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Research article

Random Caputo-Fabrizio fractional differential inclusions

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Abstract: This paper deals with some existence and Ulam stability results for Caputo-Fabrizio type fractional differential inclusions with convex and non-convex right hand side. We employ some multi-valued random fixed point theorems and the notion of the generalized Ulam-Hyers-Rassias stability. Next we present two examples in the last section.

Keywords: random functional differential inclusion; Caputo-Fabrizio fractional derivative; convex; non-convex; existence; Ulam stability; random solution; fixed point

1. Introduction

Fractional order differential equations have recently been applied in various areas of engineering, mathematics, physics and bio-engineering, and other applied sciences [32]. For some fundamental results in the theory of fractional calculus and fractional differential equations we refer the reader to the monographs [2, 4, 5, 20, 34, 35], and the references therein.

The stability of functional equations was originally raised by Ulam [33]) and then followed by Hyers [17]. In 1978, Rassias [25] provided a remarkable generalization of the Ulam-Hyers stability of mappings by considering variables. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Considerable attention has been given to the study of the Ulam-Hyers and Ulam-Hyers-Rassias stability for all kinds of functional equations; one can see the monographs of [2, 5, 18], and the papers [7, 8, 24, 26, 27] discussed the Ulam-Hyers stability for operatorial equations and inclusions. More details from historical points of view and recent developments of such stabilities are reported in [19, 26].

Recently, considerable attention has been given to the existence of solutions of initial and boundary value problems for fractional differential equations with Caputo-Fabrizio derivative; see [1, 3, 6, 9–11, 14, 15, 21, 22, 28–31].

Motivated by the above papers, in this article we discuss the existence and the Ulam stability of solutions for the following problem of Random Caputo-Fabrizio fractional differential inclusions

$$\begin{cases} \left({}^{CF}D_0^r u\right)(t,w) \in F(t,u(t,w),w),\ t \in I := [0,T],\ w \in \Omega, \\ u(t,w)|_{t=0} = \phi(w), \end{cases} \tag{1.1}$$

where $T > 0$, ${}^{CF}D_0^r$ is the Caputo-Fabrizio fractional derivative of order $r \in (0,1)$, $(\Omega, \mathcal{A})$ is a measurable space (that is, Ω is a set with a σ-algebra $\mathcal{A}$ of subsets of Ω called the measurable sets), $\phi : \Omega \to \mathbb{R}$ is a measurable, bounded function, $F : I \times \mathbb{R} \times \Omega \to \mathcal{P}(\mathbb{R})$ is a given multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

2. Preliminaries

Let C be the Banach space of all continuous functions v from I into $\mathbb{R}$ with the supremum (uniform) norm

$$\|v\|_C := \sup_{t\in I} |v(t)|.$$

As usual, $L^1(I)$ denotes the space of measurable functions $v : I \to \mathbb{R}$ which are Lebesgue integrable with the norm

$$\|v\|_1 = \int_I |v(t)|dt.$$

Let $L^\infty(I)$ be the Banach space of measurable functions $u : I \to \mathbb{R}$ which are essentially bounded, equipped with the norm

$$\|u\|_{L^\infty} = \inf\{c > 0 : |u(t)| \le c, \ a.e.\ t \in I\}.$$

Definition 2.1. *Let $P(Y)$ be the family of all nonempty subsets of $Y \subset \mathbb{R}$ and C be a mapping from Ω into $P(Y)$. A mapping $T : \{(w,u) : w \subset \Omega, y \subset C(w)\} \to Y$ is called a random operator with stochastic domain C if C is measurable (i.e for all closed $A \subset Y$, $\{w \subset \Omega, C(w) \cap A \neq \emptyset\}$ is measurable) and for all open $D \subset Y$ and all $u \subset Y, \{w \subset \Omega : u \subset C(w), T(w,u) \subset D\}$ is measurable. T will be called continuous if every $T(w)$ is continuous. For a random operator T, a mapping $u : \Omega \to Y$ is called a random (stochastic) fixed point of T if for P-almost all $w \subset \Omega, u(w) \subset C(w)$ and $T(w)u(w) = u(w)$ and for all open $D \subset Y, \{w \subset \Omega : u(w) \subset D\}$ is measurable.*

For each $u \in C$ and $w \in \Omega$, define the set of selections of F by

$$S_{F\circ u}(w) = \{v : \Omega \to L^1(I) : v(t,w) \in F(t, u(t,w), w);\ t \in I\}.$$

Let $(E, \|\cdot\|)$ be a Banach space, and denote $P_{cl}(E) = \{A \in \mathcal{P}(E) : A \text{ closed}\}$, $P_{bd}(E) = \{A \in \mathcal{P}(E) : A \text{ bounded}\}$, $P_{cp,c}(E) = \{A \in \mathcal{P}(E) : A \text{ compact and convex}\}$.

Consider $H_d : \mathcal{P}(E) \times \mathcal{P}(E) \longrightarrow [0,\infty) \cup \{\infty\}$ given by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\},$$

where $d(A,b) = \inf_{a\in A} d(a,b)$, $d(a,B) = \inf_{b\in B} d(a,b)$. Then $(\mathcal{P}_{bd,cl}(E), H_d)$ is a Hausdorff metric space.

Definition 2.2. *A multifunction $F : \Omega \to E$ is called $\mathcal{A}$-measurable if, for any open subset B of E, the set $F^{-1}(B) = \{w \in \Omega : F(w) \cap B \neq \emptyset\} \in \mathcal{A}$. Note that if $F(w) \in \mathcal{P}_{cl}(E)$ for all $w \in \Omega$, then F is measurable if and only if $F^{-1}(D) \in \mathcal{A}$ for all $D \in \mathcal{P}_{cl}(E)$. A measurable operator $u : \Omega \to E$ is called a measurable selector for a measurable multifunction $F : \Omega \to E$, if $u(w) \in F(w)$. Let $M \in \mathcal{P}_{cl}(E)$, then a mapping $f : \Omega \times M \to E$ is called a random operator if, for each $u \in M$, the mapping $f(.,u) : \Omega \to E$ is measurable. An operator $u : \Omega \to E$ is said to be a random fixed point of F if u is measurable and $u(w) \in F(w, u(w))$ for all $w \in \Omega$.*

Definition 2.3. *A multifunction $F : \Omega\times E \to \mathcal{P}(E)$ is said to be Carathéodory, if $F(\cdot, u)$ is measurable for all $u \in E$ and $F(w,\cdot)$ is continuous for all $w \in \Omega$.*

Definition 2.4. *A multivalued map $F : I \times E \times \Omega \to \mathcal{P}_{cp}(E)$ is said to be random Carathéodory if*

(i) $(t,w) \longmapsto F(t,u,w)$ *is jointly measurable for each* $u \in E$; *and*

(ii) $u \longmapsto F(t,u,w)$ *is Hausdorff continuous for almost all* $t \in I$, $w \in \Omega$.

Definition 2.5. *[16] Let E be a Banach space. If $F : I \times E \to \mathcal{P}_{cp}(E)$ is Carathéodory, then the multivalued mapping, $(t, u(t)) \to F(t, u(t))$, is jointly measurable for any measurable E-valued function u on I.*

Definition 2.6. *A multivalued random operator $N : \Omega\times E \to \mathcal{P}_{cl}(E)$ is called multivalued random contraction if there is a measurable function $k : \Omega \to [0,\infty)$ such that*

$$H_d(N(w)u, N(w)v) \le k(w)\|u - v\|_E,$$

for all $u, v \in E$ and $w \in \Omega$, where $k(w) \in [0,1)$ on Ω.

Let us recall some definitions and properties of Caputo-Fabrizio fractional operators.

Definition 2.7. *[11, 22] The Caputo-Fabrizio fractional integral of order $0 < r < 1$ for a function $w \in L^1(I)$ is defined, for $\tau \ge 0$, by*

$$^{CF}I^r w(\tau) = \frac{2(1-r)}{M(r)(2-r)} w(\tau) + \frac{2r}{M(r)(2-r)} \int_0^\tau w(x)dx.$$

where $M(r)$ is a normalization constant depending on r.

Definition 2.8. *[11, 22] The Caputo-Fabrizio fractional derivative for a function $w \in C^1(I)$ of order $0 < r < 1$, is defined, for $\tau \in I$, by*

$$^{CF}D^r w(\tau) = \frac{(2-r)M(r)}{2(1-r)} \int_0^\tau \exp\left(-\frac{r}{1-r}(\tau - x)\right) w'(x)dx.$$

Note that $(^{CF}D^r)(w) = 0$ if and only if w is a constant function.

Example 2.9. *[11]*
1- For $h(t) = t$ and $0 < r \leq 1$, we have

$$(^{CF}D^r h)(t) = \frac{M(r)}{r}\left(1 - \exp\left(-\frac{r}{1-r}t\right)\right).$$

2- For $g(t) = e^{\lambda t}$, $\lambda \geq 0$ and $0 < r \leq 1$, we have

$$(^{CF}D^r g)(t) = \frac{\lambda M(r)}{r + \lambda(1-r)} e^{\lambda t}\left(1 - \exp\left(-\lambda - \frac{r}{1-r}t\right)\right).$$

Lemma 2.10. *[10] A function u is a random solution of problem (1.1) if and only if u satisfies the following integral equation*

$$u(t,w) = C(w) + a_r v(t,w) + b_r \int_0^t v(s,w)ds \tag{2.1}$$

where $v \in S_{F\circ u}(w)$, and

$$C(w) = \phi(w) - a_r v(0,w).$$

Now, we consider the Ulam stability for the problem (1.1). Let $\epsilon > 0$ and $\Phi : I \times \Omega \to [0,\infty)$ be a continuous function. We consider the following inequalities

$$H_d((^{CF}D_0^r u)(t,w), F(t,u(t,w),w)) \leq \epsilon;\ t \in I,\ w \in \Omega. \tag{2.2}$$

$$H_d((^{CF}D_0^r u)(t,w), F(t,u(t,w),w)) \leq \Phi(t,w);\ t \in I,\ w \in \Omega. \tag{2.3}$$

$$H_d((^{CF}D_0^r u)(t,w), F(t,u(t,w),w)) \leq \epsilon\Phi(t,w);\ t \in I,\ w \in \Omega. \tag{2.4}$$

Definition 2.11. [4, 26] *The problem* (1.1) *is Ulam-Hyers stable if there exists a real number $c_F > 0$ such that for each $\epsilon > 0$ and for each random solution $u : \Omega \to C_\gamma$ of the inequality* (2.2) *there exists a random solution $v : \Omega \to C_\gamma$ of* (1.1) *with*

$$|u(t,w) - v(t,w)| \leq \epsilon c_F;\ t \in I,\ w \in \Omega.$$

Definition 2.12. [4, 26] *The problem* (1.1) *is generalized Ulam-Hyers stable if there exists $c_F \in C([0,\infty),[0,\infty))$ with $c_F(0) = 0$ such that for each $\epsilon > 0$ and for each random solution $u : \Omega \to C_\gamma$ of the inequality* (2.2) *there exists a random solution $v : \Omega \to C_\gamma$ of* (1.1) *with*

$$|u(t,w) - v(t,w)| \leq c_F(\epsilon);\ t \in I,\ w \in \Omega.$$

Definition 2.13. [4, 26] *The problem* (1.1) *is Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $c_{F,\Phi} > 0$ such that for each $\epsilon > 0$ and for each random solution $u : \Omega \to C_\gamma$ of the inequality* (2.4) *there exists a random solution $v : \Omega \to C_\gamma$ of* (1.1) *with*

$$|u(t,w) - v(t,w)| \leq \epsilon c_{F,\Phi}\Phi(t,w);\ t \in I,\ w \in \Omega.$$

Definition 2.14. [4, 26] *The problem* (1.1) *is generalized Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $c_{F,\Phi} > 0$ such that for each random solution $u : \Omega \to C_\gamma$ of the inequality* (2.3)*, there exists a random solution $v : \Omega \to C_\gamma$ of* (1.1) *with*

$$|u(t,w) - v(t,w)| \leq c_{F,\Phi}\Phi(t,w);\ t \in I,\ w \in \Omega.$$

Remark 2.15. *It is clear that*

(i) Definition 2.11 $\Rightarrow$ Definition 2.12,
(ii) Definition 2.13 $\Rightarrow$ Definition 2.14,
(iii) Definition 2.13 for $\Phi(\cdot,\cdot) = 1$ $\Rightarrow$ Definition 2.11.

One can have similar remarks for the inequalities (2.2) and (2.4).

In the sequel, we need the following random multi-valued fixed point theorems.

Theorem 2.16. *[13] Let $(\Omega, \mathcal{A})$ be a complete σ-finite measure space, X be a separable Banach space, $\mathcal{M}(\Omega, X)$ be the space of all measurable X-valued functions defined on Ω, and let $N : \Omega \times X \to \mathcal{P}_{cp,cv}(X)$ be a continuous and condensing multi-valued random operator. If the set $\{u \in \mathcal{M}(\Omega, X) : \lambda u \in N(w)u\}$ is bounded for each $w \in \Omega$ and all $\lambda > 1$, then $N(w)$ has a random fixed point.*

Theorem 2.17. *[23] Let $(\Omega, \mathcal{A})$ be a complete σ-finite measure space, E a separable Banach space, and let $N : \Omega \times E \to \mathcal{P}_{cl}(E)$ be a random multi-valued contraction. Then $N(w)$ has a random fixed point.*

3. Existence and Ulam-Hyers-Rassias stability results

In this section, we are concerned with the existence and the Ulam-Hyers-Rassias stability for problem (1.1). Let us start by defining what we mean by a random solution of the problem (1.1).

Definition 3.1. *By a random solution of the problem (1.1) we mean a measurable function $u : \Omega \to C_\gamma$ that satisfies the condition $u(0, w) = \phi(w)$, and the equation $({}^{CF}D_0^r u)(t, w) = v(t, w)$ on $I \times \Omega$, where $v \in S_{F\circ u}(w)$.*

3.1. The convex case

We present now some existence and Ulam stabilities results for the problem (1.1) with convex valued right hand side.

The following hypotheses will be used in the sequel.

(H_1) The multifunction $F : I\times\mathbb{R}\times\Omega \to \mathcal{P}_{cp,cv}(\mathbb{R})$ is random Carathéodory on $I \times \mathbb{R} \times \Omega$.

(H_2) There exists a measurable and bounded function $l : \Omega \to L^\infty(I, [0, \infty))$ satisfying, for each $w \in \Omega$, $t \in I$ and $u, \overline{u} \in \mathbb{R}$,

$$H_d(F(t, u, w), F(t, \overline{u}, w)) \leq l(t, w)|u - \overline{u}|,$$

and

$$d(0, F(t, 0, w)) \leq l(t, w);\ \text{for } t \in I,$$

with

$$l^* = \sup_{w\in\Omega} \|l(w)\|_{L^\infty}.$$

(H_3) For each bounded set $D \subset C$, the set $\{t \mapsto v(t, w) : v \in S_{F\circ u}(w) : u \in D\}$ is equicontinuous.

(H_4) There exists $\lambda_\Phi > 0$ such that for each $t \in I$, and $w \in \Omega$, we have

$$({}^{CF}I_0^r\Phi)(t, w) \leq \lambda_\Phi\Phi(t, w).$$

Theorem 3.2. *Assume that the hypotheses $(H_1) - (H_3)$ hold. If $l^*a_r < 1$, then the problem (1.1) has a random solution defined on $I \times \Omega$.*

Remark 3.3. *For each $u : \Omega \to C$, the set $S_{F,u}(w)$ is nonempty since by (H_1), F has a measurable selection (see [12], Theorem III.6).*

Remark 3.4. *The hypothesis (H_2) implies that, for every $t \in I$, $u \in \mathbb{R}$ and $w \in \Omega$, we get*

$$H_d(F(t, u, w), F(t, 0, w)) \leq l(t, w)|u|,$$

and

$$\begin{aligned} H_d(0, F(t, u, w)) &\leq H_d(0, F(t, 0, w)) \\ &\quad + H_d(F(t, u, w), F(t, 0, w)) \\ &\leq l(t, w)(1 + |u|). \end{aligned}$$

Proof. Set

$$\phi^* = \sup_{w\in\Omega} |\phi(w)|.$$

Define a multivalued operator $N : \Omega \times C \to \mathcal{P}(C)$ by,

$$(N(w)u)(t) = \Big\{h : \Omega \to C : h(t, w) = \phi(w) +$$

$$a_r(v(t, w) - v(0, w)) + b_r \int_0^t v(s, w)ds, t \in I, v \in S_{F\circ u}(w)\Big\}. \tag{3.1}$$

The map ϕ is measurable for all $w \in \Omega$. Again, as the integral is continuous on I, for each $v \in S_{F\circ u}(w)$, then $N(w)$ defines a multivalued mapping $N : \Omega\times C \to \mathcal{P}(C)$. Thus u is a random solution for the problem (1.1) if and only if $u \in N(w)u$. We shall show that the multivalued operator N satisfies all conditions of Theorem 2.16. The proof will be given in several steps.

Step 1. *$N(w)$ is a multi-valued random operator on C.*
Since $F(t, u, w)$ is strong random Carathéodory, the map $w \to F(ty, u, w)$ is measurable in view of Definition 2.5. Therefore, the map

$$w \mapsto \phi(w) + a_r(v(t, w) - v(0, w)) + b_r \int_0^t v(s, w)ds,$$

is measurable. As a result, $N(w)$ is a multi-valued random operator on C.

Step 2. *$N(w)u \in \mathcal{P}_{cv}(C)$ for each $u \in C$.*
Indeed, if h_1, h_2 belong to $N(w)u$, then there exist $v_1, v_2 \in S_{F\circ u}(w)$ such that for each $t \in I$ and $w \in \Omega$, we have for $i = 1, 2$,

$$h_i(t, w) = \phi(w) + a_r(v_i(t, w) - v_i(0, w)) + b_r \int_0^t v_i(s, w)ds.$$

Let $0 \leq d \leq 1$. Then, for each $t \in I$ and $w \in \Omega$, we get

$$(dh_1 + (1 - d)h_2)(t, w) = a_r([dv_1 + (1 - d)v_2])(t, w)$$

$$-[dv_1 + (1-d)v_2])(0,w))$$
$$+b_r \int_0^t [dv_1 + (1-d)v_2])(s,w)ds.$$

Since $S_{F\circ u}(w)$ is convex (because F has convex values), we get

$$(dh_1 + (1-d)h_2)(\cdot, w) \in N(w)u.$$

Step 3. *$N(w)$ is continuous and completely continuous.* We give the proof of this step in several claims.

Claim 1: *$N(w)$ is continuous.*

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in C. Then from (H_2), for each $t \in I$ and $w \in \Omega$, we have

$$\begin{aligned} & H_d(F(t,u_n(t,w),w), F(t,u(t,w),w)) \\ \leq\ & l(t,w)|u_n(t,w) - u(t,w)| \\ \leq\ & l^*\|u_n(\cdot,w) - u(\cdot,w)\|_C \to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus, we obtain

$$H_d(F(t,u_n(t,w),w), F(t,u(t,w),w)) \to 0 \text{ as } n \to \infty.$$

Claim 2: *$N(w)$ maps bounded sets into bounded sets in C.*

Let $B_\eta = \{u \in C : \|u\|_C \leq \eta\}$ be bounded set in C, and $u \in B_\eta$. Then for each $h \in N(w)u$, there exists $v \in S_{F\circ u}(w)$ such that

$$h(t,w) = \phi(w) + a_r(v(t,w) - v(0,w)) + b_r \int_0^t v(s,w)ds.$$

By (H_2), for each $t \in I$ and $w \in \Omega$, we obtain

$$\begin{aligned} |h(t,w)| &\leq |\phi(w)| + a_r(|v(t,w)| + |v(0,w)|) \\ &\quad + b_r \int_0^t |v(s,w)|ds \\ &\leq \phi^* + a_r l(t,w)(1 + |u(t,w)|) \\ &\quad + a_r l(t,w)(1 + |u(0,w)|) \\ &\quad + b_r \int_0^t l(s,w)(1 + |u(s,w)|)ds \\ &\leq \phi^* + a_r l(t,w)(1 + |u(t,w)|) \\ &\quad + a_r l(t,w)(1 + |\phi(w)|) \\ &\quad + b_r \int_0^t l(s,w)(1 + |u(s,w)|)ds \\ &\leq \phi^* + a_r l^*(1 + \phi^*) + a_r l^*(1 + \eta) \\ &\quad + b_r \int_0^t l^*(1 + \eta)ds \\ &\leq \phi^* + a_r l^*(1 + \phi^*) + l^*(a_r + Tb_r)(1 + \eta) \\ &:= \ell. \end{aligned}$$

Claim 3: *$N(w)$ maps bounded sets into equicontinuous sets in C.*

Let $t_1, t_2 \in I$, $t_1 < t_2$, and let B_η be a bounded set of C as in claim 2, and let $u \in B_\eta$ and $h \in N(w)u$. Then, there exists $v \in S_{F\circ u}(w)$ such that for each $w \in \Omega$, we obtain

$$\begin{aligned} |h(t_2,w) - h(t_1,w)| &\leq a_r|v(t_2,w) - v(t_1,w)| \\ &\quad + b_r \int_{t_1}^{t_2} |v(s,w)|ds \\ &\leq a_r|v(t_2,w) - v(t_1,w)| \\ &\quad + b_r l^*(1 + \eta)(t_2 - t_1). \end{aligned}$$

From (H_3), the right-hand side of the above inequality tends to zero; as $t_1 \to t_2$. As a consequence of the Claims 1 to 3, and the Arzela-Ascoli theorem, we can conclude that $N(w)$ is continuous and completely continuous multi-valued random operator.

Step 4: *The set $\mathcal{E} := \{u \in C : \lambda u \in N(w)u\}$ is bounded for some $\lambda > 1$.*

Let $u \in C$ be arbitrary and let $w \in \Omega$ be fixed such that $\lambda u \in N(w)u$ for all $\lambda > 1$. Then, there exists $v \in S_{F\circ u}(w)$ such that for each $t \in I$, we have

$$\lambda u(t,w) = \phi(w) + a_r(v(t,w) - v(0,w)) + b_r \int_0^t v(s,w)ds.$$

This implies by (H_2) that,

$$\begin{aligned} |u(t,w)| &\leq \frac{|\phi(w)|}{\lambda} + \frac{a_r}{\lambda}(|v(t,w)| + |v(0,w)|) \\ &\quad + \frac{b_r}{\lambda} \int_0^t |v(s,w)|ds \\ &\leq \phi(t,w) + a_r l(t,w)(1 + |u(t,w)|) \\ &\quad + a_r l(t,w)(1 + |u(0,w)|) \\ &\quad + b_r \int_0^t l(s,w)(1 + |u(s,w)|)ds \\ &\leq \phi^* + l^* a_r(1 + \phi^*) + l^* a_r(1 + |u(t,w)|) \\ &\quad + b_r l^* \int_0^t (1 + |u(s,w)|)ds. \end{aligned}$$

Thus

$$\begin{aligned} 1 + |u(t,w)| &\leq \frac{(1 + l^* a_r)(1 + \phi^*)}{1 - l^* a_r} \\ &\quad + \frac{b_r l^*}{1 - l^* a_r} \int_0^t (1 + |u(s,w)|)ds. \end{aligned}$$

By applying the classical Gronwall lemma, we get

$$1 + |u(t,w)| \leq \frac{(1 + l^* a_r)(1 + \phi^*)}{1 - l^* a_r} \exp\left(\frac{b_r l^*}{1 - l^* a_r} \int_0^t ds\right)$$

$$= \frac{(1+l^*a_r)(1+\phi^*)}{1-l^*a_r}\exp\left(\frac{Tb_rl^*}{1-l^*a_r}\right).$$

Hence

$$\begin{aligned}|u(t,w)| &\leq \frac{(1+l^*a_r)(1+\phi^*)}{1-l^*a_r}\exp\left(\frac{Tb_rl^*}{1-l^*a_r}\right)-1\\ &:= M.\end{aligned}$$

This gives $\|u\|_C \leq M$.

As a consequence of Steps 1 to 4, together with Theorem 2.16, N has a random fixed point u which is a random solution to problem (1.1). □

Now, we are concerned with the generalized Ulam-Hyers-Rassias stability of our problem (1.1).

Theorem 3.5. *Assume that the hypotheses $(H_1)-(H_4)$ hold. If $l^*a_r < 1$, then the problem (1.1) is generalized Ulam-Hyers-Rassias stable.*

Proof. Let u be a random solution of the inequality (2.3), and let us assume that v is a random solution of problem (1.1). Thus, we have

$$v(t,w) = \phi(w) + a_r(f_v(t,w) - f_v(0,w)) + b_r\int_0^t f_v(s,w)ds,$$

where $f_v \in S_{F\circ v}(w)$. From the inequality (2.3) for each $t \in I$, and $w \in \Omega$, we have

$$\left|u(t,w) - \phi(w) - a_r(f_u(t,w) - f_u(0,w)) - b_r\int_0^t f_u(s,w)ds\right| \leq ({}^{CF}I_0^r\Phi)(t,w),$$

where $f_u \in S_{F\circ u}(w)$. From hypotheses (H_2) and (H_4), for each $t \in I$, and $w \in \Omega$, we get

$$\begin{aligned}|u(t,w)-v(t,w)| &\leq \Big|u(t,w)-\phi(w)-a_r(f_u(t,w)\\ &\quad -f_u(0,w))-b_r\int_0^t f_u(s,w)ds\Big|\\ &\quad +a_r|f_u(t,w)-f_v(t,w)|\\ &\quad +a_r|f_u(0,w)-f_v(0,w)|\\ &\quad +b_r\int_0^t|f_u(s,w)-f_v(s,w)|ds\\ &\leq ({}^{CF}I_0^r\Phi)(t,w)\\ &\quad +l^*a_r|u(t,w)-v(t,w)|\\ &\quad +l^*a_r|u(0,w)-v(0,w)|\\ &\quad +l^*b_r\int_0^t|u(s,w)-v(s,w)|ds\\ &= ({}^{CF}I_0^r\Phi)(t,w)\\ &\quad +l^*a_r|u(t,w)-v(t,w)|\\ &\quad +l^*b_r\int_0^t|u(s,w)-v(s,w)|ds\\ &\leq \lambda_\Phi\Phi(t,w)\\ &\quad +l^*a_r|u(t,w)-v(t,w)|\\ &\quad +l^*b_r\int_0^t|u(s,w)-v(s,w)|ds.\end{aligned}$$

Thus

$$|u(t,w)-v(t,w)| \leq \frac{\lambda_\Phi}{1-l^*a_r}\Phi(t,w) + \frac{l^*b_r}{1-l^*a_r}\int_0^t|u(s,w)-v(s,w)|ds.$$

From the classical Gronwall lemma, we get

$$\begin{aligned}|u(t,w)-v(t,w)| &\leq \frac{\lambda_\Phi}{1-l^*a_r}\Phi(t,w)\exp\left(\frac{l^*b_r}{1-l^*a_r}\int_0^t ds\right)\\ &= \frac{\lambda_\Phi}{1-l^*a_r}\exp\left(\frac{Tl^*b_r}{1-l^*a_r}\right)\Phi(t,w)\\ &:= c_{F,\Phi}\Phi(t,w).\end{aligned}$$

Finally, our problem (1.1) is generalized Ulam-Hyers-Rassias stable. □

3.2. The nonconvex case

We present now some existence and Ulam stabilities results for the problem (1.1) with non-convex valued right hand side.

The following hypotheses will be used in the sequel.

(H_{01}) The multifunction $F : I \times \mathbb{R} \times \Omega \to \mathcal{P}_{cp}(\mathbb{R})$ is random Carathéodory on $I \times \mathbb{R} \times \Omega$.

(H_{02}) There exists a measurable and bounded function $l : \Omega \to L^\infty(I, [0,\infty))$ satisfying, for each $w \in \Omega$, $t \in I$ and $u, \overline{u} \in \mathbb{R}$,

$$H_d(F(t,u,w), F(t,\overline{u},w)) \leq t^{1-\gamma}l(t,w)|u-\overline{u}|.$$

Set

$$l^* = \sup_{w\in\Omega}\|l(w)\|_{L^\infty}.$$

Theorem 3.6. *Assume that the hypotheses (H_{01}) and (H_{02}) hold. If*

$$l^*(a_r + Tb_r) < 1, \tag{3.2}$$

then the problem (1.1) has at least one random solution defined on $I \times \Omega$.

Proof. Let $N : \Omega \times C \to \mathcal{P}(C)$ be the multivalued operator defined in (3.1). We know that $N(w)$ is a multi-valued random operator on C. We shall show in two steps that the multivalued operator N satisfies all conditions of Theorem 2.17.

Step 1. *$N(w)u \in \mathcal{P}_{cl}(C)$ for each $u \in C$.*
Let $\{u_n\}_{n\geq 0} \in N(w)u$ be such that $u_n \longrightarrow \tilde{u}$ in C. Then, $\tilde{u} \in C$ and there exists $f_{u_n} \in S_{F\circ u_n}(w)$ be such that, for each $t \in I$ and $w \in \Omega$, we have

$$u_n(t,w) = \phi(w) + a_r(f_{u_n}(t,w) - f_{u_n}(0,w)) + b_r \int_0^t f_{u_n}(s,w)ds.$$

Using the fact that F has compact values and from (H_{01}), we may pass to a subsequence if necessary to get that f_{u_n} converges to f_u in $L^1(I)$, and hence $f_u \in S_{F\circ u}(w)$. Then, for each $t \in I$ and $w \in \Omega$, we get

$$\begin{aligned} u_n(t,w) \longrightarrow \tilde{u}(t,w) &= \phi(w) + a_r(f_u(t,w) - f_u(0,w)) \\ &\quad + b_r \int_0^t f_u(s,w)ds. \end{aligned}$$

So, $\tilde{u} \in N(w)u$.

Step 2. *There exists $0 \leq \lambda < 1$ such that, for each $w \in \Omega$,*

$$H_d(N(w)u, N(w)\overline{u}) \leq \lambda \|u - \overline{u}\|_C \text{ for each } u, \overline{u} \in C.$$

Let $u, \overline{u} \in C$ and $h \in N(w)u$. Then, there exists $f(t,w) \in F(t,u(t,w),w)$ such that for each $t \in I$ and $w \in \Omega$, we have

$$h(t,w) = \phi(w) + a_r(f(t,w) - f(0,w)) + b_r \int_0^t f(s,w)ds.$$

From (H_{02}) it follows that

$$H_d(F(t,u(t,w),w), F(t,\overline{u}(t,w),w)) \leq l(t,w)|u(t,w) - \overline{u}(t,w)|.$$

Hence, there exists $v \in S_{F\circ u}$ such that

$$|f(t,w) - v(t,w)| \leq l(t,w)|u(t,w) - \overline{u}(t,w)|.$$

Consider $U : I \times \Omega \to \mathcal{P}(\mathbb{R})$ defined by

$$\begin{aligned} U(t,w) &= \{v(t,w) \in \mathbb{R} : |f(t,w) - v(t,w)| \\ &\quad \leq l(t,w)|u(t,w) - \overline{u}(t,w)|\}. \end{aligned}$$

Since the multivalued operator $u(t,w) = U(t,w) \cap F(t,\overline{u}(t,w),w)$ is measurable (see [12, Proposition III.4]), there exists a function $\overline{f}(t,w)$ which is a measurable selection for $\overline{u}$. So, $\overline{f}(t,w) \in F(t,\overline{u}(t,w),w)$, and for each $t \in I$ and $w \in \Omega$, we get

$$|f(t,w) - \overline{f}(t,w)| \leq l(t,w)|u(t,w) - \overline{u}(t,w)|.$$

Let us define for each $t \in I$ and $w \in \Omega$,

$$\overline{h}(t,w) = \phi(w) + a_r(\overline{f}(t,w) - \overline{f}(0,w)) + b_r \int_0^t \overline{f}(s,w)ds.$$

Then, for each $t \in I$ and $w \in \Omega$, we obtain

$$\begin{aligned} |h(t,w) - \overline{h}(t,w)| &\leq a_r|f_u(t,w) - \overline{f}(t,w)| \\ &\quad + a_r|f_u(0,w) - \overline{f}(0,w)| \\ &\quad + b_r \int_0^t |f_u(s,w) - \overline{f}(s,w)|ds \\ &\leq a_r l(t,w)|u(t,w) - \overline{u}(t,w)| \\ &\quad + b_r \int_0^t l(s,w)|u(s,w) - \overline{u}(s,w)|ds. \end{aligned}$$

Hence

$$\|h - \overline{h}\|_C \leq l^*(a_r + Tb_r)\|u - \overline{u}\|_C.$$

By an analogous relation, obtained by interchanging the roles of u and $\overline{u}$, it follows that

$$H_d(N(w)u, N(w)\overline{u}) \leq l^*(a_r + Tb_r)\|u - \overline{u}\|_C.$$

So by (3.2), N is random contraction and thus, by Theorem 2.17, N has a random fixed point u which is a random solution to problem (1.1). □

Now, we can state (without proof) the following generalized Ulam-Hyers-Rassias stability result.

Theorem 3.7. *Assume that the hypotheses (H_{01}), (H_{02}), (H_4) and the condition (3.2) hold. Then the problem (1.1) is generalized Ulam-Hyers-Rassias stable.*

4. Examples

Let $\Omega = (-\infty, 0)$ be equipped with the usual σ-algebra consisting of Lebesgue measurable subsets of $(-\infty, 0)$.

Example 4.1. *Consider the following problem of Caputo-Fabrizio fractional differential inclusion*

$$\begin{cases} ({}^{CF}D_0^{\frac{1}{2}}u)(t,w) \in F(t,u(t,w),w);\ t\in[0,1], \\ u(0,w) = 1+w^2, \end{cases} \quad w\in\Omega, \tag{4.1}$$

where

$$F(t,u(t,w),w) = \Big\{ v:\Omega \to C([0,1],\mathbb{R}) : |f_1(t,u(t,w),w)| \le |v(w)| \le |f_2(t,u(t,w),w)| \Big\},$$

$t\in[0,1],\ w\in\Omega,$ *with* $f_1, f_2 : [0,1]\times\mathbb{R}\times\Omega \to \mathbb{R}$, *such that*

$$f_1(t,u(t,w),w) = \frac{t^2u}{(1+w^2+|u|)e^{10+t}},$$

and

$$f_2(t,u(t,w),w) = \frac{t^2u}{(1+w^2)e^{10+t}}.$$

We assume that F is closed and convex valued. A simple computation shows that the conditions of Theorem 3.2 are satisfied. Hence, the problem (4.1) has at least one random solution defined on $[0,1]$.

Also, the hypothesis (H_3) *is satisfied with*

$$\Phi(t,w) = \frac{e^t}{1+w^2}, \text{ and } \lambda_\Phi = M(1/2)\left(1-e^{-1-t}\right).$$

Indeed, for each $t\in[0,1]$, *and* $w\in\Omega$, *we get*

$$\begin{aligned} ({}^{CF}D^{1/2}\Phi)(t,w) &\le M(1/2)\left(1-e^{-1-t}\right)e^t \\ &= \lambda_\Phi \Phi(t,w). \end{aligned}$$

Consequently, Theorem 3.5 implies that the problem (4.1) is generalized Ulam-Hyers-Rassias stable.

Example 4.2. . *Consider now the following problem of fractional differential inclusion*

$$\begin{cases} ({}^{CF}D_0^{\frac{1}{2}}u)(t,w) \in F(t,u(t,w),w);\ t\in[0,1], \\ u(0,w) = \frac{1}{1+w^2}, \end{cases} \quad w\in\Omega, \tag{4.2}$$

where for $t\in[0,1],\ w\in\Omega$,

$$F(t,u(t,w),w) = \frac{1+t^2}{(1+w^2+|u(t,w)|)e^{10+t}}[u(t,w)-1, u(t,w)].$$

Set $r=\frac{1}{2}$, *and assume that F is closed valued. Simple computations show that the conditions of Theorem 3.6 are satisfied. Hence, the problem (4.2) has at least one random solution defined on* $[0,1]$. *Also, Theorem 3.7 implies that the problem (4.2) is generalized Ulam-Hyers-Rassias stable.*

5. Conclusions

We have provided some sufficient conditions ensuring the existence and Ulam stability of solutions of Random Caputo-Fabrizio type fractional differential inclusions with convex and non-convex right hand side. We have used some multi-valued random fixed point theorems and a suitable Gronwall type inequality. Two examples have been presented. In a forthcoming work we shall consider the problem (1.1) on the half line and make use of the diagonalization process together with some properties in the Fréchet space.

Conflict of interest

The authors declare that they have no conflicts of interest to this work.

References

1. S. Abbas, M. Benchohra, H. Gorine, Caputo-Fabrizio fractional differential equations in Fréchet spaces, *Bulletin Transilvania Univ. Braşov*, **13** (2020), 373–386.
2. S. Abbas, M. Benchohra, J.R. Graef, J. Henderson, *Implicit Fractional Differential and Integral Equations: Existence and Stability*, De Gruyter, Berlin, 2018.
3. S. Abbas, M. Benchohra, J. Henderson, Coupled Caputo-Fabrizio fractional differential systems in generalized Banach spaces, *Malaya J. Math.*, **9** (2021), 20-25.
4. S. Abbas, M. Benchohra, G.M. N'Guérékata, *Topics in Fractional Differential Equations*, Springer, New York, 2012.
5. S. Abbas, M. Benchohra, G.M. N'Guérékata, *Advanced Fractional Differential and Integral Equations*, Nova Science Publishers, New York, 2015.
6. S. Abbas, M. Benchohra, J.J. Nieto, Caputo-Fabrizio fractional differential equations with instantaneous impulses, *AIMS Math.*, **6** (2021), 2932–2946.
7. S. Abbas, M. Benchohra, A. Petrusel, Ulam stabilities for the Darboux problem for partial fractional differential inclusions via Picard Operators, *Electron. J. Qual. Theory Differ. Equ.*, **1** (2014), 1–13.

8. S. Abbas, M. Benchohra, S. Sivasundaram, Ulam stability for partial fractional differential inclusions with multiple delay and impulses via Picard operators, *J. Nonlinear Stud.*, **20** (2013), 623–641.

9. S.M. Aydogan, J.F. Gomez Aguilar, D. Baleanu, S. Rezapour, M.E. Samei, Approximate endpoint solutions for a class of fractional q-differential inclusions by computational results, *Fractals*, **28** (2020), 2040029.

10. F. Bekada, S. Abbas, M. Benchohra, Boundary value problem for Caputo–Fabrizio random fractional differential equations, *Moroccan J. Pure Appl. Anal. (MJPAA)*, **6** (2020), 218–230.

11. M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel, *Prog. Frac. Differ. Appl.*, **1** (2015), 73–78.

12. C. Castaing, M. Valadier, *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Mathematics **580**, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

13. B.C. Dhage, Multi-valued condensing random operators and functional random integral inclusions, *Opuscula Math.*, **31** (2011), 27–48.

14. S. Etemad, S. Rezapour, M.E. Samei, On fractional hybrid and non-hybrid multi-term integro-differential inclusions with three-point integral hybrid boundary conditions, *Adv. Differ. Equ.*, **2020** (2020), 161.

15. S. Etemad, S. Rezapour, M. E. Samei, On a fractional Caputo–Hadamard inclusion problem with sum boundary value conditions by using approximate endpoint property, *Math. Methods Appl. Sciences*, **43** (2020), 9719–9734.

16. A. Granas, J. Dugundji, *Fixed Point Theory,* Springer-Verlag, New York, 2003.

17. D.H. Hyers, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci.*, **27** (1941), 222–224.

18. S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, 2001.

19. S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, Springer, New York, 2011.

20. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V., Amsterdam, 2006.

21. S. Krim, S. Abbas, M. Benchohra, M. A. Darwish, Boundary value problem for implicit Caputo–Fabrizio fractional differential equations, *Int. J. Difference Equ.*, **15** (2020), 493–510.

22. J. Losada, J. J. Nieto, Properties of a new fractional derivative without singular kernel, *Progr. Fract. Differ. Appl.*, **1** (2015), 87–92.

23. A. Nowak, Applications of random fixed point theorem in the theory of generalized random differential equations, *Bull. Polish. Acad. Sci.*, **34** (1986), 487–494.

24. T. P. Petru, A. Petrusel, J.-C. Yao, Ulam-Hyers stability for operatorial equations and inclusions via nonself operators, *Taiwanese J. Math.*, **15** (2011), 2169–2193.

25. Th. M. Rassias, On the stability of linear mappings in Banach spaces, *Proc. Amer. Math. Soc.*, **72** (1978), 297–300.

26. I. A. Rus, Ulam stability of ordinary differential equations, *Studia Univ. Babes-Bolyai, Math.*, **4** (2009), 125–133.

27. I. A. Rus, Remarks on Ulam stability of the operatorial equations, *Fixed Point Th.*, **10** (2009), 305–320.

28. M. E. Samei, V. Hedayati, S. Rezapour, Existence results for a fraction hybrid differential inclusion with Caputo-Hadamard type fractional derivative, *Adv. Differ. Equ.*, **2019** (2019), 163.

29. M. E. Samei, V. Hedayati, G. Khalilzadeh Ranjbar, The existence of solution for k-dimensional system of Langevin Hadamard-type fractional differential inclusions with $2k$ different fractional orders, *Mediterr. J. Math.*, **17** (2020), 37.

30. M. E. Samei, S. Rezapour, On a system of fractional q-differential inclusions via sum of two multi-term functions on a time scale, *Bound. Value Probl.*, **2020** (2020), 135.

31. M. E. Samei, S. Rezapour, On a fractional q-differential inclusion on a time scale via endpoints and numerical calculations, *Adv. Differ. Equ.*, **2020** (2020), 460.

32. V. E. Tarasov, *Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer, Heidelberg; Higher Education Press, Beijing, 2010.
33. S. M. Ulam, *A Collection of Mathematical Problems*, Interscience Publishers, New York, 1968.
34. Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, 2014.
35. Y. Zhou, *Fractional Evolution Equations and Inclusions: Analysis and Control*, Elsevier Science, 2016.