n=0}^{\infty} z_n(\tau) W_n(\kappa) : \ z_n(\tau) \in C[0, T], \ \displaystyle\max_{0 \le \tau \le T} |z_0(\tau)| + \left[ \sum_{n=1}^{\infty} \left( \lambda_n^2 \max_{0 \le \tau \le T} |z_n(\tau)| \right)^2 \right]^{1/2} < +\infty \Big\}$$

with the norm

$$\|z(\kappa, \tau)\|_{B_T} \equiv \max_{0 \le \tau \le T} |z_0(\tau)| + \left[ \sum_{n=1}^{\infty} \left( \lambda_n^2 \max_{0 \le \tau \le T} |z_n(\tau)| \right)^2 \right]^{1/2}.$$

This space is related to the Fourier coefficients $z_n(\tau)$ of the function $z(\kappa, \tau)$ is obtained by using the eigenfunctions $W_n(\kappa)$, $n = 0, 1, 2, \cdots$, and the space $B_T$ is a Banach space.

**II:** $E_T = C[0, T] \times C[0, T] \times B_T$ of the vector function $w = [a(\tau), b(\tau), z(\kappa, \tau)]^T$ with the norm

$$\|w\|_{E_T} = \|a(\tau)\|_{C[0,T]} + \|b(\tau)\|_{C[0,T]} + \|z(\kappa, \tau)\|_{B_T}.$$

Here, $E_T$ is also a Banach space.

**Theorem 3.1.** *Let us assume that the following assumptions hold:*

**(A$_1$)** $\varphi(\kappa) \in C^2[0, 1]$, $\varphi'(1) = \varphi'(0) = 0$;

**(A$_2$)** $h_i(\tau) \in C^1[0, T]$, $i = 1, 2$, $\int_0^1 \varphi(\kappa) d\kappa = h_1(0)$, $\int_0^1 \kappa\varphi(\kappa) d\kappa = h_2(0)$;

**(A$_3$)** $f(\kappa, \tau) \in C^{2,0}(\Omega_T)$, $f_\kappa(1, \tau) = f_\kappa(0, \tau) = 0$;

**(A$_4$)** $\Delta(\tau) = h_2(\tau) \int_0^1 f(\kappa, \tau) d\kappa - h_1(\tau) \int_0^1 \kappa f(\kappa, \tau) d\kappa \neq 0$, $\forall \tau \in [0, T]$.

*Then, the problem (2.2)–(2.5) has unique solution for $T^* > 0$ such that $T \in (0, T^*)$.*

*Proof.* Let us seek the solution of (2.2)–(2.5) in terms of the eigenfunction basis

$$z(\kappa,\tau) = \sum_{n=0}^{\infty} z_n(\tau) W_n(\kappa), \tag{3.2}$$

where

$$z_n(\tau) = \int_0^1 z(\kappa,\tau) W_n(\kappa) d\kappa.$$

Since the Eq (3.2) is the solution of the problem (2.2)–(2.5), $z_n(\tau)$ satisfies the following initial value problems:

$$\begin{cases} z_0'(\tau) = F_0(\tau; a, b, z_0), \ 0 \le \tau \le T, \\ z_0(0) = \varphi_0, \end{cases} \tag{3.3}$$

$$\begin{cases} (1+\beta\lambda_n^2) z_n'(\tau) + \alpha\lambda_n^2 z_n(\tau) = F_n(\tau; a, b, z_n), \\ \qquad\qquad 0 \le \tau \le T, \\ z_n(0) = \varphi_n, \ n = 1, 2, \cdots, \end{cases} \tag{3.4}$$

where

$$\begin{aligned} F_n(\tau; a, b, z_n) &= a(\tau) z_n(\tau) + b(\tau) f_n(\tau), \\ z_n(\tau) &= \int_0^1 z(\kappa,\tau) W_n(\kappa) d\kappa, \\ f_n(\tau) &= \int_0^1 f(\kappa,\tau) W_n(\kappa) d\kappa, \\ \varphi_n = \int_0^1 \varphi(\kappa) W_n(\kappa) d\kappa, &\quad n = 0, 1, 2, \cdots. \end{aligned}$$

We obtain the solutions of the Cauchy problems (3.3)-(3.4)

$$z_0(\tau) = \varphi_0 + \int_0^\tau F_0(s; a, b, z_0) ds, \tag{3.5}$$

$$z_n(\tau) = \varphi_n e^{-A\tau} + \frac{1}{1+\beta\lambda_n^2} \int_0^\tau e^{A(s-\tau)} F_n(s; a, b, z_n) ds, \tag{3.6}$$

where

$$A = \frac{\alpha\lambda_n^2}{1+\beta\lambda_n^2}.$$

Considering (3.5) and (3.6) into (3.2), we get

$$\begin{aligned} z(\kappa,\tau) = \quad & \varphi_0 + \textstyle\int_0^\tau F_0(s; a, b, z_0) ds + \sum_{n=1}^\infty \Big(\varphi_n e^{-A\tau} \\ & + \tfrac{1}{1+\beta\lambda_n^2} \textstyle\int_0^\tau e^{A(s-\tau)} F_n(s; a, b, z_n) ds\Big) W_n(\kappa). \end{aligned} \tag{3.7}$$

For determination of $a(\tau)$ and $b(\tau)$, we need to use the additional conditions (2.5). Integrating the Eq (2.2) from 0 to 1 with respect to $\kappa$, we obtain

$$\begin{aligned} &\int_0^1 z_\tau(\kappa,\tau) d\kappa - \beta \int_0^1 z_{\tau\kappa\kappa}(\kappa,\tau) d\kappa - \alpha \int_0^1 z_{\kappa\kappa}(\kappa,\tau) d\kappa \\ &\qquad = a(\tau) \int_0^1 z(\kappa,\tau) d\kappa + b(\tau) \int_0^1 f(\kappa,\tau) d\kappa. \end{aligned} \tag{3.8}$$

Using the first over-determination condition in this equality, we have

$$\begin{aligned} & h_1'(\tau) - \beta\left[z_{\tau\kappa}(1,\tau) - z_{\tau\kappa}(0,\tau)\right] - \alpha\left[z_\kappa(1,\tau) - z_\kappa(0,\tau)\right] \\ &\qquad = a(\tau) h_1(\tau) + b(\tau) f_{int1}(\tau), \end{aligned} \tag{3.9}$$

where

$$f_{int1}(\tau) = \int_0^1 f(\kappa,\tau) d\kappa.$$

Since $z_\kappa(0,\tau) = z_\kappa(1,\tau) = 0$, we obtain

$$a(\tau) h_1(\tau) + b(\tau) f_{int1}(\tau) = h_1'(\tau), \tag{3.10}$$

with respect to unknown terms $a(\tau)$ and $b(\tau)$.

Similarly, multiplying the Eq (2.2) by $\kappa$ and integrating the obtained equation from 0 to 1 with respect to $\kappa$, we obtain

$$\begin{aligned} &\int_0^1 \kappa z_\tau(\kappa,\tau) d\kappa - \beta \int_0^1 \kappa z_{\tau\kappa\kappa}(\kappa,\tau) d\kappa - \alpha \int_0^1 \kappa z_{\kappa\kappa}(\kappa,\tau) d\kappa \\ &= a(\tau) \int_0^1 \kappa z(\kappa,\tau) d\kappa + b(\tau) \int_0^1 \kappa f(\kappa,\tau) d\kappa. \end{aligned} \tag{3.11}$$

Using the second over-determination condition in this equality, we have

$$\begin{aligned} & h_2'(\tau) - \beta \int_0^1 \kappa z_{\tau\kappa\kappa}(\kappa,\tau) d\kappa - \alpha \int_0^1 \kappa z_{\kappa\kappa}(\kappa,\tau) d\kappa \\ &= a(\tau) h_2(\tau) + b(\tau) f_{int2}(\tau), \end{aligned} \tag{3.12}$$

where

$$f_{int2}(\tau) = \int_0^1 \kappa f(\kappa,\tau) d\kappa.$$

Since $z(\kappa,\tau) = z_0(\tau) + \sqrt{2}\sum_{n=1}^\infty z_n(\tau)\cos(\lambda_n\kappa)$, we can easily obtain

$$\begin{aligned} \int_0^1 \kappa z_{\tau\kappa\kappa}(\kappa,\tau) d\kappa &= \int_0^1 \kappa \sqrt{2} \sum_{n=1}^\infty -\lambda_n^2 z_n'(\tau) \cos(\lambda_n\kappa) d\kappa \\ &= \sqrt{2} \sum_{n=1}^\infty \left[1 - (-1)^n\right] z_n'(\tau), \end{aligned} \tag{3.13}$$

$$\int_0^1 \kappa z_{\kappa\kappa}(\kappa,\tau)d\kappa = \int_0^1 \kappa\sqrt{2}\sum_{n=1}^{\infty} -\lambda_n^2 z_n(\tau)\cos(\lambda_n\kappa)d\kappa$$
$$= \sqrt{2}\sum_{n=1}^{\infty}\left[1-(-1)^n\right]z_n(\tau). \tag{3.14}$$

Considering the last equalities into the Eq (3.12), we get

$$b(\tau)f_{int2}(\tau) + a(\tau)h_2(\tau)$$
$$= h_2'(\tau) + \sqrt{2}\sum_{n=1}^{\infty}\left[1-(-1)^n\right](\beta u_n'(\tau) + \alpha u_n(\tau)). \tag{3.15}$$

Since $\beta z_n'(\tau)+\alpha z_n(\tau) = \frac{1}{\lambda_n^2}\left[F_n(\tau;a,b,z_n) - z_n'(\tau)\right]$ from (3.4), we can rewrite (3.15) as

$$b(\tau)f_{int2}(\tau) + a(\tau)h_2(\tau)$$
$$= h_2'(\tau) + \sqrt{2}\sum_{n=1}^{\infty}\frac{\left[1-(-1)^n\right]}{\lambda_n^2}\left[F_n(\tau;a,b,z_n) - z_n'(\tau)\right]. \tag{3.16}$$

Solving Eqs (3.10) and (3.16) for the terms $a(\tau)$ and $b(\tau)$, we have

$$a(\tau) = \frac{1}{\Delta(\tau)}\Big\{ -h_1'(\tau)f_{int2}(\tau) + h_2'(\tau)f_{int1}(\tau)$$
$$+ f_{int1}(\tau)\sqrt{2}\sum_{n=1}^{\infty}\frac{\left[1-(-1)^n\right]}{\lambda_n^2}\left[F_n(\tau;a,b,z_n) - z_n'(\tau)\right]\Big\}, \tag{3.17}$$

and

$$b(\tau) = \frac{1}{\Delta(\tau)}\Big\{ -h_1'(\tau)h_2(\tau) + h_2'(\tau)h_1(\tau)$$
$$+ h_1(\tau)\sqrt{2}\sum_{n=1}^{\infty}\frac{\left[1-(-1)^n\right]}{\lambda_n^2}\left[F_n(\tau;a,b,z_n) - z_n'(\tau)\right]\Big\}, \tag{3.18}$$

where

$$\Delta(\tau) = h_2(\tau)f_{int1}(\tau) - h_1(\tau)f_{int2}(\tau) \neq 0.$$

Thereby, we obtain the Eqs (3.7), (3.17), and (3.18) with respect to unknown terms $a(\tau)$, $b(\tau)$, and $z(\kappa,\tau)$. Solving the problem (2.2)–(2.5) is equivalent to solving the Eqs (3.7), (3.17), and (3.18). Therefore, we will investigate the unique solvability of (3.7), (3.17), and (3.18) instead of the problem (2.2)–(2.5).

Denote $w = [a(\tau), b(\tau), z(\kappa,\tau)]^T$ and rewrite the Eqs (3.7), (3.17), and (3.18) in the operator form as

$$w = \widehat{\mathbf{\Lambda}}(w), \tag{3.19}$$

where $\widehat{\mathbf{\Lambda}} = [\Lambda_1, \Lambda_2, \Lambda_3]^T$, and

$$\Lambda_1(w) = \frac{1}{\Delta(\tau)}\Big\{ -h_1'(\tau)f_{int2}(\tau) + h_2'(\tau)f_{int1}(\tau)$$
$$+ f_{int1}(\tau)\sqrt{2}\sum_{n=1}^{\infty}\frac{\left[1-(-1)^n\right]}{\lambda_n^2}\left[F_n(\tau;a,b,z_n) - z_n'(\tau)\right]\Big\}, \tag{3.20}$$

$$\Lambda_2(w) = \frac{1}{\Delta(\tau)}\Big\{ -h_1'(\tau)h_2(\tau) + h_2'(\tau)h_1(\tau)$$
$$+ h_1(\tau)\sqrt{2}\sum_{n=1}^{\infty}\frac{\left[1-(-1)^n\right]}{\lambda_n^2}\left[F_n(\tau;a,b,z_n) - z_n'(\tau)\right]\Big\}, \tag{3.21}$$

$$\Lambda_3(w) = \varphi_0 + \int_0^\tau F_0(s;a,b,z_0)ds + \sum_{n=1}^{\infty}\Big(\varphi_n e^{-A\tau}$$
$$+ \frac{1}{1+\beta\lambda_n^2}\int_0^\tau e^{A(s-\tau)}F_n(s;a,b,z_n)ds\Big)W_n(\kappa). \tag{3.22}$$

To apply the Banach fixed point theorem, first we need to illustrate that $\widehat{\mathbf{\Lambda}}$ maps $E_T$ onto itself continuously. In other words, let us show that $\Lambda_1(w), \Lambda_2(w) \in C[0,T]$ and $\Lambda_3(w) \in B_T$ for arbitrary $w = [a(\tau), b(\tau), z(\kappa,\tau)]^T$ by considering $a(\tau), b(\tau) \in C[0,T]$, $z(\kappa,\tau) \in B_T$.

Before proving $\widehat{\mathbf{\Lambda}}$ maps $E_T$ onto itself continuously, let us give some equalities which are important to apply the Bessel inequality. Integrating by parts under the conditions $(A_1)$–$(A_4)$, it easy to see that

$$\varphi_n = \frac{1}{\lambda_n^2}\eta_n, \quad f_n(\tau) = \frac{1}{\lambda_n^2}\gamma_n(\tau), \tag{3.23}$$

where

$$\eta_n = -\int_0^1 \varphi''(\kappa)W_n(\kappa)d\kappa, \quad \gamma_n(\tau) = -\int_0^1 f_{\kappa\kappa}(\kappa,\tau)W_n(\kappa)d\kappa.$$

Also,

$$F_n(\tau;a,b,z_n) - z_n'(\tau) = A\varphi_n e^{-A\tau} + \frac{\beta\lambda_n^2}{1+\beta\lambda_n^2}F_n(\tau;a,b,z_n)$$
$$+ \frac{A}{1+\beta\lambda_n^2}\int_0^t e^{A(s-\tau)}F_n(s;a,b,z_n)ds.$$

Since $\lambda_n^2 = (n\pi)^2$ , $n = 1,2,\cdots$, we have the estimates

$$|A| = \left|\frac{\alpha\lambda_n^2}{1+\beta\lambda_n^2}\right| \leq \frac{\alpha}{\beta}, \left|\frac{A}{1+\beta\lambda_n^2}\right| \leq \frac{\alpha}{\beta^2}, \left|\frac{\beta\lambda_n^2}{1+\beta\lambda_n^2}\right| \leq 1,$$
$$\left(\sum_{n=1}^{\infty}\frac{1}{\lambda_n^8}\right)^{1/2} = \frac{1}{\sqrt{9450}} := m.$$

Under the assumptions $(A_1)$–$(A_4)$, using the Cauchy-Schwartz and Bessel inequalities, and estimates given above, we obtain from (3.20) and (3.21) that

$$\begin{aligned}\max_{0\le\tau\le T}|\Lambda_1(\tau)| \le& \frac{1}{\min\limits_{0\le\tau\le T}|\Delta(\tau)|}\Big[\max_{0\le\tau\le T}\left|h_2'(\tau)\right|\max_{0\le\tau\le T}|f_{int1}(\tau)| \\ &+\max_{0\le\tau\le T}\left|h_1'(\tau)\right|\max_{0\le\tau\le T}|f_{int2}(\tau)| \\ &+2\sqrt{2}m\max_{0\le\tau\le T}|f_{int1}(\tau)|\left\{\frac{\alpha}{\beta}\left(\sum_{n=1}^{\infty}|\eta_n|^2\right)^{1/2}\right. \\ &+\left(1+\frac{\alpha T}{\beta^2}\right)\Big[\max_{0\le\tau\le T} \\ &\times|a(\tau)|\left(\sum_{n=1}^{\infty}\left(\lambda_n^2\max_{0\le\tau\le T}|z_n(\tau)|\right)^2\right)^{1/2} \\ &\left.+\max_{0\le\tau\le T}|b(\tau)|\left(\sum_{n=1}^{\infty}\left(\max_{0\le\tau\le T}|\gamma_n(\tau)|\right)^2\right)^{1/2}\Big]\right\}\Big],\end{aligned}\tag{3.24}$$

and

$$\begin{aligned}&\max_{0\le\tau\le T}|\Lambda_2(\tau)| \\ \le&\frac{1}{\min\limits_{0\le\tau\le T}|\Delta(\tau)|}\Big[\max_{0\le\tau\le T}\left|h_2'(\tau)\right|\max_{0\le\tau\le T}|h_1(\tau)| \\ &+\max_{0\le\tau\le T}\left|h_1'(\tau)\right|\max_{0\le\tau\le T}|h_2(\tau)| \\ &+2\sqrt{2}m\max_{0\le\tau\le T}|h_1(\tau)|\left\{\frac{\alpha}{\beta}\left(\sum_{n=1}^{\infty}|\eta_n|^2\right)^{1/2}\right. \\ &+\left(1+\frac{\alpha T}{\beta^2}\right)\Big[\max_{0\le\tau\le T}|a(\tau)|\left(\sum_{n=1}^{\infty}\left(\lambda_n^2\max_{0\le\tau\le T}|z_n(\tau)|\right)^2\right)^{1/2} \\ &\left.+\max_{0\le\tau\le T}|b(\tau)|\left(\sum_{n=1}^{\infty}\left(\max_{0\le\tau\le T}|\gamma_n(\tau)|\right)^2\right)^{1/2}\Big]\right\}\Big].\end{aligned}\tag{3.25}$$

Since the right-hand side (RHS) of (3.24) and (3.25) are bounded, $\Lambda_1(w), \Lambda_2(w) \in C[0,T]$. Now, let us show that $\Lambda_3(w) \in B_T$; in other words, we need to show

$$J_T(\Lambda_3) = \max_{0\le\tau\le T}|\Lambda_{30}(\tau)| + \left[\sum_{n=1}^{\infty}\left(\lambda_n^2\max_{0\le\tau\le T}|\Lambda_{3n}(\tau)|\right)^2\right]^{1/2} < +\infty,$$

where

$$\begin{aligned}\Lambda_{30}(\tau) =&\varphi_0+\int_0^{\tau}F_0(s;a,b,z_0)ds,\ \Lambda_{3n}(\tau) \\ =&\varphi_n e^{-A\tau}+\frac{1}{1+\beta\lambda_n^2}\int_0^{\tau}e^{A(s-\tau)}F_n(s;a,b,z_n)ds.\end{aligned}$$

After some manipulations on the last equality under the assumptions $(A_1)$–$(A_3)$, we get

$$\begin{aligned}&\max_{0\le\tau\le T}|\Lambda_{30}(\tau)| \\ &\le|\varphi_0|+T\left[\max_{0\le\tau\le T}|a(\tau)|\max_{0\le\tau\le T}|z_0(\tau)|+\max_{0\le\tau\le T}|b(\tau)|\max_{0\le\tau\le T}|f_0(\tau)|\right],\end{aligned}\tag{3.26}$$

$$\begin{aligned}&\sum_{n=1}^{\infty}\left(\lambda_n^2\max_{0\le\tau\le T}|\Lambda_{3n}(\tau)|\right)^2 \\ \le&2\sum_{n=1}^{\infty}|\eta_n|^2+4T^2\Big[\max_{0\le\tau\le T}|a(\tau)|\sum_{n=1}^{\infty}\left(\lambda_n^2\max_{0\le\tau\le T}|z_n(\tau)|\right)^2 \\ &+\max_{0\le\tau\le T}|b(\tau)|\sum_{n=1}^{\infty}\left(\max_{0\le\tau\le T}|\gamma_n(\tau)|\right)^2\Big].\end{aligned}\tag{3.27}$$

Since the RHS of $\max\limits_{0\le\tau\le T}|\Lambda_{30}(\tau)|$ is bounded and the sums on the RHS of $\sum_{n=1}^{\infty}\left(\lambda_n^2\max\limits_{0\le\tau\le T}|\Lambda_{3n}(\tau)|\right)^2$ are convergent from the Bessel inequality, we can conclude that $J_T(\Lambda_3) < +\infty$. Thus, $\Lambda_3(w)$ belongs to the space $B_T$.

Since $\Lambda_1(w), \Lambda_2(w) \in C[0,T]$ and $\Lambda_3(w) \in B_T$ for arbitrary $w = [a(\tau), b(\tau), z(\kappa,\tau)]^T$, $\widehat{\mathbf{\Lambda}}$ maps $E_T$ onto itself.

Let us now show that $\widehat{\mathbf{\Lambda}}$ is a contraction mapping on $E_T$ . Let us choose any two elements $w_i = \left[a^i(\tau), b^i(\tau), z^i(\kappa,\tau)\right]^T$, $i = 1,2$ in $E_T$. We know that

$$\begin{aligned}&\left\|\widehat{\mathbf{\Lambda}}(w_1)-\widehat{\mathbf{\Lambda}}(w_2)\right\|_{E_T^5} \\ =&\|\Lambda_1(w_1)-\Lambda_1(w_2)\|_{C[0,T]}+\|\Lambda_2(w_1)-\Lambda_2(w_2)\|_{C[0,T]} \\ &+\|\Lambda_3(w_1)-\Lambda_3(w_2)\|_{B_T}.\end{aligned}\tag{3.28}$$

From the Eqs (3.20)–(3.22), we get

$$\begin{aligned}&\Lambda_1(w_1)-\Lambda_1(w_2) \\ =&\frac{1}{\Delta(\tau)}\Big\{f_{int1}(\tau)\sqrt{2}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{\lambda_n^2}\Big[F_n(\tau;a^1,b^1,z_n^1) \\ &-F_n(\tau;a^2,b^2,z_n^2)+\left(z_n^2(\tau)\right)'-\left(z_n^1(\tau)\right)'\Big]\Big\},\end{aligned}\tag{3.29}$$

$$\begin{aligned}&\Lambda_2(w_1)-\Lambda_2(w_2) \\ =&\frac{1}{\Delta(\tau)}\Big\{h_1(t)\sqrt{2}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{\lambda_n^2}\Big[F_n(\tau;a^1,b^1,z_n^1) \\ &-F_n(\tau;a^2,b^2,z_n^2)+\left(z_n^2(\tau)\right)'-\left(z_n^1(\tau)\right)'\Big]\Big\},\end{aligned}\tag{3.30}$$

$$\begin{aligned}
&\Lambda_3(w_1) - \Lambda_3(w_2) \\
&= \int_0^\tau (F_0(s; a^1, b^1, z_0^1) - F_0(s; a^2, b^2, z_0^2))ds \\
&+ \sum_{n=1}^{\infty} \frac{1}{1+\beta\lambda_n^2} \int_0^\tau e^{A(s-\tau)}(F_n(s; a^1, b^1, z_n^1) \\
&- F_n(s; a^2, b^2, z_n^2))dsW_n(\kappa).
\end{aligned} \tag{3.31}$$

Before estimating these differences, let us examine subtraction $F_n(\tau; a^1, b^1, z_n^1) - F_n(\tau; a^2, b^2, z_n^2)$. Since $F_n(\tau; a, b, z_n) = b(\tau)f_n(\tau) + a(\tau)z_n(\tau)$,

$$\begin{aligned}
&F_n(\tau; a^1, b^1, z_n^1) - F_n(\tau; a^2, b^2, z_n^2) \\
&= \left(a^1(\tau)z_n^1(\tau) + b^1(\tau)f_n(\tau)\right) - \left(a^2(\tau)z_n^2(\tau) + b^2(\tau)f_n(\tau)\right).
\end{aligned}$$

Adding and removing the term $a^1(\tau)z_n^2(\tau)$, we get

$$\begin{aligned}
F_n(\tau; a^1, b^1, z_n^1) - F_n(\tau; a^2, b^2, z_n^2) &= a^1(\tau)\left(z_n^1(\tau) - z_n^2(\tau)\right) \\
&+ z_n^2(\tau)\left(a^1(\tau) - a^2(\tau)\right) + f_n(\tau)\left(b^1(\tau) - b^2(\tau)\right).
\end{aligned}$$

After some manipulations in the difference equations of $\Lambda_i(w_1) - \Lambda_i(w_2)$, $i = 1, 2, 3$ under the assumptions $(A_1)$–$(A_4)$, and using the coefficient estimates that we obtained above, we obtain

$$\left\|\widehat{\mathbf{\Lambda}}(w_1) - \widehat{\mathbf{\Lambda}}(w_2)\right\|_{E_T} \le A(T)C(a^1, z^2, f)\,\|w_1 - w_2\|_{E_T}, \tag{3.32}$$

where

$$\begin{aligned}
A(T) = &2T + \frac{m}{\underset{0\le\tau\le T}{min}|\Delta(\tau)|}\left(1 + \frac{\alpha T}{\beta^2}\right) \\
&\left(\|f(\kappa,\tau)\|_{C^{2,0}(\Omega_T)} + \|h_1(\tau)\|_{C^1[0,T]}\right),
\end{aligned}$$

and $C(a^1, z^2, f)$ depends on the norms

$$\left\|a^1\right\|_{C[0,T]}, \quad \left\|z^2\right\|_{B_T}, \quad \|f(\kappa,\tau)\|_{C^{2,0}(\Omega_T)}.$$

There exists $T^*$ such as

$$T \le T^*$$

where

$$T^* = \min\left|\frac{D\beta^2(D-m)}{2D^2\beta^2 + \alpha m D\left(\|f(\kappa,\tau)\|_{C^{2,0}(\Omega_T)} + \|h_1(\tau)\|_{C^1[0,T]}\right)}\right|,$$

and $D = \underset{0\le\tau\le T}{min}|\Delta(\tau)|$. For $T \le T^*$,

$$0 < A(T) < 1,$$

and the operator $\widehat{\mathbf{\Lambda}}$ is a contraction mapping. Thus, $\widehat{\mathbf{\Lambda}}$ maps $E_T$ onto itself. Then, $\exists$ is a unique solution of (3.19) by virtue of the Banach fixed point theorem. Hereby, the problem (2.2)–(2.5) has a unique solution for sufficiently small $T$. □

Now, let us investigate the stability of the solution of the problem (2.2)–(2.5). Because of the presence of the term $a(\tau)z(\kappa,\tau)$ in (2.2), finding the triplet $\{a(\tau), b(\tau), z(\kappa,\tau)\}$ of (2.2)–(2.5) is nonlinear. Therefore, we cannot apply the standard stability criteria, but we can characterize the estimation of conditional stability. Thus, we can obtain a stability estimate under a priori assumption on the smallness of $a(\tau)$ and $b(\tau)$. These type of stability results are studied by V.G. Romanov in [52] and more recently in [53, 54].

These type of stability estimates can be obtained by setting a specific class of data $\Im(N_1, N_2, N_3, N_4)$ for the functions $\varphi(\kappa)$, $f(\kappa,\tau)$, $h_i(\tau)$, $i = 1, 2$, and a class $\aleph(M_1, M_2)$ for the functions $a(\tau)$ and $b(\tau)$ if they satisfy

$$\|h_i\|_{C^1[0,T]} \le N_i,\ i = 1, 2,\ \|\varphi\|_{C^2[0,1]} \le N_3,\ \|f\|_{C^{2,0}(\Omega_T)} \le N_4,$$

and

$$\|a(\tau)\|_{C[0,T]} \le M_1 < \frac{1}{2T},\ \|b(\tau)\|_{C[0,T]} \le M_2,$$

respectively.

Let us consider $\varphi(\kappa), f(\kappa,\tau), h_i(\tau) \in \Im(N_1, N_2, N_3, N_4)$ and $a(\tau), b(\tau) \in \aleph(M_1, M_2)$, then from the equation

$$\begin{aligned}
z(\kappa,\tau) = \quad &\varphi_0 + \textstyle\int_0^\tau F_0(s; a, b, z_0)ds + \sum_{n=1}^\infty \Big(\varphi_n e^{-A\tau} \\
&+ \tfrac{1}{1+\beta\lambda_n^2}\textstyle\int_0^\tau e^{A(s-\tau)}F_n(s; a, b, z_n)ds\Big)W_n(\kappa),
\end{aligned}$$

we can easily obtain the estimate

$$\|z(\kappa,\tau)\|_{B_T} \le L, \tag{3.33}$$

where

$$L = \frac{1}{1 - 2TM_1}(\sqrt{2}N_3 + 2TM_2N_4),$$

with $1 - 2TM_1 > 0$.

Let $\{a(\tau), b(\tau), z(\kappa,\tau)\}$ and $\left\{\overline{a}(\tau), \overline{b}(\tau), \overline{z}(\kappa,\tau)\right\}$ be the solutions of (2.2)–(2.5) corresponding to data $\varphi(\kappa)$, $f(\kappa,\tau)$, $h_i(\tau)$, $i = 1,2$ and $\overline{\varphi}(\kappa)$, $\overline{f}(\kappa,\tau)$, $\overline{h}_i(\tau)$, $i = 1,2$, respectively. Then, we obtain from (3.7), (3.17), and (3.18) that

$$\begin{aligned} a(\tau)-\overline{a}(\tau) &= \frac{1}{\Delta(\tau)}\Big\{-h_1'(\tau)f_{int2}(\tau)+h_2'(\tau)f_{int1}(\tau) \\ &\quad + f_{int1}(\tau)\sqrt{2}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{\lambda_n^2}\left[F_n(\tau;a,b,z_n)-z_n'(\tau)\right]\Big\} \\ &\quad - \frac{1}{\overline{\Delta}(\tau)}\Big\{-\overline{h}_1'(\tau)\overline{f}_{int2}(\tau)+\overline{h}_2'(\tau)\overline{f}_{int1}(\tau) \\ &\quad + \overline{f}_{int1}(\tau)\sqrt{2}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{\lambda_n^2}\left[F_n(\tau;\overline{a},\overline{b},\overline{z}_n)-\overline{z}_n'(\tau)\right]\Big\}, \end{aligned} \tag{3.34}$$

$$\begin{aligned} b(\tau)-\overline{b}(\tau) &= \frac{1}{\Delta(\tau)}\Big\{-h_1'(\tau)h_2(\tau)+h_2'(\tau)h_1(\tau) \\ &\quad + h_1(\tau)\sqrt{2}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{\lambda_n^2}\left[F_n(\tau;a,b,z_n)-z_n'(\tau)\right]\Big\} \\ &\quad - \frac{1}{\overline{\Delta}(\tau)}\Big\{-\overline{h}_1'(\tau)\overline{h}_2(\tau)+\overline{h}_2'(\tau)\overline{h}_1(\tau) \\ &\quad + \overline{h}_1(\tau)\sqrt{2}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{\lambda_n^2}\left[F_n(\tau;\overline{a},\overline{b},\overline{z}_n)-\overline{z}_n'(\tau)\right]\Big\}, \end{aligned} \tag{3.35}$$

$$\begin{aligned} z(\kappa,\tau)-\overline{z}(\kappa,\tau) &= \varphi_0+\int_0^\tau F_0(s;a,b,z_0)ds+\sum_{n=1}^{\infty}\Big(\varphi_n e^{-A\tau} \\ &\quad +\frac{1}{1+\beta\lambda_n^2}\int_0^\tau e^{A(s-\tau)}F_n(s;a,b,z_n)ds\Big)W_n(\kappa) \\ &\quad -\Big(\overline{\varphi}_0+\int_0^\tau F_0(s;\overline{a},\overline{b},\overline{z}_0)ds+\sum_{n=1}^{\infty}\Big(\overline{\varphi}_n e^{-A\tau} \\ &\quad +\frac{1}{1+\beta\lambda_n^2}\int_0^\tau e^{A(s-\tau)}F_n(s;\overline{a},\overline{b},\overline{z}_n)ds\Big)W_n(\kappa)\Big). \end{aligned} \tag{3.36}$$

Before estimating above equations, let us give some inequalities.

$$\begin{aligned} \left|\int_0^1 f(\kappa,\tau)d\kappa\right| &\le \int_0^1 |f(\kappa,\tau)|\,d\kappa \le N_4, \\ \left|\int_0^1 \kappa f(\kappa,\tau)d\kappa\right| &\le \int_0^1 |\kappa f(\kappa,\tau)|\,d\kappa \le \frac{N_4}{2}, \end{aligned} \tag{3.37}$$

$$\begin{aligned} \Delta(\tau) = h_2(\tau)f_{int1}(\tau)-h_1(\tau)f_{int2}(\tau) \ne 0 \\ \Longrightarrow 0<\varepsilon\le|\Delta(\tau)|\le P, \end{aligned} \tag{3.38}$$

$$\left|\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}\left[F_n(\tau;a,b,z_n)-z_n'(\tau)\right]\right| \le R, \tag{3.39}$$

where

$$P = N_4\left(N_2+\frac{N_1}{2}\right), R = \frac{\alpha}{\beta}N_3+\left(1+\frac{\alpha T}{\beta^2}\right)m(LM_1+M_2N_4).$$

Denote the difference between two functions with the tilde (~), i.e., $\widetilde{a} = a-\overline{a}, \widetilde{z} = z-\overline{z}$, etc. From Eq (3.34), we get

$$\begin{aligned} \widetilde{a}(\tau) = &\frac{1}{\overline{\Delta}(\tau)\Delta(\tau)}\Big\{\overline{\Delta}(\tau)G(\tau)+\overline{\Delta}(\tau)f_{int1}(\tau)S(\tau) \\ &-\Delta(\tau)\overline{G}(\tau)-\Delta(\tau)\overline{f}_{int1}(\tau)\overline{S}(\tau)\Big\}, \end{aligned}$$

where

$$\begin{aligned} G(\tau) &= h_2'(\tau)f_{int1}(\tau)-h_1'(\tau)f_{int2}(\tau), \\ S(\tau) &= \sqrt{2}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{\lambda_n^2}\left[F_n(\tau;a,b,z_n)-z_n'(\tau)\right]. \end{aligned}$$

Adding and removing the terms $\Delta(\tau)f_{int1}(\tau)S(\tau)$, $\Delta(\tau)G(\tau)$, and $\Delta(\tau)f_{int1}(\tau)\overline{S}(\tau)$, we obtain

$$\begin{aligned} \widetilde{a}(\tau) = &\frac{1}{\overline{\Delta}(\tau)\Delta(\tau)}\Big\{G(\tau)\left(\overline{\Delta}(\tau)-\Delta(\tau)\right)+\Delta(\tau)\left(G(\tau)-\overline{G}(\tau)\right) \\ &+f_{int1}(\tau)S(\tau)\left(\Delta(\tau)-\overline{\Delta}(\tau)\right)+\Delta(\tau)f_{int1}(\tau)\left(S(\tau)-\overline{S}(\tau)\right) \\ &+\Delta(\tau)\overline{S}(\tau)\left(f_{int1}(\tau)-\overline{f}_{int1}(\tau)\right)\Big\}. \end{aligned}$$

We can easily obtain the differences

$$\begin{aligned} \Delta(\tau)-\overline{\Delta}(\tau) = &f_{int1}\left(h_2-\overline{h}_2\right)+\overline{h}_2\left(f_{int1}-\overline{f}_{int1}\right) \\ &-f_{int2}\left(h_1-\overline{h}_1\right)-\overline{h}_1\left(f_{int2}-\overline{f}_{int2}\right), \end{aligned}$$

$$\begin{aligned} G(\tau)-\overline{G}(\tau) = &f_{int1}\left(h_2'-\overline{h}_2'\right)+\overline{h}_2'\left(f_{int1}-\overline{f}_{int1}\right) \\ &-f_{int2}\left(h_1'-\overline{h}_1'\right)-\overline{h}_1'\left(f_{int2}-\overline{f}_{int2}\right), \end{aligned}$$

and

$$\begin{aligned} S(\tau)-\overline{S}(\tau) = \sqrt{2}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{\lambda_n^2}\Big[&F_n(\tau;a,b,z_n)-z_n'(\tau) \\ &-F_n(\tau;\overline{a},\overline{b},\overline{z}_n)+\overline{z}_n'(\tau)\Big]. \end{aligned}$$

Then, under the conditions $(A_1)$–$(A_4)$, by using the estimates given above, we obtain from (3.34) that

$$\begin{aligned} &\left(1-\frac{m}{\varepsilon^2}PLN_4\left(1+\frac{\alpha T}{\beta^2}\right)\right)\|\widetilde{a}\|_{C[0,T]}-\frac{m}{\varepsilon^2}PN_4^2\left(1+\frac{\alpha T}{\beta^2}\right)\left\|\widetilde{b}\right\|_{C[0,T]} \\ &-\frac{m}{\varepsilon^2}PN_4M_1\left(1+\frac{\alpha T}{\beta^2}\right)\|\widetilde{z}\|_{B_T}\le C_1(T)\Big[\|\widetilde{\varphi}\|_{C^2[0,T]}+\left\|\widetilde{h}_1\right\|_{C^1[0,T]} \\ &+\left\|\widetilde{h}_2\right\|_{C^1[0,T]}+\left\|\widetilde{f}\right\|_{C^{2,0}(\Omega_T)}\Big], \end{aligned} \tag{3.40}$$

Analogously, we can obtain the estimates for the Eqs (3.35) and (3.36):

$$-\frac{m}{\varepsilon^2}PLN_1\left(1+\frac{\alpha T}{\beta^2}\right)\|\tilde{a}\|_{C[0,T]}+\left(1-\frac{m}{\varepsilon^2}PN_1N_4\left(1+\frac{\alpha T}{\beta^2}\right)\right)\left\|\tilde{b}\right\|_{C[0,T]}$$
$$-\frac{m}{\varepsilon^2}PN_1M_1\left(1+\frac{\alpha T}{\beta^2}\right)\|\tilde{z}\|_{B_T}$$
$$\le C_2(T)\Big[\|\tilde{\varphi}\|_{C^2[0,T]}+\left\|\tilde{h}_1\right\|_{C^1[0,T]}+\left\|\tilde{h}_2\right\|_{C^1[0,T]}+\left\|\tilde{f}\right\|_{C^{2,0}(\Omega_T)}\Big],$$
$$-TL\|\tilde{a}\|_{C[0,T]}-TN_4\left\|\tilde{b}\right\|_{C[0,T]}+(1-TM_1)\|\tilde{z}\|_{B_T}$$
$$\le C_3(T)\Big[\|\tilde{\varphi}\|_{C^2[0,T]}+\left\|\tilde{h}_1\right\|_{C^1[0,T]}+\left\|\tilde{h}_2\right\|_{C^1[0,T]}+\left\|\tilde{f}\right\|_{C^{2,0}(\Omega_T)}\Big],$$

where

$$C_1(T)=\max\left\{\frac{m\alpha}{\beta\varepsilon^2}PN_4,\ \frac{N_4}{\varepsilon^2}\left(P+\frac{N_4R}{2}\right),\ \frac{N_4}{\varepsilon^2}\left(2P+N_4R\right),\ \frac{1}{\varepsilon^2}\left(K+\left(1+\frac{\alpha T}{\beta^2}\right)PN_1M_2+PR+RN_4\left(\frac{N_4}{2}+N_2\right)\right)\right\},$$

$$C_2(T)=\max\left\{\frac{m\alpha}{\beta\varepsilon^2}PN_1,\ \frac{1}{\varepsilon^2}\left(P(N_4+R)+\frac{N_1N_4R}{2}\right),\ \frac{N_4}{\varepsilon^2}\left(2P+N_1R\right),\ \frac{1}{\varepsilon^2}\left(K+\left(1+\frac{\alpha T}{\beta^2}\right)PM_2N_1+RN_1\left(\frac{N_1}{2}+N_2\right)\right)\right\},$$

$$C_3(T)=\max\{1,M_2\},\quad K=2P\left(N_2+\frac{N_1}{2}\right).$$

The coefficient matrix of this system is

$$W(T)=\begin{bmatrix} 1-\frac{m}{\varepsilon^2}PLN_4\left(1+\frac{\alpha T}{\beta^2}\right) & -\frac{m}{\varepsilon^2}PN_4^2\left(1+\frac{\alpha T}{\beta^2}\right) & -\frac{m}{\varepsilon^2}PN_4M_1\left(1+\frac{\alpha T}{\beta^2}\right)\\ -\frac{m}{\varepsilon^2}PLN_1\left(1+\frac{\alpha T}{\beta^2}\right) & \left(1-\frac{m}{\varepsilon^2}PN_1N_4\left(1+\frac{\alpha T}{\beta^2}\right)\right) & -\frac{m}{\varepsilon^2}PN_1M_1\left(1+\frac{\alpha T}{\beta^2}\right)\\ -TL & -TN_4 & (1-TM_1)\end{bmatrix}$$

and $\det(W(T))\neq 0$ with $1-\frac{m}{\varepsilon^2}\left(1+\frac{\alpha T}{\beta^2}\right)PN_4(L+N_1)-TM_1\neq 0$. Thus, under this condition, the inverse matrix $W^{-1}(T)$ exists. Then we obtain the estimates

$$\begin{bmatrix}\|\tilde{a}\|_{C[0,T]}\\ \left\|\tilde{b}\right\|_{C[0,T]}\\ \|\tilde{z}\|_{B_T}\end{bmatrix}\le W^{-1}(T)\begin{bmatrix}C_1(T)\\ C_2(T)\\ C_3(T)\end{bmatrix}$$
$$\times\Big[\|\tilde{\varphi}\|_{C^2[0,T]}+\left\|\tilde{h}_1\right\|_{C^1[0,T]}+\left\|\tilde{h}_2\right\|_{C^1[0,T]}+\left\|\tilde{f}\right\|_{C^{2,0}(\Omega_T)}\Big].\quad(3.41)$$

Therefore, the following theorem is proved.

**Theorem 3.2.** *Let $\{a(\tau),b(\tau),z(\kappa,\tau)\}$ and $\{\overline{a}(\tau),\overline{b}(\tau),\overline{z}(\kappa,\tau)\}$ be any two solutions of (2.2)–(2.5) with the data $\varphi(\kappa)$, $f(\kappa,\tau)$, $h_i(\tau)$, $i=1,2$ and $\overline{\varphi}(\kappa)$, $\overline{f}(\kappa,\tau)$, $\overline{h}_i(\tau)$, $i=1,2$, respectively. Also, the conditions of the Theorem 1 are satisfied. Then, the estimate given by the inequality (3.41) is true for sufficiently small $T$. The constants $C_i(T)$, $i=1,2,3$ and inverse coefficient matrix $W^{-1}(T)$ depend only on the choice of the classes $\mathfrak{I}(N_1,N_2,N_3,N_4)$ and $\aleph(M_1,M_2)$.*

## 4. Discretization of the forward problem via CB-spline functions

We employ the CB-spline functions for numerical solution of the forward problem (2.2)-(2.3), where $\alpha$, $\beta$, $f(\kappa,\tau)$, $a(\tau)$, and $b(\tau)$ are given and $z(\kappa,\tau)$ is unknown. Let us first install a uniform partition of the temporal domain $[0,T]$ using $N+1$ knots $0=\tau_0,\tau_1,\cdots,\tau_N=T$, where $\tau_j=j\Delta\tau$ for $j=0,1,\cdots,N$ and $\Delta\tau=\frac{T}{N}$. To discretize the problem (2.2) in time direction, we utilize finite difference (FD) formulation and Crank-Nicolson (CN) scheme [55] as

$$\frac{z(\kappa,\tau_{j+1})-z(\kappa,\tau_j)}{\Delta\tau}-\beta\left[\frac{z_{\kappa\kappa}(\kappa,\tau_{j+1})-z_{\kappa\kappa}(\kappa,\tau_j)}{\Delta\tau}\right]-\alpha\left[\frac{z_{\kappa\kappa}(\kappa,\tau_{j+1})+z_{\kappa\kappa}(\kappa,\tau_j)}{2}\right]$$
$$=\frac{a(\tau_{j+1})z(\kappa,\tau_{j+1})+a(\tau_j)z(\kappa,\tau_j)}{2}+\frac{b(\tau_{j+1})f(\kappa,\tau_{j+1})+b(\tau_j)f(\kappa,\tau_j)}{2},\ j=0,1,\cdots,N-1.\quad(4.1)$$

After some simplification, Eq (4.1) takes the following form:

$$\left[2-(\Delta\tau)a(\tau_{j+1})\right]z(\kappa,\tau_{j+1})-\left[2\beta+\alpha(\Delta\tau)\right]z_{\kappa\kappa}(\kappa,\tau_{j+1})$$
$$=\left[2+(\Delta\tau)a(\tau_j)\right]z(\kappa,\tau_j)-\left[2\beta-\alpha(\Delta\tau)\right]z_{\kappa\kappa}(\kappa,\tau_j)$$
$$+\Delta\tau\left[b(\tau_{j+1})f(\kappa,\tau_{j+1})+b(\tau_j)f(\kappa,\tau_j)\right],\ j=0,1,\cdots,N-1.\quad(4.2)$$

Now, we subdivide the spatial domain $[0,1]$ into a uniform subinterval of equal length $\Delta\kappa=\frac{1}{M}$, $i=0,1,\cdots,M$. Let $z(\kappa,\tau_j)$ be the CB-spline solution to problem (2.2) at $j^{th}$ time knot s.t.

$$z(\kappa,\tau_j)=\sum_{r=-1}^{M+1}\delta_r(\tau_j)\varrho_r(\kappa),\quad(4.3)$$

where $\delta_r(\tau_j)$'s are unknown time dependant constants, to be determined and the CB-spline basis functions $\varrho_i(\kappa)$ are defined [56]

$$\varrho_i(\kappa) = \frac{1}{6(\Delta\kappa)^3}\begin{cases}(\kappa-\kappa_{i-2})^3, & \text{if } \kappa\in[\kappa_{i-2},\kappa_{i-1}),\\ (\kappa-\kappa_{i-2})^3-4(\kappa-\kappa_{i-1})^3, & \text{if } \kappa\in[\kappa_{i-1},\kappa_i),\\ 4(\kappa-\kappa_{i+1})^3-(\kappa-\kappa_{i+2})^3, & \text{if } \kappa\in[\kappa_i,\kappa_{i+1}),\\ (\kappa_{i+2}-\kappa)^3, & \text{if } \kappa\in[\kappa_{i+1},\kappa_{i+2}),\\ 0, & \text{otherwise.}\end{cases} \tag{4.4}$$

Discretizing Eq (4.2) at $\kappa=\kappa_i$ and using (4.3), we get

$$\begin{aligned}&v_1^{j+1}\sum_{r=-1}^{M+1}\delta_r(\tau_{j+1})\varrho_r(\kappa_i)-v_2\sum_{r=-1}^{M+1}\delta_r(\tau_{j+1})\varrho_r''(\kappa_i)\\ =&v_3^j\sum_{r=-1}^{M+1}\delta_r(\tau_j)\varrho_r(\kappa_i)-v_4\sum_{r=-1}^{M+1}\delta_r(\tau_j)\varrho_r''(\kappa_i)\\ &+\Delta\tau\left[b(\tau_{j+1})f(\kappa_i,\tau_{j+1})+b(\tau_j)f(\kappa_i,\tau_j)\right], i=0,1,\cdots,M,\end{aligned} \tag{4.5}$$

where

$$v_1^j=2-\Delta\tau a(\tau_j),\ v_2=2\beta+\alpha\Delta\tau,$$
$$v_3^j=2+\Delta\tau a(\tau_j),\ v_4=2\beta-\alpha\Delta\tau.$$

Hence, the full discretized form of problem (2.2) is given by

$$\begin{aligned}&v_1^{j+1}\sum_{r=-1}^{M+1}\delta_r^{j+1}\varrho_r(\kappa_i)-v_2\sum_{r=-1}^{M+1}\delta_r^{j+1}\varrho_r''(\kappa_i)\\ =&v_3^j\sum_{r=-1}^{M+1}\delta_r^j\varrho_r(\kappa_i)-v_4\sum_{r=-1}^{M+1}\delta_r^j\varrho_r''(\kappa_i)+\Delta\tau\left[b^{j+1}f_i^{j+1}+b^jf_i^j\right],\\ &i=0,1,2,\cdots,M,\end{aligned} \tag{4.6}$$

where $\delta_r^j=\delta_r(\tau_j)$, $b^j=b(\tau_j)$, and $f_i^j=f(\kappa_i,\tau_j)$. At $(j+1)$th time level, Eq (4.6) represents $M+1$ equations involving $M+3$ unknowns $\delta_{-1}^{j+1},\delta_0^{j+1},\delta_1^{j+1},\cdots,\delta_{M+1}^{j+1}$. In order to obtain a unique solution, one equation is extracted from the end condition (2.3) as

$$\sum_{r=-1}^{M+1}\delta_r^{j+1}\varrho_r'(\kappa_0)-\sum_{r=-1}^{M+1}\delta_r^{j+1}\varrho_r'(\kappa_M)=0. \tag{4.7}$$

Using (4.4), the set of Eqs (4.6) and (4.7) can be written as

$$\tilde{A}\delta^{j+1}=\tilde{B}\delta^j+D^{j+1}, \tag{4.8}$$

where

$$\tilde{A}=\begin{pmatrix}L_1 & L_2 & L_1 & 0 & 0 & 0 & \cdots & 0 & -L_1 & -L_2 & -L_1\\ L_3 & L_4 & L_3 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & L_3 & L_4 & L_3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & & \ddots & \ddots & \ddots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & L_3 & L_4 & L_3 & 0\\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & L_3 & L_4 & L_3\\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & L_5 & L_6 & -L_5\end{pmatrix},$$

$$\tilde{B}=\begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\ L_7 & L_8 & L_7 & 0 & 0 & 0 & \cdots & 0\\ 0 & L_7 & L_8 & L_7 & 0 & 0 & \cdots & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\ 0 & \cdots & 0 & 0 & L_7 & L_8 & L_7 & 0\\ 0 & \cdots & 0 & 0 & 0 & L_7 & L_8 & L_7\\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix},$$

$$\delta^j=\begin{pmatrix}\delta_{-1}^j\\ \delta_0^j\\ \delta_1^j\\ \vdots\\ \delta_{M-1}^j\\ \delta_M^j\\ \delta_{M+1}^j\end{pmatrix}, D^{j+1}=\Delta\tau\begin{pmatrix}L_9\\ b^{j+1}f_0^{j+1}+b^jf_0^j\\ b^{j+1}f_1^{j+1}+b^jf_1^j\\ \vdots\\ b^{j+1}f_M^{j+1}+b^jf_M^j\\ b^{j+1}f_{M+1}^{j+1}+b^jf_{M+1}^j\\ L_{10}\end{pmatrix},$$

$$\begin{aligned}&L_1=1/6,\quad L_2=2/3,\quad L_3=v_1/6-v_2/\Delta\kappa^2,\\ &L_4=2v_1/3+2v_2/\Delta\kappa^2, L_5=-1/2\Delta\kappa, L_6=0,\\ &L_7=v_3/6-v_4/\Delta\kappa^2, L_8=2v_3/3+2v_4/\Delta\kappa^2,\\ &L_9=0, L_{10}=0.\end{aligned} \tag{4.9}$$

The Eq (4.8) is solved for $\delta^{j+1}$ by using the Thomas algorithm. The values of $\delta_r^{j+1}$'s are plugged into Eq (4.3) to get the approximate solution at $\tau=\tau_{j+1}$. However, before using Eq (4.8), we first need $\delta^0$. For this purpose, the initial condition (2.2) is utilized to get the following set of equations:

$$\sum_{r=-1}^{M+1} \delta_r^0 \varrho_r'(\kappa_0) = \varphi'(\kappa_0), \tag{4.10}$$

$$\sum_{r=-1}^{M+1} \delta_r^0 \varrho_r(\kappa_i) = \varphi(\kappa_i), \quad i = 0, 1, 2, \cdots, M, \tag{4.11}$$

$$\sum_{r=-1}^{M+1} \delta_r^0 \varrho_r'(\kappa_M) = \varphi'(\kappa_M). \tag{4.12}$$

Using (4.4), the above set of equations can be written as

$$\begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1/6 & 2/3 & 1/6 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1/6 & 2/3 & 1/6 & 0 & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & 0 & 1/6 & 2/3 & 1/6 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1/6 & 2/3 & 1/6 \\ 0 & \cdots & 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \delta_{-1}^0 \\ \delta_0^0 \\ \delta_1^0 \\ \vdots \\ \delta_{M-1}^0 \\ \delta_M^0 \\ \delta_{M+1}^0 \end{pmatrix} = \begin{pmatrix} 2\Delta\kappa\varphi'(\kappa_0) \\ \varphi(\kappa_0) \\ \varphi(\kappa_1) \\ \vdots \\ \varphi(\kappa_{M-1}) \\ \varphi(\kappa_M)) \\ 2\Delta\kappa\varphi'(\kappa_M) \end{pmatrix}. \tag{4.13}$$

Solving Eq (4.13), we can easily calculate the unknown vector $\delta^0$.

**Example:** Consider the following example of the forward problem (2.2)-(2.3) with $T = 1$ and

$$\varphi(\kappa) = z(\kappa, 0) = (1-\kappa)^4\kappa^4, \; z_\kappa(0,\tau) - z_\kappa(1,\tau) = 0,$$
$$\alpha = 1, \; \beta = 1, \; a(\tau) = -1-\tau, \; b(\tau) = 1+\tau,$$
$$f(\kappa,\tau) = \frac{1}{1+\tau}(12\exp(-2\tau)(1-\kappa)^4\kappa^2 - 32\exp(-2\tau)(1-\kappa)^3\kappa^3 + 12\exp(-2\tau)(1-\kappa)^2\kappa^4 - 2\exp(-2\tau)(1-\kappa)^4\kappa^4 - \exp(-2\tau)(-1-\tau)(1-\kappa)^4\kappa^4). \tag{4.14}$$

The analytical exact solution is

$$z(\kappa,\tau) = \exp(-2\tau)(1-\kappa)^4\kappa^4, \quad (\kappa,\tau) \in \Omega_T, \tag{4.15}$$

and the target outputs (2.5) and (3.1) are

$$h_1(\tau) = \int_0^1 z(\kappa,\tau)d\kappa = \frac{\exp(-2\tau)}{630}, \tag{4.16}$$

$$h_2(\tau) = \int_0^1 \kappa z(\kappa,\tau)d\kappa = \frac{\exp(-2\tau)}{1260}. \tag{4.17}$$

The three-dimensional visuals of approximate and the exact solutions are shown in Figure 1 using different values of $M$ and $N$, and we can see that the computational results converge to the analytical exact solution as the mesh sizes are decreased. Table 1 depicts the numerical additional measurements in (2.5) and (3.1), in comparison of the exact (4.16) and (4.17) obtained by using the CB-spline collocation method as described in Section 3. The root mean square errors (RMSE) for $M = N = 20, 40, 80$ are also listed in Table 2. The trapezoidal rule has been utilized to figure out the integral in (2.5) and (3.1) as

$$\int_0^1 z(\kappa,\tau_j)d\kappa = \frac{1}{2N}\left(z(0,\tau_j) + 2\sum_{i=1}^{M-1} z(\kappa_i,\tau_j) + z(1,\tau_j)\right), \quad j = 0, 1, \cdots, N, \tag{4.18}$$

$$\int_0^1 \kappa z(\kappa,\tau_j)d\kappa = \frac{1}{2N}\left(\kappa(0)z(0,\tau_j) + 2\sum_{i=1}^{M-1} \kappa_i z(\kappa_i,\tau_j) + \kappa(1)z(1,\tau_j)\right), \quad j = 0, 1, \cdots, N. \tag{4.19}$$

**Table 1.** The exact (4.16) and (4.17) and approximate solutions for $h_1(\tau)$ and $h_2(\tau)$ for the direct problem with $\Delta\kappa = \Delta\tau \in \left\{\frac{1}{20}, \frac{1}{40}, \frac{1}{80}\right\}$.

| $\tau$ | 0.1 | $\cdots$ | 0.9 | $\Delta\kappa = \Delta\tau$ |
|---|---|---|---|---|
| $h_1(\tau)$ | 0.001298 | $\cdots$ | 2.6040E-4 | $\frac{1}{20}$ |
| | 0.001299 | $\cdots$ | 2.6216E-4 | $\frac{1}{40}$ |
| | 0.001299 | $\cdots$ | 2.6234E-4 | $\frac{1}{80}$ |
| | **0.001299** | $\cdots$ | **2.6237E-4** | exact |
| $h_2(\tau)$ | 6.4931E-4 | $\cdots$ | 1.3020E-4 | $\frac{1}{20}$ |
| | 6.4973E-4 | $\cdots$ | 1.3108E-4 | $\frac{1}{40}$ |
| | 6.4978E-4 | $\cdots$ | 1.3117E-4 | $\frac{1}{80}$ |
| | **6.4978E-4** | $\cdots$ | **1.3118E-4** | exact |

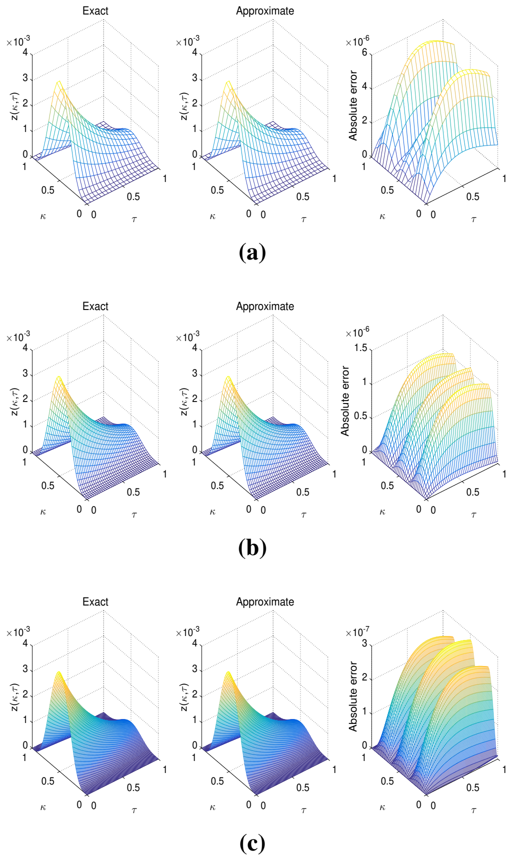



**Figure 1.** The analytical (4.15) solutions, approximate curves, and the absolute computational error for the forward problem using $\Delta\kappa = \Delta\tau = \frac{1}{20}, \frac{1}{40}, \frac{1}{80}$.

**Table 2.** The RMSE between the analytical (4.16) and approximate values for $h_1(\tau)$ and $h_2(\tau)$, with $M = N \in \{20, 40, 80\}$.

| $M = N$ | 20 | 40 | 80 |
|---|---|---|---|
| RMSE($h_1$) | 8.5E-4 | 7.2E-5 | 6.3E-7 |
| RMSE($h_2$) | 1.1E-4 | 2.2E-5 | 3.3E-7 |

## 5. Stability analysis of proposed numerical approach

This section presents the von Neumann stability analysis [57] for the pseudo-parabolic problem (2.2). For the sake of simplicity, we consider Eq (2.2) with $a(\tau) = \mu$ in the absence of source term as

$$z_\tau(\kappa,\tau) - \beta z_{\tau\kappa\kappa}(\kappa,\tau) - \alpha z_{\kappa\kappa}(\kappa,\tau) - \mu z(\kappa,\tau) = 0. \quad (5.1)$$

The full discretized form of (5.1) using FD formulation and CN scheme is given by

$$\mu_1 z_i^{j+1} - \mu_2 (z_{\kappa\kappa})_i^{j+1} = \mu_3 z_i^j - \mu_4 (z_{\kappa\kappa})_i^j,$$

$$j = 0, 1, \cdots, N-1, \quad i = 0, 1, \cdots, M, \quad (5.2)$$

where

$$z_i^j = z(\kappa_i, \tau_j), \quad \mu_1 = 2 - \mu\Delta\tau, \quad \mu_2 = 2\beta + \alpha\Delta\tau,$$
$$\mu_3 = 2 + \mu\Delta\tau, \quad \mu_4 = 2\beta - \alpha\Delta\tau.$$

Using (4.3) in (5.2), we have

$$\begin{aligned} &\mu_1 \sum_{r=-1}^{M+1} \delta_r^{j+1} \varrho_r(\kappa_i) - \mu_2 \sum_{r=-1}^{M+1} \delta_r^{j+1} \varrho_r''(\kappa_i) \\ =&\mu_3 \sum_{r=-1}^{M+1} \delta_r^{j} \varrho_r(\kappa_i) - \mu_4 \sum_{r=-1}^{M+1} \delta_r^{j} \varrho_r''(\kappa_i). \end{aligned} \quad (5.3)$$

Simplifying the above equation, we obtain

$$\begin{aligned} &\left[\frac{\mu_1}{6} - \frac{\mu_2}{(\Delta\kappa)^2}\right]\delta_{i-1}^{j+1} + \left[\frac{2\mu_1}{3} + \frac{2\mu_2}{(\Delta\kappa)^2}\right]\delta_{i}^{j+1} + \left[\frac{\mu_1}{6} - \frac{\mu_2}{(\Delta\kappa)^2}\right]\delta_{i+1}^{j+1} \\ =&\left[\frac{\mu_3}{6} - \frac{\mu_4}{(\Delta\kappa)^2}\right]\delta_{i-1}^{j} + \left[\frac{2\mu_3}{3} + \frac{2\mu_4}{(\Delta\kappa)^2}\right]\delta_{i}^{j} + \left[\frac{\mu_3}{6} - \frac{\mu_4}{(\Delta\kappa)^2}\right]\delta_{i+1}^{j}. \end{aligned} \quad (5.4)$$

Now, we plug in $\delta_i^j = \xi^j e^{\iota i\theta}$ into Eq (5.4) and simplify the result as

$$\begin{aligned} &\left[\frac{\mu_1}{6} - \frac{\mu_2}{(\Delta\kappa)^2}\right]\xi^{j+1} e^{\iota(i-1)\theta} + \left[\frac{2\mu_1}{3} + \frac{2\mu_2}{(\Delta\kappa)^2}\right]\xi^{j+1} e^{\iota(i)\theta} \\ &+ \left[\frac{\mu_1}{6} - \frac{\mu_2}{(\Delta\kappa)^2}\right]\xi^{j+1} e^{\iota(i+1)\theta} = \left[\frac{\mu_3}{6} - \frac{\mu_4}{(\Delta\kappa)^2}\right]\xi^{j} e^{\iota(i-1)\theta} \\ &+ \left[\frac{2\mu_3}{3} + \frac{2\mu_4}{(\Delta\kappa)^2}\right]\xi^{j} e^{\iota(i)\theta} + \left[\frac{\mu_3}{6} - \frac{\mu_4}{(\Delta\kappa)^2}\right]\xi^{j} e^{\iota(i+1)\theta}, \end{aligned} \quad (5.5)$$

where $\theta = \varpi\Delta x$, $\iota = \sqrt{-1}$, and $\varpi$ is the mode number. Substituting $\mu_i$'s, $\alpha = 1$, and $\beta = 1$, the last equation can be written as

$$\xi = \frac{X}{Y}, \quad (5.6)$$

where

$$X = \frac{2(2-\Delta\tau)}{(\Delta\kappa)^2} + \frac{2(2+\Delta\tau)}{3} + 2\left[-\frac{2-\Delta\tau}{(\Delta\kappa)^2} + \frac{2+\Delta\tau}{6}\right]\cos\theta,$$

$$Y = \frac{2(2+\Delta\tau)}{(\Delta\kappa)^2} + \frac{2(2-\Delta\tau)}{3} + 2\left[-\frac{2+\Delta\tau}{(\Delta\kappa)^2} + \frac{2-\Delta\tau}{6}\right]\cos\theta.$$

Now,

$$Y^2 - X^2 =$$

$$\frac{4\Delta\tau\left[216 - \cos^2\left(\frac{\theta}{2}\right)(288 - 16(\Delta\kappa)^4) + \cos^2\theta(72 - 2(\Delta\kappa)^4)\right]}{9(\Delta\kappa)^4}.$$

It is clear that $Y^2 - X^2 > 0$. Hence, $|\xi| < 1$, and the proposed numerical approach is unconditionally stable.

## 6. Inverse problem of the pseudo-parabolic equation

We want to get stable and accurate determinations of $a(\tau)$, $b(\tau)$, and $z(\kappa, \tau)$ that satisfy (2.2)–(2.5). The proposed problem is numerically solved by minimizing the following regularized cost function:

$$\mathbb{F}(a, b) = \left\| \int_0^1 z(\kappa, \tau) d\kappa - h_1(\tau) \right\|^2 + \left\| \int_0^1 \kappa z(\kappa, \tau) d\kappa - h_2(\tau) \right\|^2 + \alpha_1 \|a(\tau)\|^2 + \alpha_2 \|b(\tau)\|^2, \quad (6.1)$$

where $z$ satisfies the forward problem (2.2)-(2.3) with given $a(\tau)$, $b(\tau)$, and $\alpha_i$ for $i = 1, 2$ is the nonnegative penalty parameter. The discretized form of the above function is given by

$$\begin{aligned} \mathbb{F}(\underline{a}, \underline{b}) &= \textstyle\sum_{j=1}^N \left[\int_0^1 z(\kappa, \tau_j) d\kappa - h_1(\tau_j)\right]^2 \\ &+ \textstyle\sum_{j=1}^N \left[\int_0^1 \kappa z(\kappa, \tau_j) d\kappa - h_2(\tau_j)\right]^2 \\ &+ \alpha_1 \textstyle\sum_{j=1}^N (a^j)^2 + \alpha_2 \sum_{j=1}^N (b^j)^2, \end{aligned} \quad (6.2)$$

where $a^j = a(\tau_j)$. The cost function (6.2) is minimizing by means of the MATLAB subroutine *lsqnonlin* tool [58].

## 7. Numerical results and discussion

A couple of experiment examples have been discussed in this section to demonstrate the stability and accuracy of the CB-spline technique together with the Tikhonov regularization process. To validate the efficiency of the numerical solution, the RMSE has been calculated as:

$$\text{RMSE}(a) = \left[\frac{T}{N} \sum_{j=1}^N \left(a^{numerical}(\tau_j) - a^{exact}(\tau_j)\right)^2\right]^{1/2}, \quad (7.1)$$

$$\text{RMSE}(b) = \left[\frac{T}{N} \sum_{j=1}^N \left(b^{numerical}(\tau_j) - b^{exact}(\tau_j)\right)^2\right]^{1/2}. \quad (7.2)$$

For the sake of simplicity, we set $T = 1$. The upper and lower bounds for $a(\tau)$ and $b(\tau)$ are supposed to be $10^2$ and $-10^2$, respectively.

The problem (2.2)–(2.5) has been solved with both perturbed and exact data. The perturbed data is handled as

$$h_1^{\epsilon_1}(\tau_j) = h_1(\tau_j) + \epsilon 1_j, \quad j = 1, 2, ..., N, \quad (7.3)$$

$$h_2^{\epsilon_2}(\tau_j) = h_2(\tau_j) + \epsilon 2_j, \quad j = 1, 2, \cdots, N, \quad (7.4)$$

where $\epsilon 1_j$ and $\epsilon 2_j$ denote the random variables obtained from a Gaussian normal distribution having zero mean and standard deviations $\sigma_1$ and $\sigma_2$ given by

$$\sigma_1 = \max_{0 \le \tau \le T} |h_1(\tau)| \times p, \quad \sigma_2 = \max_{0 \le \tau \le T} |h_2(\tau)| \times p, \quad (7.5)$$

where $p$ represents the percentage of noise. For the perturbed data (7.3), (7.4), $h_1(\tau_j)$ and $h_2(\tau_j)$ are replaced by $h_1^{\epsilon_1}(\tau_j)$, $h_2^{\epsilon_2}(\tau_j)$ in (6.2).

### *7.1. Example 1*

As a first test problem, we take (2.2)–(2.5) with input (4.14)–(4.17), and the unknown smooth and linear time-dependent terms $a(\tau)$ and $b(\tau)$

$$a(\tau) = -1 - \tau, \quad b(\tau) = 1 + \tau, \quad \tau \in [0, 1]. \quad (7.6)$$

From given data, it can be observed that the assumptions of Theorem 1 are fulfilled, so the inverse problem possesses a unique solution. The initial approximation for $\underline{a}$ and $\underline{b}$ is taken as

$$a^0(\tau_j) = a(0) = -1, \quad b^0(\tau_j) = b(0) = 1, \quad j = 1, 2, \cdots, N. (7.7)$$

The piecewise defined approximate solution to the direct problem using CB-spline functions with $M = N = 20$ at

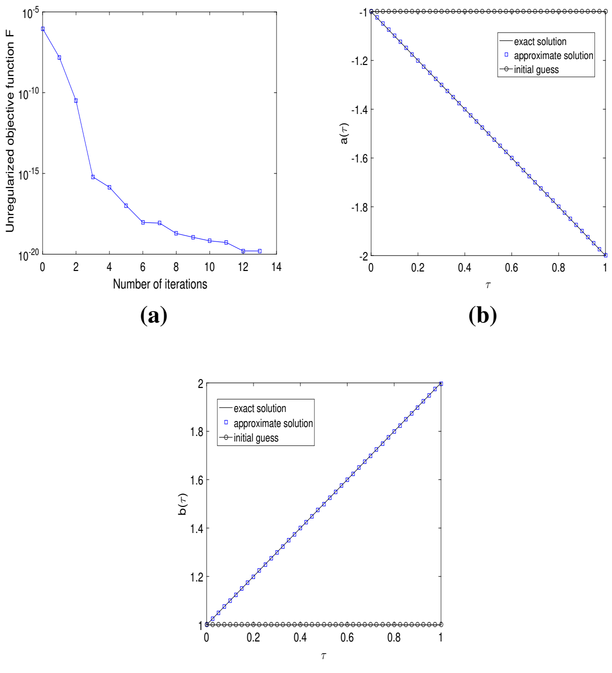
$\tau = 1$ is given by

$$z(\kappa,1)=\begin{cases}
1.45784\times10^{-7}+\kappa(-1.06767\times10^{-16}+(0.002597+0.263406\kappa)\kappa), & \text{if } \kappa\in[0.0,0.05]\\
0.000021393+\kappa(-0.00127483+(0.0280936+0.0934289\kappa)\kappa), & \text{if } \kappa\in[0.05,0.1]\\
0.00014231+\kappa(-0.00490235+(0.0643688-0.0274887\kappa)\kappa), & \text{if } \kappa\in[0.1,0.15]\\
0.000412715+\kappa(-0.0103104+(0.100423-0.107608\kappa)\kappa), & \text{if } \kappa\in[0.15,0.2]\\
0.000765543+\kappa(-0.0156029+(0.126885-0.151712\kappa)\kappa), & \text{if } \kappa\in[0.2,0.25]\\
0.00098119+\kappa(-0.0181906+(0.137236-0.165513\kappa)\kappa), & \text{if } \kappa\in[0.25,0.3]\\
0.000683206+\kappa(-0.0152108+(0.127303-0.154477\kappa)\kappa), & \text{if } \kappa\in[0.3,0.35]\\
-0.00061776+\kappa(-0.00405968+(0.0954428-0.124134\kappa)\kappa), & \text{if } \kappa\in[0.35,0.4]\\
-0.0034425+\kappa(0.0171259+(0.0424789-0.0799972\kappa)\kappa), & \text{if } \kappa\in[0.4,0.45]\\
-0.00821853+\kappa(0.0489661+(-0.0282772-0.0275853\kappa)\kappa), & \text{if } \kappa\in[0.45,0.5]\\
-0.0151148+\kappa(0.090344+(-0.111033+0.0275853\kappa)\kappa), & \text{if } \kappa\in[0.5,0.55]\\
-0.0238349+\kappa(0.137908+(-0.197513+0.0799972\kappa)\kappa), & \text{if } \kappa\in[0.55,0.6]\\
-0.0333684+\kappa(0.185575+(-0.276958+0.124134\kappa)\kappa), & \text{if } \kappa\in[0.6,0.65]\\
-0.0417014+\kappa(0.224035+(-0.336128+0.154477\kappa)\kappa), & \text{if } \kappa\in[0.65,0.7]\\
-0.0454869+\kappa(0.240259+(-0.359304+0.165513\kappa)\kappa), & \text{if } \kappa\in[0.7,0.75]\\
-0.0396644+\kappa(0.216969+(-0.328251+0.151712\kappa)\kappa), & \text{if } \kappa\in[0.75,0.8]\\
-0.0170834+\kappa(0.13229+(-0.222402+0.107608\kappa)\kappa), & \text{if } \kappa\in[0.8,0.85]\\
0.0321201+\kappa(-0.0413693+(-0.0180971+0.0274887\kappa)\kappa), & \text{if } \kappa\in[0.85,0.9]\\
0.120269+\kappa(-0.335199+(0.30838-0.0934289\kappa)\kappa), & \text{if } \kappa\in[0.9,0.95]\\
0.266003+\kappa(-0.795411+(0.792814-0.263406\kappa)\kappa), & \text{if } \kappa\in[0.95,1.0].
\end{cases}$$

We fix $\Delta\kappa = \Delta\tau = 0.025$. In Figure 2(a), the unregularized function (6.2), i.e., $\alpha_1 = \alpha_2 = 0$, has been plotted versus the number of iterations in absence of noise in measurement data. It can be noted that after six iterations, there is a quick decline in the low value of $O(10^{-20})$. The exact (7.6) and approximate solutions to the function $a(\tau)$ and $b(\tau)$ are portrayed in Figures 2(b),(c). It is observed that the numerical outcomes are very accurate with RMSE($a$) =5.8E-4 and RMSE($b$) =7.8E-4.

Next, we associate 0.01%, 0.1% noise with the simulated data (2.4) and (2.5), as in Eq (7.5). It is significant to note that the inverse problem is not well posed; therefore, we anticipate that the cost function needs to be regularized for the sake of stability and accuracy in results. Although not presented, it is illustrated that the regularized cost function $\mathbb{F}$ versus no. of iterations monotonically decreasing convergence is observed. Figures 3 and 4 show visuals of the reconstructed terms $a(\tau)$ and $b(\tau)$. From Figures 3(a),(c) and 4(a),(c), it is clear that, as expected, for $\alpha_1 = \alpha_2 = 0$ we obtain inaccurate and unstable solutions with RMSE($a$) = 0.7741 and RMSE($b$) = 0.6062 for $p = 0.01\%$, and RMSE($a$) = 7.8516 and RMSE($b$) = 8.3187 for $p = 0.1\%$, respectively, as the problem is noise sensitive and ill-posed. Hence, the regularization process is crucial for stable solutions. For this, the regularization parameters $\alpha_i$ for $i = 1, 2$ are chosen to be $10^{-15}, 10^{-14}, 10^{-13}$ for $p = 0.01\%$ noise (see Figures 3(b),(d) obtaining RMSE($a$) $\in \{0.0308, 0.0309, 0.0313\}$ and RMSE($b$) $\in \{0.0222, 0.0223, 0.0224\}$), and $\alpha_1 = \alpha_2 \in \{10^{-14}, 10^{-13}, 10^{-12}\}$ for $p = 0.1\%$ noise (see Figures 4(b),(d) obtaining RMSE($a$) $\in \{0.0577, 0.0578, 0.0581\}$ and RMSE($b$) $\in \{0.0407, 0.0402, 0.0409\}$), which provide stable and comparatively accurate approximations for the functions $a(\tau)$ and $b(\tau)$.

Other details about the RMSE values (7.1) and (7.2), and the no. of iterations, with and without regularization, are listed in Table 3. Eventually, from Figures 2–4 and Table 3, it is observed that the MATLAB simulation results are fairly stable and accurate.




(a) (b) (c)

**Figure 2.** (a) The unregularized form of objective function $\mathbb{F}$ (6.2) versus no. of iterations, and the approximate and analytical exact curves for: (b) the potential $a(\tau)$ and (c) the heat source $b(\tau)$, in absence of noise and regularization, for Example 1.

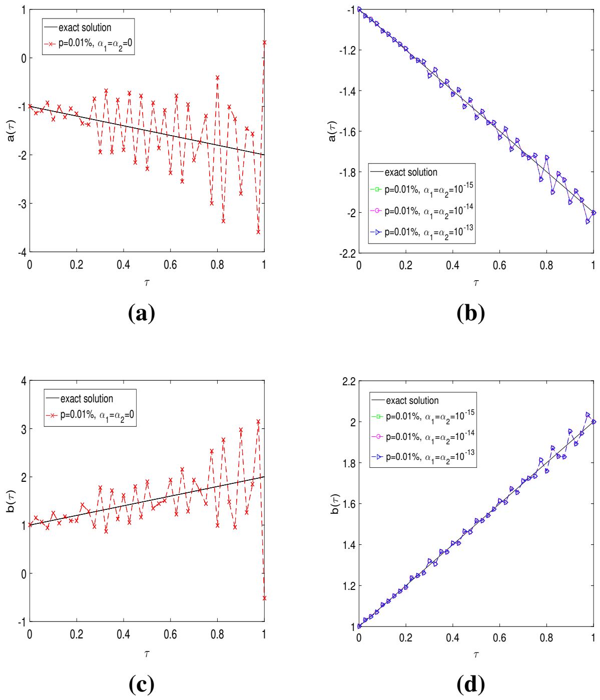



**Figure 3.** The approximate and analytical (7.6) solutions of the potential $a(\tau)$, and the heat source $b(\tau)$, for $p = 0.01\%$, with $\alpha_1 = \alpha_2 \in \{10^{-15}, 10^{-14}, 10^{-13}\}$, for Example 1.

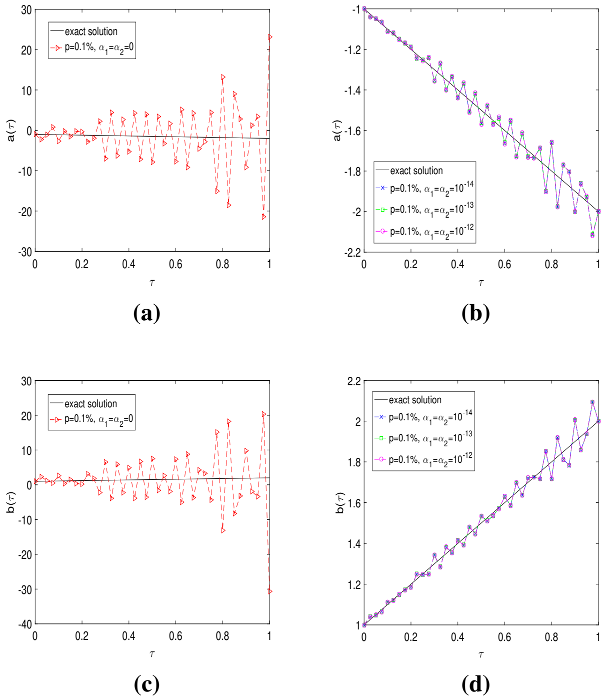



**Figure 4.** The approximate and analytical (7.6) solutions of the potential $a(\tau)$, and the heat source $b(\tau)$, for $p = 0.1\%$, with $\alpha_1 = \alpha_2 \in \{10^{-14}, 10^{-13}, 10^{-12}\}$, for Example 1.

**Table 3.** RMSE values and no. of iterations, with $p \in \{0.01\%, 0.1\%\}$ noise, with $\alpha_1 = \alpha_2 \in \{0, 10^{-16}, 10^{-15}, 10^{-14}, 10^{-13}, 10^{-12}\}$, for Example 1.

| $p$ | $\alpha_1 = \alpha_2$ | RMSE($a$) | RMSE($b$) | Iter |
|---|---|---|---|---|
| | 0 | 0.7741 | 0.6062 | 30 |
| | $10^{-16}$ | 0.0331 | 0.0239 | 10 |
| 0.01% | $10^{-15}$ | 0.0308 | 0.0222 | 10 |
| | $10^{-14}$ | 0.0309 | 0.0223 | 10 |
| | $10^{-13}$ | 0.0313 | 0.0224 | 10 |
| | 0 | 7.8516 | 8.3187 | 50 |
| | $10^{-15}$ | 0.0687 | 0.0512 | 20 |
| 0.1% | $10^{-14}$ | 0.0577 | 0.0407 | 20 |
| | $10^{-13}$ | 0.0578 | 0.0402 | 20 |
| | $10^{-12}$ | 0.0581 | 0.0409 | 20 |

### 7.2. Example 2

In Example 1, the smooth and linear coefficient given by Eq (7.6) has been inverted. Now, we recover a nonlinear function for the potential $a(\tau)$ and heat source $b(\tau)$ terms in the inverse problem (2.2)–(2.5) subject to the following input data:

$$a(\tau) = \frac{1}{-2-\tau^2}, \quad b(\tau) = \frac{1}{2+\tau^2}, \quad \tau \in [0, 1]. \tag{7.8}$$

The analytical exact solution for $z(\kappa, \tau)$ is the same as that given in (4.15). The other input data is kept the same as it was used in Example 1 and

$$f(\kappa,\tau) = -\exp(-2\tau)(-1+\kappa)^2\kappa^2(-24+112\kappa-109\kappa^2 \\ -6\kappa^3+3\kappa^4+2\tau^2(-6+28\kappa-27\kappa^2-2\kappa^3+\kappa^4)). \tag{7.9}$$

With the above data, the assumptions of Theorem 1 can also be verified to affirm that we have a unique solution to the problem. The initial approximation for $\underline{a}$ and $\underline{b}$ is supposed to be

$$a^0(\tau_j) = a(0) = -0.5,$$
$$b^0(\tau_j) = b(0) = 0.5, j = 1, 2, \cdots, N. \qquad (7.10)$$

The piecewise defined approximate solution to the direct problem using CB-spline functions with $M = N = 20$ at $\tau = 1$ is given by

$$z(\kappa, 1) = \begin{cases} -0.0000203367 + \kappa(2.95445 \times 10^{-17} + (0.00259626 + 0.263406\kappa)\kappa), & \text{if } \kappa \in [0.0, 0.05] \\ 9.10345 \times 10^{-7} + \kappa(-0.00127482 + (0.0280927 + 0.0934294\kappa)\kappa), & \text{if } \kappa \in [0.05, 0.1] \\ 0.000121827 + \kappa(-0.00490234 + (0.0643678 - 0.0274877\kappa)\kappa), & \text{if } \kappa \in [0.1, 0.15] \\ 0.00039223 + \kappa(-0.0103104 + (0.100421 - 0.107607\kappa)\kappa), & \text{if } \kappa \in [0.15, 0.2] \\ 0.000745055 + \kappa(-0.0156028 + (0.126883 - 0.15171\kappa)\kappa), & \text{if } \kappa \in [0.2, 0.25] \\ 0.0009607 + \kappa(-0.0181905 + (0.137234 - 0.165511\kappa)\kappa), & \text{if } \kappa \in [0.25, 0.3] \\ 0.000662719 + \kappa(-0.0152107 + (0.127302 - 0.154475\kappa)\kappa), & \text{if } \kappa \in [0.3, 0.35] \\ -0.000638228 + \kappa(-0.00405972 + (0.0954417 - 0.124132\kappa)\kappa), & \text{if } \kappa \in [0.35, 0.4] \\ -0.00346293 + \kappa(0.0171256 + (0.0424785 - 0.0799962\kappa)\kappa), & \text{if } \kappa \in [0.4, 0.45] \\ -0.00823891 + \kappa(0.0489654 + (-0.0282767 - 0.0275849\kappa)\kappa), & \text{if } \kappa \in [0.45, 0.5] \\ -0.0151351 + \kappa(0.0903428 + (-0.111032 + 0.0275849\kappa)\kappa), & \text{if } \kappa \in [0.5, 0.55] \\ -0.0238551 + \kappa(0.137906 + (-0.19751 + 0.0799962\kappa)\kappa), & \text{if } \kappa \in [0.55, 0.6] \\ -0.0333885 + \kappa(0.185573 + (-0.276955 + 0.124132\kappa)\kappa), & \text{if } \kappa \in [0.6, 0.65] \\ -0.0417213 + \kappa(0.224032 + (-0.336123 + 0.154475\kappa)\kappa), & \text{if } \kappa \in [0.65, 0.7] \\ -0.0455068 + \kappa(0.240256 + (-0.3593 + 0.165511\kappa)\kappa), & \text{if } \kappa \in [0.7, 0.75] \\ -0.0396844 + \kappa(0.216966 + (-0.328247 + 0.15171\kappa)\kappa), & \text{if } \kappa \in [0.75, 0.8] \\ -0.0171036 + \kappa(0.132288 + (-0.222399 + 0.107607\kappa)\kappa), & \text{if } \kappa \in [0.8, 0.85] \\ 0.0320997 + \kappa(-0.0413703 + (-0.0180952 + 0.0274877\kappa)\kappa), & \text{if } x \in [0.85, 0.9] \\ 0.120248 + \kappa(-0.335199 + (0.308381 - 0.0934294\kappa)\kappa), & \text{if } \kappa \in [0.9, 0.95] \\ 0.265982 + \kappa(-0.79541 + (0.792814 - 0.263406\kappa)\kappa), & \text{if } \kappa \in [0.95, 1.0]. \end{cases}$$

First, we use $M = N = 40$ and $p = 0$ to retrieve the unknown coefficients $a(\tau)$, $b(\tau)$, and $z(\kappa, \tau)$ for the analytical input data. In Figure 5(a), the unregularized function (6.2), i.e., $\alpha_1 = \alpha_2 = 0$, has been plotted versus the number of iterations in absence of noise in measurement data. It can be noted that after three iterations, there is a quick decline for getting a stipulated tolerance of $O(10^{-26})$. The analytical (7.6) and approximate solutions to the function $a(\tau)$ and $b(\tau)$ are portrayed in Figures 5(b),(c). It is observed that the numerical outcomes are very accurate with RMSE($a$) =1.6E-4 and RMSE($b$) =1.5E-4.

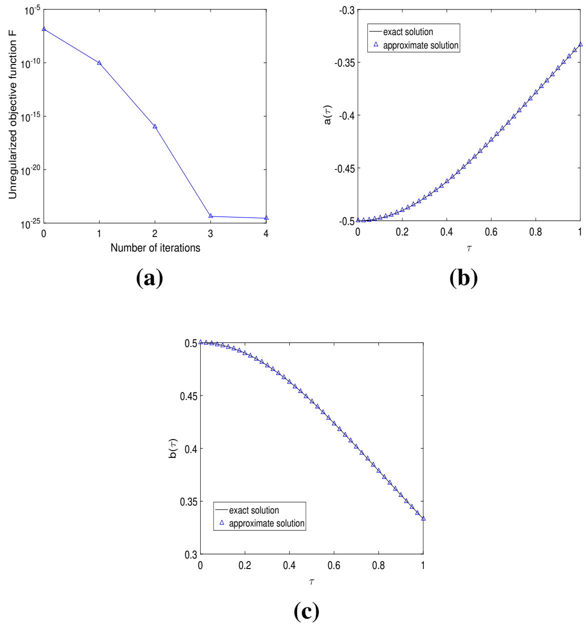



**Figure 5.** (a) The unregularized form of objective function $F$ (6.2) versus no. of iterations, and the approximate and analytical exact curves for: (b) the potential $a(\tau)$ and (c) the heat source $b(\tau)$, in absence of noise and regularization, for Example 2.

Now, we discuss the case of noisy input data with $p \in \{0.01\%, 0.1\%\}$ by including Gaussian random noise in $h_1(\tau)$ and $h_2(\tau)$. As anticipated earlier, without regularization, i.e., $\alpha_1 = \alpha_2 = 0$, the least-squares minimization returns an unstable solution. Hence, for restoration of the stability, we need to utilize the Tikhonov regularization approach by entering the stabilizer term in $\mathbb{F}$ (6.2). The analytical (7.8) and approximate solutions for $a(\tau)$ and $b(\tau)$, with and without regularization are depicted in Figures 6 and 7. From Figures 6(a) and 7(a), it can be noticed that the unstable (highly oscillatory) and inaccurate results are obtained for $a(\tau)$ and $b(\tau)$, if no regularization is installed with RMSE($a$) = 0.5523 and RMSE($b$) = 0.2182 for $p$ = 0.01%, and RMSE($a$) = 1.9582 and RMSE($b$) = 1.9701 for $p$ = 0.1%. In order to stabilize the coefficients $a(\tau)$ and $b(\tau)$, we employed regularization with $\alpha_1 = \alpha_2 \in \{10^{-15}, 10^{-14}, 10^{-13}\}$ (see From Figures 6(b),(d)), obtaining RMSE($a$) $\in \{0.0095, 0.0103, 0.0110\}$ and RMSE($b$) $\in \{0.0074, 0.0076, 0.0099\}$ for $p$ = 0.01%, and $\alpha_1 = \alpha_2 \in$

$\{10^{-14}, 10^{-13}, 10^{-12}\}$ (see From Figures 7(b),(d)), obtaining RMSE($a$) $\in$ $\{0.0257, 0.0240, 0.0404\}$ and RMSE($b$) $\in$ $\{0.0131, 0.0149, 0.0165\}$ for $p = 0.1\%$. The exact (4.15) and numerically approximated solutions for $z(\kappa, \tau)$, without and with regularization, is plotted in Figure 8, where the contribution of $\alpha_i > 0$ for $i = 1, 2$ in curtailing the instability of the recovered temperature can be noted. For completeness, other RMSE values and the number of iterations, with and without regularization, are listed in Table 4. Overall, the computational outcomes produced by the CB-spline collocation approach together with Tikhonov's regularization advocate that accurate and stable approximate solutions can be achieved for the ill-posed third-order pseudo-parabolic equation.

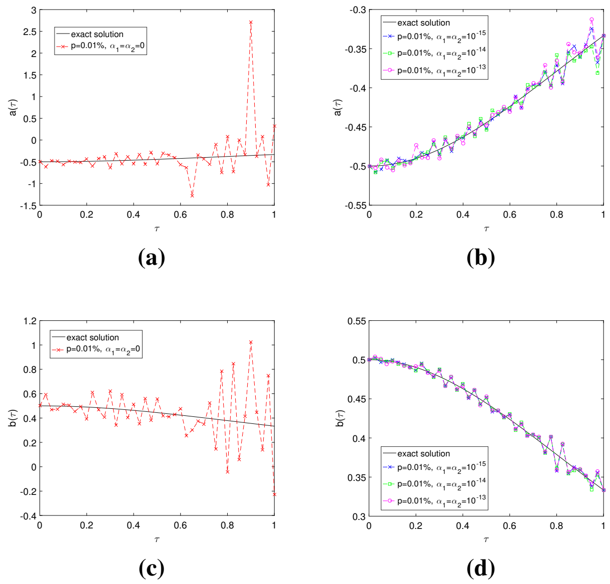



**Figure 6.** The approximate and analytical (7.8) solutions of the potential $a(\tau)$ and the heat source $b(\tau)$, for $p = 0.01\%$, with $\alpha_1 = \alpha_2 \in \{10^{-15}, 10^{-14}, 10^{-13}\}$, for Example 2.

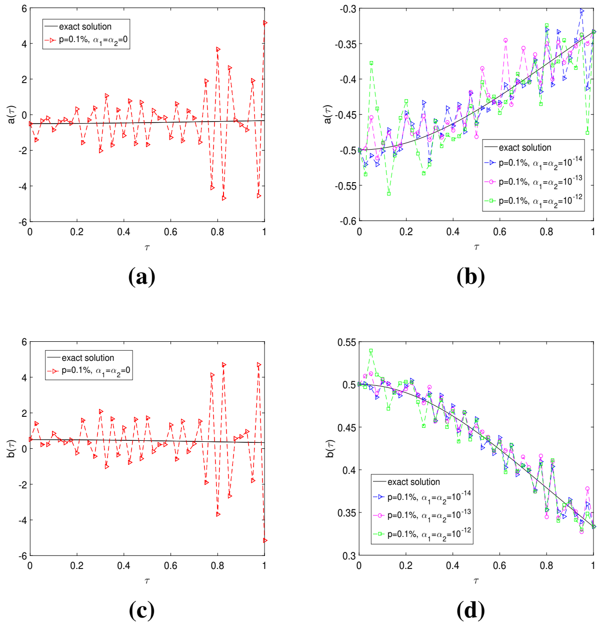



**Figure 7.** The approximate and analytical (7.8) solutions of the potential $a(\tau)$ and the heat source $b(\tau)$, for $p = 0.1\%$, with $\alpha_1 = \alpha_2 \in \{10^{-14}, 10^{-13}, 10^{-12}\}$, for Example 2.

**Table 4.** RMSE values and no. of iterations, with $p \in \{0.01\%, 0.1\%\}$ noise, with $\alpha_1 = \alpha_2 \in \{0, 10^{-16}, 10^{-15}, 10^{-14}, 10^{-13}, 10^{-12}\}$, for Example 2.

| $p$ | $\alpha_1 = \alpha_2$ | RMSE($a$) | RMSE($b$) | Iter |
|---|---|---|---|---|
| | 0 | 0.5523 | 0.2182 | 21 |
| | $10^{-16}$ | 0.0183 | 0.0078 | 10 |
| 0.01% | $10^{-15}$ | 0.0095 | 0.0074 | 10 |
| | $10^{-14}$ | 0.0103 | 0.0076 | 10 |
| | $10^{-13}$ | 0.0110 | 0.0099 | 10 |
| | 0 | 1.9582 | 1.9701 | 31 |
| | $10^{-15}$ | 0.0982 | 0.0812 | 20 |
| 0.1% | $10^{-14}$ | 0.0257 | 0.0131 | 20 |
| | $10^{-13}$ | 0.0240 | 0.0149 | 20 |
| | $10^{-12}$ | 0.0404 | 0.0165 | 20 |

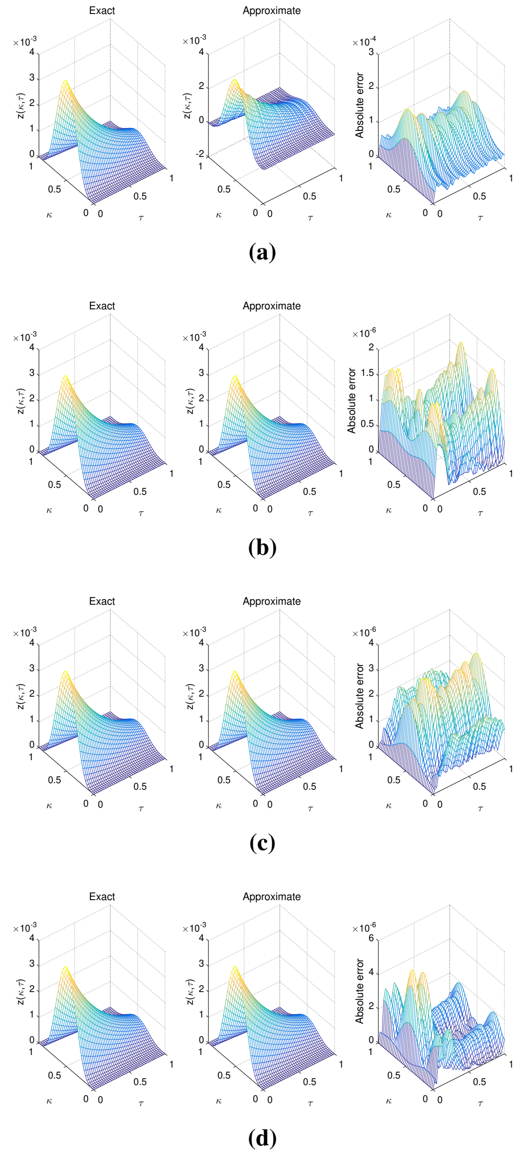



**Figure 8.** The approximate and analytical (4.15) solutions for $z(\kappa, \tau)$, with $p = 0.1\%$, $\alpha_1 = \alpha_2 \in \{0, 10^{-14}, 10^{-13}, 10^{-12}\}$, and the absolute computational error between them, for Example 2.

## 8. Conclusions

The paper considers the problem of recovering the time dependent potential and force coefficients in the third order pseudo-parabolic equation with the homogeneous boundary condition and the nonlocal integral observations. The unique solvability of the solution of the inverse problem on a sufficiently small time interval has been proved by using the contraction principle. The key step of the proof is to establish a fixed-point system, and it was presented via the Fourier series. Such a form of the system brings along computations that are technically simpler than the system in the case of the usual variational approach. The stability results for the inverse problem were presented. In previous works, we investigated various inverse problems of determining a single time-dependent source/force term for the pseudo-parabolic equations of order three or four with fractional or nonfractional derivatives. In this work, we studied the inverse problem of simultaneous recovering the time wise potential and source term, for the first time numerically and theoretically, from a third order pseudo-parabolic equation by using the known energy and barycenter of the system. The proposed work is novel and has never been solved theoretically nor numerically before. The direct solver based on the CB-spline collocation technique was employed. The stability analysis of the numerical solution has been discussed using the von Neumann method. The resulting nonlinear optimization problem was solved computationally by means of the MATLAB subroutine *lsqnonlin*. Since the problem under consideration was ill-posed, therefore, the Tikhonov regularization was utilized in order to tackle the stability. The numerical results for the problem show that stable and accurate approximate results have been obtained.

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Conflict of interest

The authors declare that there is no conflicts of interest.

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