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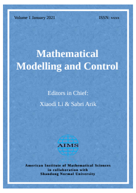

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Research article

Partitioning planar graphs with girth at least 9 into an edgeless graph and a graph with bounded size components

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Abstract: In this paper, we study the problem of partitioning the vertex set of a planar graph with girth restriction into parts, also referred to as color classes, such that each part induces a graph with components of bounded order. An $(\mathcal{I}, \mathcal{O}_k)$-partition of a graph G is the partition of $V(G)$ into two non-empty subsets V_1 and V_2, such that $G[V_1]$ is an edgeless graph and $G[V_2]$ is a graph with components of order at most k. We prove that every planar graph with girth 9 and without intersecting 9-face admits an $(\mathcal{I}, \mathcal{O}_6)$-partition. This improves a result of Choi, Dross and Ochem (2020) which says every planar graph with girth at least 9 admits an $(\mathcal{I}, \mathcal{O}_9)$-partition.

Keywords: planar graph; girth; face; vertex partition; discharging procedure

1. Introduction

All graphs considered in this paper are finite, simple, and undirected. Given a graph G, we use $V(G)$, $E(G)$, and $F(G)$ to denote the vertex set of G, edge set of G and face set of G, respectively. We say that two faces are intersecting if they have at least one vertex in common. Let $g(G)$ denote the girth of G, which is the length of a shortest cycle in G.

Given a graph G, let $\mathcal{G}_i$ be a class of graphs for $1 \leq i \leq m$. A $(\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_m)$-partition of a graph G is the partition of $V(G)$ into m sets $V_1, V_2, \ldots, V_m$, such that for all $1 \leq i \leq m$, the induced subgraph $G[V_i]$ belongs to $\mathcal{G}_i$. We use $\mathcal{I}$, $\mathcal{O}_k$, $\mathcal{P}_k$, $\mathcal{F}$ and $\mathcal{F}_d$ to denote the class of edgeless graphs (independent sets), the class of graphs whose components have order at most k, the class of graphs whose components are paths of order at most k, the class of forests and the class of forests with maximum degree d . In particular, an $(\mathcal{I}, \mathcal{O}_k)$-partition of a graph G is the partition of $V(G)$ into two non-empty subsets V_1 and V_2, such that $G[V_1]$ is an edgeless graph and $G[V_2]$ is a graph with components of order at most k. A planar graph G, equipped with a drawing in the plane so that two edges intersect only at their ends, ia called a plane graph.

There are many results on partitions of graphs. The Four Color Theorem [1, 2] implies that every planar graph has an $(\mathcal{I}, \mathcal{I}, \mathcal{I}, \mathcal{I})$-partition. Alon et al. [5] showed that there is no finite k such that every planar graph has an $(\mathcal{O}_k, \mathcal{O}_k, \mathcal{O}_k)$-partition. Poh [6] showed that every planar graphs admit an $(\mathcal{F}_2, \mathcal{F}_2, \mathcal{F}_2)$-partition. Borodin [8] proved that every planar graph admits an $(\mathcal{I}, \mathcal{F}, \mathcal{F})$-partition.

We focus on partitions of planar graphs with girth restrictions. Borodin, Kostochka, and Yancey [4] proved that a planar graph with girth at least 7 has a $(\mathcal{P}_2, \mathcal{P}_2)$-partition. Borodin and Glebov [7] showed that every planar graph with girth 5 admits an $(\mathcal{I}, \mathcal{F})$-partition. Dross, Montassier, Pinlou [9] proved that every triangle-free planar graph admits an $(\mathcal{F}_5, \mathcal{F})$-partition. Choi, Dross and Ochem [3] proved that every planar graph with girth at least 10 admits an $(\mathcal{I}, \mathcal{P}_3)$-partition and every planar graph with girth at least 9 admits an $(\mathcal{I}, \mathcal{O}_9)$-partition. Choi, Dross and Ochem [3] give an example that a planar graph with girth 7 and maximum degree 4 that has no $(\mathcal{I}, \mathcal{P}_3)$-partition.

In this paper, we establish the following result.

Theorem 1. *Every plane graph with girth at least 9 and without intersecting 9-face admits an $(\mathcal{I}, \mathcal{O}_6)$-partition.*

2. Proof of Theorem 1

2.1. Structure properties of minimum counterexample

Assume that G is the counterexample to Theorem 1 such that G is vertex-minimal. The graph G is connected, since otherwise at least one of its components would be a counterexample with fewer vertices than G. This further implies that every vertex of G has degree at least 1.

For an element $x \in V(G)\cup F(G)$, the degree of x is denoted by $d(x)$. A vertex v is called a k-vertex, k^+-vertex, or k^--vertex if $d(v) = k, d(v) \geq k$, or $d(v) \leq k$, respectively. We define a k-face, k^+-face, or k^--face analogously. Let $N(v)$ denote the set of the neighbours of v. Let $N[v]$ denote $N(v) \cup \{v\}$. A neighbour of the vertex v with degree k, at least k, or at most k is called a k-neighbour, k^+-neighbour, or k^--neighbour of v, respectively. We use $d_k(f)$, $d_{k^+}(f)$ and $d_{k^-}(f)$ to denote the number of k-vertices incident with f, k^+-vertices incident with f and k^--vertices incident with f respectively. For $f \in F(G)$, we use $b(f)$ to denote the boundary walk of f, and $f = [v_1v_2 \dots v_m]$ if $v_1, v_2, \dots, v_m$ are the boundary vertices of f in cyclic order. An $(\ell_1,\ell_2,\dots,\ell_k)$-face is a k-face $[v_1v_2 \dots v_k]$ with $d(v_i) = \ell_i$ for each $i \in \{1, 2, \dots, k\}$. An $(\ell_1,\ell_2,\dots,\ell_k)$-path is a k-path $v_1v_2 \dots v_k$ with $d(v_i) = \ell_i$ for each $i \in \{1, 2, \dots, k\}$, analogously.

Given an $(\mathcal{I}, O_k)$-partition of G, we will assume that $V(G)$ is partitioned into two parts I and O where I is an independent set and O induces a graph whose components have order at most k; we also call the sets I and O colors, and a vertex in I and O is said to have color I and O, respectively.

Claim 1. *Every vertex in G has degree at least 2.*

Proof. Let v be a vertex of degree 1 in G. Since the graph $G - v$ has fewer vertices than G, it admits an $(\mathcal{I}, O_6)$-partition, which can be extended to an $(\mathcal{I}, O_6)$-partition of G by giving to v the color distinct from that of its neighbour. This contradicts G as a counterexample. □

Claim 2. *Every* 6^-*-vertex in G has at least one* 3^+*-neighbour.*

Proof. Let v be a 6^--vertex where every neighbour has degree 2 and let $G' = G - N[v]$. Because the girth of graph G is at least 9, every 2-neighbour of v can not have neighbours in $N(v)$ and the neighbours of 2-neighbour in G' are different. Since G' has fewer vertices than G, it admits an $(\mathcal{I}, O_6)$-partition. For every neighbour u of v that has a neighbour u' in G', color u with the color distinct from that of u'. And color v with color O. Obviously, it does not give an $(\mathcal{I}, O_6)$-partition of G only when all uncoloured vertices with O. Therefore, we can recolor v with I to obtain an $(\mathcal{I}, O_6)$-partition of G, which is a contradiction. □

In G, a chain is a longest induced path whose internal vertices all have degree 2. A chain with k internal vertices is a k-chain. Every end-vertex of a chain is a 3^+-vertex. By Claim 2, there are no 3-chains in G. A 3-vertex is weak if it has two 2-neighbours; a 3-vertex is bad if it is weak and incident with a 2-chain; and a 3-vertex is good otherwise.

Remark 3. *Let v_1,v_2,v_3,v_4 be four vertices of 2-chain, where v_2 and v_3 are 2-vertices. Whether v_1 has been colored I or O, we choose one of the four coloring methods in Table 1 to color the other three uncolored vertices of the 2-chain in the following proofs.*

Table 1. Four coloring methods of 2-chain.

v_1	v_2	v_3	v_4
I	O	I	O
I	O	O	I
O	I	O	O
O	I	O	I

Claim 4. *Every $d(v)$-vertex v($3 \leq d(v) \leq 6$)in G is incident with at most $(d(v) - 2)$ 2-chain.*

Proof. By Claim 2, v has at least one 3^+-neighbour v_1. Assume to the contrary that v is incident with $(d(v) - 1)$ 2-chains. Let graph G' be a graph obtained from G by deleting v and all 2-vertices of 2-chains incident with v. By the minimality of G, G' has an $(\mathcal{I}, O_6)$-partition. For all the 3^+-vertices other than v of 2-chains that have been colored, we let them correspond to v_1 in the Table 1. Now we color the uncolored vertices. Firstly, we color v with the color distinct from that of v_1. Then according to Remark 3, no matter what color v and all the 3^+-vertices other than v of 2-chains are colored, we can always choose appropriate methods from Table 1 to color all the uncolored 2-vertices such that G admits an $(\mathcal{I}, O_6)$-partition. This is a contradiction. □

Claim 5. *Every 3-vertex v is adjacent to at most one weak 3-vertex.*

Proof. Let v_1, v_2 and v_3 be the neighbours of v. Assume to the contrary that v is adjacent to 2 weak 3-vertices. That is, $d(v_1) = d(v_2) = 3$ and $d(v_3) \geq 2$. Let u_1 and u_2 be two 2-neighbours of v_1. Let w_1 and w_2 be two 2-neighbours of v_2. Let z_1 and z_2 be the neighbours other than v_1 of u_1 and u_2, respectively. By the minimality of G, $G^{'} = G - \{v, v_1, v_2, u_1, u_2, w_1, w_2\}$ has an $(\mathcal{I}, O_6)$-partition. Now we color the uncolored vertices. Firstly, we color v with the color distinct from that of v_3. Secondly, we consider the coloring methods of v_1, u_1 and u_2. We give the following three coloring methods. If z_1 and z_2 are colored O, then we assign I to u_1, u_2 and assign O to v_1. If z_1 and z_2 are colored I, then we assign O to u_1, u_2 and assign O to v_1. If z_1 and z_2 are colored I and O respectively, then we assign O to v_1, u_1 and assign I to u_2. In all of the above cases, we can assign O to v_1. The coloring methods of v_2, w_1 and w_2 are similar to those of v_1, u_1 and u_2. We can color the remaining uncolored vertices according to the given three coloring methods. It does not give an $(\mathcal{I}, O_6)$-partition of G only when every vertex in $\{v, v_1, v_2, u_1, u_2, w_1, w_2\}$ with O. Therefore, we can recolor v_1 and v_2 with I to obtain an $(\mathcal{I}, O_6)$-partition of G, which is a contradiction. □

Claim 6. *Every 4-vertex v incident with two 2-chains can not be adjacent to a weak 3-vertex.*

Proof. Let v_1, v_2, v_3 and v_4 be the neighbours of v. Assume to the contrary that v is adjacent to at least weak 3-vertex. That is, $d(v_1) = d(v_2) = 2$, $d(v_3) = 3$ and $d(v_4) \geq 2$. Let u_i be the 2-vertex adjacent to v_i for $i = 1, 2$. Let w_1 and w_2 be two 2-neighbours of v_3. By the minimality of $G, G^{'} = G - \{v, v_1, v_2, v_3, u_1, u_2, w_1, w_2\}$ has an $(\mathcal{I}, O_6)$-partition. Now we color the uncolored vertices. Firstly, we color v with the color distinct from that of v_4. Then according to Remark 3 and the given three coloring methods of v_1, u_1 and u_2 in the proof of Claim 5, we can always choose appropriate methods to color the remaining uncolored vertices such that G admits an $(\mathcal{I}, O_6)$-partition. This is a contradiction. □

Claim 7. *Let v_1 and v_2 be two adjacent 3-vertices.*

(1)These two vertices can not both be weak 3-vertices.

(2)If v_1 is a weak 3-vertex, then v_2 can not be incident with 2-chain.

Proof. (1)Assume to the contrary that v_1 and v_2 be two weak 3-vertices. Let u_1 and u_2 be two 2-neighbours of v_1. Let w_1 and w_2 be two 2-neighbours of v_2. By the minimality of G, $G^{'} = G - \{v_1, v_2, u_1, u_2, w_1, w_2\}$ has an $(\mathcal{I}, O_6)$-partition. Now we color the uncolored vertices. According to the given three coloring methods of v_1, u_1 and u_2 in the proof of Claim 5, we can always choose appropriate methods to color all uncolored vertices such that G admits an $(\mathcal{I}, O_6)$-partition. This is a contradiction.

(2)By (1), we know v_2 has a 3^+-neighbour z_1. Assume to the contrary that v_2 is incident with a 2-chain. Let u_1 and u_2 be two 2-neighbours of v_1. Let w_1 and w_2 be two 2-vertices of 2-chain. Here w_1 is a neighbour of v_2. By the minimality of $G, G^{'} = G - \{v_1, v_2, u_1, u_2, w_1, w_2\}$ has an $(\mathcal{I}, O_6)$-partition. Now we color the uncolored vertices. Firstly, we color v_2 with the color distinct from that of z_1. Then according to Remark 3 and the given three coloring methods of v_1, u_1 and u_2 in the proof of Claim 5, we can always choose appropriate methods to color the remaining uncolored vertices such that G admits an $(\mathcal{I}, O_6)$-partition. This is a contradiction. □

Claim 8. *Let v_1, v_2 and v_3 be three 3-vertices such that $v_i v_{i+1} \in E(G)$, where $i = 1, 2$.*

(1)If v_1 is a weak 3-vertex and v_2 is incident with one 1-chain, then v_3 can not be incident with 2-chain.

(2)If v_2 is adjacent to a 2-vertex, then v_1 and v_3 can not both be incident with 2-chain.

Proof. (1)By Claim 5, we know v_3 has at least a 3^+-neighbour z_1. Assume to the contrary that v_3 is incident with a 2-chain. Let u_1 and u_2 be two 2-neighbours of v_1. Let w_1 and w_2 be two 2-vertices of 2-chain. Let y_1 and y_2 be the neighbours other than v_1 of u_1 and u_2, respectively. Let x_1 be one 2-neighbour of v_2. Let x_2 be an another 3^+-vertex of 1-chain incident with v_2. By the minimality of G, $G^{'} = G - \{v_1, v_2, v_3, u_1, u_2, w_1, w_2, x_1\}$ has an $(\mathcal{I}, O_6)$-partition. Now we color the uncolored vertices. Firstly, we color v_3 with the color distinct from that of z_1. Secondly, we consider the coloring methods of x_1 and v_2. We give the following two coloring methods. If x_2 is colored I, then we assign O to x_1 and assign O to v_2. If x_2 is colored O, then we assign I to x_1 and assign O to v_2. So, we can assign O to

v_2 whatever x_2 has been colored I or O. Then we consider the coloring methods of v_1, u_1 and u_2. We give the following three coloring methods. If y_1 and y_2 are colored O, then we assign I to u_1, u_2 and assign O to v_1. If y_1 and y_2 are colored I, then we assign O to u_1, u_2 and assign I to v_1. If y_1 and y_2 are colored I and O respectively, then we assign O to v_1, u_1 and assign I to u_2. Then according to Remark 3 and the given these coloring methods , we can always choose appropriate methods to color the remaining uncolored vertices such that G admits an $(\mathcal{I}, O_6)$-partition. This is a contradiction.

(2)Assume to the contrary that v_1 and v_3 are both incident with a 2-chain. Let u_1 and u_2 be two 2-vertices of 2-chain incident with v_1. Let w_1 and w_2 be two 2-vertices of 2-chain incident with v_3. Let x_1 be one 2-neighbour of v_2. Let z_1 and z_2 be other neighbours of v_1 and v_3 respectively. By the minimality of G, $G' = G - \{v_1, v_2, v_3, u_1, u_2, w_1, w_2, x_1\}$ has an $(\mathcal{I}, O_6)$-partition. Now we color the uncolored vertices. According to Remark 3 and the given two coloring methods of x_1 and v_2 in the proof of Claim 8(1), we can always choose appropriate methods to color all uncolored vertices such that G admits an $(\mathcal{I}, O_6)$-partition. This is a contradiction. □

Claim 9. *Let v_1 and v_3 be two 3-vertices and v_2 is the common 2-neighbor of v_1 and v_3. Then v_1 and v_3 can not both be incident with 2-chain.*

Proof. Assume to the contrary that v_1 and v_3 are both incident with a 2-chain. Let u_1 and u_2 be two 2-vertices of 2-chain incident with v_1. Let w_1 and w_2 be two 2-vertices of 2-chain incident with v_3. By the minimality of G, $G' = G - \{v_1, v_2, v_3, u_1, u_2, w_1, w_2\}$ has an $(\mathcal{I}, O_6)$-partition. Now we color the uncolored vertices. Firstly, we assign O to v_2. Then by Remark 3, we can color all uncolored vertices such that G admits an $(\mathcal{I}, O_6)$-partition. This is a contradiction. □

Claim 10. *Let v_1, v_2, v_3 and v_4 be four 3-vertices such that $v_iv_{i+1} \in E(G)$, where $i = 1, 2, 3$. If v_2 and v_3 are both incident with a 1-chain, then v_1 and v_4 can not both be weak 3-vertices.*

Proof. Assume to the contrary that v_1 and v_4 be two weak 3-vertices. Let u_1 and u_2 be two 2-neighbours of v_1. Let w_1 and w_2 be two 2-neighbours of v_4. Let z_1 and z_2 be 2-neighbours of v_2 and v_3 respectively. By the minimality of G, $G' = G - \{v_1, v_2, v_3, v_4, u_1, u_2, w_1, w_2, z_1, z_2\}$ has an $(\mathcal{I}, O_6)$-partition. Now we color the uncolored vertices. We can color all uncolored vertices according to the given two coloring methods of x_1, v_2 and three coloring methods of v_1, u_1 and u_2 in the proof of Claim 8(1). Obviously, it does not give an $(\mathcal{I}, O_6)$-partition of G only when v_1, v_2, v_3 and v_4 are colored with O and at least one of z_1 and z_2 is colored with O, say z_1. Then we recolor 3-neighbour v_2 of z_1 with I. Therefore, we can know G admits an $(\mathcal{I}, O_6)$-partition. This is a contradiction. □

Claim 11. *If f is a 9-face with $d_3(f) = 9$, then these 3-vertices on f can not all be incident with 2-chain.*

Proof. Assume to the contrary that these 3-vertices all be incident with 2-chain. According to Claim 8(2), this situation is impossible. □

Claim 12. *There are no $(3, 2, 2, 3, 2, 3, 2, 3, 2)$-faces in G.*

Proof. Suppose to the contrary that G contains such a $(3, 2, 2, 3, 2, 3, 2, 3, 2)$-face f. By Claim 2, we know the neighbours of these 3-vertices that are not on f are 3^+-vertices. Let graph G' be a graph obtained from G by deleting all vertices on f. By the minimality of G, G' has an $(\mathcal{I}, O_6)$-partition. Now we color the uncolored vertices. Firstly, we color these 3-vertices on f with the color distinct from that of their 3^+-neighbours. We know these 3-vertices on f color either I or O. Then we consider the coloring of 2-vertices on f. If two 3-vertices of 1-chain are colored $O(I)$, then we assign $I(O)$ to 2-vertex. If two 3-vertices of 1-chain are colored O and I respectively, then we assign O to 2-vertex. And we assign O to two 2-vertices of 2-chain. In this way, we can get an $(\mathcal{I}, O_6)$-partition of graph G, a contradiction. □

Claim 13. *If f is a $(3, 2, 2, 3, 3, 2, 3, 2, 3)$-face, then the neighbours of v_1 and v_4 that are not on f can not both be 2-vertices.*

Proof. Let v_1' and v_4' be neighbours of v_1 and v_4 that are not on f, respectively. Assume to the contrary that v_1' and v_4' are both 2-vertices. Let z_1 be another neighbour other than v_1 of v_1'. By Claim 7(2), we know the neighbours of v_5 and v_9 that are not on f are 3^+-vertices. By Claim 2, we know the neighbour of v_7 that is not on f is a 3^+-vertex. Let graph G' be a graph obtained from G by deleting v_1', v_4' and all vertices on f. By the minimality of G, G' has an $(\mathcal{I}, O_6)$-partition. Now we color the uncolored vertices. Firstly, we color v_5, v_7 and v_9 to make their colors different from their 3^+-neighbours that are not on f. Then we consider the coloring of v_1 and v_1'. If z_1 is colored I, then we assign O to v_1' and assign O to v_1. If z_1 is colored O, then we assign I to v_1' and assign O to v_1. So, we can assign O to v_1 whatever z_1 has been colored I or O. The coloring methods of v_4 and v_4' are similar to those of v_1 and v_1'. Finally, we consider the coloring of 2-vertices on f. We assign O and I to v_2 and v_3, respectively. If two 3-vertices of 1-chain are colored $O(I)$, then we assign $I(O)$ to 2-vertex. If two 3-vertices of 1-chain are colored O and I respectively, then we assign O to 2-vertex. In this way, we can get an $(\mathcal{I}, O_6)$-partition of graph G, a contradiction. □

Claim 14. *If f is a $(3,3,2,3,2,3,2,3,2)$-face, then the neighbors of v_1 and v_2 that are not on f are both 3^+-vertices.*

Proof. Let v_1' and v_2' be neighbours of v_1 and v_2 that are not on f, respectively. By Claim 7(1), we know that one of v_1' and v_2' is a 3^+-vertex. Without loss of generality, let v_1' be a 3^+-vertex. Assume to the contrary that v_2' is a 2-vertex. By Claim 2, we know the neighbours of v_4, v_6 and v_8 that are not on f are 3^+-vertices. Let graph G' be a graph obtained from G by deleting v_2' and all vertices on f. By the minimality of G, G' has an $(\mathcal{I}, O_6)$-partition. Now we color the uncolored vertices. Firstly, we color v_1, v_4, v_6 and v_8 to make their colors different from their 3^+-neighbours that are not on f. Then we consider the coloring of v_2 and v_2'. According to the coloring methods of v_1 and v_1' in the proof of Claim 13, we can color v_2 and v_2'. Finally, we consider the coloring of 2-vertices on f. If two 3-vertices of 1-chain are colored $O(I)$, then we assign $I(O)$ to 2-vertex. If two 3-vertices of 1-chain are colored O and I respectively, then we assign O to 2-vertex. In this way, we can get an $(\mathcal{I}, O_6)$-partition of graph G, a contradiction. □

Claim 15. *If f is a $(3,3,2,3,3,2,3,3,2)$-face, then at least a pair of adjacent 3-vertices on f have two 3^+-neighbours that are not on f.*

Proof. By Claim 7(1), we know one neighbour of each pair of adjacent 3-vertices that is not on f is a 3^+-vertex. Assume to the contrary that the other neighbour of each pair of adjacent 3-vertices that is not on f is a 2-vertex. Let graph G' be a graph obtained from G by deleting all vertices on f and 2-vertices which are not on f and are incident with 3-vertices on f. By the minimality of G, G' has an $(\mathcal{I}, O_6)$-partition. Now we color the uncolored vertices. Firstly, we color non-weak 3-vertices on f to make their colors different from their 3^+-neighbours that are not on f. Then, we consider the coloring of weak 3-vertices and their 2-neighbours that are not f. According to the coloring methods of v_1 and v_1' in the proof of Claim 13, we can color weak 3-vertices and their 2-neighbours that are not f. Finally, we consider the coloring of 2-vertices on f. If two 3-vertices of 1-chain are colored $O(I)$, then we assign $I(O)$ to 2-vertex. If two 3-vertices of 1-chain are colored O and I respectively, then we assign O to 2-vertex. In this way, we can get an $(\mathcal{I}, O_6)$-partition of graph G, a contradiction. □

Claim 16. *If f is a $(3,3,2,3,3,2,3,2,2)$-face, then at least a pair of adjacent 3-vertices on f have two 3^+-neighbours that are not on f.*

Proof. By Claim 2, we know the neighbour of v_7 that is not on f is a 3^+-vertex. By Claim 7(2), we know the neighbour of v_2 that is not on f is a 3^+-vertex. By Claim 7(1), we know one neighbour of each pair of adjacent 3-vertices that is not on f is a 3^+-vertex. Assume to the contrary that the other neighbour of each pair of adjacent 3-vertices that is not on f is a 2-vertex. Let graph G' be a graph obtained from G by deleting all vertices on f and 2-vertices which are not on f and are incident with 3-vertices on f. By the minimality of G, G' has an $(\mathcal{I}, O_6)$-partition. Now we color the uncolored vertices. Firstly, we color non-weak 3-vertices on f to make their colors different from their 3^+-neighbours that are not on f. Then, we consider the coloring of weak 3-vertices and their 2-neighbours that are not f. According to the coloring methods of v_1 and v_1' in the proof of Claim 13, we can color weak 3-vertices and their 2-neighbours that are not f. Finally, we consider the coloring of 2-vertices on f. If two 3-vertices of 1-chain are colored $O(I)$, then we assign $I(O)$ to 2-vertex. If two 3-vertices of 1-chain are colored O and I respectively, then we assign O to 2-vertex. And we assign O and I to v_8 and v_9, respectively. In this way, we can get an $(\mathcal{I}, O_6)$-partition of graph G, a contradiction. □

2.2. Discharging procedure

To prove Theorem 1, we will get a contradiction by a discharging procedure. For all $x \in V(G) \cup F(G)$, we define an initial weight function ω: if $v \in V(G)$, let $\omega(v) = 2d(v) - 5$; if $f \in F(G)$, let $\omega(f) = \frac{1}{2}d(f) - 5$. The total initial charge is negative, since Euler's formula implies

$$\sum_{v \in V(G)} (2d(v) - 5) + \sum_{f \in F(G)} (\frac{1}{2}d(f) - 5) = -10. \qquad (2.1)$$

We then redistribute the charge at the vertices and faces according to carefully designed discharging rules, which preserve the total charge sum. Once the discharging is finished, a new charge function $\omega^{'}$ is produced. Finally, we can show that the final charge $\omega^{'}$ on $V(G) \cup F(G)$ satisfies $\sum_{x \in V(G) \cup F(G)} \omega^{'}(x) \geq 0$. Then it leads to a contradiction in the inequality:

$$0 \leq \sum_{x \in V(G) \cup F(G)} \omega^{'}(x) = \sum_{x \in V(G) \cup F(G)} \omega(x) = -10. \qquad (2.2)$$

and thus this completes the proof of Theorem 1. The discharging rules are as follows.

(R1)Every 2-vertex that belongs to a 1-chain gets charge $\frac{1}{2}$ from its each ends, while each 2-vertex that belongs to a 2-chain gets charge 1 from its neighbour of degree greater than 3.

(R2)Every weak 3-vertex sends charge $\frac{1}{2}$ to its adjacent 2-vertex on 2-chain.

(R3)Every good 3-vertex sends charge 1 to its adjacent 2-vertex on 2-chain.

(R4)Each 3^{+}-vertex along its adjacent bad 3-vertex v sends charge $\frac{1}{2}$ to 2-vertex on 2-chain adjacent v

For each 3-vertex v, let $\alpha(v)$ be the remaining charge of v after rules $R1 - R4$.

(R5)Each 3-vertex v sends charge $\alpha(v)$ to each incident 9-face.

(R6)Each 4^{+}-vertex sends charge $\frac{1}{2}$ to each incident 9-face.

In the following, we will prove that $\omega^{'}(x) \geq 0$ for all $x \in V(G) \cup F(G)$.

Claim 17. *Every vertex v has non-negative final charge.*

Proof. Let v be a 2-vertex, which has initial charge -1. If v belongs to a 1-chain, then $\omega^{'}(v) = -1 + \frac{1}{2} \times 2 = 0$ by $R1$. If v belongs to a 2-chain, then $\omega^{'}(v) = -1 + \frac{1}{2} + \frac{1}{2} = 0$ or $\omega^{'}(v) = -1 + 1 = 0$ by $R1$, $R2$, $R3$, and $R4$.

Let v be a 3-vertex, which has initial charge 1. By the discharging rules, we only need to show that $\alpha(v) \geq 0$. By Claim 2, we know v has at least a 3^{+}-neighbour v_1. By Claim 4, we know v is incident with at most one 2-chain. Suppose v is a weak 3-vertex. By Claim 7(1), we know v_1 can not be a weak 3-vertex. Then $\alpha(v) = 1 - \frac{1}{2} - \frac{1}{2} = 0$ by $R1$ and $R2$. Suppose v is a good 3-vertex. By Claim 5, we know every 3-vertex v is adjacent to at most one weak 3-vertex. If v is not incident with chains, then $\alpha(v) \geq 1 - \frac{1}{2} = \frac{1}{2}$ by $R4$. If v is incident with a 1-chain, then $\alpha(v) \geq 1 - \frac{1}{2} - \frac{1}{2} = 0$ by $R1$ and $R4$. If v is incident with a 2-chain, then we know v can not be adjacent to weak 3-vertices by Claim 7(2). Thus, $\alpha(v) = 1 - 1 = 0$ by $R3$.

Let v be a 4-vertex, which has initial charge 3. By Claim 2, we know v has at least a 3^{+}-neighbour v_1. By Claim 4, we know v is incident with at most two 2-chains. We also know v incident with two 2-chains can not be adjacent to weak 3-vertex by Claim 6. Then $\omega^{'}(v) \geq 3 - max\{1 \times 2 + \frac{1}{2} + \frac{1}{2}, 1 + \frac{1}{2} \times 2 + \frac{1}{2} + \frac{1}{2}\} = 0$ by $R1$, $R4$ and $R6$.

Let v be a 5-vertex, which has initial charge 5. By Claim 2, we know v has at least a 3^{+}-neighbour v_1. By Claim 4, we know v is incident with at most three 2-chains. Then $\omega^{'}(v) \geq 5 - 1 \times 3 - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} = \frac{1}{2}$ by $R1$, $R4$ and $R6$.

Let v be a 6-vertex, which has initial charge 7. By Claim 2, we know v has at least a 3^{+}-neighbour v_1. By Claim 4, we know v is incident with at most four 2-chains. Then $\omega^{'}(v) \geq 7 - 1 \times 4 - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} = \frac{3}{2}$ by $R1$, $R4$ and $R6$.

Each 7^{+}-vertex with degree $d(v)$ has initial charge $2d(v) - 5$. Then $\omega^{'}(v) \geq 2d(v) - 5 - d(v) - \frac{1}{2} = d(v) - \frac{11}{2} \geq \frac{3}{2}$ by $R1$ and $R6$. □

Claim 18. *Every* 10^{+}*-face f has non-negative final charge.*

Proof. Let f be a 10^{+}-face. We know that a 10^{+}-face is not involved in discharging rules, so $\omega^{'}(f) = \omega(f) = \frac{1}{2}d(f) - 5 \geq \frac{1}{2} \times 10 - 5 = 0$. □

Before discussing 9-faces, we give the following two useful Lemmas.

Lamma 19. *If there is a* $(3, 2, 2, 3, 3, 3)$*-path on* f*, then* $\omega^{'}(f) \geq 0$.

Proof. Let v_i' be a neighbour of v_i that is not on $f(i = 4, 5)$. By Claim 5, we know every 3-vertex is adjacent to at most one weak 3-vertex.

Suppose v_4 is a weak 3-vertex. If v_5' is a 3^+-vertex, then $\alpha(v_5) = 1 - \frac{1}{2} = \frac{1}{2}$ by $R4$. So $\omega'(f) \geq -\frac{1}{2} + \frac{1}{2} = 0$ by $R5$. If v_5' is a 2-vertex, then we can know v_5' belongs to 1-chain by Claim 7(2). Then we consider the case of v_6. According to Claim 5 and Claim 8(2), we know v_6 is not a weak 3-vertex and v_6 can not be incident with 2-chain. If v_6 is not adjacent 2-vertex, then $\alpha(v_6) \geq 1 - \frac{1}{2} = \frac{1}{2}$ by $R4$. So $\omega'(f) \geq -\frac{1}{2} + \frac{1}{2} = 0$ by $R5$. If v_6 is incident with a 1-chain, then we can know another 3^+-vertex other than v_5 of v_6 is not a weak 3-vertex by Claim 10. Then $\alpha(v_6) = 1 - \frac{1}{2} = \frac{1}{2}$ by $R1$. So $\omega'(f) \geq -\frac{1}{2} + \frac{1}{2} = 0$ by $R5$.

Suppose v_4 is a good 3-vertex. If v_5' is a 3^+-vertex, then $\alpha(v_5) \geq 1 - \frac{1}{2} = \frac{1}{2}$ by $R4$. Thus, $\omega'(f) \geq -\frac{1}{2} + \frac{1}{2} = 0$. So we can assume v_5' is a 2-vertex. If v_5' belongs to 1-chain, then we can know v_6 is not a weak 3-vertex by Claim 8(1). Then $\alpha(v_5) = 1 - \frac{1}{2} = \frac{1}{2}$ by $R1$. So $\omega'(f) \geq -\frac{1}{2} + \frac{1}{2} = 0$ by $R5$. If v_5' belongs to 2-chain, then we consider the case of v_6. According to Claim 7(2) and Claim 8(2), we know v_6 is not a weak 3-vertex and v_6 can not be incident with 2-chain. If v_6 is not adjacent 2-vertex, Then $\alpha(v_6) \geq 1 - \frac{1}{2} = \frac{1}{2}$ by $R4$. So $\omega'(f) \geq -\frac{1}{2} + \frac{1}{2} = 0$ by $R5$. If v_6 is incident with a 1-chain, then we can know another 3^+-vertex other than v_5 of v_6 is not a weak 3-vertex by Claim 8(1). Then $\alpha'(v_6) = 1 - \frac{1}{2} = \frac{1}{2}$ by $R1$. So $\omega'(f) \geq -\frac{1}{2} + \frac{1}{2} = 0$ by $R5$. □

Lamma 20. *If there is a* $(3, 2, 3, 3, 3)$*-path on* f*, then* $\omega'(f) \geq 0$.

Proof. Let v_i' be a neighbour of v_i that is not on $f(i = 3, 4,)$. By Claim 5, we know every 3-vertex is adjacent to at most one weak 3-vertex.

Suppose v_3 is a weak 3-vertex. If v_4' is a 3^+-vertex, then $\alpha(v_4) \geq 1 - \frac{1}{2} = \frac{1}{2}$ by $R4$. So $\omega'(f) \geq -\frac{1}{2} + \frac{1}{2} = 0$ by $R5$. If v_4' is a 2-vertex, then we can know v_4' belongs to 1-chain by Claim 7(2). Then we consider the case of v_5. According to Claim 5 and Claim 8(1), we know v_5 is not a weak 3-vertex and v_5 can not be incident with 2-chain. If v_5 is not adjacent to 2-vertex, then $\alpha(v_5) \geq 1 - \frac{1}{2} = \frac{1}{2}$ by $R4$. So $\omega'(f) \geq -\frac{1}{2} + \frac{1}{2} = 0$ by $R5$. If v_5 is incident with a 1-chain, then we can know another 3^+-vertex other than v_4 of v_5 is not a weak 3-vertex by Claim 10. Then $\alpha(v_5) = 1 - \frac{1}{2} = \frac{1}{2}$ by $R1$. So $\omega'(f) \geq -\frac{1}{2} + \frac{1}{2} = 0$ by $R5$.

Suppose v_3 is a good 3-vertex. If v_3' is a 3^+-vertex but not a bad 3-vertex, then $\alpha(v_3) = 1 - \frac{1}{2} = \frac{1}{2}$ by $R1$. So $\omega'(f) \geq -\frac{1}{2} + \frac{1}{2} = 0$ by $R5$. If v_3' is a bad 3-vertex, then we consider the case of v_4. According to Claim 5 and Claim 8(2), we know v_4 is not a weak 3-vertex and v_4 can not be incident with 2-chain. If v_4 is not an adjacent 2-vertex, then $\alpha(v_4) \geq 1 - \frac{1}{2} = \frac{1}{2}$ by $R4$. So $\omega'(f) \geq -\frac{1}{2} + \frac{1}{2} = 0$ by $R5$. If v_4 is incident with a 1-chain, then we can know v_5 is not a weak 3-vertex by Claim 10. Then $\alpha(v_4) = 1 - \frac{1}{2} = \frac{1}{2}$ by $R1$. So $\omega'(f) \geq -\frac{1}{2} + \frac{1}{2} = 0$ by $R5$. □

Claim 21. *Every* 9*-face* f *has non-negative final charge.*

Proof. If f is incident with at least a 4^+-vertex, then $\omega'(f) \geq \frac{1}{2} \times 9 - 5 + \frac{1}{2} = 0$ by $R6$. Now we only need to consider the situation that f is only incident with 2-vertices and 3-vertices.

Case 1. $d_2(f) = 0$.

By Claim 11, we know these 3-vertices on f can not all be incident with 2-chain. So there is at least one vertex v not incident with 2-chain. If v is incident with a 1-chain, then $\alpha(v) = 1 - \frac{1}{2} = \frac{1}{2}$ by $R1$. If v is adjacent to a 3^+-vertex that is not on f, then $\alpha(v) \geq 1 - \frac{1}{2} = \frac{1}{2}$ by $R4$. So $\omega'(f) \geq -\frac{1}{2} + \frac{1}{2} = 0$ by $R5$.

Case 2. $d_2(f) = 1$.

By Lemma 20, we know $\omega'(f) \geq 0$.

Case 3. $d_2(f) = 2$.

For $(3, 2, 2, 3, 3, 3, 3, 3, 3)$-face, we have $\omega'(f) \geq 0$ by Lemma 19.

For $(3, 2, 3, 2, 3, 3, 3, 3, 3)$-face, $(3, 2, 3, 3, 2, 3, 3, 3, 3)$-face and $(3, 2, 3, 3, 3, 2, 3, 3, 3)$-face, we also have $\omega'(f) \geq 0$ by Lemma 20.

Case 4. $d_2(f) = 3$.

For $(3, 2, 2, 3, 2, 3, 3, 3, 3)$-face, $(3, 2, 2, 3, 3, 2, 3, 3, 3)$-face and $(3, 2, 2, 3, 3, 3, 2, 3, 3)$-face, we have $\omega'(f) \geq 0$ by Lemma 19.

For $(3, 2, 3, 2, 3, 2, 3, 3, 3)$-face and $(3, 2, 3, 2, 3, 3, 2, 3, 3)$-face, we also have $\omega'(f) \geq 0$ by Lemma 20.

For $(3, 3, 2, 3, 3, 2, 3, 3, 2)$-face, we can know at least a pair of adjacent 3-vertices on f have two 3^+-neighbours that are not on f by Claim 15. By Claim 10, we know these two 3^+-neighbours can not both be weak 3-vertices. So there is a 3-vertex v on f such that $\alpha(v) = 1 - \frac{1}{2} = \frac{1}{2}$ by $R1$. So $\omega^{'}(f) \geq -\frac{1}{2} + \frac{1}{2} = 0$ by $R5$.

Case 5. $d_2(f) = 4$.

For $(3,2,2,3,2,3,2,3,3)$-face, we have $\omega^{'}(f) \geq 0$ by Lemma 19.

For $(3,2,2,3,3,2,3,2,3)$-face, we can know the neighbours of v_1 and v_4 that are not on f can not both be 2-vertices by Claim 13. Without loss of generality, let the neighbour of v_4 that is not on f is a 3^+-vertex. By Claim 7(2), v_5 is not a weak 3-vertex. By Claim 8(1), we know 3^+-neighbour of v_5 that is not on f can not be a weak 3-vertex. Then $\alpha(v_5) = 1 - \frac{1}{2} = \frac{1}{2}$ by $R1$. So $\omega^{'}(f) \geq -\frac{1}{2} + \frac{1}{2} = 0$ by $R5$.

For $(3,3,2,3,2,3,2,3,2)$-face, we can know the neighbors of v_1 and v_2 that are not on f are both 3^+-vertices by Claim 14. By Claim 10, we know these two 3^+-neighbours can not both be weak 3-vertices. So there is a 3-vertex v on f such that $\alpha(v) = 1 - \frac{1}{2} = \frac{1}{2}$ by $R1$. So $\omega^{'}(f) \geq -\frac{1}{2} + \frac{1}{2} = 0$ by $R5$.

For $(3,3,2,3,3,2,3,2,2)$-face, we can know at least a pair of adjacent 3-vertices on f,say v_1 and v_2, have two 3^+-neighbours that are not on f by Claim 16. By Claim 8(1), we know 3^+-neighbour of v_2 that is not on f can not be a weak 3-vertex. Then $\alpha(v_2) = 1-\frac{1}{2} = \frac{1}{2}$ by $R1$. So $\omega^{'}(f) \geq -\frac{1}{2}+\frac{1}{2} = 0$ by $R5$.

For $(3,3,3,2,2,3,3,2,2)$-face, we know v_2 can not have a 2-neighbour that is not on f by Claim 8(2). By Claim 5, we know every 3-vertex is adjacent to at most one weak 3-vertex. then $\alpha(v_2) \geq 1-\frac{1}{2} = \frac{1}{2}$ by $R1$. So $\omega^{'}(f) \geq -\frac{1}{2}+\frac{1}{2} = 0$ by $R5$.

For $(3,2,3,3,3,2,3,2,2)$-face, there is a $(3,2,3,3,3)$-path. For $(3,2,3,3,3)$-path, we can conclude that $\omega^{'}(f) \geq 0$ by using the same analysis method as Lemma 20.

By Claim 12, there are no $(3,2,2,3,2,3,2,3,2)$-faces in G. We know there are no $(3,2,2,3,3,2,2,3,2)$-faces in G by Claim 9. So there is no case of $d_2(f) = 5$.

According to Claim 2, we know that there are only 1-chains and 2-chains in G. According to Claim 4, every 3-vertex v in G is incident with at most one 2-chain. So there is no case of $d_2(f) \geq 6$. □

3. Conclusions

Every planar graph with girth 9 and without intersecting 9-face admits an $(\mathcal{I}, O_6)$-partition.

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Conflict of interest

The authors declare there is no conflict of interests.

References

1. K. Appel, W. Haken, Every planar map is four colorable, part I. Discharging, *Illinois J. Math.,* **21** (1977), 429–490.
2. K. Appel, W. Haken, Every planar map is four colorable, part II. Reducibility, *Illinois J. Math.,* **21** (1977), 491–567.
3. I. Choi, F. Dross, P. Ochem, Partitioning sparse graphs into an independent set and a graph with bounded size components, *Discrete Mathematics,* **343** (2020), 111921.
4. O.V. Borodin, A.V. Kostochka, M. Yancey , On 1-improper 2-coloring of sparse graphs, *Discrete Math.,* **313** (2013), 2638–2649
5. N. Alon, G. Ding, B. Oporowski, D. Vertigan, Partitioning into graphs with only small components, *J. Combin. Theory Ser. B,* **87** (2003), 231–243.
6. K.S. Poh, On the linear vertex-arboricity of a planar graph, *J. Graph Theory,* **14** (1990), 73–75.
7. O.V. Borodin, A.N. Glebov, On the partition of a planar graph of girth 5 into an empty and an acyclic subgraph, *Diskretn. Anal. Issled. Oper. Ser. 1,* **8** (2001), 34–53.
8. O.V. Borodin, A proof of B. Grünbaum's conjecture on the acyclic 5-colorability of planar graphs, *Dokl. Akad. Nauk SSSR,* **231** (1976), 18–20.

9. F. Dross, M. Montassier, A. Pinlou, Partitioning a triangle-free planar graph into a forest and a forest of bounded degree, *European J. Combin.,* **66** (2017), 81–94.