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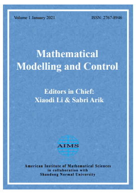

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Research article

New criteria for oscillation of damped fractional partial differential equations

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Abstract: In this paper, we consider a class of fractional partial differential equations with damping term subject to Robin and Dirichlet boundary value conditions. We derive several new sufficient criteria for oscillation of solutions of the equations by using the integral averaging technique and generalized Riccati type transformations. Some applications of the main results are illustrated by some examples.

Keywords: oscillation; fractional partial differential equation; damping term; Riemann-Liouville derivative; Riccati transformation technique

1. Introduction

Differential equations of fractional order are generalizations of classical differential equations of integer order and have found varied applications in wide spread fields of science and engineering. In the last few decades, the theory of fractional differential equations and its applications have been investigated in some monographs [1–5]. In recent years, the research on the oscillatory behavior of fractional partial differential equations is a very interesting topic and some effort has been made to establish oscillation criteria for these equations which involve the Riemann-Liouville fractional partial derivative [6–15].

However, the study of oscillatory behavior of fractional partial differential equation is still in its infancy. To develop the qualitative properties of fractional partial differential equations, it is of great interest to study the oscillatory behavior of fractional partial differential equation. In this paper, we consider the oscillatory behavior of the damped fractional partial differential equations (1.1) under boundary value conditions (1.2) and (1.3), respectively. By using Riccati type transformations and the integral averaging technique, we establish some new sufficient conditions which guarantee the oscillation of solutions of the problems (1.1), (1.2) and (1.1), (1.3). We also provide four examples to illustrate the relevance of the main results.

$$\begin{aligned}\frac{\partial}{\partial t}(r(t)D^{\alpha}_{+,t}u(x,t)) + p(t)D^{\alpha}_{+,t}u(x,t) &= a(t)\Delta u(x,t)\\ &+ b(t)\Delta u(x,t-\tau) - q(x,t)f(E(x,t)),\\ &(x,t)\in\Omega\times R_+ = G,\end{aligned} \tag{1.1}$$

$$\frac{\partial u(x,t)}{\partial N} + \beta(x,t)u(x,t) = 0, \quad (x,t)\in\partial\Omega\times R_+, \tag{1.2}$$

$$u(x,t) = 0, \quad (x,t)\in\partial\Omega\times R_+, \tag{1.3}$$

where $\alpha\in(0,1)$ is a constant, $D^{\alpha}_{+,t}u(x,t)$ is the Riemann-Liouville fractional derivative of order α of u with respect to t, $E(x,t)=\int_0^t(t-\xi)^{-\alpha}u(x,\xi)d\xi$, Ω is a bounded domain in R^n with a piecewise smooth boundary $\partial\Omega$, $R_+=(0,\infty)$, N is the unit exterior normal vector to $\partial\Omega$, Δ is the Laplacianin in R^n and $\beta(x,t)$ is a nonnegative continuous function on $\partial\Omega\times R_+$.

Throughout this paper, we assume that the following conditions hold:

(C_1) $r \in C^1([0,\infty), R_+)$, $a,\ b,\ p \in C([0,\infty),[0,\infty))$, $\tau \geq 0$ is a constant;

(C_2) $q(x,t) \in C(\overline{G},[0,\infty))$ and $q(t) = \min_{x\in\overline{\Omega}} q(x,t)$;

(C_3) $f(u) \in C(R,R)$ and there exists a constant $K > 0$ such that $\frac{f(u)}{u} \geq K$ for all $u \neq 0$.

By a solution of the problem (1.1), (1.2) (or (1.1), (1.3)), we mean a function $u(x,t) \in C^{1+\alpha}(\overline{\Omega}\times[0,\infty))$ such that $D^{\alpha}_{+,t}u(x,t),\ E(x,t) \in C^1(\overline{G},R)$ and satisfies (1.1) on $\overline{G}$ along with the boundary condition (1.2) (or (1.3)).

A solution of the problem (1.1), (1.2) (or (1.1), (1.3)) is said to be oscillatory in G if it is neither eventually positive nor eventually negative. Otherwise it is said to be nonoscillatory.

In this paper, we always assume that the solutions of the problems (1.1), (1.2) (or (1.1), (1.3)) under consideration exist globally.

The organization of the rest of this paper is as follows. In Section 2, we briefly state some basic definitions and a lemma which will be used in Section 3. In Section 3, we obtain several oscillation criteria of solutions of the problem (1.1), (1.2). In Section 4, we obtain several oscillation criteria of solutions of the problem (1.1), (1.3). In Section 5, we give some examples to illustrate the efficiency of our results.

2. Preliminaries

In this section, we give the definitions of fractional derivatives and integrals and a lemma which are useful throughout this paper. There are several kinds of definitions of fractional derivatives and integrals. In this paper, we use the Riemann-Liouville left-sided definition on the half-axis R_+.

Definition 2.1([1]). The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $y : R_+ \to R$ on the half-axis R_+ is given by

$$I^{\alpha}_{+}y(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds \ \text{ for } \ t > 0$$

provided the right hand side exists pointwise on R_+, where $\Gamma = \int_0^{\infty} e^{-t}t^{\alpha-1}dt$ is the Gamma function.

Definition 2.2([1]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ for a function $y : R_+ \to R$ on the half-axis R_+ is given by

$$\begin{aligned} D^{\alpha}_{+}y(t) &:= \frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}}(I^{\lceil\alpha\rceil-\alpha}_{+}y(t)) \\ &= \frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)}\frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}}\int_0^t (t-s)^{\lceil\alpha\rceil-\alpha-1}y(s)ds \ \text{ for } \ t > 0 \end{aligned}$$

provided the right hand side exists pointwise on R_+, where $\lceil\alpha\rceil := \min\{z \in Z : z \geq \alpha\}$ is the ceiling function of α.

Definition 2.3([3]). The Riemann-Liouville fractional derivative of order $0 < \alpha < 1$ with respect to t of a function $u(x,t)$ is given by

$$D^{\alpha}_{+,t}u(x,t) := \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_0^t (t-s)^{-\alpha}u(x,s)ds$$

provided the right hand side is pointwise defined on R_+.

Lemma 2.1([6]). Let

$$E(t) = \int_0^t (t-\xi)^{-\alpha}y(\xi)d\xi \text{ for } \alpha \in (0,1) \text{ and } t > 0.$$

Then $E'(t) = \Gamma(1-\alpha)D^{\alpha}_{+}y(t)$.

3. Oscillation of the problem (1.1), (1.2)

In this section, we give some sufficient conditions under which all solutions of the problem (1.1), (1.2) are oscillatory.

Theorem 3.1. Suppose that the conditions (C_1)-(C_3) hold. Furthermore, assume that for some $t_0 > 0$,

$$\int_{t_0}^{\infty} \frac{1}{r(t)v(t)}dt = \infty, \tag{3.1}$$

and

$$\int_{t_0}^{\infty} q(t)v(t)dt = \infty, \tag{3.2}$$

then every solution $u(x,t)$ of the problem (1.1), (1.2) is oscillatory in G, where

$$v(t) = \exp(\int_{t_0}^{t}\frac{p(s)}{r(s)}ds) > 0,\ t \geq t_0.$$

Proof. Assume to the contrary that $u(x,t)$ is a nonoscillat -ory solution of the problem (1.1), (1.2). Without loss of generality, we may assume that $u(x,t) > 0,\ u(x,t-\tau) > 0$ and $E(x,t) > 0$ in $\Omega\times[t_1,\infty)$ for $t_1 \geq t_0$.

Integrating (1.1) with respect to x over the domain Ω, we have

$$\begin{aligned}&\frac{d}{dt}(r(t)\int_\Omega D^\alpha_{+,t}u(x,t)dx)+p(t)\int_\Omega D^\alpha_{+,t}u(x,t)dx\\&+\int_\Omega q(x,t)f(E(x,t))dx\\&=a(t)\int_\Omega \Delta u(x,t)dx+b(t)\int_\Omega \Delta u(x,t-\tau)dx,\ \ t\ge t_1.\end{aligned}\tag{3.3}$$

By Green's formula and the boundary condition (1.2) yield

$$\begin{aligned}\int_\Omega \Delta u(x,t)dx&=\int_\Omega \frac{\partial u(x,t)}{\partial N}dS\\&=-\int_{\partial\Omega}\beta(x,t)u(x,t)dx\le 0,\qquad t\ge t_1,\end{aligned}\tag{3.4}$$

$$\begin{aligned}&\int_\Omega \Delta u(x,t-\tau)dx\\&=\int_{\partial\Omega}\frac{\partial u(x,t-\tau)}{\partial N}dS\\&=-\int_{\partial\Omega}\beta(x,t-\tau)u(x,t-\tau)dS\ \le 0,\ \ t\ge t_1,\end{aligned}\tag{3.5}$$

where dS is the surface element on $\partial\Omega$.

By (C_2) and (C_3) , we have

$$\begin{aligned}&\int_\Omega q(x,t)f(E(x,t))dx\\&\ge Kq(t)\int_\Omega E(x,t)dx\\&=Kq(t)\int_0^t (t-\xi)^{-\alpha}(\int_\Omega (u(x,\xi)dx)d\xi,\ \ t\ge t_1.\end{aligned}\tag{3.6}$$

Let

$$U(t)=\int_\Omega u(x,t)dx,$$

then $U(t)>0,\ t\ge t_1$. Combining (3.3)-(3.6), we have

$$(r(t)D^\alpha_+U(t))'+p(t)D^\alpha_+U(t)+Kq(t)E(t)\le 0,\qquad t\ge t_1,\tag{3.7}$$

where

$$E(t)=\int_0^t (t-\xi)^{-\alpha}U(\xi)d\xi>0.\tag{3.8}$$

It follows from (3.7) and (3.8) that

$$\begin{aligned}((r(t)D^\alpha_+U(t))v(t))'&=(r(t)D^\alpha_+U(t))'v(t)+(p(t)D^\alpha_+U(t))v(t)\\&=-Kq(t)E(t)v(t)<0,\qquad t\ge t_1.\end{aligned}\tag{3.9}$$

Then $(r(t)D^\alpha_+U(t))v(t)$ is strictly decreasing on $[t_1,\infty)$ and thus $D^\alpha_+U(t)$ is eventually of one sign. We claim that $D^\alpha_+U(t)>0$ on $[t_2,\infty)$, for $t_2>t_1$ sufficiently large. In fact, if there exists a sufficiently large $t_3>t_2$ such that $D^\alpha_+U(t_3)<0$, then we have

$$(r(t)D^\alpha_+U(t))v(t)<(r(t_3)D^\alpha_+U(t_3))v(t_3):=C<0,\qquad t\ge t_3.\tag{3.10}$$

Thus by Lemma 2.1 and (3.10), we have

$$\frac{E'(t)}{\Gamma(1-\alpha)}=D^\alpha_+U(t)<\frac{C}{r(t)v(t)},\qquad t\ge t_3.\tag{3.11}$$

Integrating (3.11) from t_3 to $t(t>t_3)$, we obtain

$$E(t)<E(t_3)+C\Gamma(1-\alpha)\int_{t_3}^t \frac{1}{r(s)v(s)}ds.\tag{3.12}$$

Let $t\to\infty$ in (3.12), by the assumption (3.1), we get $\lim_{t\to\infty}E(t)=-\infty$, which contradicts with the fact $E(t)>0$. Hence $D^\alpha_+U(t)>0$ for $t>t_2$.

Define

$$W(t)=\frac{r(t)D^\alpha_+U(t)v(t)}{E(t)},\ \ t\ge t_2.\tag{3.13}$$

Then $W(t)>0,\ \ t\ge t_2$. Using Lemma 2.1, from (3.7) and (3.13), we have

$$\begin{aligned}W'(t)&=\frac{[(r(t)D^\alpha_+U(t))v(t)]'}{E(t)}-\frac{(r(t)D^\alpha_+U(t))v(t)E'(t)}{E^2(t)}\\&=\frac{(r(t)D^\alpha_+U(t))'v(t)+(p(t)D^\alpha_+U(t))v(t)}{E(t)}\\&\quad-\frac{\Gamma(1-\alpha)r(t)v(t)(D^\alpha_+U(t))^2}{E^2(t)}\\&\le -Kq(t)v(t),\ \ t\ge t_2.\end{aligned}\tag{3.14}$$

Integrating (3.14) from t_2 to $t(t>t_2)$, we obtain

$$W(t)\le W(t_2)-K\int_{t_2}^t q(s)v(s)ds.\tag{3.15}$$

Let $t\to\infty$ in (3.15), by the assumption (3.2), we obtain a contradiction with $W(t)>0$. This completes the proof of Theorem 3.1.

Theorem 3.2. Suppose that the conditions (C_1)-(C_3) and (3.1) hold, and additionally

$$\int_{t_0}^\infty [Kq(t)-\frac{p^2(t)}{4\Gamma(1-\alpha)r(t)}]dt=\infty,\tag{3.16}$$

then every solution $u(x,t)$ of the problem (1.1), (1.2) is oscillatory in G.

Proof. We prove this theorem by contradiction. Let $u(x,t)$ be a nonoscillatory solution of the problem (1.1), (1.2). One can proceed a similar proof of Theorem 3.1 to obtain (3.7) and $E(t) > 0,\ D_+^\alpha U(t) > 0$ on $\Omega \times [t_2, \infty)$ for some $t_2 \geq t_1 \geq t_0$.

Define

$$\tilde{W}(t) = \frac{r(t)D_+^\alpha U(t)}{E(t)}, \quad t \geq t_2. \tag{3.17}$$

Then $\tilde{W}(t) > 0$ for $t \geq t_2$. Using Lemma 2.1, from (3.7) and (3.17), we have

$$\begin{aligned}
\tilde{W}'(t) &= \frac{(r(t)D_+^\alpha U(t))'}{E(t)} - \frac{(r(t)D_+^\alpha U(t))E'(t)}{E^2(t)} \\
&\leq \frac{-p(t)D_+^\alpha U(t) - Kq(t)E(t)}{E(t)} \\
&\quad - \frac{\Gamma(1-\alpha)r(t)(D_+^\alpha U(t))^2}{E^2(t)} \\
&= -Kq(t) - \frac{p(t)}{r(t)}\tilde{W}(t) - \frac{\Gamma(1-\alpha)}{r(t)}\tilde{W}^2(t) \\
&= -Kq(t) - [\sqrt{\frac{\Gamma(1-\alpha)}{r(t)}}\tilde{W}(t) + \frac{p(t)}{2\sqrt{\Gamma(1-\alpha)r(t)}}]^2 \\
&\quad + \frac{p^2(t)}{4\Gamma(1-\alpha)r(t)} \\
&\leq -[Kq(t) - \frac{p^2(t)}{4\Gamma(1-\alpha)r(t)}], \quad t \geq t_2.
\end{aligned} \tag{3.18}$$

Integrating (3.18) from t_2 to $t (t > t_2)$, we obtain

$$\tilde{W}(t) \leq \tilde{W}(t_2) - K\int_{t_2}^{t}[Kq(s) - \frac{p^2(s)}{4\Gamma(1-\alpha)r(s)}]ds. \tag{3.19}$$

Let $t \to \infty$ in (3.19), by the assumption (3.16), we obtain a contradiction with $\tilde{W}(t) > 0$. This completes the proof of Theorem 3.2.

Theorem 3.3. Suppose that the conditions (C_1)-(C_3) hold. Furthermore, assume that there exists a function $\Psi(t) \in C^1([t_0,\infty),[0,\infty))$ such that

$$\int_{t_0}^{\infty}\frac{dt}{\Psi(t)r(t)}dt = \infty, \quad t_0 > 0, \tag{3.20}$$

$$\lim_{t\to\infty}\Phi(t) = \infty, \tag{3.21}$$

then every solution $u(x,t)$ of the problem (1.1), (1.2) is oscillatory in G, where

$$\begin{aligned}
\Phi(t) = &\int_{t_0}^{t}\{K\Gamma(1-\alpha)\Psi(s)q(s) - \frac{[\Psi'(s)r(s) - \Psi(s)p(s)]^2}{4\Psi(s)r(s)}\}ds \\
&+ \frac{\Psi'(t)r(t) - \Psi(t)p(t)}{2}.
\end{aligned}$$

Proof. We prove by contradiction again. Suppose that $u(x,t)$ is a nonoscillatory solution of the problem (1.1), (1.2). We proceed as in the similar proof of Theorem 3.1 to get (3.7) and $E(t) > 0,\ D_+^\alpha U(t) > 0$ on $[t_2,\infty)$ for some $t_2 \geq t_1 \geq t_0$.

By Lemma 2.1, from (3.7), we can get

$$(r(t)E'(t))' + p(t)E'(t) + K\Gamma(1-\alpha)q(t)E(t) \leq 0, \quad t \geq t_2. \tag{3.22}$$

Define

$$\overline{W}(t) = -\Psi(t)\frac{r(t)E'(t)}{E(t)}, \quad t \geq t_2. \tag{3.23}$$

From (3.23) and combining (3.22), it follows that

$$\begin{aligned}
\overline{W}'(t) &= -\Psi'(t)\frac{r(t)E'(t)}{E(t)} - \Psi(t)[\frac{(r(t)E'(t))'}{E(t)} - \frac{(r(t)(E'(t))^2}{E^2(t)}] \\
&\geq \frac{\Psi'(t)r(t) - \Psi(t)p(t)}{\Psi(t)r(t)}\overline{W}(t) + \frac{\overline{W}^2(t)}{\Psi(t)r(t)} \\
&\quad + K\Gamma(1-\alpha)\Psi(t)q(t) \\
&= \frac{1}{\Psi(t)r(t)}\{[\overline{W}(t) + \frac{\Psi'(t)r(t) - \Psi(t)p(t)}{2}]^2 \\
&\quad - [\frac{\Psi'(t)r(t) - \Psi(t)p(t)}{2}]^2\} + K\Gamma(1-\alpha)\Psi(t)q(t) \\
&= \frac{H^2(t)}{\Psi(t)r(t)} + \{K\Gamma(1-\alpha)\Psi(t)q(t) \\
&\quad - \frac{[\Psi'(t)r(t) - \Psi(t)p(t)]^2}{4\Psi(t)r(t)}\}, \quad t \geq t_2,
\end{aligned} \tag{3.24}$$

where

$$H(t) = \overline{W}(t) + \frac{\Psi'(t)r(t) - \Psi(t)p(t)}{2}, \quad t \geq t_2.$$

Integrating (3.24) from t_2 to $t (t > t_2)$, we obtain

$$H(t) \geq \overline{W}(t_2) + \int_{t_2}^{t}\frac{H^2(s)}{\Psi(s)r(s)}ds + \Phi(t). \tag{3.25}$$

From (3.21), we can choose a sufficiently large t_3 such that

$$H(t) \geq \int_{t_2}^{t}\frac{H^2(s)}{\Psi(s)r(s)}ds, \quad t \geq t_3.$$

Let

$$R(t) = \int_{t_2}^{t}\frac{H^2(s)}{\Psi(s)r(s)}ds, \quad t \geq t_3, \tag{3.26}$$

then $H(t) \geq R(t) > 0,\ t \geq t_3$.

From (3.26), we get

$$R'(t) = \frac{H^2(t)}{\Psi(t)r(t)} > \frac{R^2(t)}{\Psi(t)r(t)}, \quad t \geq t_3, \tag{3.27}$$

Divide (3.27) by $R^2(t)$ and integrate it from t_3 to $t(t > t_3)$

$$\int_{t_3}^{t} \frac{ds}{\Psi(s)r(s)} < \frac{1}{R(t_3)} - \frac{1}{R(t)} < \frac{1}{R(t_3)}. \qquad (3.28)$$

Let $t \to \infty$ in (3.28) , we obtain $\int_{t_3}^{\infty} \frac{dt}{\Psi(t)r(t)} < \infty$, which contradicts (3.20). This completes the proof of Theorem 3.3.

In Theorem 3.3, if $r(t) \equiv 1$, $\Psi(t) \equiv 1$, then we have the following corollary.

Corollary 3.1. Suppose that the conditions (C_1)-(C_3) hold. If

$$\lim_{t\to\infty}\{\int_{t_0}^{t} [K\Gamma(1-\alpha)q(s) - \frac{p^2(s)}{4}]ds - \frac{p(t)}{2}\} = \infty, \quad t_0 > 0,$$

then every solution $u(x,t)$ of the problem (1.1), (1.2) is oscillatory in G.

Theorem 3.4. Suppose that the conditions (C_1)-(C_3) hold. Furthermore, assume that there exists a function $\Psi(t) \in C^1([t_0,\infty),[0,\infty))$, such that

$$\int_{t_0}^{\infty} (\int_{t_0}^{t} \Psi(s)r(s)ds)^{-1}dt = \infty, \quad t_0 > 0, \qquad (3.29)$$

$$\lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \Phi(s)ds = \infty, \qquad (3.30)$$

then every solution $u(x,t)$ of the problem (1.1), (1.2) is oscillatory in G.

Proof. Similar with the proof of Theorem 3.3, we get (3.25) for $t \ge t_2$. Integrating (3.25) from t_2 to $t(t > t_2)$ and dividing by t, we obtain

$$\frac{1}{t}\int_{t_2}^{t} H(s)ds \ge \overline{W}(t_2)\frac{t-t_2}{t} + \frac{1}{t}\int_{t_2}^{t} R(s)ds + \frac{1}{t}\int_{t_2}^{t}\Phi(s)ds. \qquad (3.31)$$

By (3.30), we can choose a sufficiently large t_3 such that

$$\int_{t_2}^{t} H(s)ds \ge \int_{t_2}^{t} R(s)ds, \quad t \ge t_3.$$

Let

$$A(t) = \int_{t_2}^{t} R(s)ds, \quad t \ge t_3.$$

By using Hölder's inequality, we get

$$\begin{aligned} A^2(t) &\le (\int_{t_2}^{t} H(s)ds)^2 = (\int_{t_2}^{t} \sqrt{\Psi(s)r(s)}\frac{H(s)}{\sqrt{\Psi(s)r(s)}}ds)^2 \\ &\le (\int_{t_2}^{t} \Psi(s)r(s)ds)(\int_{t_2}^{t} \frac{H^2(s)}{\Psi(s)r(s)}ds) \\ &= R(t)\int_{t_2}^{t} \Psi(s)r(s)ds \\ &= A'(t)\int_{t_2}^{t} \Psi(s)r(s)ds, \quad t \ge t_3. \end{aligned} \qquad (3.32)$$

Divide (3.32) by $A^2(t)\int_{t_2}^{t}\Psi(s)r(s)ds$ and integrate it from t_3 to $t(t > t_3)$

$$\int_{t_3}^{t} (\int_{t_2}^{s} \Psi(v)r(v)dv)^{-1}ds < \frac{1}{A(t_3)} - \frac{1}{A(t)} < \frac{1}{A(t_3)}. \qquad (3.33)$$

Let $t \to \infty$ in (3.33) we obtain $\int_{t_3}^{\infty} (\int_{t_2}^{s} \Psi(v)r(v)dv)^{-1}ds < \infty$, which contradicts (3.29). This completes the proof of Theorem 3.4.

4. Oscillation of the problem (1.1), (1.3)

The following fact in [16] will be used.

The first eigenvalue λ_0 of the Dirichlet eigenvalue problem

$$\begin{cases} \Delta\phi(x) + \lambda\phi(x) = 0, & x \in \Omega, \\ \phi(x) = 0, & x \in \partial\Omega \end{cases}$$

is positive and the corresponding eigenfunction $\phi(x)$ is also positive in Ω.

Theorem 4.1. Suppose that the conditions of Theorem 3.1 hold. Then every solution $u(x,t)$ of the problem (1.1), (1.3) is oscillatory in G.

Proof. As $u(x,t)$ is a nonoscillatory solution of the problem (1.1), (1.3). Without loss of generality, we can assume that $u(x,t) > 0$, $u(x,t-\tau) > 0$ and $E(x,t) > 0$ in $[t_1,\infty)$ for some $t_1 \ge t_0$.

By multiplying $\phi(x)$ on both sides of (1.1) by and integrating with respect to x over the domain Ω, we have

$$\begin{aligned} &\frac{d}{dt}(r(t)\int_{\Omega} D^{\alpha}_{+,t}u(x,t)\phi(x)dx) + p(t)\int_{\Omega} D^{\alpha}_{+,t}u(x,t)\phi(x)dx \\ &\quad + \int_{\Omega} q(x,t)f(E(x,t))\phi(x)dx \\ &= a(t)\int_{\Omega}\phi(x)\Delta u(x,t)dx + b(t)\int_{\Omega}\phi(x)\Delta u(x,t-\tau)dx, \quad t \ge t_1. \end{aligned} \qquad (4.1)$$

Using Green's formula and the boundary condition (1.3) yield

$$\begin{aligned} \int_{\Omega}\phi(x)\Delta u(x,t)dx &= \int_{\Omega} u(x,t)\Delta\phi(x)dx \\ &= -\lambda_0\int_{\Omega} u(x,t)\phi(x)dx \le 0, \quad t \ge t_1, \end{aligned} \qquad (4.2)$$

$$\begin{aligned}
&\int_\Omega \phi(x)\Delta u(x,t-\tau)dx\\
&= \int_\Omega u(x,t-\tau)\Delta\phi(x)dx \qquad (4.3)\\
&= -\lambda_0 \int_\Omega u(x,t-\tau)\phi(x)dx \le 0, \quad t \ge t_1.
\end{aligned}$$

By the assumption (C_2) and (C_3) that

$$\begin{aligned}
&\int_\Omega q(x,t)f(E(x,t))\phi(x)dx\\
&\ge Kq(t)\int_\Omega E(x,t)\phi(x)dx \qquad (4.4)\\
&= Kq(t)\int_0^t (t-\xi)^{-\alpha}(\int_\Omega (u(x,\xi)\phi(x)dx)d\xi, \quad t \ge t_1.
\end{aligned}$$

Let

$$\tilde{U}(t) = \int_\Omega u(x,t)\phi(x)dx,$$

then $\tilde{U}(t) > 0,\ t \ge t_1$. Combining (4.1)-(4.4), we have

$$(r(t)D_+^\alpha \tilde{U}(t))' + p(t)D_+^\alpha \tilde{U}(t) + Kq(t)\tilde{E}(t) \le 0, \qquad t \ge t_1, \quad (4.5)$$

where

$$\tilde{E}(t) = \int_0^t (t-\xi)^{-\alpha}\tilde{U}(\xi)d\xi > 0.$$

The remainder of the proof is similar to that of Theorem 3.1 and we omit it here. The proof of Theorem 4.1 is complete.

Theorem 4.2. Suppose that the conditions of Theorem 3.2 hold. Then every solution $u(x,t)$ of the problem (1.1), (1.3) is oscillatory in G.

Theorem 4.3. Suppose that the conditions of Theorem 3.3 hold. Then every solution $u(x,t)$ of the problem (1.1), (1.3) is oscillatory in G.

Corollary 4.1. Suppose that the conditions of Corollary 4.2 hold. Then every solution $u(x,t)$ of the problem (1.1), (1.3) is oscillatory in G.

Theorem 4.4. Suppose that the conditions of Theorem 3.4 hold. Then every solution $u(x,t)$ of the problem (1.1), (1.3) is oscillatory in G.

The proofs of Theorem 4.2, Theorem 4.3, Corollary 4.1 and Theorems 4.4 are similar to that of in Section 3 and hence the details are omitted.

5. Examples

In this section, we show four examples as applications of our main results.

Example 5.1. Consider the following fractional partial differential equation

$$\begin{aligned}
&\frac{\partial}{\partial t}(\frac{1}{t^2}D_{+,t}^{\frac{1}{2}}u(x,t)) + \frac{1}{t^3}D_{+,t}^{\frac{1}{2}}u(x,t)\\
&= e^t\Delta u(x,t) + 2t\Delta u(x,t-\frac{1}{3}) - (x^3+\frac{1}{t^2})E(x,t), \qquad (5.1)\\
&\qquad (x,t) \in (0,\pi)\times R_+
\end{aligned}$$

with the boundary value condition

$$u_x(0,t) = u_x(\pi,t) = 0,\ t > 0, \qquad (5.2)$$

where $\alpha = \frac{1}{2},\ \Omega = (0,\pi),\ n = 1,\ E(x,t) = \int_0^t (t-\xi)^{-\frac{1}{2}}u(x,\xi)d\xi$, $a(t) = e^t,\ b(t) = 2t, \tau = \frac{1}{3},\ r(t) = \frac{1}{t^2}, p(t) = \frac{1}{t^3}, q(x,t) = x^3 + \frac{1}{t^2}$ and $f(E(x,t)) = E(x,t)$. Hence $q(t) = \frac{1}{t^2}$.

Take $t_0 > 0$ and $K = 1$, we see that

$$v(t) = \exp(\int_{t_0}^t \frac{p(s)}{r(s)}ds) = \exp(\int_{t_0}^t \frac{1}{s}ds) = \frac{t}{t_0},$$

$$\int_{t_0}^\infty \frac{1}{r(t)v(t)}dt = \int_{t_0}^\infty \frac{1}{\frac{1}{t^2}\frac{t}{t_0}}dt = \int_{t_0}^\infty t_0 t dt = \infty,$$

and

$$\int_{t_0}^\infty q(t)v(t)dt = \int_{t_0}^\infty \frac{1}{t^2}\frac{t}{t_0}dt = \int_{t_0}^\infty \frac{1}{t_0 t}dt = \infty.$$

Therefore, the conditions in Theorem 3.1 hold. Then every solution of problem (5.1), (5.2) oscillates in $(0,\pi)\times R_+$.

Example 5.2. Consider the following fractional partial differential equation

$$\begin{aligned}
&\frac{\partial}{\partial t}(\frac{1}{t}D_{+,t}^{\frac{1}{2}}u(x,t)) + \frac{1}{t^2}D_{+,t}^{\frac{1}{2}}u(x,t)\\
&= e^{-t}\Delta u(x,t) + \frac{t}{2}\Delta u(x,t-1) - (2x^2+\frac{1}{t})e^{E(x,t)}E(x,t),\\
&\qquad (x,t) \in (0,\pi)\times R_+ \qquad (5.3)
\end{aligned}$$

with the boundary value condition

$$u_x(0,t) = u_x(\pi,t) = 0,\ t > 0, \qquad (5.4)$$

where $\alpha = \frac{1}{2},\ \Omega = (0,\pi),\ n = 1,\ E(x,t) = \int_0^t (t-\xi)^{-\frac{1}{2}}u(x,\xi)d\xi$, $a(t) = e^{-t},\ b(t) = \frac{t}{2},\ \tau = 1,\ r(t) = \frac{1}{t},\ p(t) = \frac{1}{t^2}, q(x,t) = 2x^2 + \frac{1}{t}$ and $f(E(x,t)) = e^{E(x,t)}E(x,t)$. Hence $q(t) = \frac{1}{t}$ and $\Gamma(1-\alpha) = \Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Take $t_0 > 0$ and $K = 1$, we have

$$v(t) = \exp(\int_{t_0}^{t} \frac{p(s)}{r(s)} ds) = \exp(\int_{t_0}^{t} \frac{1}{s} ds) = \frac{t}{t_0},$$

$$\int_{t_0}^{\infty} \frac{1}{r(t)v(t)} dt = \int_{t_0}^{\infty} \frac{1}{\frac{1}{t}\frac{t}{t_0}} dt = \int_{t_0}^{\infty} t_0 dt = \infty,$$

and

$$\int_{t_0}^{\infty} [Kq(t) - \frac{p^2(t)}{4\Gamma(1-\alpha)r(t)}] dt = \int_{t_0}^{\infty} (\frac{1}{t} - \frac{1}{4\sqrt{\pi t^3}}) dt = \infty.$$

Therefore, the conditions in Theorem 3.2 hold. Then every solution of problem (5.3), (5.4) oscillates in $(0, \pi) \times R_+$.

Example 5.3. Consider the following fractional partial differential equation

$$\begin{aligned} &\frac{\partial}{\partial t}(t^2 D_{+,t}^{\frac{1}{2}} u(x,t)) + t D_{+,t}^{\frac{1}{2}} u(x,t) = \frac{t}{2}\Delta u(x,t) \\ &\quad + \Delta u(x, t - \frac{1}{2}) - \frac{2}{\sqrt{\pi}} e^{E(x,t)} E(x,t), \quad (x,t) \in (0,\pi) \times R_+ \end{aligned} \tag{5.5}$$

with the boundary value condition

$$u(0,t) = u(\pi,t) = 0, \ t > 0, \tag{5.6}$$

where $\alpha = \frac{1}{2}$, $\Omega = (0,\pi)$, $n = 1$, $E(x,t) = \int_0^t (t-\xi)^{-\frac{1}{2}} u(x,\xi) d\xi$, $a(t) = \frac{t}{2}$, $b(t) = 1$, $\tau = \frac{1}{2}$, $r(t) = t^2$, $p(t) = t$, $q(x,t) = \frac{2}{\sqrt{\pi}}$ and $f(E(x,t)) = e^{E(x,t)} E(x,t)$. Hence $q(t) = \frac{2}{\sqrt{\pi}}$ and $\Gamma(1-\alpha) = \Gamma(\frac{1}{2}) = \sqrt{\pi}$. It is obvious that $\lambda_0 = 1$, $\Phi(x) = \sin x$, $x \in \Omega$.

Take $t_0 > 0$, $\Psi(t) = \frac{1}{t}$ and $K = 1$, we have

$$\int_{t_0}^{\infty} \frac{dt}{\Psi(t)r(t)} = \int_{t_0}^{\infty} \frac{dt}{\frac{1}{t}t^2} = \infty,$$

$$\begin{aligned} \Phi(t) &= \int_{t_0}^{t} \{K\Gamma(1-\alpha)\Psi(s)q(s) - \frac{[\Psi'(s)r(s) - \Psi(s)p(s)]^2}{4\Psi(s)r(s)}\} ds \\ &\quad + \frac{\Psi'(t)r(t) - \Psi(t)p(t)}{2} \\ &= \int_{t_0}^{t} [\sqrt{\pi}\frac{1}{s}\frac{2}{\sqrt{\pi}} - \frac{(-\frac{1}{s^2}s^2 - \frac{1}{s}s)^2}{4\frac{1}{s}s^2}] ds + \frac{-\frac{1}{t^2}t^2 - \frac{1}{t}t}{2} \\ &= \int_{t_0}^{t} (\frac{2}{s} - \frac{1}{s}) ds - 1 \\ &= \ln t - \ln t_0 - 1, \end{aligned}$$

$$\lim_{t\to\infty} \Phi(t) = \infty,$$

Therefore, the conditions in Theorem 4.3 hold. Then every solution of problem (5.5), (5.6) oscillates in $(0, \pi) \times R_+$.

Example 5.4. Consider the following fractional partial differential equation

$$\begin{aligned} &\frac{\partial}{\partial t}(e^{-t} D_{+,t}^{\frac{1}{2}} u(x,t)) + 3e^{-t} D_{+,t}^{\frac{1}{2}} u(x,t) = \Delta u(x,t) \\ &\quad + \frac{t}{3}\Delta u(x, t-1) - \frac{2e^{-t}}{\sqrt{\pi}} E(x,t), \quad (x,t) \in (0,\pi) \times R_+ \end{aligned} \tag{5.7}$$

with the boundary value condition

$$u(0,t) = u(\pi,t) = 0, \ t > 0, \tag{5.8}$$

where $\alpha = \frac{1}{2}$, $\Omega = (0,\pi)$, $n = 1$, $E(x,t) = \int_0^t (t-\xi)^{-\frac{1}{2}} u(x,\xi) d\xi$, $a(t) = 1$, $b(t) = \frac{t}{3}$, $\tau = 1$, $r(t) = e^{-t}$, $p(t) = 3e^{-t}$, $q(x,t) = \frac{2e^{-t}}{\sqrt{\pi}}$ and $f(E(x,t)) = E(x,t)$. Hence $q(t) = \frac{2e^{-t}}{\sqrt{\pi}}$ and $\Gamma(1-\alpha) = \Gamma(\frac{1}{2}) = \sqrt{\pi}$. It is obvious that $\lambda_0 = 1$, $\Phi(x) = \sin x$, $x \in \Omega$.

Take $t_0 > 0$, $\Psi(t) = e^t$ and $K = 1$, we see that

$$\int_{t_0}^{\infty} (\int_{t_0}^{t} \Psi(s)r(s) ds)^{-1} dt = \int_{t_0}^{\infty} (\int_{t_0}^{t} e^s e^{-s} ds)^{-1} dt = \infty,$$

$$\begin{aligned} \Phi(t) &= \int_{t_0}^{t} \{K\Gamma(1-\alpha)\Psi(s)q(s) - \frac{[\Psi'(s)r(s) - \Psi(s)p(s)]^2}{4\Psi(s)r(s)}\} ds \\ &\quad + \frac{\Psi'(t)r(t) - \Psi(t)p(t)}{2} \\ &= \int_{t_0}^{t} [\sqrt{\pi} e^s \frac{2e^{-s}}{\sqrt{\pi}} \\ &\quad - \frac{(e^s e^{-s} - e^s(3e^{-s}))^2}{4e^s e^{-s}}] ds + \frac{e^t e^{-t} - e^t(3e^{-t})}{2} \\ &= \int_{t_0}^{t} ds - 1 \\ &= t - (t_0 + 1), \end{aligned}$$

$$\lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \Phi(s) ds = \infty.$$

Therefore, the conditions in Theorem 4.4 hold. Then every solution of problem (5.7), (5.8) oscillates in $(0, \pi) \times R_+$.

6. Conclusions

In this paper, we have studied the oscillation of a class of damped fractional partial differential equations (1.1) with the Robin boundary value conditions (1.2) and the Dirichlet boundary value conditions (1.3). We have also given some new oscillation conditions by using generalized Riccati transformation method and inequality technique. We illustrated our main results by providing suitable examples. We believe that there is extensive research space on this topic.

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Conflict of interest

All authors declare that they have no competing interests.

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