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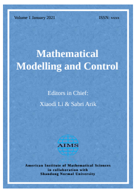

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Research article

Constrainted least squares solution of Sylvester equation

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Abstract: In this paper, we study several constrainted least squares solutions of quaternion Sylvester matrix equation. We first propose a real vector representation of quaternion matrix and study its properties. By using this real vector representation, semi-tensor product of matrices, swap matrix and Moore-Penrose inverse, we derive compatible conditions and the expressions of several constrainted least squares solutions of quaternion Sylvester equation.

Keywords: quaternion matrix equation; least squares solution; semi-tensor product of matrices; real vector representation; (anti)η-Hermitian matrix

1. Introduction

First some necessary notations are given to make this paper more fluid. $\mathbb{R}\backslash\mathbb{Q}$ represent the real number field and quaternion skew-field, respectively. $\mathbb{R}^t$ represents the set of all real column vectors with order t. $\mathbb{R}^{m\times n}\backslash\mathbb{Q}^{m\times n}$ represent the set of all $m\times n$ real \quaternion matrices, respectively. $\eta\mathbb{HQ}^{n\times n}\backslash\ \eta\mathbb{AQ}^{n\times n}$ represent the set of all $n\times n$ quaternion $\eta-Hermitian\ matrix$ and quaternion $\eta-anti-Hermitian\ matrices$, respectively. I_n represents the unit matrix with order n. δ_n^i represents the ith column of unit matrix I_n. $A^T, A^H, A^\dagger$ stands for the transpose, the conjugate transpose, Moore-Penrose(MP) inverse of matrix A, respectively. $\otimes$ represents the Kronecker product of matrices. $\ltimes$ represents the semi-tensor product of matrices. $\|\cdot\|$ represents the Frobenius norm of a matrix or Euclidean norm of a vector.

In the process of studying the theory and numerical calculation of mathematical and physical problems, it is often necessary to solve the approximate solution of quaternion linear system, which also have wide applications in computer science, quantum physics, statistic, signal and color image processing, rigid mechanics, quantum mechanics, control theory, field theory and so on [1–9]. Many researchers are interested in quaternion linear system and use different methods to get a lot of results [10, 11]. In this paper, we are interested in the Sylvester equation

$$AXB+CYD=E \tag{1.1}$$

over quaternion algebra. $\eta-Hermitian$ matrix and $\eta-anti-Hermitian$ matrix are two kind of important matrices in linear modeling and convergence analysis in statistical signal processing [12, 13]. As for the special Hermitian solution of the Sylvester equation, the following literatures are available. Ling et al came up with iterative algorithms for the η -Hermitian and η-bi Hermitian solutions with minimal norm for quaternion least squares problem [14]. Yuan et al. studied η -Hermitian and η-anti-Hermitian solutions to the quaternion matrix equations [15, 16]. Liu considered the η-anti-Hermitian solution for the quaternion matrix equations $AX=B$, $AXB=C$, $AXA^{\eta*}=B$, $EXE^{\eta*}+FYF^{\eta*}=H$, and established general expressions of solutions [17]. Rehman et al. mentioned some necessary and sufficient conditions for the existence of the solution to the system of real quaternion matrix equations including $\eta-Hermicity$ and also constructed the general solution to the system when it is consistent [18].

In this paper, we will propose a new method to solve the special least squares problems of (1.1) by using a powerful tool-the semi-tensor product of matrices. The semi-tensor product(STP) is a new matrix product, which generalizes the conventional matrix product to two arbitrary matrices. The conventional multiplication of matrix is limited of dimension and non-commutativity. The semi-tensor product breaks through the limitation of dimension and satisfies quasi-commutative. It has been proved to be extremely useful in many fields such as the coloring problem [19], the design of shifting register [20], the fault detection [21] and so on. In addition, since the dynamics of a finite game can be modeled as a logical network [22], the semi-tensor product method has also been applied to the study of game theory [23, 24]. In this paper, we will convert the least squares problems of quaternion matrix equation to the corresponding real problems by using the semi-tensor product. Our specific problem is as follows:

Problem 1. Let $A \in \mathbb{Q}^{m\times n}$, $B \in \mathbb{Q}^{n\times s}$, $C \in \mathbb{Q}^{m\times k}$, $D \in \mathbb{Q}^{k\times s}$, $E \in \mathbb{Q}^{m\times s}$, and

$$S_M=\left\{(X,Y)|X \in \eta\mathbb{HQ}^{n\times n}, Y \in \eta\mathbb{AQ}^{k\times k}, \|AXB+CYD-E\|=min\right\}.$$

Find out $(\hat{X},\hat{Y}) \in S_M$ such that

$$\left\|\hat{X}\right\|^2+\left\|\hat{Y}\right\|^2=\min_{\substack{X\in\eta\mathbb{HQ}^{n\times n}\\ Y\in\eta\mathbb{AQ}^{k\times k}}}\left\{\|X\|^2+\|Y\|^2\right\}$$

$(\hat{X},\hat{Y})$ is called minimal norm least squares mixed solution of (1.1).

This paper is arranged as follows. In Section 2, we recall some preliminary results on quaternion matrix and STP used in the paper. In Section 3, we propose a new kind of real vector representation of a quaternion matrix and survey its properties. In Section 4, we study the solutions of Problem 1 by applying the real vector representation of quaternion matrix, the special structure of solutions and STP. In Section 5, we give a numerical experiment to illustrate the effectiveness of the method. In Section 6, we make some concluding remarks.

2. Preliminaries

Definition 2.1. *[25] A quaternion $q \in \mathbb{Q}$ is expressed as*

$$q=a+b\mathbf{i}+c\mathbf{j}+d\mathbf{k},$$

where $a,b,c,d \in \mathbb{R}$, and three imaginary units $\mathbf{i},\mathbf{j},\mathbf{k}$ satisfy

$$\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=\mathbf{ijk}=-1,\ \ \mathbf{ij}=-\mathbf{ji}=\mathbf{k},$$

$$\mathbf{jk}=-\mathbf{kj}=\mathbf{i},\ \mathbf{ki}=-\mathbf{ik}=\mathbf{j}.$$

$\mathbb{Q}$ is clearly an associative but non-commutative algebra of rank four over $\mathbb{R}$, called quaternion skew-field.

Let $A=A_1+A_2\mathbf{i}+A_3\mathbf{j}+A_4\mathbf{k} \in \mathbb{Q}^{k\times k}$, where $A_i \in \mathbb{R}^{k\times k}$ $(i=1,2,3,4)$. The matrix $A^{\mathbf{i}H}, A^{\mathbf{j}H}, A^{\mathbf{k}H}$ are defined as below

$$A^{\mathbf{i}H}=-\mathbf{i}A^H\mathbf{i}=A_1^T-A_2^T\mathbf{i}+A_3^T\mathbf{j}+A_4^T\mathbf{k},$$

$$A^{\mathbf{j}H}=-\mathbf{j}A^H\mathbf{j}=A_1^T+A_2^T\mathbf{i}-A_3^T\mathbf{j}+A_4^T\mathbf{k},$$

$$A^{\mathbf{k}H}=-\mathbf{k}A^H\mathbf{k}=A_1^T+A_2^T\mathbf{i}+A_3^T\mathbf{j}-A_4^T\mathbf{k}.$$

Definition 2.2. *[26] Let $A \in \mathbb{Q}^{k\times k}$, $\eta=\mathbf{i},\mathbf{j},\mathbf{k}$. If $A^{\eta H}=A$, then A is η-Hermitian. If $A^{\eta H}=-A$, then A is η-anti-Hermitian. For $A=A_1+A_2\mathbf{i}+A_3\mathbf{j}+A_4\mathbf{k} \in \mathbb{Q}^{k\times k}$, by Definition 2.1, we can obtain*

(1)*For* $\eta=\mathbf{i}$, $A \in \eta\mathbb{HQ}^{k\times k} \Longleftrightarrow A_2^T=-A_2, A_s^T=A_s, s=1,3,4.$
(2)*For* $\eta=\mathbf{j}$, $A \in \eta\mathbb{HQ}^{k\times k} \Longleftrightarrow A_3^T=-A_3, A_s^T=A_s, s=1,2,4.$
(3)*For* $\eta=\mathbf{k}$, $A \in \eta\mathbb{HQ}^{k\times k} \Longleftrightarrow A_4^T=-A_4, A_s^T=A_s, s=1,2,3.$

Similarly, we have

(4)*For* $\eta=\mathbf{i}$, $A \in \eta\mathbb{AQ}^{k\times k} \Longleftrightarrow A_2^T=A_2, A_s^T=-A_s, s=1,3,4.$
(5)*For* $\eta=\mathbf{j}$, $A \in \eta\mathbb{AQ}^{k\times k} \Longleftrightarrow A_3^T=A_3, A_s^T=-A_s, s=1,2,4.$
(6)*For* $\eta=\mathbf{k}$, $A \in \eta\mathbb{AQ}^{k\times k} \Longleftrightarrow A_4^T=A_4, A_s^T=-A_s, s=1,2,3.$

Definition 2.3. *[27] Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times q}$, the semi-tensor product of A and B is denoted by*

$$A \ltimes B=(A \otimes I_{t/n})(B \otimes I_{t/p}),$$

where $t=lcm(n,p)$ is the least common multiple of n and p.

If $n=p$, the semi-tensor product of matrices reduces to the conventional matrix product.

Theorem 2.1. *[27]Assume that A, B, C are real matrix with appropriate sizes , $a,b \in \mathbb{R}$, then*
(1) (Distributive law)

$$A \ltimes (aB \pm bC)=aA \ltimes B \pm bA \ltimes C,$$

$$(aA \pm bB) \ltimes C=aA \ltimes C \pm bB \ltimes C.$$

(2) (Associative law)

$$(A \ltimes B) \ltimes C = A \ltimes (B \ltimes C).$$

(3) Assume that $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, then

$$x \ltimes y = x \otimes y.$$

The semi-tensor product of a matrix and a vector has the following properties of quasi-commutativity.

Theorem 2.2. *[27] Let $x \in \mathbb{R}^t$, $A \in \mathbb{R}^{m\times n}$, then*

$$x \ltimes A = (I_t \otimes A) \ltimes x.$$

Definition 2.4. *[28] Let $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, then*

$$W_{[m,n]}(x \ltimes y) = y \ltimes x,$$

in which

$$W_{[m,n]} = \delta_{mn}[1, \cdots, (n-1)m+1, \cdots, m, \cdots, nm],$$

where $\delta_k[i_1, \cdots, i_s]$ is an abbreviation of $[\delta_k^{i_1}, \cdots, \delta_k^{i_s}]$. Especially, when $m = n$, we denote $W_{[n]} := W_{[n,n]}$.

The following results are the well-known conclusions of matrix equations.

Theorem 2.3. *[29] The least squares solutions of the matrix equation $Ax = b$, with $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, can be represented as*

$$x = A^\dagger b + (I - A^\dagger A)y,$$

where $y \in \mathbb{R}^n$ is an arbitrary vector. The minimal norm least squares solution of the matrix equation $Ax = b$ is $A^\dagger b$.

Theorem 2.4. *[29] The matrix equation $Ax = b$, with $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, has a solution $x \in \mathbb{R}^n$ if and only if*

$$AA^\dagger b = b.$$

In this case it has the general solution

$$x = A^\dagger b + (I - A^\dagger A)y,$$

where $y \in \mathbb{R}^n$ is an arbitrary vector.

3. A new kind of real vector representation of a quaternion matrix and its properties

In this section, we will propose the concept of real vector representation of a quaternion matrix and study its properties. First we define real staking form of $x \in \mathbb{Q}$.

Definition 3.1. *Let $x = x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k} \in \mathbb{Q}$, denote*

$$v^R(x) = (x_1, x_2, x_3, x_4)^T,$$

$v^R(x)$ is called as the real staking form of x.

By means of structure matrix and the real stacking form, we can express the product of two quaternions by the semi-tensor product of matrices.

Theorem 3.1. *Let $x, y \in \mathbb{Q}$, then*

$$v^R(xy) = M_Q \ltimes v^R(x) \ltimes v^R(y), \tag{3.1}$$

where

$$M_Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

is the structure matrix of multiplication of quaternion.

Combining the real stacking form of a quaternion with vec operator of a real matrix, we propose a new kind of real vector representation of a quaternion matrix. For this purpose, we first propose the real stacking form of a quaternion vector as follows.

Definition 3.2. *Let $x = (x^1, \cdots, x^n)$, $y = (y^1, \cdots, y^n)^T$ be quaternion vectors. Denote*

$$v^R(x) = \begin{pmatrix} v^R(x^1) \\ \vdots \\ v^R(x^n) \end{pmatrix}, \quad v^R(y) = \begin{pmatrix} v^R(y^1) \\ \vdots \\ v^R(y^n) \end{pmatrix}$$

$v^R(x)$ and $v^R(y)$ are called as the real staking form of quaternion vector x and y, respectively.

Now we define the concepts of the real column stacking form and the real row stacking form of a quaternion matrix A.

Definition 3.3. *For $A \in \mathbb{Q}^{m\times n}$, denote*

$$v_c^R(A) = \begin{pmatrix} v^R(Col_1(A)) \\ v^R(Col_2(A)) \\ \vdots \\ v^R(Col_n(A)) \end{pmatrix}, \quad v_r^R(A) = \begin{pmatrix} v^R(Row_1(A)) \\ v^R(Row_2(A)) \\ \vdots \\ v^R(Row_m(A)) \end{pmatrix},$$

$v_c^R(A)$ and $v_r^R(A)$ are called the real column stacking form and the real row stacking form of A, respectively.

We can prove that this real vector representation has the following properties with respect to vector or matrix operations.

Theorem 3.2. *Let $x = (x^1, x^2, \cdots, x^n)$, $\check{x} = (\check{x}^1, \check{x}^2, \cdots, \check{x}^n)$, $y = (y^1, y^2, \cdots, y^n)^T$, x^i, $\check{x}^i$ $y^i \in \mathbb{Q}$, $a \in \mathbb{R}$, then*

(1) $v^R(x + \check{x}) = v^R(x) + v^R(\check{x})$,

(2) $v^R(ax) = av^R(x)$,

(3) $v^R(xy) = M_Q \ltimes \left(\sum_{i=1}^{n} (\delta_n^i)^T \ltimes (I_{4n} \otimes (\delta_n^i)^T)\right) \ltimes v^R(x) \ltimes v^R(y)$.

Proof. By simply computing, we know (1), (2) hold. We only give a detailed proof of (3). Using (3.1), we have

$$\begin{aligned} v^R(xy) =& v^R(x^1y^1 + \cdots + x^ny^n) \\ =& M_Q \ltimes v^R(x^1) \ltimes v^R(y^1) + \cdots + M_Q \ltimes v^R(x^n) \ltimes v^R(y^n) \\ =& M_Q \ltimes \left(\sum_{i=1}^{n} v^R(x^i) \ltimes v^R(y^i)\right) \\ =& M_Q \ltimes \left(\sum_{i=1}^{n} (\delta_n^i)^T \ltimes v^R(x) \ltimes (\delta_n^i)^T \ltimes v^R(y)\right) \\ =& M_Q \ltimes \left(\sum_{i=1}^{n} (\delta_n^i)^T \ltimes (I_{4n} \otimes (\delta_n^i)^T)\right) \ltimes v^R(x) \ltimes v^R(y). \end{aligned}$$

By using Theorem 3.2, we can drive the following result on the real vector representation of multiplication of two quaternion matrices.

Theorem 3.3. *Let A, $\check{A} \in \mathbb{Q}^{m\times n}$, $B \in \mathbb{Q}^{n\times p}$, $\alpha \in \mathbb{R}$, then*

(1) $v_r^R(A + \check{A}) = v_r^R(A) + v_r^R(\check{A})$, $v_c^R(A + \check{A}) = v_c^R(A) + v_c^R(\check{A})$,

(2) $\|A\| = \|v_r^R(A)\| = \|v_c^R(A)\|$,

(3) $v_r^R(AB) = G(v_r^R(A) \ltimes v_c^R(B))$,

in which

$$F = M_Q \ltimes \left(\sum_{i=1}^{n} (\delta_n^i)^T \ltimes (I_{4n} \otimes (\delta_n^i)^T)\right),$$

$$G = \begin{pmatrix} F\ltimes(\delta_m^1)^T \ltimes [I_{4mn}\otimes(\delta_p^1)^T] \\ \vdots \\ F\ltimes(\delta_m^1)^T \ltimes [I_{4mn}\otimes(\delta_p^p)^T] \\ \vdots \\ F\ltimes(\delta_m^m)^T \ltimes [I_{4mn}\otimes(\delta_p^1)^T] \\ \vdots \\ F\ltimes(\delta_m^m)^T \ltimes [I_{4mn}\otimes(\delta_p^p)^T] \end{pmatrix}.$$

Proof. We only prove the equality in (3). We partition A and B with its rows or columns as follows ,

$$A = \begin{pmatrix} Row_1(A) \\ Row_2(A) \\ \vdots \\ Row_m(A) \end{pmatrix}, \ B = \left(Col_1(B)\ Col_2(B)\ \cdots, Col_p(B)\right).$$

Then we have

$$\begin{aligned} v_r^R(AB) \ &= \begin{pmatrix} v^R(Row_1(A)Col_1(B)) \\ \vdots \\ v^R(Row_1(A)Col_p(B)) \\ \vdots \\ v^R(Row_m(A)Col_1(B)) \\ \vdots \\ v^R(Row_m(A)Col_p(B)) \end{pmatrix} = \begin{pmatrix} F\ltimes v^R(Row_1(A))\ltimes v^R(Col_1(B)) \\ \vdots \\ F\ltimes v^R(Row_1(A))\ltimes v^R(Col_p(B)) \\ \vdots \\ F\ltimes v^R(Row_m(A))\ltimes v^R(Col_1(B)) \\ \vdots \\ F\ltimes v^R(Row_m(A))\ltimes v^R(Col_p(B)) \end{pmatrix} \\ &= \begin{pmatrix} F\ltimes[(\delta_m^1)^T\ltimes v_r^R(A)]\ltimes[(\delta_p^1)^T\ltimes v_c^R(B)] \\ \vdots \\ F\ltimes[(\delta_m^1)^T\ltimes v_r^R(A)]\ltimes[(\delta_p^p)^T\ltimes v_c^R(B)] \\ \vdots \\ F\ltimes[(\delta_m^m)^T\ltimes v_r^R(A)]\ltimes[(\delta_p^1)^T\ltimes v_c^R(B)] \\ \vdots \\ F\ltimes[(\delta_m^m)^T\ltimes v_r^R(A)]\ltimes[(\delta_p^p)^T\ltimes v_c^R(B)] \end{pmatrix} \\ &= \begin{pmatrix} F\ltimes(\delta_m^1)^T\ltimes[I_{4mn}\otimes(\delta_p^1)^T] \\ \vdots \\ F\ltimes(\delta_m^1)^T\ltimes[I_{4mn}\otimes(\delta_p^p)^T] \\ \vdots \\ F\ltimes(\delta_m^m)^T\ltimes[I_{4mn}\otimes(\delta_p^1)^T] \\ \vdots \\ F\ltimes(\delta_m^m)^T\ltimes[I_{4mn}\otimes(\delta_p^p)^T] \end{pmatrix} (v_r^R(A) \ltimes v_c^R(B)). \end{aligned}$$

4. The solutions of Problem 1

In this section, we study the solutions of Problem 1. First, Through the structural characteristics of η-Hermitian matrix and anti-η-Hermitian matrix, we can find a large number of repeated elements in the matrices. In order to reduce the calculation order of quaternion matrix equation (1.1), we can extract some elements as independent elements, and express the whole matrix by independent elements. The specific contents are as follows.

Theorem 4.1. *Let $X \in \eta\mathbb{HQ}^{n\times n}$ $\eta = \mathbf{i}, \mathbf{j}, \mathbf{k}$, denote*

$$LX_i = \begin{pmatrix} x_{ii} \\ x_{i(i+1)} \\ \vdots \\ x_{in} \end{pmatrix}, (i = 1, 2, \cdots, n), \quad v_s^R(X) = \begin{pmatrix} v^R(LX_1) \\ v^R(LX_2)) \\ \vdots \\ v^R(LX_n) \end{pmatrix}.$$

Then

$$v_c^R(X) = J^\eta v_s^R(X),$$

where

$$J^\eta = \begin{pmatrix} J_1^\eta \\ \vdots \\ J_m^\eta \\ \vdots \\ J_n^\eta \end{pmatrix} \text{ and } J_m^\eta = \begin{pmatrix} J_{1m}^\eta \\ \vdots \\ J_{rm}^\eta \\ \vdots \\ J_{nm}^\eta \end{pmatrix} \quad m = 1, 2, \cdots, n,$$

when $\eta = \mathbf{i}$, $J_m^{\mathbf{i}}$ is as follows

$$J_{rm}^{\mathbf{i}} = \begin{cases} \left(\delta_{n(n+1)/2}^{\frac{(r-1)(2n-r+2)}{2}+m-r+1}\right)^T \otimes R_4 & r < m \\ \left(\delta_{n(n+1)/2}^{\frac{(m-1)(2n-m+2)}{2}+r-m+1}\right)^T \otimes I_4 & r \geq m \end{cases}, \ R_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Similarly we have $J_m^{\mathbf{j}}$, $J_m^{\mathbf{k}}$.

$$J_{rm}^{\mathbf{j}} = \begin{cases} \left(\delta_{n(n+1)/2}^{\frac{(r-1)(2n-r+2)}{2}+m-r+1}\right)^T \otimes L_4 & r < m \\ \left(\delta_{n(n+1)/2}^{\frac{(m-1)(2n-m+2)}{2}+r-m+1}\right)^T \otimes I_4 & r \geq m \end{cases}, \ L_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$J_{rm}^{\mathbf{k}} = \begin{cases} \left(\delta_{n(n+1)/2}^{\frac{(r-1)(2n-r+2)}{2}+m-r+1}\right)^T \otimes S_4 & r < m \\ \left(\delta_{n(n+1)/2}^{\frac{(m-1)(2n-m+2)}{2}+r-m+1}\right)^T \otimes I_4 & r \geq m \end{cases}, \ S_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

We can also find the relationship of $v_c^R(X)$ and $v_s^R(X)$ for $\eta - anti - Hermitian$ matrix.

Theorem 4.2. *Let $X \in \eta\mathbb{AQ}^{n\times n}$ $\eta = \mathbf{i}, \mathbf{j}, \mathbf{k}$, $v_s^R(X)$ is defined in Theorem 4.1 Then*

$$v_c^R(X) = R^\eta v_s^R(X),$$

where

$$R^\eta = \begin{pmatrix} R_1^\eta \\ \vdots \\ R_m^\eta \\ \vdots \\ R_n^\eta \end{pmatrix} \text{ and } R_m^\eta = \begin{pmatrix} R_{1m}^\eta \\ \vdots \\ R_{rm}^\eta \\ \vdots \\ R_{nm}^\eta \end{pmatrix} \quad m = 1, 2, \cdots n,$$

when $\eta = \mathbf{i}$, $R^{\mathbf{i}}$ is as follows

$$R_{rm}^{\mathbf{i}} = \begin{cases} \left(\delta_{n(n+1)/2}^{\frac{(r-1)(2n-r+2)}{2}+m-r+1}\right)^T \otimes R_4' & r < m \\ \left(\delta_{n(n+1)/2}^{\frac{(m-1)(2n-m+2)}{2}+r-m+1}\right)^T \otimes I_4 & r \geq m \end{cases}, \ R_4' = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Similarly we have $R^{\mathbf{j}}$, $R^{\mathbf{k}}$.

$$R_{rm}^{\mathbf{j}} = \begin{cases} \left(\delta_{n(n+1)/2}^{\frac{(r-1)(2n-r+2)}{2}+m-r+1}\right)^T \otimes L_4' & r < m \\ \left(\delta_{n(n+1)/2}^{\frac{(m-1)(2n-m+2)}{2}+r-m+1}\right)^T \otimes I_4 & r \geq m \end{cases}, \ L_4' = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$R_{rm}^{\mathbf{k}} = \begin{cases} \left(\delta_{n(n+1)/2}^{\frac{(r-1)(2n-r+2)}{2}+m-r+1}\right)^T \otimes S_4' & r < m \\ \left(\delta_{n(n+1)/2}^{\frac{(m-1)(2n-m+2)}{2}+r-m+1}\right)^T \otimes I_4 & r \geq m \end{cases}, \ S_4' = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Based on the above discussion, we give the solution of problem 1 by feat of the real vector representation of quaternion matrix and STP.

Theorem 4.3. *Let $A \in \mathbb{Q}^{m\times n}$, $B \in \mathbb{Q}^{n\times s}$, $C \in \mathbb{Q}^{m\times k}$, $D \in \mathbb{Q}^{k\times s}$, $E \in \mathbb{Q}^{m\times s}$, G_i has the same structure as G except for the dimension, denote*

$$M_1 = G_2 \ltimes G_3 \ltimes v_r^R(A) \ltimes W_{[4ns,4n^2]} \ltimes v_c^R(B) \ltimes J^\eta,$$

$$M_2 = G_4 \ltimes G_5 \ltimes v_r^R(C) \ltimes W_{[4ks,4k^2]} \ltimes v_c^R(D) \ltimes R^\eta,$$

$$\hat{M} = (M_1, \ M_2).$$

Then the set S_M of Problem 1 is represented as

$$S_M = \left\{ (X,Y) \middle| \begin{pmatrix} v_s^R(X) \\ v_s^R(Y) \end{pmatrix} = \hat{M}^\dagger v_r^R(E) + \left(I_{2(n^2+k^2)+2(n+k)} - \hat{M}^\dagger \hat{M}\right) y \right\} \quad (4.1)$$

where, $y \in \mathbb{R}^{2(n^2+k^2)+2(n+k)}$. And then, the minimal norm least squares mixed solution $(\hat{X}, \hat{Y})$ of (1.1) *satisfies*

$$\begin{pmatrix} v_s^R(\hat{X}) \\ v_s^R(\hat{Y}) \end{pmatrix} = \hat{M}^\dagger v_r^R(E). \quad (4.2)$$

Proof.

$$\begin{aligned}
&\|AXB + CYD - E\| \\
&= \left\|v_r^R\left(AXB + CYD\right) - v_r^R(E)\right\| \\
&= \left\|M_1 \ltimes v_s^R(X) + M_2 \ltimes v_s^R(Y) - v_r^R(E)\right\| \\
&= \left\|(M_1,\ M_2)\begin{pmatrix} v_s^R(X) \\ v_s^R(Y) \end{pmatrix} - v_r^R(E)\right\| \\
&= \left\|\hat{M}\begin{pmatrix} v_s^R(X) \\ v_s^R(Y) \end{pmatrix} - v_r^R(E)\right\|.
\end{aligned}$$

Thus

$$\|AXB + CYD - E\| = min$$

if and only if

$$\left\|\hat{M}\begin{pmatrix} v_s^R(X) \\ v_s^R(Y) \end{pmatrix} - v_r^R(E)\right\| = min.$$

For the real matrix equation

$$\hat{M}\begin{pmatrix} v_s^R(X) \\ v_s^R(Y) \end{pmatrix} = v_r^R(E).$$

According to Theorem 2.3, its least squares solutions can be represented as

$$\begin{pmatrix} v_s^R(X) \\ v_s^R(Y) \end{pmatrix} = \hat{M}^\dagger v_r^R(E) + (I_{2(n^2+k^2)+2(n+k)} - \hat{M}^\dagger \hat{M})y,$$

where $y \in \mathbb{R}^{2(n^2+k^2)+2(n+k)}$. Thus we get the formula in (4.1).

Notice

$$\min_{(X,Y)\in S_M} \|X\|^2 + \|Y\|^2 \Longleftrightarrow \min_{(X,Y)\in S_M} \left\|\begin{pmatrix} v_s^R(X) \\ v_s^R(Y) \end{pmatrix}\right\|^2,$$

so we have that the minimal norm least squares mixed solution $(\hat{X}, \hat{Y})$ of (1.1) satisfies

$$\begin{pmatrix} v_s^R(\hat{X}) \\ v_s^R(\hat{Y}) \end{pmatrix} = \hat{M}^\dagger v_r^R(E).$$

Therefore, (4.2) holds.

Corollary 4.4. *Let $A \in \mathbb{Q}^{m\times n}$, $B \in \mathbb{Q}^{n\times s}$, $C \in \mathbb{Q}^{m\times k}$, $D \in \mathbb{Q}^{k\times s}$, $\hat{M}$ is defined in Theorem 4.3. Then $AXB + CYD = E$ has a mixed solution (X, Y) if and only if*

$$\left(\hat{M}\hat{M}^\dagger - I_{4ms}\right) v_r^R(E) = 0. \tag{4.3}$$

Moreover, if (4.3) *holds, the mixed solution set of $AXB + CYD = E$ can be represented as*

$$\widetilde{S}_M = \left\{(X,Y) \,\middle|\, \begin{pmatrix} v_s^R(X) \\ v_s^R(Y) \end{pmatrix} = \hat{M}^\dagger v_r^R(E) + (I_{2(n^2+k^2)+2(n+k)} - \hat{M}^\dagger \hat{M})y\right\} \tag{4.4}$$

where $y \in \mathbb{R}^{2(n^2+k^2)+2(n+k)}$. We can obtain the minimal norm mixed solution $(\hat{X}, \hat{Y})$ satisfying

$$\begin{pmatrix} v_s^R(\hat{X}) \\ v_s^R(\hat{Y}) \end{pmatrix} = \hat{M}^\dagger v_r^R(E). \tag{4.5}$$

Proof. $AXB + CYD = E$ has a mixed solution (X, Y) if and only if

$$\|AXB + CYD - E\| = 0.$$

Using (2) in Theorem 3.3 and the properties of the MP inverse, we get

$$\begin{aligned}
&\|AXB + CYD - E\| \\
&= \left\|\hat{M}\begin{pmatrix} v_s^R(X) \\ v_s^R(Y) \end{pmatrix} - v_r^R(E)\right\| \\
&= \left\|\hat{M}\hat{M}^\dagger\hat{M}\begin{pmatrix} v_s^R(X) \\ v_s^R(Y) \end{pmatrix} - v_r^R(E)\right\| \\
&= \left\|\hat{M}\hat{M}^\dagger v_r^R(E) - v_r^R(E)\right\| \\
&= \left\|(\hat{M}\hat{M}^\dagger - I_{4ms})v_r^R(E)\right\|.
\end{aligned}$$

Therefore, for (X, Y), we obtain

$$\begin{aligned}
&\|AXB + CYD - E\| = 0 \\
\Longleftrightarrow\ &\left\|(\hat{M}\hat{M}^\dagger - I_{4ms})v_r^R(E)\right\| = 0 \\
\Longleftrightarrow\ &(\hat{M}\hat{M}^\dagger - I_{4ms})v_r^R(E) = 0.
\end{aligned}$$

When $AXB + CYD = E$ is compatible, its mixed solution $[X, Y] \in \widetilde{S}_M$ satisfies

$$\hat{M}\begin{pmatrix} v_s^R(X) \\ v_s^R(Y) \end{pmatrix} = v_r^R(E).$$

Moreover, according to Theorem 2.4, the mixed solution $[X, Y]$ satisfies

$$\begin{pmatrix} v_s^R(X) \\ v_s^R(Y) \end{pmatrix} = \hat{M}^\dagger v_r^R(E) + (I_{2(n^2+k^2)+2(n+k)} - \hat{M}^\dagger \hat{M})y,$$

where $y \in \mathbb{R}^{2(n^2+k^2)+2(n+k)}$ and the minimal norm mixed $(\hat{X}, \hat{Y})$, satisfies

$$\begin{pmatrix} v_s^R(\hat{X}) \\ v_s^R(\hat{Y}) \end{pmatrix} = \hat{M}^\dagger v_r^R(E).$$

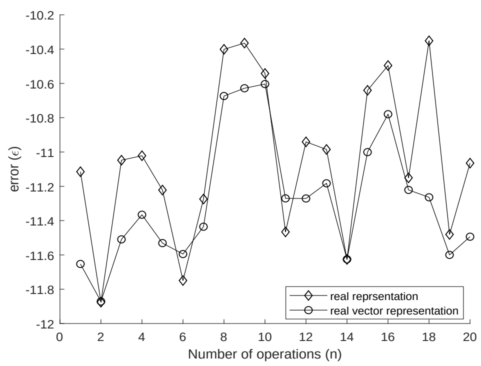
So, we can get the formula in (4.4), (4.5).

5. Algorithm and numerical experiment

In this section, using the results in Section 4, we propose the algorithm of solving Problem 1.

Algorithm 5.1. *(Problem 1)*
(1) Input $A, B, C, D, E, \in \mathbb{Q}^{n\times n}, (i = 1 : 4)$, *output* $v_r^R(A), v_r^R(C), v_c^R(B), v_c^R(D), v_r^R(E)$,
(2) Input $G, W_{[}m, n], J^\eta, R^\eta$ *output the matrix* $\hat{M}$,
(3) According to (4.2), *output the minimal norm least squares mixed solution* $(\hat{X}, \hat{Y})$ *of* (1.1).

Example 5.1. *Consider the quaternion matrix equation* $AXB + CYD = E$. *Using the 'rand' and 'quaternion' in Matlab, the quaternion matrix* A, B, C, D *are created. Suppose* $X \in \eta\mathbb{HQ}^{n\times n}$, $Y \in \eta\mathbb{AHQ}^{k\times k}$, $\eta = \mathbf{i}$. *Let* $m = n = k = s = 8$, *and randomly generate 20 groups of matrices* A, B, C, D, X, Y. *Compute quaternion matrix equation 1.1. we get a solution* (X_T, Y_T) *of Problem 1 by Algorithm 5.1 and the method in [30], respectively. and the error* $\varepsilon = log_{10}([X_T, Y_T] - [X, Y])$ *is shown in the Figure below.*



$\eta = i$

Here, two methods are used for comparing the **i**-Hermitian and **i**-anti-Hermitian mixed solutions with the real solutions. It can be seen that the real vector representation method based on the semi tensor product of matrix has more times than the real representation method. A large number of numerical experiments show that the real vector representation method has a dominant probability of more than 50% when calculating the same quaternion matrix equation (1.1).

Remark 5.1. *(i) There are many kinds of mixed solutions. In Example 5.1, only the* **i***-Hermitian and* **i***-anti-Hermitian cases are studied.*
(ii) Because the comparison with the real representation method in [30], In order to ensure the number of effective elements calculated is the same, the $J^{\mathbf{i}}$ *and* $R^{\mathbf{i}}$ *which are used to find rules are changed before.*

6. Conclusions

In this paper, we proposed a real vector representation of quaternion matrix and combined this real vector representation with semi-tensor product of matrices. We solved the least squares problems as in Problem 1. It is not hard to find that with the help of this real vector representation and semi-tensor product of matrices, we can transform the problems of solving matrices with some special structure on quaternion skew-field into the corresponding problems on real number field. It is very helpful for us to solve the quaternion matrix equation.

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Conflict of interest

The authors declare they have no conflicts of interest to this work.

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