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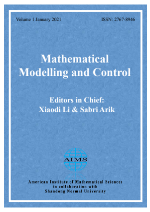

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Research article

Pareto optimal filter design with hybrid H_2/H_∞ optimization

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Abstract: In this article, we study the Pareto optimal H_2 /H_∞ filter design problem for a generalization of discrete-time stochastic systems. By constructing the estimation equation of the given systems with the estimated signal, a filter error estimation system is obtained. The aim is to obtain a gain matrix $K^\star$ that optimizes both performance indicators we set. To deal with this problem, two different upper bounds for two performance indicators are given respectively. The optimal problem therefore is transformed into a Pareto optimal problem with linear matrix inequalities ($LMIs$) which can be addressed through the LMI toolbox in $MATLAB$.

Keywords: $LMIs$; H_2/H_∞ filter; Pareto optimality solutions; multi-objective problem

1. Introduction

How to design a filter is a synthesis problem of high concern. It has developed from classical filters to the present various filters which can be designed according to different characteristics. Therefore, the filtering design plays a great role in dealing with engineering [1–5]. For example, designing a suitable filter for a certain system can reduce the damage of external disturbances to the observed signals. However, in practice, it is hard to design an optimal filter in order to optimize the system, which requires strong theoretical and tool support.

In recent years, the designs of H_∞ filter and H_2/H_∞ filter have become hot research issues and attracted extensive attention [6–11]. H_∞ filter design is a single-objective filter design problem under the restricted condition of a prescribed H_∞ performance index. The vast majority of studies now still focus on the H_∞ filter. As for the advantage of the H_∞ filter, [12] explained that one does not need to know explicitly the statistical nature of the external interference, and it is only needed to assume that the external disturbance has bounded energy. In this paper, we consider Pareto optimal filter design with H_2/H_∞ constraints. It requires the filter reaches the given H_2 and H_∞ performance indices at the same time. Significantly, this simultaneous optimization is optimal not in the usual sense, but in the sense of Pareto optimality. It can be seen as a kind of ideal state of resource allocation. In particular, if a group of people and natural resources are allocated, it should make a person better, at least not make anyone to get worse when a distribution state changes from one to another. In fact, it can be thought of a cooperative game.

Since the concept of Pareto optimality was put forward, many scholars have explored this hot issue [13–18]. Generally speaking, three methods are mainly adopted to solve Pareto Optimization problem: variational method, Pontryagin maximum principle and Bellman dynamic programming method. For continuous or discrete stochastic systems, necessary and sufficient conditions of the existence of Pareto optimal strategy have been studied deeply, and the optimal problem under the LQ performance index in a finite and an infinite time domain have also been dealt with, which provide some basic theoretical support for the subsequent discussions of related problems [19,20]. Pareto optimization filter design has received a lot of attention in recent years [21–23]. To design a multi-objective H_2/H_∞ filter for nonlinear systems, [24] developed an evolutionary algorithm based on $LMIs$ to derive the Pareto optimal solution.

In this paper, we attempt to construct an estimation equation for a general linear system with perturbations. Through a series of constraints, we will get that the filtering error estimation system meets the H_2 and H_∞ performance indices in the sense of Pareto optimality. We just convert the inequality constrained problem into a Pareto optimal problem with $LMIs$ and find out the optimal gain matrix K^*. To this end, by analyzing the linear quadratic objective function and the robustness, a sufficient condition by means of a constraint optimization is derived in the first place.

For convenience, the notation is given as follows: R^n is the space of all real n-dimensional vectors; $x(k)$, $\hat{x}(k)$ and $\tilde{x}(k)$ represent the state vector, the state estimation and the estimation error, respectively; $Tr(A)$ denotes the trace of a matrix A; $min(\alpha^*, \beta^*)$ is the simultaneous minimization of α and β.

2. Problem formulation and preliminary

Consider the following discrete-time linear stochastic system with perturbations and multiplicative noises:

$$\begin{cases} x(k+1) = Ax(k) + Bv(k) + [Cx(k) + Dv(k)]\omega(k), \\ y(k) = Lx(k) + Gv(k), \\ z(k) = Mx(k), \end{cases} \tag{2.1}$$

where $x(k) \in R^n$, $y(k) \in R^q$ and $z(k) \in R^m$ are the system state, the measurement output and the state combination to be estimated, respectively. A, B, C, D, L, G and M are constant matrices of appropriate dimensions. Let $\{\omega(k),\ k = 1, 2, \cdots\}$ be a sequence of real random variables defined on the filtered probability space (Ω, F, P, F_k) with $F_k = \sigma\{\omega(s),\ s = 1, 2, \cdots, k\}$, satisfying $E(\omega(s)) = 0$ and $E(\omega(s_1)\omega(s_2))=\delta_{s_1 s_2}$, where $\delta_{s_1 s_2}$ is the Kronecker operator. $v \in l^2(R_+; R^{n_v}) := \{v_k$ is F_k-measurable, and $E(\sum_{k=0}^{\infty} \|v(k)\|^2)^{\frac{1}{2}} < \infty\}$ is the exogenous disturbance. Moreover, assume that $v(k)$ and $\omega(k)$ are independent of each other.

Construct the estimated equation for $z(k)$ as follows:

$$\begin{cases} \hat{x}(k+1) = A\hat{x}(k) + C\hat{x}(k)\omega(k) + K[y(k) - L\hat{x}(k)], \\ \hat{z}(k) = M\hat{x}(k). \end{cases} \tag{2.2}$$

Subtracting (2.2) from (2.1),we can obtain the filtering error estimation equation:

$$\begin{cases} \tilde{x}(k+1) = (A - KL)\tilde{x}(k) + (B - KG)v(k) + [C\tilde{x}(k) + Dv(k)]\omega(k), \\ \tilde{z}(k) = M\tilde{x}(k), \end{cases} \tag{2.3}$$

where $\tilde{x}(k) = x(k) - \hat{x}(k)$ stands for the state estimation error and $\tilde{z}(k) = z(k) - \hat{z}(k)$ stands for the signal estimation error.

We express H_∞ and H_2 performance indices of system (2.1) in the following form:

$$J_1(K) = \sup_{v \in l^2(R_+; R^{n_v}), v \neq 0, x(0)=0} \frac{E\left\{\sum_{k=0}^{\infty} \tilde{x}^{\mathrm{T}}(k) M^{\mathrm{T}} R_1 M \tilde{x}(k)\right\}}{E\left\{\sum_{k=0}^{\infty} v(k)^{\mathrm{T}} v(k)\right\}},$$
$$J_2(K) = \mathrm{Tr}\left(E\left\{\tilde{z}(k+1) R_2 \tilde{z}^{\mathrm{T}}(k+1)\right\}\right),$$

where R_1 and R_2 are given weighted matrices with $R_2 \geq 0$. The problem of multi-objective filter design can be represented as follows:

$$min_K (J_1(K), J_2(K))\ . \tag{2.4}$$

Remark 2.1. *The traditional H_2/H_∞ filter design focuses on minimizing the H_2 filter performance index $J_2(K) \leqslant \beta$ with a given expected H_∞ performance, which is usually viewed as a single-objective problem with the limit of H_∞ index. For the multi-objective H_2/H_∞ filter problem in* (2.4)*, the filtering performance indices $J_1(K)$ and $J_2(K)$ need to be minimized at the same time. This is the difference between the multi-objective H_2/H_∞ filter design problem and the traditional one.*

With the above analysis, the design of the multi-objective filter including at least two objectives in (2.4), we utilize an indirect method to solve this meaningful question. To this end, we consider the following upper bound for each indicator:

$$J_1(K) \leqslant \alpha, \tag{2.5}$$

$$J_2(K) \leqslant \beta, \tag{2.6}$$

where α and β are positive scalars.

On the basis of (2.5) and (2.6), the multi-objective optimization problem (2.4) can be converted to a solvable form:

$$\begin{aligned} & \min_K (\alpha, \beta) \\ s.t.\quad & J_1(K) \leqslant \alpha, \\ & J_2(K) \leqslant \beta. \end{aligned} \tag{2.7}$$

The definitions and lemma that will be shown are essential to the later discussions.

Definition 2.1 ([16]). *(Pareto optimality) Let $\mathcal{U}$ be the admissible set of all gain matrices. If there is no $K \in \mathcal{U}$ satisfying $J_i(K) \leqslant J_i(K^*)$, $i = 1,2$ and an inequality is strictly true, then $K^* \in \mathcal{U}$ is called Pareto efficient. The corresponding point $J_i(K^*)$ is called a Pareto solution. The set of all Pareto efficient solutions is called the Pareto frontier.*

Definition 2.2 ([16]). *(Pareto dominance) For two solutions (α_1,β_1) and (α_2,β_2), if at least one inequality in $\alpha_1 \leqslant \alpha_2$ and $\beta_1 \leqslant \beta_2$ is strictly true, then we call that (α_1,β_1) is dominate solutions.*

Lemma 2.1. *In the sense of Pareto optimality, the multi-objective optimization problem in* (2.4) *is the same as the one in* (2.7).

Proof. Just state that the two inequality constraints in (2.7) are of the form of Pareto optimal solution. Assume that the 3-element optimal solution of the multi-objective optimization problem in (2.7) is (K^*,α^*,β^*), and the inequality in (2.7) is treated as a strict inequality in the sense of Pareto solutions. If we set $J_1(K^*) < \alpha^*$, then there is an α^1 such that $\alpha^1 < \alpha^*$ and $J_1(K^*) = \alpha^1$ are satisfied for the same K^*, and then (α^1,β^*) dominates the optimal solution (α^*,β^*), which contradicts the hypothesis. So, the conclusion is correct. □

Definition 2.3 ([7]). *(asymptotically mean square stability) Let $v = 0$ in* (2.1). *Stochastic system* (2.1) *is said to be asymptotically stable in the sense of mean square for any initial state $x(0) = x_0 \in R^n$, if*

$$lim_{k\to\infty} E[x(k)x^{\mathrm{T}}(k)] = 0.$$

The fundamental purpose of this article is to get a filter gain matrix K so that the two conditions which will be shown are satisfied:

(i) The equilibrium point $\tilde{x} \equiv 0$ of the filtering error estimation system with $v = 0$ is globally mean square asymptotically stable;

(ii)$J_1(K) \leqslant \alpha$, $J_2(K) \leqslant \beta$ hold for the given disturbance attenuation level α and β, namely, the upper bounds of the performance indices.

From (2.5), it can be obtained that:

$$E\left\{\sum_{k=0}^{\infty} \tilde{x}^{\mathrm{T}}(k)M^{\mathrm{T}}R_1M\tilde{x}(k)\right\} \leqslant \alpha E\left\{\sum_{k=0}^{\infty} v(k)^{\mathrm{T}}v(k)\right\}. \quad (2.8)$$

Because of the impact of initial conditions $\tilde{x}(0)$ on the performance indices of H_∞, the above should be corrected as follows:

$$\begin{aligned} &E\left\{\sum_{k=0}^{\infty} \tilde{x}^{\mathrm{T}}(k)M^{\mathrm{T}}R_1M\tilde{x}(k)\right\} \\ \leqslant &E\tilde{x}^{\mathrm{T}}(0)P\tilde{x}(0) + \alpha E\left\{\sum_{k=0}^{\infty} v(k)^{\mathrm{T}}v(k)\right\}, \end{aligned} \quad (2.9)$$

where P is some positive definite matrix.

Meanwhile, (2.8) can be denoted by

$$J_2(K) = \mathrm{Tr}\left(E\left\{\tilde{z}(k+1)R_2\tilde{z}^{\mathrm{T}}(k+1)\right\}\right) \leqslant \beta. \quad (2.10)$$

3. Main results

Through the introductory analysis, the H_2/H_∞ filter design problem treated with in the sense of Pareto optimality has been fully presented. In this section, we will describe our main results.

Theorem 3.1. *The H_2/H_∞ filter design issue in (2.6) can be transformed into the following multi-objective optimization problem:*

$$\min_{P>0,K} (\alpha,\beta), \quad (3.1)$$

s.t. (3.2),(3.3),(3.4) *(These three inequalities are detailed at the upward side of the page down.), and in* (3.4) *m is the dimension of M.*

Before proving Theorem 3.1, we recall the following useful lemma.

Lemma 3.1 ([24]). *(Schur complement) The LMI*

$$\begin{pmatrix} R_1(x) & S(x) \\ S^{\mathrm{T}}(x) & R_2(x) \end{pmatrix} > 0 \quad (3.5)$$

is equivalent to $R_2(x) > 0$, $R_1(x) - S(x)R_2^{-1}(x)S^T(x) > 0$, where $R_1(x) = R_1^T(x)$ and $R_2(x) = R_2^T(x)$.

Proof. Considering (2.7) and remembering the performance indicator (2.9) in mind, we have

$$E\{\sum_{k=0}^{T} \tilde{x}^{\mathrm{T}}(k)M^{\mathrm{T}}R_1M\tilde{x}(k)\}$$

$$\begin{bmatrix} P - M^{\mathrm{T}}R_1M & 0 & 0 & 0 & (A-KL)^{\mathrm{T}}P \\ 0 & \alpha I & 0 & 0 & (B-KG)^{\mathrm{T}}P \\ 0 & 0 & I & 0 & C^{\mathrm{T}} \\ 0 & 0 & 0 & I & D^{\mathrm{T}} \\ P(A-KL) & P(B-KG) & C & D & P \end{bmatrix} > 0, \tag{3.2}$$

$$\begin{bmatrix} P & P(B-KG)\sqrt{R_2} & PD\sqrt{R_2} & PC\sqrt{R_2} & P(A-KL)\sqrt{R_2} \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & P & 0 \\ \left(P(B-KG)\sqrt{R_2}\right)^T & \left(PD\sqrt{R_2}\right)^T & \left(PC\sqrt{R_2}\right)^T & \left(P(A-KL)\sqrt{R_2}\right)^T & P \end{bmatrix} > 0, \tag{3.3}$$

$$\begin{bmatrix} \beta I & \sqrt{m}M \\ \sqrt{m}M^T & P \end{bmatrix} > 0, \tag{3.4}$$

$$\begin{aligned} = \quad & E\{\tilde{x}^{\mathrm{T}}(0)P\tilde{x}(0)\} - E\{\tilde{x}^{\mathrm{T}}(T+1)P\tilde{x}(T+1)\} \\ & +E\{\sum_{k=0}^{T}(\tilde{x}^{\mathrm{T}}(k)M^{\mathrm{T}}R_1M\tilde{x}(k) + \tilde{x}^{\mathrm{T}}(k+1)P \\ & \cdot\tilde{x}(k+1) - \tilde{x}^{\mathrm{T}}(k)P\tilde{x}(k))\} \\ & \leq E\{\tilde{x}^{\mathrm{T}}(0)P\tilde{x}(0) + \alpha\sum_{k=0}^{T}[v(k)^{\mathrm{T}}v(k)]\} \\ & +E\{\sum_{k=0}^{T}\tilde{x}^{\mathrm{T}}(k)M^{\mathrm{T}}R_1M\tilde{x}(k) - \tilde{x}^{\mathrm{T}}(k)P\tilde{x}(k) \\ & +[(A-KL)\tilde{x}(k) + (B-KG)v(k) \\ & +[C\tilde{x}(k) + Dv(k)]\omega(k)]^{\mathrm{T}}P[(A-KL)\tilde{x}(k) \\ & +(B-KG)v(k) + [C\tilde{x}(k) + Dv(k)]\omega(k)] \\ & -\alpha v(k)^{\mathrm{T}}v(k)\} \\ & = E\{\tilde{x}^{\mathrm{T}}(0)P\tilde{x}(0)\} + \alpha\sum_{k=0}^{T}E[v(k)^{\mathrm{T}}v(k)] \\ & +E\sum_{k=0}^{T}\{\tilde{x}^{\mathrm{T}}(k)(MR^{\mathrm{T}}R_1M - P)\tilde{x}(k) \\ & -\alpha v(k)^{\mathrm{T}}v(k) + [(A-KL)\tilde{x}(k) + (B-KG)v(k) \\ & +[C\tilde{x}(k) + Dv(k)]\omega(k)]^{\mathrm{T}}P[(A-KL)\tilde{x}(k) \\ & +(B-KG)v(k) + [C\tilde{x}(k) + Dv(k)]\omega(k)]\}. \end{aligned}$$

Let $T \to \infty$ in the above inequality, one can infer by Definition 2.3

$$\begin{aligned} & E\{\sum_{k=0}^{\infty}[\tilde{x}^{\mathrm{T}}(k)(M^{\mathrm{T}}R_1M - P)\tilde{x}(k)] - \alpha v(k)^{\mathrm{T}}v(k) \\ & +[(A-KL)\tilde{x}(k) + (B-KG)v(k) + [C\tilde{x}(k) \\ & +Dv(k)]\omega(k)]^{\mathrm{T}}P[(A-KL)\tilde{x}(k) + (B-KG)\cdot v(k) \\ & +[C\tilde{x}(k) + Dv(k)]\omega(k)]\} < 0, \end{aligned}$$

which is equivalent to the following inequality being true:

$$\begin{bmatrix} \tilde{x}(k) \\ v(k) \\ \tilde{x}(k)\omega(k) \\ \omega(k) \end{bmatrix}^{\mathrm{T}} \left\{ \begin{bmatrix} (A-KL)^{\mathrm{T}} \\ (B-KG)^{\mathrm{T}} \\ C^{\mathrm{T}} \\ D^{\mathrm{T}} \end{bmatrix} P \begin{bmatrix} (A-KL)^{\mathrm{T}} \\ (B-KG)^{\mathrm{T}} \\ C^{\mathrm{T}} \\ D^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} - \begin{bmatrix} P - M^{\mathrm{T}}R_1M & 0 & 0 & 0 \\ 0 & \alpha I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \right\} \begin{bmatrix} \tilde{x}(k) \\ v(k) \\ \tilde{x}(k)\omega(k) \\ \omega(k) \end{bmatrix} < 0. \tag{3.6}$$

Furthermore, we have from (3.6)

$$\begin{bmatrix} (A-KL)^{\mathrm{T}} \\ (B-KG)^{\mathrm{T}} \\ C^{\mathrm{T}} \\ D^{\mathrm{T}} \end{bmatrix} P \begin{bmatrix} (A-KL)^{\mathrm{T}} \\ (B-KG)^{\mathrm{T}} \\ C^{\mathrm{T}} \\ D^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} - \begin{bmatrix} P - M^{\mathrm{T}}R_1M & 0 & 0 & 0 \\ 0 & \alpha I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} < 0.$$

Using Schur complement (Lemma 3.1), (3.2) is established. At this point, the H_∞ performance index is met, that is

$$E\left\{\sum_{k=0}^{\infty}\tilde{x}^{\mathrm{T}}(k)M^{\mathrm{T}}R_1M\tilde{x}(k)\right\} \leq E\left\{\tilde{x}^{\mathrm{T}}(0)P\tilde{x}(0) + \alpha\sum_{k=0}^{\infty}v(k)^{\mathrm{T}}v(k)\right\},$$

which is equivalent to saying that if (3.2) is guaranteed, then H_∞ performance index has upper bound α.

Next, taking the H_2 performance index (2.5) and (2.3) into account, one gets

$$\begin{aligned}J_2(K) &= \mathrm{Tr}(E\{M[\tilde{x}(k+1)R_2\tilde{x}^{\mathrm{T}}(k+1)]M^{\mathrm{T}}\})\\ &= \mathrm{Tr}(M\{E[\tilde{x}(k+1)R_2\tilde{x}^{\mathrm{T}}(k+1)]\}M^{\mathrm{T}})\\ &= \mathrm{Tr}(M\{E[(A-KL)\tilde{x}(k)+(B-KG)v(k)\\ &\quad +[C\tilde{x}(k)+Dv(k)]\omega(k)]R_2[(A-KL)\tilde{x}(k)\\ &\quad +(B-KG)v(k)+[C\tilde{x}(k)+Dv(k)]\omega(k)]^{\mathrm{T}}\}M^{\mathrm{T}})\\ &= \mathrm{Tr}\{M[(A-KL)E(\tilde{x}(k)\tilde{x}^{\mathrm{T}}(k))R_2(A-KL)^{\mathrm{T}}\\ &\quad +(B-KG)R_2(B-KG)^{\mathrm{T}}+CE(\tilde{x}(k)\tilde{x}^{\mathrm{T}}(k))R_2C^{\mathrm{T}}\\ &\quad +DR_2D^{\mathrm{T}}]M^{\mathrm{T}}\}.\end{aligned}$$

Letting $Q = E\left(\tilde{x}(k)\tilde{x}^{\mathrm{T}}(k)\right)$, then the above representation becomes

$$\begin{aligned}J_2(K) &= \mathrm{Tr}\left\{M\left[(A-KL)QR_2(A-KL)^{\mathrm{T}}-Q+(B-KG)\right.\right.\\ &\quad \left.\left.\cdot R_2(B-KG)^{\mathrm{T}}+CQR_2C^{\mathrm{T}}+DR_2D^{\mathrm{T}}\right]M^{\mathrm{T}}\right\}\\ &\quad +\mathrm{Tr}\left(MQM^{\mathrm{T}}\right).\end{aligned}$$

From above, it is shown that if the inequality

$$\begin{aligned}&(A-KL)QR_2(A-KL)^{\mathrm{T}}-Q+(B-KG)R_2\\ &\quad \cdot(B-KG)^{\mathrm{T}}+CQR_2C^{\mathrm{T}}+DR_2D^{\mathrm{T}}<0\end{aligned} \tag{3.7}$$

holds, then the H_2 performance index has upper bound β

$$J_2(K) < \mathrm{Tr}\left(MQM^{\mathrm{T}}\right) \triangleq \beta. \tag{3.8}$$

Set $P = Q^{-1}$. Multiplying by P on both sides of (3.7), it can be derived that

$$\begin{aligned}&P(A-KL)P^{-1}R_2(A-KL)^{\mathrm{T}}P-P+P(B-KG)R_2\\ &\quad \cdot(B-KG)^{\mathrm{T}}P+PCQR_2C^{\mathrm{T}}P+PDR_2D^{\mathrm{T}}P<0,\end{aligned}$$

which is equivalent to the following inequality holding

$$\begin{bmatrix}\left(P(B-KG)\sqrt{R_2}\right)^{\mathrm{T}}\\ \left(PD\sqrt{R_2}\right)^{\mathrm{T}}\\ \left(PC\sqrt{R_2}\right)^{\mathrm{T}}\\ \left(P(A-KL)\sqrt{R_2}\right)^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}I&0&0&0\\0&I&0&0\\0&0&P&0\\0&0&0&P\end{bmatrix}\cdot\begin{bmatrix}\left(P(B-KG)\sqrt{R_2}\right)^{\mathrm{T}}\\ \left(PD\sqrt{R_2}\right)^{\mathrm{T}}\\ \left(PC\sqrt{R_2}\right)^{\mathrm{T}}\\ \left(P(A-KL)\sqrt{R_2}\right)^{\mathrm{T}}\end{bmatrix}-P<0.$$

Therefore, (3.3) is obtained by the lemma of Schur complement. Moreover, (3.4) can be obtained by (3.8) directly. □

Further, if we let $Z = PK$ (or $K = P^{-1}Z$), (3.2) and (3.3) are respectively equivalent to the following *LMIs* (3.9)and (3.10) (These two inequalities are detailed at the top of the next page.) which can be addressed through *LMI* toolbox in *MATLAB*.

Therefore, the multi-objective optimization problem (3.1) with limits (3.2), (3.3) and (3.4) is no different to the following one:

$$\begin{aligned}(\alpha^*,\beta^*) &= \min_{P>0,Z}(\alpha,\beta)\\ s.t.\quad &LMIs\ (3.9),(3.10)\ and\ (3.4).\end{aligned} \tag{3.11}$$

Corollary 3.1. (*i*) *When only the H_∞ performance is considered, the multi-objective optimization problem (3.11) degenerates into a single-objective optimization problem of H_∞ filter design:*

$$\begin{aligned}\alpha^0 &= \min_{P>0,Z}\alpha\\ s.t.\quad &(3.9).\end{aligned}$$

(*ii*) *When only the H_2 performance is considered, the multi-objective optimization problem (3.11) degenerates into a single-objective optimization problem of H_2 filter design:*

$$\begin{aligned}\beta^0 &= \min_{P>0,Z}\beta\\ s.t.\quad &(3.10)\ and\ (3.4).\end{aligned}$$

(*iii*) *When considering the traditional H_2/H_∞ filter design, we are going to solve a single-objective optimization problem that gives the disturbance attenuation level α of H_∞ filter:*

$$\begin{aligned}\beta_* &= \min_{P>0,Z}\beta\\ s.t.\quad &(3.9),(3.10)\ and\ (3.4).\end{aligned}$$

Theorem 3.2. *The weighting sum method described below can be used to solve the multi-objective H_2/H_∞ filter design problem.*

$$\begin{aligned}&\min_{P>0,Z}\ \eta_1\alpha+\eta_2\beta\\ &s.t.\ LMIs\ (3.9),(3.10)\ and\ (3.4),\end{aligned} \tag{3.12}$$

where $\eta_1 \geq 0$, $\eta_2 \geq 0$, and $\eta_1 + \eta_2 = 1$.

Proof. By replacing (α,β) with the weighted sum $\omega_1\alpha+\omega_2\beta$, a conclusion can be drawn from Theorem 3.1. □

$$\begin{bmatrix} P-M^{\mathrm{T}}R_1M & 0 & 0 & 0 & A^{\mathrm{T}}P-LZ^{\mathrm{T}} \\ 0 & \alpha I & 0 & 0 & B^{\mathrm{T}}P-GZ^{\mathrm{T}} \\ 0 & 0 & I & 0 & C^{\mathrm{T}} \\ 0 & 0 & 0 & I & D^{\mathrm{T}} \\ PA-ZL & PB-ZG & C & D & P \end{bmatrix} > 0 \tag{3.9}$$

$$\begin{bmatrix} P & (PB-ZG)\sqrt{R_2} & PD\sqrt{R_2} & PC\sqrt{R_2} & (PA-ZL)\sqrt{R_2} \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & P & 0 \\ \left((PB-ZG)\sqrt{R_2}\right)^{\mathrm{T}} & \left(PD\sqrt{R_2}\right)^{\mathrm{T}} & \left(PC\sqrt{R_2}\right)^{T} & \left((PA-ZL)\sqrt{R_2}\right)^{\mathrm{T}} & P \end{bmatrix} > 0 \tag{3.10}$$

From above, the multi-objective H_2/H_∞ filter design problem can be converted into a weighted sum filter design problem (3.12) which has a single objective. In order to get diverse Pareto optimal solutions, it is necessary to use different weights in (3.12) several times, and different weighted solutions will be obtained.

Remark 3.1. *Generally speaking, so as to find a feasible set of α and β, it is necessary to introduce the upper bounds α^1 and β^1 and the lower bounds α^0 and β^0 of α and β respectively, that is, $\alpha^0 \le \alpha \le \alpha^1$ and $\beta^0 \le \beta \le \beta^1$.*

Assuming $(P,Z) \in \Omega$, where Ω represents a feasible set of solutions under the constraint, the multi-objective H_2/H_∞ filter design problem (3.11) can be expressed in the following form:

$$\min_{P,Z\in\Omega} \ (\alpha,\beta)$$
$$s.t.\ LMIs\ (3.9), (3.10)\ and\ (3.4).$$

Remark 3.2. *So as to obtain the optimal gain matrix K, the most appropriate P, Z must be found. Assuming that the corresponding solutions of the target values (α_1,β_1) and (α_2,β_2) are (P_1,Z_1) and (P_2,Z_2), respectively, if an inequality in $\alpha_1 \le \alpha_2$, $\beta_1 \le \beta_2$ holds, then we can know that the solutions (P_1,Z_1) dominate the solutions (P_2,Z_2), that is to say, the solutions (P_1,Z_1) are better than the solutions (P_2,Z_2). For the solutions (P^*,Z^*) derived from the target values (α^*,β^*), if there is no other solution (P_1,Z_1) with the target values (α_1,β_1) such that the target values (α_1,β_1) dominate (α^*,β^*), then we say (P^*,Z^*) is the Pareto optimal solution of* (3.11). *The set of Pareto optimal solutions is the Pareto boundary.*

4. An example

In this section, an example is used to illustrate the effectiveness of our obtained results.

Example 4.1. *Consider the following one-dimensional discrete-time linear stochastic systems:*

$$\begin{cases} x(k+1) = -2x(k) + \frac{1}{2}v(k) + [\sqrt{2}x(k) + \frac{1}{\sqrt{2}}v(k)]\omega(k), \\ y(k) = -4x(k) + \frac{1}{4}v(k), \\ z(k) = 2x(k). \end{cases} \tag{4.1}$$

Taking $R_1 = R_2 = \frac{1}{2}$, (3.9), (3.10) *and* (3.4) *can be written as follows:*

$$\alpha\left[\left(P - M^2R_1\right)\left(P - D^2 - C^2\right) - (PA - ZL)^2\right]$$
$$- \left(P - M^2R_1\right)(PB - ZG)^2 > 0,$$
$$P^3 - R_2\left(P^2AB - ZLPB - ZGPA + Z^2GL\right) > 0,$$
$$\beta P - M^2 > 0.$$

In consideration of the parameters of system (4.1), *we get*

$$\alpha > \frac{(P-2)(2P-Z)^2}{16[(P-2)\left(P-\frac{5}{4}\right) - (ZP+4Z)^2]},$$
$$P^3 + \frac{1}{2}P^2 - \frac{5}{4}ZP + \frac{1}{2}Z^2 > 0,$$
$$\beta > \frac{M^2}{P}.$$

By calculating, $P = \frac{3}{2}$, $Z = 2$ and $K = \frac{4}{3}$ can be derived. That is, we can find the optimality α and β meet the constraints.

5. Conclusions

In the sense of Pareto optimality, the H_2/H_∞ filter design problem for stochastic systems with perturbations has been dealt with in this paper. The multi-objective optimization problem with some inequalities constrained was transformed into an optimization problem having some $LMIs$ constraint, which simplifies the filter design as an $LMIs$-constrained multi-objective optimization problem. Notice that so far few conclusions involve Pareto optimal H_2/H_∞ filter design problems for general nonlinear systems with perturbations and multiplicative noises. These challenging issues will be our future research topics.

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Conflict of interest

The authors declare no conflicts of interest to this work.

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