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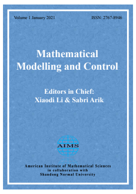

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*Theory article*

# Skew-symmetric games and symmetric-based decomposition of finite games

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**Abstract:** In this paper, skew-symmetric games and a symmetric-based decomposition of finite games are investigated. First, necessary and sufficient conditions for testing skew-symmetric games are obtained by the semi-tensor product method based on adjacent transpositions. By using the obtained conditions for skew-symmetric games, a basis of the skew-symmetric game subspace is constructed. Then, the discriminant equations for a skew-symmetric game with the minimum number are derived. Furthermore, based on the basis of the skew-symmetric game subspace and that of the symmetric game subspace, a basis of the asymmetric game subspace is constructed, which completely solves the problem of symmetric-based decomposition of finite games. Finally, an illustrative example is provided to validate the obtained theoretical results.

**Keywords:** skew-symmetric games; asymmetric games; decomposition of finite games; semi-tensor product of matrices; adjacent transpositions

## 1. Introduction

Game theory is a mathematical theory studying competitive phenomena. Since John von Neumann proved the basic principles of game theory, modern game theory was formally established [1, 2], which has been paid wide attention and applied to biology, economics, computer science, and many other fields. For example, biologists use game theory to predict certain outcomes of evolution. Economists regard the game theory as one of the standard analysis tools of economics.

The concept of symmetric games is first proposed by John von Neumann in [2]. The symmetry of a game means that all players have the same set of strategies and fair payoffs, that is, the payoffs depend only on the strategies employed, not on who is playing them. Because fair games are more realistic and acceptable, many common games are symmetric games such as the well-known games rock-paper-scissors and prisoner's dilemma. In recent years, many problems about symmetric games have been investigated in [3], [4], [5], and [6]. In addition, based on the definition of symmetric games, the concepts of skew-symmetric games, asymmetric games and the symmetric-based decomposition of finite games have been proposed in [4]. Although the bases of the symmetric game subspace and the skew-symmetric game subspace have been constructed in [4], the vector space structure of the asymmetric game subspace has not been revealed. Therefore, the motivation of this paper is to explore the vector space structure of the asymmetric game subspace and thoroughly solve the problem of symmetric-based decomposition of finite games. In our recent paper [6], a new method to construct a basis of the symmetric game subspace has been proposed, which gives us great inspiration for the study of skew-symmetric games, asymmetric games, and symmetric-based decomposition of finite games.

In the past decade, the semi-tensor product (STP) of matrices has been successfully applied to game theory by Cheng and his collaborators [7], which enables a game to be expressed in an algebraic form. In this paper, we still use the matrix method based on STP to investigate skew-symmetric games, asymmetric games and symmetric-based decomposition of finite games. First, by the semi-tensor product method based on adjacent transpositions, necessary and sufficient conditions for testing skew-symmetric games are obtained. Then, based on the necessary and sufficient conditions, a basis of the skew-symmetric game subspace is constructed explicitly. In addition, the discriminant equations for skew-symmetric games with the minimum number are derived concretely. According to the construction methods of the basis of the symmetric game subspace in [6] and the basis of the skew-symmetric game subspace in this paper, a basis of the asymmetric game subspace is constructed for the first time. Therefore, the problem of symmetric-based decomposition of finite games is completely solved.

The rest of this paper is organized as follows: Section 2 gives some preliminaries. Section 3 studies skew-symmetric games and skew-symmetric game subspace. Section 4 studies asymmetric games and solves the problem of symmetric-based decomposition of finite games. Section 5 is a brief conclusion.

## 2. Preliminaries

In this section, some necessary preliminaries are given. Firstly, we list the following notations.

- $\mathcal{D} = \{0, 1\}$ : the set of values of logical variables;
- $\delta_k^i$ : the $i$-th column of $I_k$;
- $\Delta_k := \{\delta_k^i : i = 1, 2, \cdots, k\}$;
- $\delta_k[i_1\ i_2\ \cdots i_n] := [\delta_k^{i_1}\ \delta_k^{i_2}\ \cdots\ \delta_k^{i_n}]$;
- $\mathcal{M}_{m\times n}$ : the set of $m \times n$ matrices;
- $\mathcal{L}_{m\times n} := \{L \in \mathcal{M}_{m\times n} | \mathrm{Col}(L) \subseteq \Delta_m\}$;
- $\ltimes$ : the left semi-tensor product of matrices;
- $\mathbf{1}_n$: the $n$-dimensional column vector of ones;
- $0_{m\times n}$: the $m \times n$ matrix with zero entries;
- $\mathbf{S}_n$: the $n$-th order symmetric group, i.e., a permutation group that is composed of all the permutations of $n$ things;
- $\mathbb{R}$: the set composed of all the real numbers.

**Definition 2.1** ([7]). *Let $A \in \mathcal{M}_{m\times n}$, $B \in \mathcal{M}_{p\times q}$. The left semi-tensor product of A and B is defined as*

$$A \ltimes B = (A \otimes I_{\frac{\alpha}{n}})(B \otimes I_{\frac{\alpha}{p}}), \tag{2.1}$$

*where $\otimes$ is the Kronecker product and $\alpha = \mathrm{lcm}(n, p)$ is the least common multiple of n and p. When no confusion may arise it is usually called the semi-tensor product (STP).*

If $n$ and $p$ in Definition 2.1 satisfy $n = p$, the STP is reduced to the traditional matrix product. So, the STP is a generalized operation of the traditional matrix product. Therefore, one can directly write $A \ltimes B$ as $AB$.

**Definition 2.2** ([7]). *A swap matrix $W_{[m,n]} = (w_{ij}^{IJ})$ is an $mn \times mn$ matrix, defined as follows:*

*Its rows and columns are labeled by double indices. The columns are arranged by the ordered multi-index $Id(i_1, i_2; m, n)$, and the rows are arranged by the ordered multi-index $Id(i_2, i_1; n, m)$. The element at the position with row index $(I, J)$ and column index $(i, j)$ is*

$$w_{ij}^{IJ} = \begin{cases} 1, & I = i \text{ and } J = j, \\ 0, & \text{otherwise.} \end{cases}$$

When $m = n$, matrix $W_{[m,n]}$ is denoted by $W_{[m]}$. Swap matrices have the following properties:

$$(I_k \otimes W_{[k]})(W_{[k]} \otimes I_k)(I_k \otimes W_{[k]}) = (W_{[k]} \otimes I_k)(I_k \otimes W_{[k]})(W_{[k]} \otimes I_k). \tag{2.2}$$

**Definition 2.3** ([8]). *A finite game is a triple $G = (N, S, C)$, where*

**1)** *$N = \{1, 2, \cdots, n\}$ is the set of n players;*

**2)** *$S = S_1 \times S_2 \times \cdots \times S_n$ is the set of strategy profiles, where $S_i = \{s_1^i, s_2^i, \cdots, s_{k_i}^i\}$ is the set of strategies of player i;*

**3)** *$C = \{c_1, c_2, \cdots, c_n\}$ is the set of payoff functions, where $c_i : S \to \mathbb{R}$ is the payoff function of player i.*

Denote the set composed of all the games above by $\mathcal{G}_{[n;k_1,k_2,\cdots,k_n]}$. When $|S_i| = k$ for each $i = 1, 2, \cdots, n$, we denote it by $\mathcal{G}_{[n;k]}$.

STP is a convenient tool for investigating games. Given a game $G \in \mathcal{G}_{[n;k]}$, by using the STP method [9], each strategy $x_i$ can be written into a vector form $x_i \in \Delta_k$, and every payoff function $c_i$ can be expressed as

$$c_i(x_1, x_2, \cdots, x_n) = V_i^c \ltimes_{j=1}^n x_j,\ i = 1, 2, \cdots, n, \tag{2.3}$$

where $\ltimes_{j=1}^{n} x_j \in \Delta_{k^n}$ is called the STP form of the strategy profile, and $V_i^c$ is called the structure vector of $c_i$.

**Definition 2.4** ([10]). *A game $G \in \mathcal{G}_{[n;k]}$ is called a symmetric game if for any permutation $\sigma \in \mathbf{S}_n$*

$$c_i(x_1, x_2, \cdots, x_n) = c_{\sigma(i)}(x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \cdots, x_{\sigma^{-1}(n)}) \quad (2.4)$$

*for any $i = 1, 2, \cdots, n$.*

## 3. Skew-symmetric game and skew-symmetric game subspace

**Definition 3.1** ([4]). *A game $G \in \mathcal{G}_{[n;k]}$ is called a skew-symmetric game if for any permutation $\sigma \in \mathbf{S}_n$*

$$c_i(x_1, x_2, \cdots, x_n) = \mathrm{sgn}(\sigma) c_{\sigma(i)}(x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \cdots, x_{\sigma^{-1}(n)}) \quad (3.1)$$

*for any $i = 1, 2, \cdots, n$.*

The set composed of all the skew-symmetric games in $\mathcal{G}[n;k]$ is denoted as $\mathcal{K}[n;k]$.

**Lemma 3.1** ([11]). *The set of all the adjacent transpositions $(r, r+1), 1 \le r \le n-1$ is generator of the symmetric group $\mathbf{S_n}$.*

In the following, adjacent transpositions $(r, r+1), 1 \le r \le n-1$ are represented as $\mu_r$.

**Lemma 3.2.** *Consider $G \in \mathcal{G}_{[n;k]}$. For any $\sigma_1, \sigma_2 \in \mathbf{S_n}$, if $\sigma_1$ and $\sigma_2$ satisfy*

$$c_i(x_1, x_2, \cdots, x_n) = \mathrm{sgn}(\sigma) c_{\sigma(i)}(x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \cdots, x_{\sigma^{-1}(n)}) \quad (3.2)$$

*for any $i = 1, 2, \cdots, n$ and any $x_1, x_2, \cdots, x_n \in \Delta_k$, the compound permutation $\sigma_2 \circ \sigma_1$ also satisfies (3.2).*

*Proof.* For any given $x_i \in \Delta_k$, $i = 1, 2, \ldots, n$, let $y_i = x_{\sigma_1^{-1}(i)}$. Then

$$\begin{aligned}
& c_i(x_1, x_2, \cdots, x_n) \\
=& \mathrm{sgn}(\sigma_1) c_{\sigma_1(i)}(x_{\sigma_1^{-1}(1)}, x_{\sigma_1^{-1}(2)}, \cdots, x_{\sigma_1^{-1}(n)}) \\
=& \mathrm{sgn}(\sigma_1) c_{\sigma_1(i)}(y_1, y_2, \cdots, y_n) \\
=& \mathrm{sgn}(\sigma_2)\mathrm{sgn}(\sigma_1) c_{\sigma_2(\sigma_1(i))}(y_{\sigma_2^{-1}(1)}, y_{\sigma_2^{-1}(2)}, \cdots, y_{\sigma_2^{-1}(n)}) \\
=& \mathrm{sgn}(\sigma_2 \circ \sigma_1) c_{\sigma_2 \circ \sigma_1(i)}(x_{\sigma_1^{-1}(\sigma_2^{-1}(1))}, x_{\sigma_1^{-1}(\sigma_2^{-1}(2))}, \cdots, x_{\sigma_1^{-1}(\sigma_2^{-1}(n))}),
\end{aligned}$$

which implies that $\sigma_2 \circ \sigma_1$ satisfies (3.2).

According to Definition 3.1, Lemma 3.1 and Lemma 3.2, the following lemma follows:

**Lemma 3.3.** *Consider $G \in \mathcal{G}_{[n;k]}$. Game $G$ is a skew-symmetric game if and only if*

$$c_i(x_1, x_2, \cdots, x_n) = -c_{\mu_r(i)}(x_{\mu_r(1)}, x_{\mu_r(2)}, \cdots, x_{\mu_r(n)}) \quad (3.3)$$

*for any adjacent transposition $\mu_r$, $1 \le r \le n-1$, $i = 1, 2, \cdots, n$.*

**Proposition 3.1.** *Consider $G \in \mathcal{G}_{[n;k]}$. Game $G$ is a skew-symmetric game if and only if*

$$V_i^c = -V_{\mu_r(i)}^c T_{\mu_r}, \ \forall i = 1, 2, \cdots, n, \ 1 \le r \le n-1, \quad (3.4)$$

*where $T_{\mu_r} = I_{k^{r-1}} \otimes W_{[k]} \otimes I_{k^{n-r-1}}$.*

*Proof.* For any $i = 1, 2, \cdots, n$ and any $1 \le r \le n-1$, we have

$$\begin{aligned}
& c_{\mu_r(i)}(x_{\mu_r(1)}, x_{\mu_r(2)}, \cdots, x_{\mu_r(n)}) \\
=& V_{\mu_r(i)}^c x_{\mu_r(1)} x_{\mu_r(2)} \cdots x_{\mu_r(n)} \\
=& V_{\mu_r(i)}^c (x_1 x_2 \cdots x_{r-1})(x_{r+1} x_r)(x_{r+2} \cdots x_n) \\
=& V_{\mu_r(i)}^c (x_1 x_2 \cdots x_{r-1})(W_{[k]} x_r x_{r+1})(x_{r+2} \cdots x_n) \\
=& V_{\mu_r(i)}^c T_{\mu_r} x_1 x_2 \cdots x_n.
\end{aligned} \quad (3.5)$$

From (2.3) and (3.5), it follows that (3.4) is equivalent to (3.3). Therefore, the proposition is proved.

**Theorem 3.1.** *Consider $G \in \mathcal{G}_{[n;k]}$. Game $G$ is a skew-symmetric game if and only if*

$$\begin{bmatrix}
I_{k^n} & T_{\mu_1} & & & \\
& I_{k^n} & T_{\mu_2} & & \\
& & \ddots & \ddots & \\
& & & I_{k^n} & T_{\mu_{n-1}} \\
& & & & I_{k^n} + T_{\mu_1} \\
& & & & I_{k^n} + T_{\mu_2} \\
& & & & \vdots \\
& & & & I_{k^n} + T_{\mu_{n-2}}
\end{bmatrix} (V_G)^{\mathrm{T}} = 0, \quad (3.6)$$

*where $T_{\mu_r} = I_{k^{r-1}} \otimes W_{[k]} \otimes I_{k^{n-r-1}}$, $V_G = [V_1^c \ V_2^c \ \cdots \ V_n^c]$, and the omitted elements in the coefficient matrix of (3.6) are all zeros.*

*Proof.* Since $(W_{[k]})^{-1} = W_{[k]}$, we have $(T_{\mu_r})^{-1} = T_{\mu_r}$ for any $1 \le r \le n-1$. Then, the equation $V_i^c = -V_{\mu_r(i)}^c T_{\mu_r}$ is equivalent to $V^c_{\mu_r(i)} = -V^c_i T_{\mu_r}$. According to Proposition 3.1, $G$ is a skew-symmetric game if and only if

$$\begin{bmatrix} I_{k^n} & T_{\mu_1} & & & & \\ & I_{k^n} & T_{\mu_2} & & & \\ & & \ddots & \ddots & & \\ & & & I_{k^n} & T_{\mu_{n-1}} & \\ B_1 & & & & & \\ & B_2 & & & & \\ & & \ddots & & & \\ & & & B_{n-1} & & \\ & & & & B_n & \end{bmatrix} (V_G)^{\mathrm{T}} = 0, \tag{3.7}$$

where

$$B_1 = \begin{bmatrix} I_{k^n} + T_{\mu_2} \\ I_{k^n} + T_{\mu_3} \\ \vdots \\ I_{k^n} + T_{\mu_{n-1}} \end{bmatrix}, B_2 = \begin{bmatrix} I_{k^n} + T_{\mu_3} \\ I_{k^n} + T_{\mu_4} \\ \vdots \\ I_{k^n} + T_{\mu_{n-1}} \end{bmatrix}, \tag{3.8}$$

$$B_{n-1} = \begin{bmatrix} I_{k^n} + T_{\mu_1} \\ I_{k^n} + T_{\mu_2} \\ \vdots \\ I_{k^n} + T_{\mu_{n-3}} \end{bmatrix}, B_n = \begin{bmatrix} I_{k^n} + T_{\mu_1} \\ I_{k^n} + T_{\mu_2} \\ \vdots \\ I_{k^n} + T_{\mu_{n-2}} \end{bmatrix}, \tag{3.9}$$

$$B_r = \begin{bmatrix} I_{k^n} + T_{\mu_1} \\ I_{k^n} + T_{\mu_2} \\ \vdots \\ I_{k^n} + T_{\mu_{r-2}} \\ I_{k^n} + T_{\mu_{r+1}} \\ I_{k^n} + T_{\mu_{r+2}} \\ \vdots \\ I_{k^n} + T_{\mu_{n-1}} \end{bmatrix} \quad (3 \le r \le n-2). \tag{3.10}$$

Let the coefficient matrix of equation (3.7) be

$$\begin{bmatrix} A_1 & A_2 \\ B & 0_{(n-2)^2 k^n \times k^n} \\ 0_{(n-2)k^n \times (n-1)k^n} & B_n \end{bmatrix} \tag{3.11}$$

where

$$A_1 = \begin{bmatrix} I_{k^n} & T_{\mu_1} & & & \\ & I_{k^n} & T_{\mu_2} & & \\ & & \ddots & \ddots & \\ & & & I_{k^n} & T_{\mu_{n-2}} \\ & & & & I_{k^n} \end{bmatrix}, \tag{3.12}$$

$$A_2 = \begin{bmatrix} 0_{(n-2)k^n \times k^n} \\ T_{\mu_{n-1}} \end{bmatrix}, \tag{3.13}$$

$$B = \begin{bmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_{n-1} \end{bmatrix}. \tag{3.14}$$

Since $A_1$ is an invertible matrix, we can perform the following row transformation on the coefficient matrix of (3.7)

$$\begin{bmatrix} I_{(n-1)k^n} & & \\ -BA_1^{-1} & I_{(n-2)(n-1)k^n} & \\ & & I_{(n-2)k^n} \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ B & 0_{(n-2)^2 k^n \times k^n} \\ 0_{(n-2)k^n \times (n-1)k^n} & B_n \end{bmatrix}$$
$$= \begin{bmatrix} A_1 & A_2 \\ 0_{(n-2)^2 k^n \times (n-1)k^n} & -BA_1^{-1}A_2 \\ 0_{(n-2)k^n \times (n-1)k^n} & B_n \end{bmatrix}, \tag{3.15}$$

where

$$-BA_1^{-1}A_2 = \begin{bmatrix} (-1)^{n-1} B_1 T_{\mu_1} T_{\mu_2} \cdots T_{\mu_{n-1}} \\ (-1)^{n-2} B_2 T_{\mu_2} T_{\mu_3} \cdots T_{\mu_{n-1}} \\ \vdots \\ -B_{n-1} T_{\mu_{n-1}} \end{bmatrix}. \tag{3.16}$$

Let

$$F_1 = I_{n-2} \otimes (T_{\mu_{n-1}} T_{\mu_{n-2}} \cdots T_{\mu_1}),$$

$$F_r = I_{n-3} \otimes (T_{\mu_{n-1}} T_{\mu_{n-2}} \cdots T_{\mu_r}), \ \forall\, 2 \le r \le n-1.$$

We perform the following row transformation on matrix $-BA_1^{-1}A_2$

$$\begin{bmatrix} (-1)^{n-1} F_1 & & & \\ & (-1)^{n-2} F_2 & & \\ & & \ddots & \\ & & & -F_n \end{bmatrix} (-BA_1^{-1}A_2)$$
$$= \begin{bmatrix} F_1 B_1 T_{\mu_1} T_{\mu_2} \cdots T_{\mu_{n-1}} \\ F_2 B_2 T_{\mu_2} T_{\mu_3} \cdots T_{\mu_{n-1}} \\ \vdots \\ F_{n-1} B_{n-1} T_{\mu_{n-1}} \end{bmatrix}. \tag{3.17}$$

Therefore, the equivalent form of (3.7) is as follows

$$\begin{bmatrix} I_{k^n} & T_{\mu_1} & & & \\ & I_{k^n} & T_{\mu_2} & & \\ & & \ddots & \ddots & \\ & & & I_{k^n} & T_{\mu_{n-1}} \\ & & & & F_1B_1T_{\mu_1}\cdots T_{\mu_{n-1}} \\ & & & & F_2B_2T_{\mu_2}\cdots T_{\mu_{n-1}} \\ & & & & \vdots \\ & & & & F_{n-1}B_{n-1}T_{\mu_{n-1}} \\ & & & & B_n \end{bmatrix}(V_G)^{\mathrm{T}}=0. \quad (3.18)$$

From the property of $W_{[k]}$ shown in (2.2), it follows that

$$\begin{aligned} & T_{\mu_{n-1}}\cdots T_{\mu_{r+1}}T_{\mu_r}T_{\mu_i}T_{\mu_r}T_{\mu_{r+1}}\cdots T_{\mu_{n-1}} \\ =& \begin{cases} T_{\mu_i} & \forall 1\le i\le r-2, \\ T_{\mu_{i-1}} & \forall r+1\le i\le n-1. \end{cases} \end{aligned} \quad (3.19)$$

Then, (3.18) is equivalent to (3.6). Thus, the proof is complete.

We see that the key of solving equation (3.6) is computing the solution space of the following linear equation:

$$\begin{bmatrix} I_{k^n}+T_{\mu_1} \\ I_{k^n}+T_{\mu_2} \\ \vdots \\ I_{k^n}+T_{\mu_{n-2}} \end{bmatrix} x=0, \quad (3.20)$$

where $x$ is the $k^n$-dimensional unknown vector. Considering

$$\begin{aligned} \begin{bmatrix} I_{k^n}+T_{\mu_1} \\ I_{k^n}+T_{\mu_2} \\ \vdots \\ I_{k^n}+T_{\mu_{n-2}} \end{bmatrix} &= \begin{bmatrix} I_{k^n}+W_{[k]}\otimes I_{k^{n-2}} \\ I_{k^n}+I_k\otimes W_{[k]}\otimes I_{k^{n-3}} \\ \vdots \\ I_{k^n}+I_{k^{n-3}}\otimes W_{[k]}\otimes I_k \end{bmatrix} \\ &= \begin{bmatrix} I_{k^{n-1}}+W_{[k]}\otimes I_{k^{n-3}} \\ I_{k^{n-1}}+I_k\otimes W_{[k]}\otimes I_{k^{n-4}} \\ \vdots \\ I_{k^{n-1}}+I_{k^{n-3}}\otimes W_{[k]} \end{bmatrix}\otimes I_k, \end{aligned} \quad (3.21)$$

we only need to solve the linear equations as follows:

$$\begin{bmatrix} I_{k^{n-1}}+W_{[k]}\otimes I_{k^{n-3}} \\ I_{k^{n-1}}+I_k\otimes W_{[k]}\otimes I_{k^{n-4}} \\ \vdots \\ I_{k^{n-1}}+I_{k^{n-3}}\otimes W_{[k]} \end{bmatrix} x=0, \quad (3.22)$$

where $x$ is the $k^{n-1}$-dimensional unknown vector. Let $x=(x_{l_1l_2\cdots l_{n-1}})$ be arranged by the ordered multi-index $\mathrm{Id}(i_1,i_2,\ldots,i_{n-1};k,k,\ldots,k)$, that is,

$$\begin{aligned} x=(&x_{11\cdots11},x_{11\cdots12},\ldots,x_{11\cdots1k},x_{11\cdots21},x_{11\cdots22},\ldots,x_{11\cdots2k}, \\ &\ldots,x_{kk\cdots k1},x_{kk\cdots k2},\ldots,x_{kk\cdots kk})^{\mathrm{T}}. \end{aligned} \quad (3.23)$$

Then, by the property of $W_{[k]}$, vector $x$ is a solution of (3.22) if and only if, for any $1\le l_1,l_2,\cdots,l_{n-1}\le k$, the following equations hold:

$$\begin{aligned} x_{l_1l_2l_3\cdots l_{n-1}} &= -x_{l_2l_1l_3\cdots l_{n-1}}, \\ x_{l_1l_2l_3\cdots l_{n-1}} &= -x_{l_1l_3l_2\cdots l_{n-1}}, \\ &\vdots \\ x_{l_1l_2l_3\cdots l_{n-1}} &= -x_{l_1\cdots l_{n-3}l_{n-1}l_{n-2}}, \end{aligned} \quad (3.24)$$

i.e.

$$x_{l_1l_2\cdots l_{n-1}}=\mathrm{sgn}(\pi)x_{\pi(l_1l_2\cdots l_{n-1})},\ \forall\pi\in\mathbf{S}_{n-1}. \quad (3.25)$$

Thus, for any $1\le r\le n-2$, if $l_r=l_{r+1}$, then

$$x_{l_1\cdots l_rl_rl_{r+2}\cdots l_{n-1}}=-x_{l_1\cdots l_rl_rl_{r+2}\cdots l_{n-1}},$$

that is,

$$x_{l_1\cdots l_rl_rl_{r+2}\cdots l_{n-1}}=0.$$

Therefore, all the free variables of the linear equations (3.22) are

$$x_{l_1l_2\cdots l_{n-1}},\ \forall\ 1\le l_1<l_2<\cdots<l_{n-1}\le k, \quad (3.26)$$

whose number is $C_k^{n-1}$. That is, the dimension of the solution space of linear equations (3.22) is $C_k^{n-1}$.

For every given repeatable combination $s_1s_2\cdots s_{n-1}$, ($1\le s_1\le s_2\le\cdots\le s_{n-1}\le k$), denote by $P_{s_1s_2\cdots s_{n-1}}$ the set composed of all the repeatable permutation of $s_1s_2\cdots s_{n-1}$. For example, $P_{122}=\{122,212,221\}$. For every given unrepeatable combination $l_1l_2\cdots l_{n-1}$, ($1\le l_1<l_2<\cdots<l_{n-1}\le k$), denote by $R_{l_1l_2\cdots l_{n-1}}$ the set composed of all the unrepeatable permutation of $l_1l_2\cdots l_{n-1}$. For example, $R_{123}=\{123,132,213,231,312,321\}$. Let

$$Q=\Big(\bigcup_{1\le s_1\le s_2\le\cdots\le s_{n-1}\le k}P_{s_1s_2\cdots s_{n-1}}\Big)\setminus\Big(\bigcup_{1\le l_1<l_2<\cdots<l_{n-1}\le k}R_{l_1l_2\cdots l_{n-1}}\Big).$$

Then, any permutation in $Q$ is a repeated permutation.

**Lemma 3.4.** *For every given unrepeatable combination $l_1 l_2 \cdots l_{n-1}$ ($1 \le l_1 < l_2 < \cdots < l_{n-1} \le k$), define a vector $\theta^{l_1 l_2 \cdots l_{n-1}} = x$ with the form (3.23) by*

$$x_{t_1 t_2 \cdots t_{n-1}} = \begin{cases} \mathrm{sgn}(t_1 t_2 \cdots t_{n-1}), & t_1 t_2 \cdots t_{n-1} \in R_{l_1 l_2 \cdots l_{n-1}}, \\ 0, & \text{otherwise}. \end{cases}$$

*Then the set*

$$\left\{\theta^{l_1 l_2 \cdots l_{n-1}} \mid 1 \le l_1 < l_2 < \cdots < l_{n-1} \le k\right\} \tag{3.27}$$

*is a basis of the solution space $\bar{X}_{n-1}$ of (3.22). For every $l_1 l_2 \cdots l_{n-1}$ ($1 \le l_1 < l_2 < \cdots < l_{n-1} \le k$), we define $|R_{l_1 l_2 \cdots l_{n-1}}| - 1$ number of vectors $v^{r_1 r_2 \cdots r_{n-1}}_{l_1 l_2 \cdots l_{n-1}} = x$ with $r_1 r_2 \cdots r_{n-1} \in R_{l_1 l_2 \cdots l_{n-1}}$ and $r_1 r_2 \cdots r_{n-1} \ne l_1 l_2 \cdots l_{n-1}$ by*

$$x_{t_1 t_2 \cdots t_{n-1}} = \begin{cases} 1, & t_1 t_2 \cdots t_{n-1} = l_1 l_2 \cdots l_{n-1}, \\ -\mathrm{sgn}(t_1 t_2 \cdots t_{n-1}), & t_1 t_2 \cdots t_{n-1} = r_1 r_2 \cdots r_{n-1}, \\ 0, & \text{otherwise}. \end{cases}$$

*We define $|Q|$ number of vectors $\lambda^{h_1 h_2 \cdots h_{n-1}} = x$ ($h_1 h_2 \cdots h_{n-1} \in Q$) by*

$$x_{t_1 t_2 \cdots t_{n-1}} = \begin{cases} 1, & t_1 t_2 \cdots t_{n-1} = h_1 h_2 \cdots h_{n-1}, \\ 0, & \text{otherwise}. \end{cases}$$

*Then the set of $v^{r_1 r_2 \cdots r_{n-1}}_{l_1 l_2 \cdots l_{n-1}}$ ($1 \le l_1 < l_2 < \cdots < l_{n-1} \le k, r_1 r_2 \cdots r_{n-1} \in R_{l_1 l_2 \cdots l_{n-1}}, r_1 r_2 \cdots r_{n-1} \ne l_1 l_2 \cdots l_{n-1}$) and $\lambda^{h_1 h_2 \cdots h_{n-1}}$ ($h_1 h_2 \cdots h_{n-1} \in Q$) is a basis of the orthogonal complementary space $\bar{X}^{\perp}_{n-1}$. Denote by $M_{\mathcal{W}}$ the matrix whose columns are composed of a basis of subspace $\mathcal{W}$. Then the linear system (3.22) is equivalent to $M^{\mathrm{T}}_{\bar{X}^{\perp}_{n-1}} x = 0$.*

*Proof.* For any $1 \le l_1 < l_2 < \cdots < l_{n-1} \le k$, $\mathrm{sgn}(l_1 l_2 \cdots l_{n-1}) = 1$. From the equivalent equations (3.25) and the free variables shown by (3.26), it follows that the set of $\theta^{l_1 l_2 \cdots l_{n-1}}$ ($1 \le l_1 < l_2 < \cdots < l_{n-1} \le k$) is a basis of the solution space $\bar{X}_{n-1}$. By the construction of $v^{r_1 r_2 \cdots r_{n-1}}_{l_1 l_2 \cdots l_{n-1}}$ and $\lambda^{h_1 h_2 \cdots h_{n-1}}$, it is straightforward to see that each $v^{r_1 r_2 \cdots r_{n-1}}_{l_1 l_2 \cdots l_{n-1}}$ and each $\lambda^{h_1 h_2 \cdots h_{n-1}}$ are orthogonal to $\bar{X}_{n-1}$. The total number of $v^{r_1 r_2 \cdots r_{n-1}}_{l_1 l_2 \cdots l_{n-1}}$ is

$$\sum_{1 \le l_1 < l_2 < \cdots < l_{n-1} \le k} (|R_{l_1 l_2 \cdots l_{n-1}}| - 1) = \sum_{1 \le l_1 < l_2 < \cdots < l_{n-1} \le k} |R_{l_1 l_2 \cdots l_{n-1}}| - C^{n-1}_k.$$

The total number of $\lambda^{h_1 h_2 \cdots h_{n-1}}$ is

$$\sum_{1 \le s_1 \le s_2 \le \cdots \le s_{n-1} \le k} |P_{s_1 s_2 \cdots s_{n-1}}| - \sum_{1 \le l_1 < l_2 < \cdots < l_{n-1} \le k} |R_{l_1 l_2 \cdots l_{n-1}}|$$

$$= k^{n-1} - \sum_{1 \le l_1 < l_2 < \cdots < l_{n-1} \le k} |R_{l_1 l_2 \cdots l_{n-1}}|.$$

Then the total number of $v^{r_1 r_2 \cdots r_{n-1}}_{l_1 l_2 \cdots l_{n-1}}$ and $\lambda^{h_1 h_2 \cdots h_{n-1}}$ is $k^{n-1} - C^{n-1}_k$, i.e. $k^{n-1} - \dim(\bar{X}_{n-1})$. Therefore, we conclude that the set of $v^{r_1 r_2 \cdots r_{n-1}}_{l_1 l_2 \cdots l_{n-1}}$ ($1 \le l_1 < l_2 < \cdots < l_{n-1} \le k, r_1 r_2 \cdots r_{n-1} \in R_{l_1 l_2 \cdots l_{n-1}}, r_1 r_2 \cdots r_{n-1} \ne l_1 l_2 \cdots l_{n-1}$) and $\lambda^{h_1 h_2 \cdots h_{n-1}}$ ($h_1 h_2 \cdots h_{n-1} \in Q$) is a basis of $\bar{X}^{\perp}_{n-1}$. Then, the linear system (3.22) is equivalent to $M^{\mathrm{T}}_{\bar{X}^{\perp}_{n-1}} x = 0$.

According to the above basis of the solution space of linear equations (3.22), we can construct a basis of skew-symmetric game subspace $\mathcal{K}_{[n;k]}$.

**Theorem 3.2.** *The dimension of the skew-symmetric game subspace $\mathcal{K}_{[n;k]}$ is $kC^{n-1}_k$. A basis of $\mathcal{K}_{[n;k]}$ is composed of the columns of matrix*

$$\begin{bmatrix} (-1)^{n-1} W_{[k^{n-1},k]} \\ (-1)^{n-2} I_k \otimes W_{[k^{n-2},k]} \\ (-1)^{n-3} I_{k^2} \otimes W_{[k^{n-3},k]} \\ \vdots \\ (-1)^2 I_{k^{n-3}} \otimes W_{[k^2,k]} \\ -I_{k^{n-2}} \otimes W_{[k]} \\ I_{k^n} \end{bmatrix} (M_{\bar{X}_{n-1}} \otimes I_k), \tag{3.28}$$

*where $M_{\bar{X}_{n-1}}$ is composed of the basis of the solution space of (3.22). Moreover, the linear equations with the minimum number to test skew-symmetric games in $\mathcal{K}_{[n;k]}$ are*

$$\begin{bmatrix} I_{k^n} & T_{\mu_1} & & & & \\ & I_{k^n} & T_{\mu_2} & & & \\ & & \ddots & \ddots & & \\ & & & I_{k^n} & T_{\mu_{n-1}} & \\ & & & & M^{\mathrm{T}}_{\bar{X}^{\perp}_{n-1}} \otimes I_k & \end{bmatrix} (V_G)^{\mathrm{T}} = 0, \tag{3.29}$$

*where the omitted elements in the coefficient matrix of (3.29) are all zeros.*

*Proof.* By Theorem 3.1 and Lemma 3.4, we can easily get the dimension of skew-symmetric game subspace $\mathcal{K}_{[n;k]}$ is $kC^{n-1}_k$. Using $M_{\bar{X}_{n-1}}$ whose columns are composed of a basis of the solution space of (3.22), we get a basis of the solution space of (3.6) as follows:

$$\begin{bmatrix} (-1)^{n-1}T_{\mu_1}\cdots T_{\mu_{n-1}}(M_{\bar{X}_{n-1}}\otimes I_k) \\ (-1)^{n-2}T_{\mu_2}\cdots T_{\mu_{n-1}}(M_{\bar{X}_{n-1}}\otimes I_k) \\ (-1)^{n-3}T_{\mu_3}\cdots T_{\mu_{n-1}}(M_{\bar{X}_{n-1}}\otimes I_k) \\ \vdots \\ -T_{\mu_{n-1}}(M_{\bar{X}_{n-1}}\otimes I_k) \\ M_{\bar{X}_{n-1}}\otimes I_k \end{bmatrix}. \tag{3.30}$$

By the property of swap matrices shown in (2.2), we have

$$T_{\mu_s}T_{\mu_{s+1}}\cdots T_{\mu_{n-1}} = I_{k^{s-1}}\otimes W_{[k^{n-s},k]}$$

for each $1 \le s \le n-1$. Then, (3.30) is equivalent to (3.28). That is, the set of the columns of matrix (3.28) is a basis of $\mathcal{K}_{[n;k]}$. Since (3.29) is equivalent to (3.6) and the coefficient matrix of (3.29) has a full row rank, the equations in (3.29) have the minimum number for testing skew-symmetric games in $\mathcal{K}_{[n;k]}$.

**Remark 3.1.** *The coefficient matrix of (3.29) has $nk^n - kC_k^{n-1}$ number of rows and each row has at most two nonzero elements. Since $C_k^{n-1} \le k^{n-1}$, $(n-1)k^n \le nk^n - kC_k^{n-1} \le nk^n$. Therefore, the computational complexity is just $O(nk^n)$ due to*

$$\lim_{n\to\infty}\frac{nk^n}{(n-1)k^n} = \lim_{n\to\infty}\frac{n}{n-1} = 1.$$

## 4. Symmetric-based decomposition of finite games

**Definition 4.1** ([4])**.** *A game $G \in \mathcal{G}_{[n;k]}$ is called an asymmetric game if its structure vector*

$$V_G \in [\mathcal{S}_{[n;k]} \oplus \mathcal{K}_{[n;k]}]^{\perp}.$$

*The set of asymmetric games is denoted by $\mathcal{E}_{[n;k]}$.*

**Lemma 4.1** ([6])**.** *The dimension of the symmetric game subspace $\mathcal{S}_{[n;k]}$ is $kC_{k+n-2}^{n-1}$. A basis of $\mathcal{S}_{[n;k]}$ is composed of the columns of matrix*

$$\begin{bmatrix} W_{[k^{n-1},k]} \\ I_k\otimes W_{[k^{n-2},k]} \\ I_{k^2}\otimes W_{[k^{n-3},k]} \\ \vdots \\ I_{k^{n-2}}\otimes W_{[k]} \\ I_{k^n} \end{bmatrix}(M_{X_{n-1}}\otimes I_k), \tag{4.1}$$

*where $X_{n-1}$ is the solution space of linear equations*

$$\begin{bmatrix} I_{k^{n-1}} - W_{[k]}\otimes I_{k^{n-3}} \\ I_{k^{n-1}} - I_k\otimes W_{[k]}\otimes I_{k^{n-4}} \\ \vdots \\ I_{k^{n-1}} - I_{k^{n-3}}\otimes W_{[k]} \end{bmatrix} x = 0, \tag{4.2}$$

*and $M_{X_{n-1}}$ is the matrix composed of a basis of $X_{n-1}$.*

Let

$$A = \begin{bmatrix} W_{[k^{n-1},k]} \\ I_k\otimes W_{[k^{n-2},k]} \\ I_{k^2}\otimes W_{[k^{n-3},k]} \\ \vdots \\ I_{k^{n-2}}\otimes W_{[k]} \\ I_{k^n} \end{bmatrix}(M_{X_{n-1}}\otimes I_k), \tag{4.3}$$

$$B = \begin{bmatrix} (-1)^{n-1}W_{[k^{n-1},k]} \\ (-1)^{n-2}I_k\otimes W_{[k^{n-2},k]} \\ (-1)^{n-3}I_{k^2}\otimes W_{[k^{n-3},k]} \\ \vdots \\ -I_{k^{n-2}}\otimes W_{[k]} \\ I_{k^n} \end{bmatrix}(M_{\bar{X}_{n-1}}\otimes I_k). \tag{4.4}$$

It is easy to check that

$$\begin{aligned} &\begin{bmatrix} W_{[k^{n-1},k]} \\ I_k\otimes W_{[k^{n-2},k]} \\ I_{k^2}\otimes W_{[k^{n-3},k]} \\ \vdots \\ I_{k^{n-2}}\otimes W_{[k]} \\ I_{k^n} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} (-1)^{n-1}W_{[k^{n-1},k]} \\ (-1)^{n-2}I_k\otimes W_{[k^{n-2},k]} \\ (-1)^{n-3}I_{k^2}\otimes W_{[k^{n-3},k]} \\ \vdots \\ -I_{k^{n-2}}\otimes W_{[k]} \\ I_{k^n} \end{bmatrix} \\ &= \sum_{i=1}^{n}(I_{k^{i-1}}\otimes W_{[k,k^{n-i}]})((-1)^{n-i}I_{k^{i-1}}\otimes W_{[k^{n-i},k]}) \\ &= \sum_{i=1}^{n}(-1)^{n-i}I_{k^n} \end{aligned} \tag{4.5}$$

Since the number of odd permutations of any combination is the same as the number of even permutations, according to the construction of a basis of $X_{n-1}$ in [6], we have

$$\begin{aligned} A^{\mathrm{T}}B &= (M_{X_{n-1}}^{\mathrm{T}}\otimes I_k)(\sum_{i=1}^{n}(-1)^{n-i}I_{k^n})(M_{\bar{X}_{n-1}}\otimes I_k) \\ &= \sum_{i=1}^{n}(-1)^{n-i}(M_{X_{n-1}}^{\mathrm{T}}M_{\bar{X}_{n-1}}\otimes I_k) \\ &= 0_{p\times q}, \end{aligned} \tag{4.6}$$

where $p = kC_{k+n-2}^{n-1}, q = kC_{k}^{n-1}$. That is, $\mathcal{S}_{[n;k]}$ and $\mathcal{K}_{[n;k]}$ are orthogonal. Therefore,

$$\mathcal{G}_{[n;k]} = \mathcal{S}_{[n;k]} \oplus \mathcal{K}_{[n;k]} \oplus \mathcal{E}_{[n;k]}. \tag{4.7}$$

So far, we have constructed a basis of the symmetric game subspace and that of the skew-symmetric game subspace, respectively. Next, according to the two bases, we investigate the vector space structure of the asymmetric game subspace.

Consider the following linear equations

$$\begin{bmatrix} A^{\mathrm{T}} \\ B^{\mathrm{T}} \end{bmatrix} x = 0, \tag{4.8}$$

where $A$ and $B$ are shown in (4.3) and (4.4), composed of the bases of $\mathcal{S}_{[n;k]}$ and $\mathcal{K}_{[n;k]}$, respectively. Therefore, (4.8) is the discriminant equation with the minimum number for asymmetric games, and a basis of the solution space of (4.8) is also a basis of the asymmetric game subspace $\mathcal{E}_{[n;k]}$.

Construct matrices $M_{\mathcal{X}_{n-1}}$ and $M_{\bar{\mathcal{X}}_{n-1}}$ as follows:

$$M_{\mathcal{X}_{n-1}}=[\eta_1\ \eta_2\ \cdots\ \eta_\beta\ \eta_{\beta+1}\ \cdots\ \eta_\alpha],\ M_{\bar{\mathcal{X}}_{n-1}}=[\theta_1\ \theta_2\ \cdots\ \theta_\beta], \tag{4.9}$$

where $\alpha = C_{k+n-2}^{n-1}, \beta = C_{k}^{n-1}$, and

$$\begin{aligned} &\forall\ 1 \le i \le \beta,\ \exists\ 1 \le l_1 < l_2 < \cdots < l_{n-1} \le k, \\ &\qquad \text{s.t. } \eta_i = \eta^{l_1 l_2 \cdots l_{n-1}},\ \ \theta_i = \theta^{l_1 l_2 \cdots l_{n-1}}, \end{aligned} \tag{4.10}$$

$$\begin{aligned} &\forall\ \beta+1 \le i \le \alpha,\ \exists\ 1 \le l_1 \le l_2 \le \cdots \le l_{n-1} \le k, \\ &\qquad \text{and } l_1 l_2 \cdots l_{n-1} \in Q,\ \text{s.t. } \eta_i = \eta^{l_1 l_2 \cdots l_{n-1}}. \end{aligned} \tag{4.11}$$

Let

$$x = \left[(x^1)^{\mathrm{T}},\ (x^2)^{\mathrm{T}}, \ldots, (x^n)^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathbb{R}^{nk^n},$$

where $x^j \in \mathbb{R}^{k^n}$. Then, (4.8) is equivalent to

$$\begin{cases} \sum\limits_{j=1}^{n} [(\eta_i^{\mathrm{T}} + (-1)^{n-j+1}\theta_i^{\mathrm{T}}) \otimes I_k](I_{k^{j-1}} \otimes W_{[k,k^{n-j}]})x^j = 0, \\ \sum\limits_{j=1}^{n} [(\eta_i^{\mathrm{T}} + (-1)^{n-j}\theta_i^{\mathrm{T}}) \otimes I_k](I_{k^{j-1}} \otimes W_{[k,k^{n-j}]})x^j = 0, \end{cases} \tag{4.12}$$
$$(1 \le i \le \beta)$$

and

$$\sum_{j=1}^{n} (\eta_i^{\mathrm{T}} \otimes I_k)(I_{k^{j-1}} \otimes W_{[k,k^{n-j}]})x^j = 0 \tag{4.13}$$
$$(\beta + 1 \le i \le \alpha).$$

According to the construction of $\eta_i = \eta^{l_1 l_2 \cdots l_{n-1}}$ and $\theta_i = \theta^{l_1 l_2 \cdots l_{n-1}}$ $(1 \le i \le \beta)$, we conclude that (4.12) is equivalent to

$$\begin{cases} \sum\limits_{\substack{t_1 t_2 \cdots t_{n-1} \in R_{l_1 l_2 \cdots l_{n-1}} \\ \mathrm{sgn}(t_1 t_2 \cdots t_{n-1})=1}} \sum\limits_{\substack{1 \le j \le n \\ j \text{ is odd}}} x^j_{t_1 t_2 \cdots t_{j-1} l_n t_{j+1} \cdots t_{n-1}} \\ \quad + \sum\limits_{\substack{t_1 t_2 \cdots t_{n-1} \in R_{l_1 l_2 \cdots l_{n-1}} \\ \mathrm{sgn}(t_1 t_2 \cdots t_{n-1})=-1}} \sum\limits_{\substack{1 \le j \le n \\ j \text{ is even}}} x^j_{t_1 t_2 \cdots t_{j-1} l_n t_{j+1} \cdots t_{n-1}} = 0, \\ \sum\limits_{\substack{t_1 t_2 \cdots t_{n-1} \in R_{l_1 l_2 \cdots l_{n-1}} \\ \mathrm{sgn}(t_1 t_2 \cdots t_{n-1})=-1}} \sum\limits_{\substack{1 \le j \le n \\ j \text{ is odd}}} x^j_{t_1 t_2 \cdots t_{j-1} l_n t_{j+1} \cdots t_{n-1}} \\ \quad + \sum\limits_{\substack{t_1 t_2 \cdots t_{n-1} \in R_{l_1 l_2 \cdots l_{n-1}} \\ \mathrm{sgn}(t_1 t_2 \cdots t_{n-1})=1}} \sum\limits_{\substack{1 \le j \le n \\ j \text{ is even}}} x^j_{t_1 t_2 \cdots t_{j-1} l_n t_{j+1} \cdots t_{n-1}} = 0 \end{cases} \tag{4.14}$$

for any $1 \le l_1 < l_2 < \cdots < l_{n-1} \le k$, $1 \le l_n \le k$, and (4.13) is equivalent to

$$\sum_{t_1 t_2 \cdots t_{n-1} \in P_{l_1 \cdots l_{n-1}}} \sum_{1 \le j \le n} x^j_{t_1 t_2 \cdots t_{j-1} l_n t_{j+1} \cdots t_{n-1}} = 0 \tag{4.15}$$

for any $1 \le l_1 \le l_2 \le \cdots \le l_{n-1} \le k$, $l_1 l_2 \cdots l_{n-1} \in Q$ and any $1 \le l_n \le k$.

We first construct two sets of solution vectors of (4.14):

$$\left\{\mu^{l_1 l_2 \cdots l_n;1}_{t_1 t_2 \cdots t_{n-1};j}\right\},\ \ \left\{\mu^{l_1 l_2 \cdots l_n;-1}_{t_1 t_2 \cdots t_{n-1};j}\right\}.$$

If $n$ is odd, let

$$\mu^{l_1 l_2 \cdots l_n;1}_{t_1 t_2 \cdots t_{n-1};j} = \left[(x^1)^{\mathrm{T}},\ (x^2)^{\mathrm{T}}, \ldots, (x^n)^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathbb{R}^{nk^n}$$

with each $x^p = (x^p_{r_1 r_2 \cdots r_n})$,

$$x^p_{r_1 r_2 \cdots r_n} = \begin{cases} 1, & p = n, r_1 r_2 \cdots r_n = l_1 l_2 \cdots l_n, \\ -1, & p = j, r_1 r_2 \cdots r_n = t_1 t_2 \cdots t_{j-1} l_n t_j \cdots t_{n-1}, \\ 0, & \text{otherwise}, \end{cases} \tag{4.16}$$

where $1 \le l_1 < l_2 < \cdots < l_{n-1} \le k$, $1 \le l_n \le k$, $t_1 t_2 \cdots t_{n-1} \in R_{l_1 \cdots l_{n-1}}$, $1 \le j \le n$ satisfy one of following conditions:

**(i)** $j = n$, $t_1 t_2 \cdots t_{n-1} \ne l_1 l_2 \cdots l_{n-1}$ and $(-1)^{j+1} = \mathrm{sgn}(t_1 t_2 \cdots t_{n-1})$,

**(ii)** $j \ne n$, $(-1)^{j+1} = \mathrm{sgn}(t_1 t_2 \cdots t_{n-1})$.

Similarly, let

$$\mu^{l_1 l_2 \cdots l_n;-1}_{t_1 t_2 \cdots t_{n-1};j} = \left[(x^1)^{\mathrm{T}},\ (x^2)^{\mathrm{T}}, \ldots, (x^n)^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathbb{R}^{nk^n},$$

with each $x^p = (x^p_{r_1 r_2 \cdots r_n})$,

$$x^p_{r_1 r_2 \cdots r_n} = \begin{cases} 1, & p = n, r_1 r_2 \cdots r_n = \tilde{l}_1 \tilde{l}_2 \cdots \tilde{l}_{n-1} l_n, \\ -1, & p = j, r_1 r_2 \cdots r_n = t_1 t_2 \cdots t_{j-1} l_n t_j \cdots t_{n-1}, \\ 0, & \text{otherwise}, \end{cases} \tag{4.17}$$

where $t_1t_2\cdots t_{n-1}$ and $j$ satisfy one of the following conditions:

**(i)** $j = n$, $t_1t_2\cdots t_{n-1} \neq \tilde{l}_1\tilde{l}_2\cdots\tilde{l}_{n-1} = l_2l_1l_3\cdots l_{n-1}$ and $(-1)^j = \mathrm{sgn}(t_1\cdots t_{n-1})$,

**(ii)** $j \neq n$, $(-1)^j = \mathrm{sgn}(t_1t_2\cdots t_{n-1})$.

If $n$ is even, let

$$\mu^{l_1l_2\cdots l_n;1}_{t_1t_2\cdots t_{n-1};j} = \left[(x^1)^{\mathrm{T}},\ (x^2)^{\mathrm{T}},\ldots,(x^n)^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathbb{R}^{nk^n},$$

with

$$x^p_{r_1r_2\cdots r_n} = \begin{cases} 1, & p = n, r_1r_2\cdots r_n = l_1l_2\cdots l_n,\\ -1, & p = j, r_1r_2\cdots r_n = t_1t_2\cdots t_{j-1}l_nt_j\cdots t_{n-1},\\ 0, & \text{otherwise}, \end{cases} \tag{4.18}$$

where $1 \leq l_1 < l_2 < \cdots < l_{n-1} \leq k$, $1 \leq l_n \leq k$, $t_1t_2\cdots t_{n-1} \in R_{l_1l_2\cdots l_{n-1}}$, $1 \leq j \leq n$ satisfying one of the following conditions,

**(i)** $j = n$, $t_1t_2\cdots t_{n-1} \neq l_1l_2\cdots l_{n-1}$ and $(-1)^j = sgn(t_1t_2\cdots t_{n-1})$,

**(ii)** $j \neq n$, $(-1)^j = sgn(t_1t_2\cdots t_{n-1})$,

Similarly, let

$$\mu^{l_1l_2\cdots l_n;-1}_{t_1t_2\cdots t_{n-1};j} = \left[(x^1)^{\mathrm{T}},\ (x^2)^{\mathrm{T}},\ldots,(x^n)^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathbb{R}^{nk^n},$$

with

$$x^p_{r_1\cdots r_n} = \begin{cases} 1, & p = n, r_1r_2\cdots r_n = \tilde{l}_1\tilde{l}_2\cdots\tilde{l}_{n-1}l_n,\\ -1, & p = j, r_1r_2\cdots r_n = t_1t_2\cdots t_{j-1}l_nt_j\cdots t_{n-1},\\ 0, & \text{otherwise}, \end{cases} \tag{4.19}$$

where $t_1t_2\cdots t_{n-1}$ and $j$ satisfy one of following conditions:

**(i)** $j = n$, $t_1t_2\cdots t_{n-1} \neq \tilde{l}_1\tilde{l}_2\cdots\tilde{l}_{n-1} = l_2l_1l_3\cdots l_{n-1}$ and $(-1)^{j+1} = \mathrm{sgn}(t_1t_2\cdots t_{n-1})$,

**(ii)** $j \neq n$, $(-1)^{j+1} = \mathrm{sgn}(t_1t_2\cdots t_{n-1})$.

Then we construct a set of solution vectors of (4.15):

$$\left\{\gamma^{l_1l_2\cdots l_n}_{t_1t_2\cdots t_{n-1};j}\right\}.$$

Let

$$\gamma^{l_1l_2\cdots l_n}_{t_1t_2\cdots t_{n-1};j} = \left[(x^1)^{\mathrm{T}},\ (x^2)^{\mathrm{T}},\ldots,(x^n)^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathbb{R}^{nk^n}$$

with

$$x^p_{r_1\cdots r_n} = \begin{cases} 1, & p = n, r_1r_2\cdots r_n = l_1l_2\cdots l_n,\\ -1, & p = j, r_1r_2\cdots r_n = t_1t_2\cdots t_{j-1}l_nt_j\cdots t_{n-1},\\ 0, & \text{otherwise}, \end{cases} \tag{4.20}$$

where $1 \leq l_1 \leq l_2 \leq \cdots \leq l_{n-1} \leq k$, $l_1l_2\cdots l_{n-1} \in Q$, $1 \leq l_n \leq k$, $t_1t_2\cdots t_{n-1} \in P_{l_1l_2\cdots l_{n-1}}$, $1 \leq j \leq n$ satisfy one of the following conditions:

**(i)** $j = n$, $t_1t_2\cdots t_{n-1} \neq l_1l_2\cdots l_{n-1}$ ,

**(ii)** $j \neq n$.

**Theorem 4.1.** *The sets* $\{\mu^{l_1l_2\cdots l_n;1}_{t_1t_2\cdots t_{n-1};j}\}$, $\{\mu^{l_1l_2\cdots l_n;-1}_{t_1t_2\cdots t_{n-1};j}\}$ *and* $\{\gamma^{l_1l_2\cdots l_n}_{t_1t_2\cdots t_{n-1};j}\}$ *form a basis of the asymmetric game subspace* $\mathcal{E}_{[n;k]}$.

*Proof.* According to the construction method of $\mu^{l_1l_2\cdots l_n;1}_{t_1t_2\cdots t_{n-1};j}$, $\mu^{l_1l_2\cdots l_n;-1}_{t_1t_2\cdots t_{n-1};j}$ and $\gamma^{l_1l_2\cdots l_n}_{t_1t_2\cdots t_{n-1};j}$, all the vectors in $\{\mu^{l_1l_2\cdots l_n;1}_{t_1t_2\cdots t_{n-1};j}\}$, $\{\mu^{l_1l_2\cdots l_n;-1}_{t_1t_2\cdots t_{n-1};j}\}$ and $\{\gamma^{l_1l_2\cdots l_n}_{t_1t_2\cdots t_{n-1};j}\}$ are linearly independent and satisfy both (4.14) and (4.15). Moreover, we have

$$\begin{aligned}&\left|\{\mu^{l_1l_2\cdots l_n;1}_{t_1t_2\cdots t_{n-1};j}\}\right| = \left|\{\mu^{l_1l_2\cdots l_n;-1}_{t_1t_2\cdots t_{n-1};j}\}\right|\\ &= \sum_{1\leq l_1<l_2<\cdots<l_{n-1}\leq k} k(n\frac{|R_{l_1l_2\cdots l_{n-1}}|}{2} - 1)\\ &= \sum_{1\leq l_1<l_2<\cdots<l_{n-1}\leq k} k(n\frac{|R_{l_1l_2\cdots l_{n-1}}|}{2}) - kC^{n-1}_k.\end{aligned}$$

$$\begin{aligned}&\left|\{\gamma^{l_1l_2\cdots l_n}_{t_1t_2\cdots t_{n-1};j}\}\right|\\ &= \sum_{\substack{1\leq l_1\leq l_2\leq\cdots\leq l_{n-1}\leq k\\ l_1l_2\cdots l_{n-1}\in Q}} k(n\,|\,P_{l_1l_2\cdots l_{n-1}}\,|-1)\\ &= \sum_{\substack{1\leq l_1\leq l_2\leq\cdots\leq l_{n-1}\leq k\\ l_1l_2\cdots l_{n-1}\in Q}} k(n\,|\,P_{l_1l_2\cdots l_{n-1}}\,|) - (kC^{n-1}_{k+n-2} - kC^{n-1}_k).\end{aligned}$$

So,

$$\begin{aligned}&|\{\mu^{l_1l_2\cdots l_n;1}_{t_1t_2\cdots t_{n-1};j}\}| + |\{\mu^{l_1l_2\cdots l_n;1}_{t_1t_2\cdots t_{n-1};j}\}| + |\{\gamma^{l_1l_2\cdots l_n}_{t_1t_2\cdots t_{n-1};j}\}|\\ &=2\sum_{1\leq l_1<l_2<\cdots<l_{n-1}\leq k} k(n\frac{|R_{l_1l_2\cdots l_{n-1}}|}{2}) - 2kC^{n-1}_k\\ &\quad+\sum_{\substack{1\leq l_1\leq l_2\leq\cdots\leq l_{n-1}\leq k\\ l_1l_2\cdots l_{n-1}\in Q}} k(n\,|\,P_{l_1l_2\cdots l_{n-1}}\,|) - (kC^{n-1}_{k+n-2} - kC^{n-1}_k)\\ &=\sum_{1\leq l_1<l_2<\cdots<l_{n-1}\leq k} k(n\,|\,R_{l_1l_2\cdots l_{n-1}}\,|)\\ &\quad+\sum_{\substack{1\leq l_1\leq l_2\leq\cdots\leq l_{n-1}\leq k\\ l_1l_2\cdots l_{n-1}\in Q}} k(n\,|\,P_{l_1l_2\cdots l_{n-1}}\,|) - kC^{n-1}_{k+n-2} - kC^{n-1}_k\\ &=nk^n - kC^{n-1}_{k+n-2} - kC^{n-1}_k\\ &=nk^n - \dim(\mathcal{S}_{[n;k]}) - \dim(\mathcal{K}_{[n;k]}).\end{aligned}$$

Therefore, $\{\mu^{l_1l_2\cdots l_n;1}_{t_1t_2\cdots t_{n-1};j}\}$, $\{\mu^{l_1l_2\cdots l_n;1}_{t_1t_2\cdots t_{n-1};j}\}$ and $\{\gamma^{l_1l_2\cdots l_n}_{t_1t_2\cdots t_{n-1};j}\}$ form a basis of $\mathcal{E}_{[n;k]}$.

**Remark 4.1.** *We have given the bases of skew-symmetric game subspace $\mathcal{K}_{[n;k]}$ and asymmetric game subspace $\mathcal{E}_{[n;k]}$. In our recently published paper [6], a basis of symmetric game subspace $\mathcal{S}_{[n;k]}$ has also been given. Let the bases of $\mathcal{S}_{[n;k]}$, $\mathcal{K}_{[n;k]}$, $\mathcal{E}_{[n;k]}$ be $\{\alpha_1,\alpha_2,\cdots,\alpha_s\}$, $\{\beta_1,\beta_2,\cdots,\beta_t\}$, $\{\gamma_1,\gamma_2,\cdots,\gamma_l\}$ respectively, where $s=kC^{n-1}_{k+n-2}$, $t=kC^{n-1}_k$, $l=nk^n-kC^{n-1}_{k+n-2}-kC^{n-1}_k$. For any $G\in\mathcal{G}_{[n;k]}$, there are real numbers $p_1,\cdots,p_s,q_1,\cdots,q_t,r_1,\cdots,r_l$ such that*

$$V_G=p_1\alpha_1+\cdots+p_s\alpha_s+q_1\beta_1+\cdots+p_t\beta_t+r_1\gamma_1+\cdots+r_l\gamma_l.$$

*Thus, $[p_1,\cdots,p_s,q_1,\cdots,q_t,r_1,\cdots,r_l]^{\mathrm{T}}$ is a solution of equation*

$$[\alpha_1^{\mathrm{T}}\ \cdots\ \alpha_s^{\mathrm{T}}\ \beta_1^{\mathrm{T}}\ \cdots\ \beta_t^{\mathrm{T}}\ \gamma_1^{\mathrm{T}}\ \cdots\ \gamma_l^{\mathrm{T}}]x=V_G^{\mathrm{T}}. \tag{4.21}$$

*Since the coefficient matrix of (4.21) is a nonsingular matrix and each row has less than $3n!$ nonzero elements, the computational complexity of game decomposition is less than or equal to $O(n!nk^n)$.*

**Example 4.1.** *Consider $\mathcal{G}_{[3;2]}$. If $1\le l_1<l_2\le 2$, we have $l_1=1,l_2=2$. Then $M_{\bar{X}_2}=\begin{bmatrix}0, & 1, & -1 & 0\end{bmatrix}^{\mathrm{T}}$. If $1\le l_1\le l_2\le 2$, then $l_1l_2=11$, $l_1l_2=12$ or $l_1l_2=22$. Therefore,*

$$M_{X_2}=\begin{bmatrix}1&0&0\\0&1&0\\0&1&0\\0&0&1\end{bmatrix}.$$

*According to (3.28), a basis of $\mathcal{K}_{[3;2]}$ is composed of the columns of matrix*

$$\begin{bmatrix}W_{[2^2,2]}\\-I_2\otimes W_{[2,2]}\\I_{2^3}\end{bmatrix}(M_{\bar{X}_2}\otimes I_2) \tag{4.22}$$

*According to (4.1), a basis of $\mathcal{S}_{[3;2]}$ is composed of the columns of matrix*

$$\begin{bmatrix}W_{[2^2,2]}\\I_2\otimes W_{[2,2]}\\I_{2^3}\end{bmatrix}(M_{X_2}\otimes I_2) \tag{4.23}$$

*According (4.18)-(4.20), the basis of $\mathcal{E}_{[3;2]}$ and all non-zero elements in each vector are as follows:*

$$\mu^{121,1}_{12;1},\ x^3_{121}=1,x^1_{112}=-1;$$
$$\mu^{121,1}_{21;2},\ x^3_{121}=1,x^2_{211}=-1;$$
$$\mu^{122,1}_{12;1},\ x^3_{122}=1,x^1_{212}=-1;$$
$$\mu^{122,1}_{21;2},\ x^3_{122}=1,x^2_{221}=-1;$$
$$\mu^{121,-1}_{12;2},\ x^3_{211}=1,x^2_{112}=-1;$$
$$\mu^{121,-1}_{21;1},\ x^3_{211}=1,x^1_{121}=-1;$$
$$\mu^{122,-1}_{12;2},\ x^3_{212}=1,x^2_{122}=-1;$$
$$\mu^{122,-1}_{21;1},\ x^3_{212}=1,x^1_{221}=-1;$$
$$\gamma^{111}_{11;1},\ x^3_{111}=1,x^1_{111}=-1;$$
$$\gamma^{111}_{11;2},\ x^3_{111}=1,x^2_{111}=-1;$$
$$\gamma^{112}_{11;1},\ x^3_{112}=1,x^1_{211}=-1;$$
$$\gamma^{112}_{11;2},\ x^3_{112}=1,x^2_{121}=-1;$$
$$\gamma^{221}_{22;1},\ x^3_{221}=1,x^1_{122}=-1;$$
$$\gamma^{221}_{22;2},\ x^3_{221}=1,x^2_{212}=-1;$$
$$\gamma^{222}_{22;1},\ x^3_{222}=1,x^1_{222}=-1;$$
$$\gamma^{222}_{22;2},\ x^3_{222}=1,x^2_{222}=-1.$$

*It is easy to verify that the basis of $\mathcal{E}_{[3;2]}$ are orthogonal to the columns of the matrices shown in (4.22) and (4.23).*

## 5. Conclusions

This paper mainly investigates skew-symmetric game, asymmetric game and the problem of symmetric-based decomposition of finite games. By the semi-tensor product of matrices method with adjacent transpositions, necessary and sufficient conditions for testing skew-symmetric games are obtained. Based on the necessary and sufficient conditions of skew-symmetric games, a basis of skew-symmetric game subspace is constructed explicitly. In addition, the discriminant equations for skew-symmetric games with the minimum number are derived concretely, which reduce the computational complexity. Benefiting from the construction methods of the bases of symmetric game subspace and skew-symmetric game subspace given by us, a basis of asymmetric game subspace is constructed for the first time. Then, any game in $\mathcal{G}_{[n;k]}$ can be linear represented by the bases of symmetric game subspace, skew-symmetric game subspace and asymmetric game subspace given by this paper and our previous work. Therefore, the problem of symmetric-based decomposition of finite games is completely solved. Some other kind of games can also be investigated in the frame of semi-tensor product of matrices [12, 13, 14, 15]. We will try to generalize the obtained results in our future work.

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**Conflict of interest**

The authors declare no conflict of interest.

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