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Research article

Output controllability and observability of mix-valued logic control networks

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Abstract: This paper focuses on output controllability and observability of mix-valued logic control networks (MLCNs), of which the updating of outputs is determined by both inputs and states via logical rules. First, as for output controllability, the number of different control sequences are derived to steer a MLCN from a given initial state to a destination output in a given number of time steps via semi-tensor product method. By construsting the output controllability matrix, criteria for the output controllability are obtained. Second, to solve the problem of observability, we construct an augmented MLCN with the same transition matrix, and use the set controllability approach to determine the observability of MLCNs. Finally, a hydrogeological example is presented to verify the obtained results.

Keywords: M-valued logic control networks; output controllability; observability; semi-tensor product

1. Introduction

Boolean networks (BNs) were proposed by Kauffman in 1969 for the first time to model gene regulatory networks [1]. In BNs, the state of each gene can only take values from Boolean variables, where 1 (or 0) represents active (or inactive, respectively). The state evolution of each gene is determined by a corresponding Boolean function at each discrete-time instant. Moreover, if external inputs are added to manipulate the network, BNs can be naturally extended to Boolean control networks (BCNs) [2].

The semi-tensor product (STP) of matrices, firstly proposed by Cheng et al. [3], is an effective technique in expression and analysis of Boolean (control) networks. As a generalization of conventional matrix product, STP enables multiplication of two matrices with arbitrary dimensions. Via STP, a logical function can be converted into its algebraic form, and then the logical dynamic of a Boolean (control) network can be transformed into a discrete-time linear system [3]. Based on this approach, tremendous breakthroughs have been made in the study of BNs and BCNs, including stability and stabilization [4–6], observability [7–9], controllability [10–13], output tracking [14], disturbance decoupling [15] and so on.

Among the above problems, controllability is a basic and vital issue in control theory. The state controllability of BCNs has been deeply investigated by means of reachable set [7], input-state incidence matrix [8], Perron-Frobenius theory [10], etc. Furthermore, output controllability, the ability of steering the output between any initial and final condition via an external input, has drawn lots of concentration recently. In [16], a sufficient condition for the output controllability of BCNs was put forward by constructing topological adjacency matrix. Besides, for temporal Boolean networks (TBCNs), some necessary and sufficient conditions on output controllability are derived in [17] by referring to reachable set.

The observability is also an interesting and challenging problem. Several kinds of observability have been investigated in [7, 8, 18–20]. Some necessary and sufficient conditions of observability are presented in [7] based on observability matrix. In [19], a graph-theoretic approach is provided to solve observability, and the computational complexity is analyzed.

Note that mix-valued logic control networks (MLCNs), a generalization of Boolean control networks, are more intricate and wider applied in real life, such as the modeling of cognitive sciences [21], game theory [22], etc. As addressed in [23], a contex-aware system can be expressed as a MLCN. It should be noticed that the contex-aware system is composed of the contex and monitoring system, both of which can be regarded as a MLCN seperately, and the ultimate output is in consonance with the output of monitoring system. Despite of the context state which acts as an output of the Contex system but an input to the monitoring system, there may exist additional inputs to monitoring system. Thus the ultimate output of contex-aware system depends on not only states but also inputs, which is different from the conventional MLCN (of which the output depends on states only). In [23], the authors used an avalanche landslide alert system as an example and investigated the case of constant inputs. With the help of STP method and the algebraic representation, equilibria, observability and reconstructibility corresponding to constant inputs, and the problem of fault detection have been sucessfully studied.

However, to the authors' best knowledge, there is little literature available about output controllability and observability of such MLCNs under general inputs. In this paper, output controllability and observability of the specific MLCNs by utlizing output controllability matrix and set controllability approach are investigated. Motivated by the above discussions, this paper makes the following main contributions:

(1) In order to study output controllability for MLCNs, the number of different control sequences are derived to steer a MLCN from a given initial state to a destination output in a given number of time steps, based on which the output controllability matrix is provided and a series of output controllability criteria are obtained;

(2) The observability of MLCNs is equivalently transformed into the corresponding set controllability. Further, to utlize set controllability technique, an augmented MLCN with the same transition matrix is constructed, then a necessary and sufficient condition for observability is derived;

(3) A comparison between the conventional and the considered MLCNs is made.

The rest of this paper is organized as follows. Section 2 reviews some necessary preliminries on STP and the algebraic representation of MLCNs. Section 3 and Section 4 respectively study some necessary and sufficient conditions for output controllability and observability of MLCNs. In Section 5, we make a comparison between the conventional and the considered MLCNs. In Section 6, an illustrative example is given to clarify our results. Section 7 is a brief conclusion.

2. Preliminaries and problem formulation

In this section, some preliminaries about STP of matrices and the algebraic form of MLCNs will be presented.

2.1. Notations and STP of matrices

1) $\mathbb{R}$: the sets of real numbers;
2) $\mathbb{N}_+$: the sets of positive integers;
3) $\mathcal{D}_k := \{1, \frac{k-2}{k-1}, \cdots, \frac{1}{k-1}, 0\}$;
4) $\mathcal{M}_{m\times n}$: the set of $m \times n$-dimensional real matrices;
5) δ_k^i: the i-th column of identity matrix I_k;
6) $\Delta_k := \{\delta_k^i | 1 \le i \le k\}$;
7) $\mathrm{Col}_i(A)$ ($\mathrm{Row}_i(A)$): the i-th column (row) of A;
8) $\mathrm{Col}(A)$ ($\mathrm{Row}(A)$): the collection of columns (rows) of A;
9) A_{ij}: the (i, j)-th element of a matrix A;
10) $\mathcal{B}_{m\times n} := \left\{B \in \mathcal{M}_{m\times n} \mid B_{ij} \in \mathcal{D}\right\}$ is the set of $m \times n$ Boolean matrices;
11) $\mathcal{L}_{m\times n} := \{L \in \mathcal{B}_{m\times n} \mid \mathrm{Col}_i(L) \in \Delta_m, i = 1, 2, \ldots, n\}$ is the set of $m \times n$ logical matrices;
12) $\delta_m[i_1, i_2, \cdots, i_n]$: a matrix $[\delta_m^{i_1}, \delta_m^{i_2}, \cdots, \delta_m^{i_n}] \in \mathcal{L}_{m\times n}$;
13) $M +_{\mathcal{B}} N := \left(M_{ij} \vee N_{ij}\right)_{m\times n} \in \mathcal{B}_{m\times n}$, $M, N \in \mathcal{L}_{m\times n}$;
14) $M \times_{\mathcal{B}} N := \sum_{\mathcal{B}}{}_{k=1}^{n} \left(M_{ik} \wedge N_{kj}\right) \in \mathcal{B}_{n\times n}$, $M, N \in \mathcal{L}_{n\times n}$;
15) $A^{(k)} := \underbrace{A \times_{\mathcal{B}} \cdots \times_{\mathcal{B}} A}_{k}$;
16) $|V|$: the cardinality of set V;
17) $\mathbf{1}_{m\times n}$: an $m \times n$ matrix with all elements 1;
18) $\mathbf{1}_k = [\underbrace{1, 1, \ldots, 1}_{k}]^{\mathrm{T}}$.

Definition 2.1. *[3] Given two matrices $X \in \mathcal{M}_{m\times n}$ and $Y \in \mathcal{M}_{p\times q}$, the semi-tensor product (STP) of X and Y, denoted* *by $X \ltimes Y$, is defined as*

$$X \ltimes Y = (X \otimes I_{\alpha/n})(Y \otimes I_{\alpha/p}),$$

where $\alpha = lcm(n, p)$ represents the least common multiple of n and p, and $\otimes$ is the Kronecker product.

Remark 2.1. *When $n = p$, the semi-tensor product becomes the conventional matrix product. In this paper, the default matrix product is assumed as STP, and thus the symbol "$\ltimes$" is mostly omitted without confusion.*

Definition 2.2. *[3] Given two matrices $X \in \mathcal{M}_{m\times n}$ and $Y \in \mathcal{M}_{p\times n}$, the Khatri-Rao product of X and Y, denoted by $X * Y$, is defined as*

$$Col_j(X * Y) = Col_j(X) \ltimes Col_j(Y), j = 1, 2, \cdots, n.$$

Using vector form expression of k-valued logical variables, $\frac{i}{k-1}$ is equivalent to δ_k^{k-i}, $i = 1, 2, \cdots, k$. Thus, $\mathcal{D}_k$ is equivalent to Δ_k. Based on this, we have the following result.

Lemma 2.1. *[3] Let $x_i \in \Delta_{k_i}$, $i = 1, 2, \cdots, r$ be k_i-valued logical variables. Consider a mix-valued logical function $f(x_1, x_2, \cdots, x_r) : \Delta_{k_1} \times \Delta_{k_2} \times \cdots \times \Delta_{k_r} \to \Delta_{k_0}$, there exists a unique matrix $L_f \in \mathcal{L}_{k_0 \times \prod_{i=1}^{r} k_i}$, called the structure matrix of f, such that $f(x_1, x_2, \cdots, x_r) = L_f \ltimes x_1 \ltimes x_2 \ltimes \cdots \ltimes x_r$.*

Next, some fundamental concepts and properties of STP are presented as follows.

Lemma 2.2. *[3] Let $A \in \mathcal{M}_{m\times n}$ and $x \in \mathcal{M}_{t\times 1}$ is a column vector. Then $x \ltimes A = (I_t \otimes A) \ltimes x$.*

Lemma 2.3. *[3] Let $x \in \Delta_n$ and $y \in \Delta_m$. Then $x \ltimes y = W_{[m,n]} \ltimes y \ltimes x$, where $W_{[m,n]} := [I_n \otimes \delta_m^1, I_n \otimes \delta_m^2, \cdots, I_n \otimes \delta_m^m]$ is called a swap matrix.*

Lemma 2.4. *[3] Let $x \in \Delta_n$. Then $x \ltimes x = \Phi_n x$, where $\Phi_n = [\delta_n^1 \ltimes \delta_n^1, \delta_n^2 \ltimes \delta_n^2, \cdots, \delta_n^n \ltimes \delta_n^n]$ is called a power-reducing matrix.*

2.2. MLCN and its algebraic representation

Consider a MLCN with n nodes, m control inputs and p outputs as

$$\begin{cases} x_1(t+1) = f_1(x_1(t), \cdots, x_n(t); u_1(t), \cdots, u_m(t)), \\ x_2(t+1) = f_2(x_1(t), \cdots, x_n(t); u_1(t), \cdots, u_m(t)), \\ \quad \vdots \\ x_n(t+1) = f_n(x_1(t), \cdots, x_n(t); u_1(t), \cdots, u_m(t)), \\ y_1(t) = h_1(x_1(t), \cdots, x_n(t); u_1(t), \cdots, u_m(t)), \\ y_2(t) = h_2(x_1(t), \cdots, x_n(t); u_1(t), \cdots, u_m(t)), \\ \quad \vdots \\ y_p(t) = h_p(x_1(t), \cdots, x_n(t); u_1(t), \cdots, u_m(t)), \end{cases} \tag{2.1}$$

where $x_i \in \Delta_{N_i}, i = 1, \cdots, n$ are state variables; $u_k \in \Delta_{M_k}, k = 1, \cdots, m$ are inputs (or controls); $y_j \in \Delta_{P_j}, j = 1, \cdots, p$ are outputs; $f_i : \prod_{i=1}^{n} \Delta_{N_i} \times \prod_{k=1}^{m} \Delta_{M_k} \to \Delta_{N_i}, i = 1, \cdots, n$ and $h_j : \prod_{i=1}^{n} \Delta_{N_i} \times \prod_{k=1}^{m} \Delta_{M_k} \to \Delta_{P_j}, j = 1, \cdots, p$ are logical functions.

Let $x(t) = \ltimes_{i=1}^{n} x_i(t) \in \Delta_N$, $u(t) = \ltimes_{k=1}^{m} u_k(t) \in \Delta_M$ and $y(t) = \ltimes_{j=1}^{p} y_j(t) \in \Delta_P$, where $N = \prod_{i=1}^{n} N_i$, $M = \prod_{k=1}^{m} M_k$ and $P = \prod_{j=1}^{p} P_j$. By Lemma 2.1, for every logical function f_i, h_j, we can obtain their unique structure matrices $L_{f_i} \in \mathcal{L}_{N_i \times MN}$ and $L_{h_j} \in \mathcal{L}_{P_j \times MN}$, $i = 1, \cdots, n$, $j = 1, \cdots, p$. Thus, system (2.1) can be transformed into a vector form as

$$\begin{cases} x_1(t+1) = L_{f_1} u(t)x(t), \\ x_2(t+1) = L_{f_2} u(t)x(t), \\ \quad \vdots \\ x_n(t+1) = L_{f_n} u(t)x(t), \\ y_1(t) = L_{h_1} u(t)x(t), \\ y_2(t) = L_{h_2} u(t)x(t), \\ \quad \vdots \\ y_p(t) = L_{h_p} u(t)x(t), \end{cases} \tag{2.2}$$

Furthermore, (2.2) can be expressed into an algebraic form as

$$\begin{cases} x(t+1) = Lu(t)x(t), \\ y(t) = Hu(t)x(t), \end{cases} \tag{2.3}$$

where $L \in \mathcal{L}_{N\times MN}$ and $H \in \mathcal{L}_{P\times MN}$. We call L, H the network transition matrices of MLCN (2.1), which can be calculated as $L = L_{f_1} * L_{f_2} * \cdots * L_{f_n}$ and $H = L_{h_1} * L_{h_2} * \cdots * L_{h_p}$.

Remark 2.2. *Compared with ordinary MLCNs [15], the main difference of the considered system* (2.1) *is that the output of MLCN* (2.1) *is not only determined by states x_i, but also external inputs u_k, via logical functions.*

3. Output controllability via a free control sequence

In this subsection, the output controllability of MLCN (2.1), equivalently (2.3), via a free control sequence is investigated. First, we introduce the concept of output controllability below.

Definition 3.1. *[17] Consider system* (2.3)*:*

1) *Given initial state $x_0 \in \Delta_N$, the destination output $y_d \in \Delta_P$ and the finite time $s \in \mathbb{N}_+$, MLCN* (2.3) *is said to be output controllable from x_0 to y_d at the sth step if there exist an input sequence $\{u(0), u(1), \cdots, u(s)\}$, such that $y(s) = y_d$.*

2) *MLCN* (2.3) *is said to be output controllable from x_0 to y_d if there exist a $s \in \mathbb{N}_+$ and an input sequence $\{u(0), u(1), \cdots, u(s)\}$, such that $y(s) = y_d$.*

3) *MLCN* (2.3) *is said to be output controllable at x_0 if it is output controllable from x_0 to each $y_d \in \Delta_P$.*

4) *MLCN* (2.3) *is said to be output controllable if it is output controllable at each $x_0 \in \Delta_N$.*

Inspired by [8] and [10], we propose a formula for the number of different control sequences steering a MLCN (2.3) between initial states and objective outputs in a finite time, based on which the output controllability matrix can be derived.

Lemma 3.1. *The number of different control sequences that steer MLCN* (2.3) *from $x_0 \in \Delta_N$ to $y_d \in \Delta_P$ in sth step is*

$$l(s; x_0, y_d) = y_d^T (H \ltimes \mathbf{1}_M)(L \ltimes \mathbf{1}_M)^s x_0. \tag{3.1}$$

Proof. Denote matrix $\tilde{L} = LW_{[N,M]}$, $\tilde{H} = HW_{[N,M]}$, and thus system (2.3) can be converted into

$$\begin{cases} x(t+1) = \tilde{L}x(t)u(t), \\ y(t) = \tilde{H}x(t)u(t). \end{cases} \tag{3.2}$$

For simplicity, let vectors $U(t) = \ltimes_{i=0}^{t} u(i) \in \Delta_{M^{t+1}}, t \in \mathbb{N}_+$. By mathematical induction, we have

$$x(i) = \tilde{L}^i x(0) \ltimes_{t=0}^{i-1} u(t) = \tilde{L}^i x(0) U(i-1).$$

Substituting it into the second equation of (3.2), we get

$$y(i) = \tilde{H}\tilde{L}^s x(0) \ltimes_{t=0}^{i} u(t) = \tilde{H}\tilde{L}^s x(0) U(i).$$

Let $W^1(s), W^2(s), \cdots, W^{l(s;x_0,y_d)}(s) \in \Delta_{M^{s+1}}$ be the different control sequences steering MLCN (2.3) from x_0 to y_d at the sth step, i.e.,

$$y_d = \tilde{H}\tilde{L}^s x_0 W^i(s), i = 1, 2, \cdots, l(s; x_0, y_d). \tag{3.3}$$

Since the total number of control sequences $U(s)$ in s time steps is M^{s+1}, there must be $V^j(s) \in \Delta_{M^{s+1}}, |j| = M^{s+1} - l(s; x_0, y_d)$, such that

$$y_d \neq \tilde{H}\tilde{L}^s x_0 V^j(s), j = 1, 2, \cdots, M^{s+1} - l(s; x_0, y_d). \tag{3.4}$$

Multiply (3.3) and (3.4) from the left by y_d^T and sum up this set of M^{s+1} equations yields

$$l(s; x_0, y_d) = y_d^T \tilde{H}\tilde{L}^s x_0 \ltimes \mathbf{1}_{M^{s+1}}. \tag{3.5}$$

In order to convert (3.5) into the form of (3.1), we use the properties of STP and swap matrices as follows.

$$\begin{aligned}
&\tilde{L}^s x_0 \ltimes \mathbf{1}_{M^{s+1}} \\
&= (LW_{[N,M]})^s x_0 \ltimes_{i=1}^{s+1} \mathbf{1}_M \\
&= (LW_{[N,M]})^{s-1} LW_{[N,M]} x_0 \ltimes \mathbf{1}_M \ltimes_{i=1}^{s} \mathbf{1}_M \\
&= (LW_{[N,M]})^{s-1} L \ltimes \mathbf{1}_M \ltimes x_0 \ltimes_{i=1}^{s} \mathbf{1}_M \\
&= (LW_{[N,M]})^{s-2} L \ltimes \mathbf{1}_M \ltimes (L \ltimes \mathbf{1}_M \ltimes x_0) \ltimes_{i=1}^{s-1} \mathbf{1}_M \\
&= \cdots \\
&= (L \ltimes \mathbf{1}_M)^s \ltimes x_0 \ltimes \mathbf{1}_M.
\end{aligned}$$

By straightforward computation, the right side of (3.5) can be rewrited as $y_d^T \tilde{H}(L \ltimes \mathbf{1}_M)^s \ltimes x_0 \ltimes \mathbf{1}_M = y_d^T HW_{[N,M]}(L \ltimes \mathbf{1}_M)^s x_0 \mathbf{1}_M = y_d^T (H \ltimes \mathbf{1}_M)(L \ltimes \mathbf{1}_M)^s x_0$. Then (3.1) can be obtained. □

Remark 3.1. *Formula* (3.1) *reflects the precise number of different paths from a given state to an objective output. But as for output controllability task, we only focus on the existence of paths instead of the precise number. Hence, the matrix algebra above can simply be replaced by Boolean algebra.*

For the simplification of expression, we define the sth step input-output transfer matrix of MLCN (2.3) as

$$C_s := \left({\sum_{\mathscr{B}}}_{i=1}^{M} H\delta_M^i \right) \times_{\mathscr{B}} \left({\sum_{\mathscr{B}}}_{i=1}^{M} L\delta_M^i \right)^{(s)} \in \mathcal{B}_{P \times N}, \tag{3.6}$$

and set

$$C := {\sum_{\mathscr{B}}}_{s=1}^{MN} C_s \in \mathcal{B}_{P \times N}, \tag{3.7}$$

which is called the output controllability matrix.

Resorting to the definitions given in this subsection, some necessary and sufficient conditions on output controllability of MLCN (2.3) can be obtained as follows.

Theorem 3.1. *MLCN* (2.3) *is*

1) *output controllable from δ_N^j to δ_P^i at the sth step, if and only if $(C_s)_{ij} > 0$.*

2) *output controllable from δ_N^j to δ_P^i, if and only if $(C)_{ij} > 0$.*

3) *output controllable at δ_N^j, if and only if $Col_j(C) > 0$.*

4) *output controllable, if and only if $C > 0$.*

Proof. 1) Referring to Lemma 3.1, $(C_s)_{ij} > 0$ is equivalent to $l(s; \delta_N^j, \delta_P^i) = (\delta_P^i)^T (H \ltimes \mathbf{1}_M)(L \ltimes \mathbf{1}_M)^s \delta_N^j > 0$, which means that there exists at least one control sequence $\{u(0), u(1), \cdots, u(s)\}$ that steer MLCN (2.3) from $x_0 = \delta_N^j$ to $y_d = \delta_P^i$ in sth step, in other words, MLCN (2.3) is output controllable from δ_N^j to δ_P^i at the sth step.

2) According to 1) and Definition 3.1, MLCN (2.3) is output controllable from δ_N^j to δ_P^i, if and only if there exists a positive integer S, such that $\left(\sum_{\mathscr{B}}{}_{s=1}^{S} C_s\right)_{ij} > 0$. When H and L are given, the matrix C_s is determined only by the index s. Noting that the matrix M given by (14) in [8] is equal to $\sum_{\mathscr{B}}{}_{i=1}^{2^m} L\delta_{2^m}^i$, and from Corollary 3.2 of [8], we get that the upper bound of S is MN.

The discussions of 3)-4) are similar to 1)-2), and they can be easily obtained based on Definition 3.1. Thus, we omit them.

The proof is completed. □

Next, an algorithm (Algorithm 1) is proposed to find a control, which steers δ_N^j to δ_P^i. Since there can be different integer k satisfying $\mathrm{Col}_k\left((\delta_P^i)^T \tilde{H} \tilde{L}^s \delta_N^j\right) \neq 0$, it leads to several control sequences. In this paper, we just care about the existence of control sequence.

Example 3.1. *Consider a reduced BCN model [24] for the lac operon in the bacterium Escherichia coli:*

$$\begin{cases} x_1(t+1) = \neg u_1(t) \wedge (x_2(t) \vee x_3(t)), \\ x_2(t+1) = \neg u_1(t) \wedge u_2(t) \wedge x_1(t), \\ x_3(t+1) = \neg u_1(t) \wedge (u_2(t) \vee (u_3(t) \wedge x_1(t))), \end{cases} \tag{3.8}$$

Algorithm 1: An algorithm for finding a control sequence to steer δ_N^j to δ_P^i

Input: δ_N^j, δ_P^i
Output: $\{u(0), u(1), \cdots, u(s)\}$
1 **Initialization**
2 $s = 1$.
3 **If** $s \leq MN$, **do** step 4;
4 **If** $(C_s)_{ij} > 0$, **do** step 6;
5 **else** $s \leftarrow s + 1$, **do** step 3.
6 Calculate $\tilde{H}, \tilde{L}^s$, and $(\delta_P^i)^T \tilde{H} \tilde{L}^s \delta_N^j$.
7 **For** $k = 1 \to M^{s+1}$, **do** step 8;
8 **If** $\mathrm{Col}_k\left((\delta_P^i)^T \tilde{H} \tilde{L}^s \delta_N^j\right) \neq 0$, then **return** $\{u(0), u(1), \cdots, u(s)\}$ satisfying $\ltimes_{i=0}^{s} u(i) = \delta_{M^{s+1}}^k$;
9 **else end**.
10 **end for**.
11 **else end**.
12 **end**.

where x_1, x_2 and x_3 are Boolean state variables which represent lac mRNA, lactose in high concentrations, and lactose in medium concentrations, respectively; u_1, u_2 and u_3 are Boolean control inputs which denote extracellular glucose, high extracellular lactose, and the medium extracellular lactose, respectively.

In this example, the outputs are assumed as

$$\begin{cases} y_1(t) = x_1(t) \wedge u_2(t), \\ y_2(t) = x_2(t). \end{cases} \tag{3.9}$$

Its algebric form is

$$x(t+1) = Lu(t)x(t), \ y(t) = Hu(t)x(t),$$

where state $x \in \Delta_8$, input $u \in \Delta_8$, output $y \in \Delta_4$,

$$\begin{aligned} L = \delta_8[&8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8, \\ &8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8, \\ &1,1,1,5,3,3,3,7,1,1,1,5,3,3,3,7, \\ &3,3,3,7,4,4,4,8,4,4,4,8,4,4,4,8], \\ H = \delta_4[&1,1,2,2,3,3,4,4,1,1,2,2,3,3,4,4, \\ &3,3,4,4,3,3,4,4,3,3,4,4,3,3,4,4, \\ &1,1,2,2,3,3,4,4,1,1,2,2,3,3,4,4, \\ &3,3,4,4,3,3,4,4,3,3,4,4,3,3,4,4]. \end{aligned}$$

Then the output controllability matrix can be calculated as

$$C = \left(\sum_{\mathscr{B}\, i=1}^{8} H\delta_8^i \right) \times_{\mathscr{B}} \sum_{\mathscr{B}\, s=1}^{64} \left(\sum_{\mathscr{B}\, i=1}^{8} L\delta_8^i \right)^{(s)} = \mathbf{1}_{4\times 8}.$$

Hence, system (3.8) *with output* (3.9) *is output controllable according to Theorem 3.1. More precisely, we have* $C_3 = \mathbf{1}_{4\times 8}$*, while* $C_2 \neq \mathbf{1}_{4\times 8}$*, which indicates that it's output controllable within three steps. Based on Algorithm 1, control inputs can be find to steer each initial state to each destination output. Taking initial state* $x_0 = \delta_8^1$ *for example, when destination output* $y_d = \delta_4^1$*, we find out a control sequence* $\{u(0) = \delta_8^3, u(1) = \delta_8^3, u(2) = \delta_8^3, u(3) = \delta_8^1\}$*; When* $y_d = \delta_4^2$*, we have* $\{u(0) = \delta_8^1, u(1) = \delta_8^3, u(2) = \delta_8^3, u(3) = \delta_8^1\}$*; When* $y_d = \delta_4^3$*, we have* $\{u(0) = \delta_8^3, u(1) = \delta_8^3, u(2) = \delta_8^3, u(3) = \delta_8^3\}$*; When* $y_d = \delta_4^3$*, we have* $\{u(0) = \delta_8^1, u(1) = \delta_8^1, u(2) = \delta_8^1, u(3) = \delta_8^1\}$*. The discussion of control sequence is essentially the same as other* $x_0 \in \Delta_8$*, and here we omit them.*

4. Observability analysis based on set controllability

In this section, in order to dicuss the problem of observability of MLCN (2.1), we first recall the set controllability approach.

Let $\mathcal{N} := \left\{\delta_N^1, \delta_N^2, \cdots, \delta_N^N\right\}$ and $s \in 2^{\mathcal{N}}$, where $2^{\mathcal{N}}$ is the power set of $\mathcal{N}$. Now we define the index vector of s, which is denoted by $V(s) \in \mathbb{R}^N$, as

$$[V(s)]_i = \begin{cases} 1, \delta_N^i \in s, \\ 0, \delta_N^i \notin s. \end{cases}$$

The family of initial sets P^0 and the family of destination sets P^d are defined as

$$\begin{aligned} P^0 &:= \left\{s_1^0, s_2^0, \cdots, s_\alpha^0\right\} \subset 2^{\mathcal{N}}, \\ P^d &:= \left\{s_1^d, s_2^d, \cdots, s_\beta^d\right\} \subset 2^{\mathcal{N}}, \end{aligned} \tag{4.1}$$

where α and β are any positive integers.

Definition 4.1. *Consider system* (2.3) *with the initial and destination sets defined in* (4.1)*. MLCN* (2.3) *is*

1) *set controllability from* s_j^0 *to* s_i^d*, if it is controllable from some* $x^0 \in s_j^0$ *to some* $x^d \in s_i^d$*.*
2) *set controllability at* s_j^0*, if it is set controllability from* s_j^0 *to each* $s_i^d \in P^d$*.*
3) *set controllability, if it is set controllability at each* $s_j^0 \in P^0$ *.*

Based on the families of initial and destination sets, namely P^0 and P^d, we define the initial index matrix J^0 and the destination index matrix J^d respectively as

$$\begin{aligned} J^0 &:= [V(s_1^0), V(s_2^0), \cdots, V(s_\alpha^0)] \subset \mathcal{B}_{N\times\alpha}, \\ J^d &:= [V(s_1^d), V(s_2^d), \cdots, V(s_\beta^d)] \subset \mathcal{B}_{N\times\beta}. \end{aligned} \tag{4.2}$$

Next, we define the set controllability matrix as

$$\mathcal{S} := J_d^T \times_{\mathscr{B}} M \times_{\mathscr{B}} J_0 \in \mathcal{B}_{\beta\times\alpha}, \tag{4.3}$$

where $M := \sum_{\mathscr{B} s=1}^{N} \left(\sum_{\mathscr{B} i=1}^{M} L\delta_M^i\right)^{(s)}$ is called the control transfer matrix of MLCN (2.3).

According to the definition of set controllability, the following proposition is easily verifiable.

Proposition 1. *Consider MLCN* (2.3) *with the family of initial sets* P^0 *and the family of destination sets* P^d *defined in* (4.1) *as well as the corresponding set controllability matrix defined in* (4.3)*. Then MLCN* (2.3) *is*

1) *set controllable from* s_j^0 *to* s_i^d*, if and only if* $(\mathcal{S})_{ij} = 1$*;*
2) *set controllable at* s_j^0*, if and only if* $Col_j(\mathcal{S}) = 1_\beta$*;*
3) *set controllable, if and only if* $\mathcal{S} = 1_{\beta\times\alpha}$*.*

Definition 4.2. *MLCN* (2.3) *is observable, if for any two different initial states* $x(0)$ *and* $x'(0)$*, there exist an integer* $t \in \mathbb{N}_+$ *and an input sequence* $\{u(0), u(1), \cdots, u(t)\}$*, such that the output sequence* $\{y(0), y(1), \cdots, y(t)\}$ *is distinct to* $\{y'(0), y'(1), \cdots, y'(t)\}$*.*

Definition 4.3. *Consider MLCN* (2.3)*. A state pair* $(x, x') \in \Delta_N \times \Delta_N$ *is distinguishable if* $x \neq x'$ *and there exist an input* $u \in \Delta_M$*, such that* $Hux \neq Hux'$*. Otherwise,* (x, x') *is called indistinguishable. We denote* Θ*,* Ξ *as the set of distinguishable and indistinguishable state pairs, respectively.*

Lemma 4.1. *Split* H *into* M *square blocks as* $H = [H_1, H_2, \cdots, H_M]$*. The state pair* (δ_N^i, δ_N^j) *is digtinguishable, if and only if there exist an integer* $k \in [1, M]$*, such that* $(H_k^T H_k)_{ij} = 0 = (H_k^T H_k)_{ji}$*.*

To investigate the relationship between two different initial states and their output trajectories integrally, we introduce an augmented MLCN as

$$\begin{cases} x'(t+1) = Lu(t)x'(t), \\ y(t) = Hu(t)x'(t). \end{cases} \quad (4.4)$$

Let $z(t) = x(t) \ltimes x'(t)$ and $g(t) = y(t) \ltimes y'(t)$. Exploting STP method, we can combine system (2.3) and (4.4) into a new MLCN, for which the algebraic form can be expressed as

$$\begin{cases} z(t+1) = Eu(t)z(t), \\ g(t) = Gu(t)z(t), \end{cases} \quad (4.5)$$

where

$$E := L(I_{MN} \otimes L)(I_M \otimes W_{[M,N]})\Phi_M \in \mathcal{L}_{N^2 \times MN^2},$$

$$G := H(I_{MN} \otimes H)(I_M \otimes W_{[M,N]})\Phi_M \in \mathcal{L}_{P^2 \times MN^2}.$$

According to Defnition 4.3, we partition the product state space $\Delta_N \times \Delta_N$ into three disjoint subsets as

$$\begin{aligned} S_n &:= \{z := x \ltimes x' | x \neq x', H_k x = H_k x', \forall k \in [1, M]\}; \\ S_d &:= \{z := x \ltimes x' | x \neq x', H_k x \neq H_k x', \exists k \in [1, M]\}; \\ S_e &:= \{z := x \ltimes x' | x = x'\}. \end{aligned} \quad (4.6)$$

Then the observability problem of system (2.3) can be converted into a set controllability problem of system (4.5). To utilize the set controllability technique, we set $P^0 := \bigcup_{z \in S_n}\{z\}$ and $P^d := S_d$. Then the corresponding index matrices J_0 and J_d can be obtained. The set controllability matrix can be calculated as

$$\mathcal{S} = J_d^T \times_{\mathscr{B}} M \times_{\mathscr{B}} J_0 \in \mathcal{L}_{1 \times |S_n|},$$

where $M = \sum_{\mathscr{B}s=1}^{N^2} \left(\sum_{\mathscr{B}i=1}^{M} E\delta_M^i\right)^{(s)}$ is the control transfer matrix of MLCN (4.5).

Theorem 4.1. *MLCN* (2.3) *is observable, if and only if MLCN* (4.5) *is set controllable from P^0 to P^d as defined above (e.i., $\mathcal{S} = \mathbf{1}_{|S_n|}^T$).*

Proof. (Necessity.) Suppose that MLCN (2.3) is observable, but $\mathcal{S} \neq \mathbf{1}_{|S_n|}^T$. Without loss of generality, we assume that there exists an integer $i \in [1, |S_n|]$, such that $\mathcal{S}_i = 0$. Then, the ith entry $z = x \ltimes x' \in P^0$ can never be driven to P^d under any possibile control sequences. According to the state-space partition (4.6), this means that the state $z = x \ltimes x' \in S_n$ can only stay in S_n or be transferred into S_e without passing S_d by any input sequence. In this case, the output sequences starting from two initial states $x \neq x'$ are the same all the time by any input sequence. Hence, MLCN (2.3) is not observable, which is in contradiction with the assumption.

(Sufficiency.) If system (4.5) is set controllable from P^0 to P^d, for any indistingushable state pair $(x_0, x_0') \in \Theta$, $x_0 \neq x_0'$, there must exist an integer $k \in \mathbb{N}_+$ and an input sequence $\{u(0), u(1), \cdots, u(k-1)\}$, steering $(x_0, x_0') \in \Theta$ to $(x_d, x_d') \in \Xi$. Without loss of generality, we just assume that (x_d, x_d') is distinguishable under control $u_d \in \Delta_M$. Take $u(k) = u_d$, then the output sequences stemming from x_0 and x_0' satisfy $\{y(0), y(1), \cdots, y(k)\} \neq \{y'(0), y'(1), \cdots, y'(k)\}$ by control sequence $\{u(0), u(1), \cdots, u(k)\}$, which proves that MLCN (2.3) is observable. □

Remark 4.1. *Suppose that MLCN* (2.3) *is observable. From the proof above, the input sequence $\{u(0), u(1), \cdots, u(k)\}$ that distinguish between x_0 and x_0' can also be obtained.*

Example 4.1. *Reconsider the reduced lac operon model in Example 3.1. First, the matrices E and G of the combined system can be easily computed as*

$$E = \delta_{64}[64, 64, 64, \cdots, 60, 60, 60, 64] \in \mathcal{L}_{64 \times 512},$$
$$G = \delta_{16}[1, 1, 2, 2, \cdots, 15, 15, 16, 16] \in \mathcal{L}_{16 \times 512}.$$

Second, we can obtain

$$\begin{aligned} S_n &= \left\{\delta_{64}^2, \delta_{64}^9, \delta_{64}^{20}, \delta_{64}^{27}, \delta_{64}^{38}, \delta_{64}^{45}, \delta_{64}^{56}, \delta_{64}^{63}\right\}; \\ S_e &= \left\{\delta_{64}^1, \delta_{64}^{10}, \delta_{64}^{19}, \delta_{64}^{28}, \delta_{64}^{37}, \delta_{64}^{46}, \delta_{64}^{55}, \delta_{64}^{64}\right\}; \\ S_d &= \left\{\delta_{64}^i : i \in [1, 64], \delta_{64}^i \notin S_n \cup S_e\right\}. \end{aligned}$$

Utlizing the family of initial set $P^0 = \bigcup_{z \in S_n}\{z\}$ and the destination set $P^d = S_d$, we have

$$\begin{aligned} J^0 &= \delta_{64}[2, 9, 20, 27, 38, 45, 56, 63]; \\ J^d &= \textstyle\sum_{\delta_{64}^i \in S_d} \delta_{64}^i. \end{aligned}$$

It follows that

$$\mathcal{S} = J_d^T \times_{\mathscr{B}} M \times_{\mathscr{B}} J_0 = [0\ 0\ 1\ 1\ 0\ 0\ 1\ 1] \neq \mathbf{1}_8^T,$$

where $M = \sum_{\mathscr{B}s=1}^{64} \left(\sum_{\mathscr{B}i=1}^{8} E\delta_8^i\right)^{(s)}$.

According to Theorem 4.1, system (3.8) *with output* (3.9) *is not observable.*

5. Comparisons with conventional MLCNs

In the above sections, we have investigated the output controllability of a specific MLCN (2.3), of which the upating of outputs is determined by both inputs and states. Note that if the output evlution depends on states only, then MLCN (2.3) will turn into an ordinary MLCN. Thus, in the following sequel, we will make comparisons between them.

Recall a conventional and widely studied MLCN, with n nodes, m control inputs and p outputs as

$$\begin{cases} x_1(t+1) = f_1(x_1(t), \cdots, x_n(t); u_1(t), \cdots, u_m(t)), \\ x_2(t+1) = f_2(x_1(t), \cdots, x_n(t); u_1(t), \cdots, u_m(t)), \\ \quad \vdots \\ x_n(t+1) = f_n(x_1(t), \cdots, x_n(t); u_1(t), \cdots, u_m(t)), \\ y_1(t) = \hat{h}_1(x_1(t), \cdots, x_n(t)), \\ y_2(t) = \hat{h}_2(x_1(t), \cdots, x_n(t)), \\ \quad \vdots \\ y_p(t) = \hat{h}_p(x_1(t), \cdots, x_n(t)), \end{cases} \tag{5.1}$$

where f_i $(i = 1, 2, \cdots, n)$, $\hat{h}_j$ $(j = 1, 2, \cdots, p)$ are Boolean functions, and f_i $(i = 1, 2, \cdots, n)$ are the same as MLCN (2.1).

Let $x(t) = \ltimes_{i=1}^{n} x_i(t) \in \Delta_N$, $u(t) = \ltimes_{k=1}^{m} u_k(t) \in \Delta_M$ and $y(t) = \ltimes_{j=1}^{p} y_j(t) \in \Delta_P$. Using STP method, we can obtain its equivalent algebraic equations:

$$\begin{cases} x(t+1) = Lu(t)x(t), \\ y(t) = \hat{H}x(t), \end{cases} \tag{5.2}$$

where $L \in \mathcal{L}_{N \times MN}$ and $\hat{H} \in \mathcal{L}_{P \times N}$.

According to [17] , the output controllability matrix of MLCN (5.2) is

$$\hat{C} := \sum\nolimits_{\mathscr{B}\, i=1}^{MN} \hat{C}_s \in \mathcal{B}_{P \times N}, \tag{5.3}$$

where

$$\hat{C}_s := \hat{H} \times_{\mathscr{B}} \left(\sum\nolimits_{\mathscr{B}\, i=1}^{M} L\delta_M^i \right)^{(s)} \in \mathcal{B}_{P \times N}, \tag{5.4}$$

represents the sth step input-output transfer matrix.

Note that the only difference between MLCN (2.3) and MLCN (5.2) is the evolution of outputs. In order to estabilsh connections between MLCN (2.3) and MLCN (5.2), we split H into M square blocks as $H = [H_1, H_2, \cdots, H_M]$ and assume that there exists $k \in [1, M]$, such that $H_k = \hat{H}$.

Based on equation (3.6) and (5.4), the sth step input-output transfer matrix of MLCN (2.3) can be computed as

$$\begin{aligned} C_s &= \left(\sum\nolimits_{\mathscr{B}\, i=1}^{M} H_i \right) \times_{\mathscr{B}} \left(\sum\nolimits_{\mathscr{B}\, i=1}^{M} L\delta_M^i \right)^{(s)} \\ &= \hat{C}_s +_{\mathscr{B}} \left(\sum\nolimits_{\mathscr{B}\, i=1}^{k-1} H_i +_{\mathscr{B}} \sum\nolimits_{\mathscr{B}\, i=k+1}^{M} H_i \right) \times_{\mathscr{B}} \left(\sum\nolimits_{\mathscr{B}\, i=1}^{M} L\delta_M^i \right)^{(s)}. \end{aligned}$$

Referring to the definition of output controllability in [17], the following result can be verified easily.

Theorem 5.1. *Consider MLCN* (2.3) *and MLCN* (5.2). *Suppose that there exists $k \in [1, M]$, such that $H_k = \hat{H}$. If MLCN* (5.2) *is output controllable, then MLCN* (2.3) *is output controllable.*

Next, an illustrate biological example is given.

Example 5.1. *Reconsider the lac operon regulatory network model* (3.8) *in Example 3.1 and Example 4.1. Now, assume that the outputs are*

$$\begin{cases} y_1(t) = x_1(t), \\ y_2(t) = x_2(t). \end{cases} \tag{5.5}$$

Its algebric form is

$$x(t+1) = Lu(t)x(t), \ y(t) = \hat{H}x(t),$$

where state $x \in \Delta_8$, input $u \in \Delta_8$, output $y \in \Delta_4$,

$$\hat{H} = \delta_4[1, 1, 2, 2, 3, 3, 4, 4].$$

Firstly, we study the output controllability of system (3.8) *with output* (5.5). *By straightforward computation, we have*

$$\hat{C} = \hat{H} \times_{\mathscr{B}} \sum\nolimits_{\mathscr{B}\, s=1}^{64} \left(\sum\nolimits_{\mathscr{B}\, i=1}^{8} L\delta_8^i \right)^{(s)} = \mathbf{1}_{4 \times 8}.$$

Hence, system (3.8) *with output* (5.5) *is output controllable. More precisely, we have $\hat{C}_3 = \mathbf{1}_{4\times 8}$, while $\hat{C}_2 \neq \mathbf{1}_{4\times 8}$, which indicates that it's output controllable within three steps.*

Compared with outputs (3.9)*, and split the network transition matrix H into 8 square blocks as $H = [H_1, H_2, \cdots, H_8]$, thus we have $\hat{H} = H_1 = H_2 = H_5 = H_6$. According to Theorem 5.1, we conclude that system* (3.8) *with output* (3.9) *is output controllable, which matches the result in Example 3.1.*

Remark 5.1. *The converse proposition of Theorem 5.1 does not hold. A counterexample with regard to system* (3.8) *is presented as follows. Assume that the algebric form of outputs* (5.5) *is replaced by* $y(t) = \hat{H}x(t) = \delta_4[3,3,4,4,3,3,4,4]x(t)$*. It's obvious that* $\hat{H} = H_3 = H_4 = H_7 = H_8$*, but*

$$\hat{C} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

which indicates that it is not output controllable. Thus, the converse proposition of Theorem 5.1 does not hold generally.

Next, we consider the observability problem of these systems. The observability of MLCN (5.2), deeply dicussed in [7] and [20], can be deduced from the definitions and theorem propsosed in Section 4 as well. Following the progress shown in (4.4)-(4.6), we construct the combined system for MLCN (5.2) as

$$\begin{cases} z(t+1) = Eu(t)z(t), \\ g(t) = \hat{G}z(t), \end{cases} \tag{5.6}$$

where $E := L(I_{MN} \otimes L)(I_M \otimes W_{[M,N]})\Phi_M \in \mathcal{L}_{N^2 \times MN^2}$, $\hat{G} := \hat{H}(I_N \otimes \hat{H}) \in \mathcal{L}_{P^2 \times N^2}$.

And the product state space $\Delta_N \times \Delta_N$ can be divided into three disjoint subsets as

$$\begin{aligned} \hat{S}_n &:= \left\{ z := x \ltimes x' | x \neq x', \hat{H}x = \hat{H}x' \right\}; \\ \hat{S}_d &:= \left\{ z := x \ltimes x' | x \neq x', \hat{H}x \neq \hat{H}x' \right\}; \\ \hat{S}_e &:= \{ z := x \ltimes x' | x = x' \}. \end{aligned} \tag{5.7}$$

Correspondingly, we set $\hat{P}^0 := \bigcup_{z \in \hat{S}_n} \{z\}$ and $\hat{P}^d := \hat{S}_d$, as well as the index matrices $\hat{J}_0$ and $\hat{J}_d$. The set controllability matrix of MLCN (5.2) can be obtained as:

$$\hat{\mathcal{S}} := \hat{J}_d^T \times_{\mathscr{B}} M \times_{\mathscr{B}} \hat{J}_0 \in \mathcal{B}_{\beta \times \alpha}, \tag{5.8}$$

where $M := \sum_{\mathscr{B}}{}_{s=1}^{N} \left(\sum_{\mathscr{B}}{}_{i=1}^{M} L\delta_M^i \right)^{(s)}$ is the same as the control transfer matrix of MLCN (2.3).

Referring to the definition of observability in [20], we have the following result.

Lemma 5.1. *MLCN* (5.2) *is observable, if and only if MLCN* (5.6) *is set controllable from* $\hat{P}^0$ *to* $\hat{P}^d$ *as defined above (e.i.,* $\hat{\mathcal{S}} = \mathbf{1}^T_{|\hat{S}_n|}$*).*

Theorem 5.2. *Consider MLCN* (2.3) *and MLCN* (5.2)*, supposing that there exists* $k \in [1, M]$*, such that* $H_k = \hat{H}$*. If MLCN* (5.2) *is observable, then MLCN* (2.3) *is observable.*

Proof. According to the partition of the product state space and the assumption that $H_k = \hat{H}$, we have $S_e = \hat{S}_e$, $\hat{S}_d \subset S_d$, and thus $S_n \subset \hat{S}_n$. Since MLCN (5.2) is observable, that is MLCN (5.6) is set controllable from $\hat{P}^0$ to $\hat{P}^d$, then MLCN (4.5) is set controllable from P^0 to P^d, which means MLCN (2.3) is observable. □

Example 5.2. *Reconsider the observability of lac operon regulatory network model* (3.8) *in Example 5.1.*

As discussed in Example 3.1, system (3.8) *with output* (3.9) *is not observable. Thus, according to the inverse negative proposition of Theorem 5.2, system* (3.8) *with output* (5.5) *is unobservable.*

Remark 5.2. *The inverse proposition of Theorem 5.2 does not hold. With regard to system* (3.8)*, assume that the algebraic form of outputs is* $y(t) = Hu(t)x(t)$*, where*

$$\begin{aligned} H = \delta_4[&1,1,2,2,3,3,4,4,1,1,2,2,3,3,4,4, \\ &1,3,2,4,1,3,2,4,1,3,2,4,1,3,2,4, \\ &1,1,2,2,3,3,4,4,1,1,2,2,3,3,4,4, \\ &1,3,2,4,1,3,2,4,1,3,2,4,1,3,2,4]. \end{aligned}$$

It's obvious that every state pair (x, x')*,* $x \neq x'$*, is distinguishable in this case. Hence, it is observable. Compared with system* (3.8) *with output* (5.5)*, although* $\hat{H} = H_1 = H_2 = H_5 = H_6$*, system* (3.8) *with output* (5.5) *is unobservable. Therefore, when MLCN* (2.3) *is observable, we can't always conclude that MLCN* (5.2) *is observable.*

6. An illustrative example

In this section, we consider a hydrogeological example, proposed in [23] originally, to illustrate the main results.

Example 6.1. *Consider the algebraic representation of a hydrogeological example in [23],*

$$\begin{cases} \mathbf{c}(t+1) = \mathbf{C} \ltimes \mathbf{u}(t) \ltimes \mathbf{c}(t), \\ \mathbf{a}(t+1) = \mathbf{A} \ltimes \mathbf{v_3}(t) \ltimes \mathbf{a}(t), \\ \mathbf{v_4}(t) = \mathbf{H_c} \ltimes \mathbf{c}(t), \\ \mathbf{m}(t) = \mathbf{M} \ltimes \mathbf{v}(t) \ltimes \mathbf{a}(t), \end{cases} \tag{6.1}$$

where $\mathbf{C} = \delta_5[2,3,4,5,5,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1]$, $\mathbf{A} = \delta_3[2,3,3,1,1,1]$, $\mathbf{H_c} = \delta_2[2,2,2,2,1]$, $\mathbf{M} = \delta_3[2,2,1,2,2,2,2,2,2,\cdots,2,2,2,3,3,3,3,3,3,3,3,3,3,3,3,3] \in \mathcal{L}_{3\times 48}$; $\mathbf{c}(t) \in \Delta_5$ *and* $\mathbf{a}(t) \in \Delta_3$ *represent the corresponding counters of contex system and monitoring system;* $\mathbf{m}(t) \in \Delta_3$ *is the output of monitoring system, which divides the situation into three typies as "alarm", "attention" and "nomal", according to the obtained data;* $\mathbf{u}(t) = \mathbf{u_1}(t) \ltimes \mathbf{u_2}(t) \in \Delta_4$ *and* $\mathbf{v}(t) = \mathbf{v_1}(t) \ltimes \mathbf{v_2}(t) \ltimes \mathbf{v_3}(t) \ltimes \mathbf{v_4}(t) \in \Delta_{16}$ *represent the corresponding contex input vector and monitoring input vector. Here,* $\mathbf{u_1}(t)$, $\mathbf{u_2}(t)$, $\mathbf{v_1}(t)$, $\mathbf{v_2}(t)$, $\mathbf{v_3}(t) \in \Delta_2$ *are used to descirbe inputs representing earthquake, snow, terrain temperature, snow height and accelerometer, respectively;* $\mathbf{v_4}(t) \in \Delta_2$, *namely contex-alert, is both contex output and monitoring input, which can forecast danger or quiet of the contex system.*

Denote $u(t) := \mathbf{u}(t) \ltimes \mathbf{v_1}(t) \ltimes \mathbf{v_2}(t) \ltimes \mathbf{v_3}(t) \in \Delta_{32}$ *as input,* $x(t) := \mathbf{c}(t) \ltimes \mathbf{a}(t) \in \Delta_{15}$ *as state vector, and* $y(t) := \mathbf{m}(t) \in \Delta_3$ *as output, then system* (6.1) *can be converted into a standard MLCN in the form of* (2.3)*, and the corresponding structure matrices can be computed as* $L = [(\mathbf{C} \otimes \mathbf{1}_8^T) \otimes (\mathbf{1}_{16}^T \otimes \mathbf{A})](I_{32} \otimes W_{[32,5]})\Phi_{32} \in \mathcal{L}_{15\times 480}$, $H = (\mathbf{1}_4^T \otimes M)(I_{32} \otimes \mathbf{H_c}) \in \mathcal{L}_{3\times 480}$.

Now, we investigate the output controllability of system (6.1)*. The sth step input-output transfer matrix is*

$$C_s = \left(\sum_{\mathscr{B}\, i=1}^{32} H\delta_M^i\right) \times_{\mathscr{B}} \left(\sum_{\mathscr{B}\, i=1}^{32} L\delta_M^i\right)^{(s)} \in \mathcal{B}_{3\times 15}.$$

By straightforward computation, we have $C_4 = \mathbf{1}_{3\times 15}$*, while* $C_3 \neq \mathbf{1}_{3\times 15}$*. Therefore, according to Theorem 3.1, we conclude that system* (6.1) *with a free control sequence is output controllable at the 4th step, and it's also output controllable. Meanwhile, different control inputs can be obtained to drive each initial state to destination output by Algorithm 1.*

Moreover, observability of system (6.1) *can also be verified by Theorem 4.1. According to the definition of distinguishable state pairs,* $(\delta_{15}^i, \delta_{15}^{15})$, $i = 1,2,\cdots,14$ *is digtinguishable under input* $u = \delta_{32}^1$*, while the rest state pairs are indigtinguishable. Hence, the product state space can be partition into the following three subsets:*

$$\begin{aligned} S_d = \{&\delta_{225}^{15}, \delta_{225}^{30}, \delta_{225}^{45}, \delta_{225}^{60}, \delta_{225}^{75}, \delta_{225}^{90}, \delta_{225}^{105}, \delta_{225}^{120}, \delta_{225}^{135}, \delta_{225}^{150},\\ &\delta_{225}^{165}, \delta_{225}^{180}, \delta_{225}^{195}, \delta_{225}^{210}, \delta_{225}^{211}, \delta_{225}^{212}, \delta_{225}^{213}, \delta_{225}^{214}, \delta_{225}^{215}, \delta_{225}^{216},\\ &\delta_{225}^{217}, \delta_{225}^{218}, \delta_{225}^{219}, \delta_{225}^{220}, \delta_{225}^{221}, \delta_{225}^{222}, \delta_{225}^{223}, \delta_{225}^{224}\};\\ S_e = \{&\delta_{225}^{1}, \delta_{225}^{4}, \delta_{225}^{9}, \delta_{225}^{16}, \delta_{225}^{25}, \delta_{225}^{36}, \delta_{225}^{49}, \delta_{225}^{64}, \delta_{225}^{81}, \delta_{225}^{100},\\ &\delta_{225}^{121}, \delta_{225}^{144}, \delta_{225}^{169}, \delta_{225}^{196}, \delta_{225}^{225}\};\\ S_n = \{&\delta_{225}^i : i \in [1,225], \delta_{225}^i \notin S_d \cup S_e\}. \end{aligned}$$

Subsequently, the family of initial sets $P^0 := \bigcup_{z\in S_n}\{z\}$ *and the family of destination sets* $P^d := S_d$ *can be obtained, as well as the corresponding index matrices* J_0 *and* J_d*, according to* (4.2)*. What's more, the network transition matrix E of the combined system can be computed as* $E = L(I_{480} \otimes L)(I_{32} \otimes W_{[32,15]})\Phi_{32} \in \mathcal{L}_{225\times 7200}$*. Therefore, we get the set controllability matrix as*

$$\mathcal{S} = J_d^T \times_{\mathscr{B}} M \times_{\mathscr{B}} J_0 \in \mathcal{L}_{1\times 182},$$

where $M = \sum_{\mathscr{B}s=1}^{225} \left(\sum_{\mathscr{B}i=1}^{32} E\delta_{32}^i\right)^{(s)} \in \mathcal{L}_{225\times 225}$.

By calculation, we have $\mathcal{S} \neq \mathbf{1}_{182}^T$*, which implies that system* (6.1) *is not observable.*

7. Conclusions

In this paper, output controllability and observability of MLCNs have been investigated. Utilizing the effective technique of semi-tensor product and swap matrices, we have obtained a formula for the number of different control sequences that steers a MLCN from a given initial state to an objective output in a given number of time steps. Then the corresponding output controllablity matrix has been derived, based on which we obtain some necessary and sufficient conditions for output controllability. Additionally, we introduce the augmented system and convert the observability problem of the original MLCN into the set controllability task of the combined system, thus criteria are obtained accordingly. Furthermore, we make a comparison between the conventional and the considered MLCNs. Finally, a hydrogeological example has been studied to demonstrate the efficiency of the theoretical results.

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Conflict of interest

The authors declare there is no conflicts of interest to this work.

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