https://www.aimspress.com/journal/mine

Mathematics in Engineering, 8(4): 477–518.
DOI:10.3934/mine.2026015
Received: 25 February 2026
Revised: 14 July 2026
Accepted: 23 July 2026
Published: 31 July 2026

Research article

Non-divergence evolution operators modeled on Hörmander vector fields with Dini continuous coefficients

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Abstract: In this paper we analyze operators $H = \partial_t - \sum_{i,j} a_{ij}(x,t)X_iX_j$, where the X_i's are Hörmander vector fields generating a Carnot group and $A = [a_{ij}]$ is a symmetric and uniformly positive-definite matrix whose entries satisfy double Dini continuity, a strictly weaker condition than Hölder continuity. For these operators, we build a fundamental solution and show a two-sided Gaussian estimate for the latter, as well as upper Gaussian estimates for its derivatives up to weight 2. As a consequence, we prove an existence result for the related Cauchy problem, under a Dini-type condition on the source.

Keywords: ultraparabolic operators; Hörmander vector fields; fundamental solution; parametrix method; Carnot groups; Dini-continuity

1. Introduction

In this paper we build a fundamental solution Γ for nonvariational evolution operators of the (degenerate) form

$$H = \partial_t - \sum_{i,j=1}^{m} a_{ij}(x,t)\, X_iX_j = \partial_t - a^{ij}(x,t)\, X_iX_j, \tag{1.1}$$

where $X_1, ..., X_m$ are (smooth) Hörmander vector fields generating a Carnot group in $\mathbb{R}^n$ ($n > m$) and the matrix $[a_{ij}(x,t)]$ is symmetric and uniformly positive definite (in $\mathbb{R}^m$), with double Dini continuous entries in $\mathbb{R}^{n+1}$ (precise definitions are given below). We further show that Γ, together with its derivatives (up to weight 2) along the vector fields appearing in H, satisfy Gaussian estimates similar to those enjoyed by the fundamental solutions of constant coefficients' heat operators; these yield also a reproduction formula for Γ and existence of classical solutions for the Cauchy problem.

We follow a procedure known in literature as *parametrix method*, firstly developed by E. E. Levi in [1, 2] to construct fundamental solutions of uniformly elliptic operators with Hölder-continuous coefficients; later this procedure has been adapted by Friedman to obtain existence results in the context of uniformly parabolic operators (under the same regularity assumption on the coefficients), as discussed in [3].

In the last decades, several important extensions of the parabolic version of Levi's method have been developed. In [4], Polidoro constructed a fundamental solution for ultraparabolic operators of Kolmogorov-Fokker-Planck type with Hölder continuous coefficients, exploiting the knowledge of an explicit fundamental solution for the model operator with constant coefficients, built by Lanconelli and Polidoro in [5]. In [6], Bonfiglioli, Lanconelli and Uguzzoni extended the applicability of the procedure to nonvariational degenerate parabolic operators, still with Hölder continuous coefficients, structured on Hörmander vector fields on Carnot groups. This was the first case when Levi method was implemented in a situation where the fundamental solution for the model operator with constant coefficients was not explicitly known. In [7], Bramanti, Brandolini, Lanconelli, Uguzzoni built a fundamental solution for similar operators built with general Hörmander vector fields (without any underlying structure of translations and dilations), with local results. In [8], Biagi and Bramanti generalized the results in [7] again to similar operators built on homogeneous Hörmander vector fields (without any underlying group structure of translations), with global results. A different type of generalization of the parabolic parametrix method is related to the weakening of the regularity assumptions on the leading coefficients. In [9], Lucertini, Pagliarani and Pascucci built a fundamental solution for Kolmogorov-Fokker-Planck operators of the kind studied in [5], but with coefficients $a_{ij}(x,t)$ Hölder continuous in space and only bounded measurable in time. In this case the model operator is the KFP operator with coefficients $a_{ij}(t)$ only depending on time, in a bounded measurable way; for these operators, an explicit fundamental solution has been previously built in [10]. Recently, in [11], Zhenyakova and Cherepova adapted the parabolic parametrix method to uniformly parabolic operators with double Dini continuous coefficients. In the present paper, we combine the extension made in [6] about the operator class with the generalization in [11] about the coefficients' regularity, in order to prove the existence of a fundamental solution for H and the relative Gaussian estimates under these weaker assumptions on the considered differential operator.

In order to state in detail our assumptions, we briefly recall some notions. We call *Carnot group* on $\mathbb{R}^n$ an algebraic structure $(\mathbb{R}^n, \circ, \delta_\lambda)$ with

(i) $\circ$ Lie group law on $\mathbb{R}^n$ (that is, a group law such that both the "translation" $(x, y) \mapsto x \circ y$ and the inversion $x \mapsto x^{-1}$ are smooth maps);

(ii) $\{\delta_\lambda\}_{\lambda>0}$ family of Lie group automorphisms of $(\mathbb{R}^n, \circ)$ (typically called "dilations") acting as $\delta_\lambda(x) = (\lambda^{\sigma_1} x_1, ..., \lambda^{\sigma_n} x_n)$, for integers $1 = \sigma_1 \leq ... \leq \sigma_n$;

(iii) the Lie algebra $\mathfrak{a}$ of left-invariant smooth vector fields on $(\mathbb{R}^n, \circ)$ stratified as $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus ... \oplus \mathfrak{a}_s$, where $\mathfrak{a}_1$ is spanned by a set of 1-homogeneous vector fields (w.r.t. δ_λ), $\mathfrak{a}_k = [\mathfrak{a}_1, \mathfrak{a}_{k-1}]$ for all $2 \leq k \leq s$ and $[\mathfrak{a}_1, \mathfrak{a}_s] = 0$.

The vector fields spanning $\mathfrak{a}_1$ are called *(Lie) generators* of the group, whereas the number s in (iii) is the *step* of the group and stands for the length of commutators of the vector fields in $\mathbf{X} = \{X_1, X_2, ..., X_m\}$ that are needed to fulfill Hörmander condition at any point.

According to this notation, our first assumption is:

(H1) $X_1, ..., X_m$ are smooth vector fields on $\mathbb{R}^n$, agreeing with the partial derivatives $\partial_{x_1}, ..., \partial_{x_m}$ (respectively) in the origin and generating a Carnot group $\mathbb{G} = (\mathbb{R}^n, \circ, \delta_\lambda)$, with

$$\delta_\lambda(x) = (\lambda^{\sigma_1} x_1, ..., \lambda^{\sigma_n} x_n) \quad \text{for integers } 1 = \sigma_1 \leq ... \leq \sigma_n.$$

Let $\|\cdot\|$ denote an *homogeneous norm* on $\mathbb{G}$, namely a continuous function $\|\cdot\| : \mathbb{R}^n \to [0, +\infty)$ with the following properties:

(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|\delta_\lambda(x)\| = \lambda\|x\|$ for every $x \in \mathbb{R}^n$, $\lambda > 0$;
(iii) there exists a constant $c \geq 1$ such that $\|x\| \leq c\|x^{-1}\|$ for all $x \in \mathbb{R}^n$;
(iv) there exists a constant $\kappa \geq 1$ such that $\|x \circ y\| \leq \kappa(\|x\| + \|y\|)$ for all $x, y \in \mathbb{R}^n$.

We introduce also the induced quasisymmetric quasidistance $d : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$, $d(x, y) := \|y^{-1} \circ x\|$, which inherits from $\|\cdot\|$ the properties:

(i') $d(x, y) = 0$ if and only if $x = y$;
(ii') $d(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d(x, y)$ for every $x,\ y \in \mathbb{R}^n$, $\lambda > 0$;
(iii') $d(x, y) \leq cd(y, x)$ for a constant $c \geq 1$ (*quasi-symmetry*);
(iv') $d(x, z) \leq \kappa(d(x, y) + d(y, z))$ for every $x, y, z \in \mathbb{R}^n$, with $\kappa \geq 1$ (*quasi-triangle inequality*);

analogous results hold (in $\mathbb{R}^{n+1}$) for the associated "parabolic" quasidistance

$$d_p((x, t), (y, s)) := d(x, y) + \sqrt{|t - s|}, \quad x, y \in \mathbb{R}^n,\ t, s \in \mathbb{R}.$$

Finally, we define the *homogeneous dimension* of $\mathbb{G}$ as

$$Q := \sum_{i=1}^{n} \sigma_i,$$

recalling also its relation with volume growth in $\mathbb{G}$:

$$x' = \delta_\lambda(x) \implies dx' = \lambda^Q dx$$

(see, e.g., [12]); we finally set

$$E(x, t) := t^{-\frac{Q}{2}} \exp\left\{-\frac{\|x\|^2}{t}\right\} \quad x \in \mathbb{R}^n,\ t > 0, \tag{1.2}$$

which in the Euclidean context, i.e., when $Q = n$, is simply the fundamental solution of the classical heat operator (up to scaling constants).

Let us now turn to the assumptions on the coefficients a_{ij}.

We call *modulus of continuity* a continuous non-decreasing function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ with $\omega(0) = 0$; we say that the latter satisfies the *Dini condition* if

$$\widetilde{\omega}(r) := \int_0^r \frac{\omega(x)}{x}\, dx < +\infty, \quad \forall r > 0 \tag{1.3}$$

and the *double Dini condition* if moreover

$$\widetilde{\widetilde{\omega}}(r) := \int_0^r \frac{\widetilde{\omega}(x)}{x}\, dx = \int_0^r \frac{dx}{x} \int_0^x \frac{\omega(y)}{y}\, dy < +\infty, \quad \forall r > 0. \tag{1.4}$$

Note that, given a uniformly continuous function f in a set Ω, the map

$$\omega_f(r) := \sup_{x,y\in\Omega:\, d(x,y)\leq r} |f(x) - f(y)|, \quad r \in [0, +\infty)$$

is precisely a modulus of continuity; such a function f is said to be *Dini continuous* (in Ω) if ω_f satisfies (1.3), and *double Dini continuous* if ω_f satisfies also (1.4). Instead, f is said to be *locally Dini continuous* if it is Dini continuous on every compact subset of Ω; local double Dini continuity is defined analogously.

Given $\Lambda > 1$ and introduced the set of symmetric constant matrices

$$\mathcal{M}_\Lambda := \{B \in \mathbb{R}^{m\times m} : B = B^T,\ \Lambda^{-1}|v|^2 \leq \sum_{i,j=1}^{m} b_{ij}v_iv_j \leq \Lambda|v|^2 \ \ \forall v \in \mathbb{R}^m\},$$

we assume that:

(H2) $A(z) = [a_{ij}(z)]_{1\leq i,j\leq m} \in \mathcal{M}_\Lambda$ for every $z = (x,t) \in \mathbb{R}^{n+1}$ and

$$|a_{ij}(z) - a_{ij}(z')| \leq \omega(d_p(z,z')), \quad \forall z, z' \in \mathbb{R}^{n+1}, \tag{1.5}$$

where ω is a modulus of continuity satisfying the double Dini condition (1.4) and, for suitable constants $c > 0$ and $\delta \in (0,1)$, the inequality:

$$r_2^{-\delta}\,\omega(r_2) \leq c\, r_1^{-\delta}\,\omega(r_1), \quad \forall r_2 \geq r_1 > 0; \tag{1.6}$$

(H3) the modulus of continuity ω is such that

$$\int_0^r \frac{\omega^{1/2}(x)}{x}\,dx < +\infty, \quad \forall r \geq 0. \tag{1.7}$$

In particular, (H2) implies that the coefficients a_{ij} are double Dini continuous over $\mathbb{R}^{n+1}$ w.r.t. the parabolic quasi-distance d_p. Moreover, it is easy to see that $A(z) \in \mathcal{M}_\Lambda$ implies $|a_{ij}(z)| \leq \Lambda$, hence

$$|a_{ij}(z) - a_{ij}(z')| \leq min\{\omega(d_p(z,z')), 2\Lambda\} = \omega'(d_p(z,z')),$$

where $\omega'(r) := min\{\omega(r), 2\Lambda\}$ is a modulus of continuity satisfying (1.4) and (1.6); thus we can assume w.l.o.g. that

$$\omega(r) \leq 2\Lambda, \quad \forall r \geq 0. \tag{1.8}$$

Let us make a comment about the relation between the "Dini assumptions" (H2)-(H3) and the usual requirement of Hölder-continuity for a_{ij}:

Remark 1.1. *Hölder continuity trivially implies double Dini continuity (as well as (1.6)-(1.7)), but the converse is not true: e.g., for any $\alpha > 2$, the function $f_\alpha(x,t) := \omega_\alpha(\|x\| + \sqrt{|t|})$ with*

$$\omega_\alpha(r) := \begin{cases} 0, & if\, r = 0, \\ |log(r)|^{-\alpha}, & if\, 0 < r < \frac{1}{2}, \\ |log(\frac{1}{2})|^{-\alpha}, & if\, r \geq \frac{1}{2}, \end{cases}$$

is double Dini continuous on $\mathbb{G} \times \mathbb{R}$, but not Hölder continuous therein. For this reason, this paper is a generalization of analogous results discussed in [6].

For the sake of clarity, here below we explicitly point out some conventions that will be used throughout the whole paper.

Notation 1.2. *Whenever convenient, it is implicitly assumed that any repeated index (appearing once as apex and once as subscript) is summed from* 1 *up to m, according to Einstein's convention (as done, e.g., in (1.1)).*

We will not explicit the dependence of constants from our "data", i.e., the numbers δ, Λ and the structure of $\mathbb{G}$; a constant that depends also on other parameters $f_1, ..., f_n$ will be denoted by $c(f_1, ..., f_n)$. Moreover, the symbol c (or $c_0, c_1, ...$) may be used several times for possibly different constants; instead, symbols $\overline{c}, \widehat{c}, \check{c}, \widetilde{c}$ (or $\overline{c}_i$, $\widehat{c}_i$, ...) are used to denote specific constants, fixed once and for all, hence each of them denotes a unique constant at every occurrence.

x, ξ, y denote "spatial" variables in $\mathbb{R}^n$; t, τ, η denote time variables (in $\mathbb{R}$); $z = (x, t)$, $\zeta = (\xi, \tau)$ and $\chi = (y, \eta)$ are compact notations for points in $\mathbb{R}^{n+1}$.

D denotes the diagonal of $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$, i.e., $D = \{(z, \zeta) \in \mathbb{R}^{2n+2} : z = \zeta\}$.

We conclude this introductory section by stating our main result:

Theorem 1.3 (Fundamental solution and Gaussian estimates)**.** *Under assumptions (H1)–(H3), the operator (1.1) admits a* fundamental solution Γ, *namely a function* $\Gamma = \Gamma(x, t; \xi, \tau) : (\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}) \setminus D \to \mathbb{R}$ *such that:*

(i) *Γ is continuous and admits continuous Lie derivatives up to order two along $X_1, ..., X_m$ w.r.t. x and continuous derivative w.r.t. t (for any $(x, t) \neq (\xi, \tau)$); furthermore, for any fixed $(\xi, \tau) \in \mathbb{R}^{n+1}$, there holds*

$$H\Gamma(\cdot, \cdot; \xi, \tau) = 0 \quad \text{in } \mathbb{R}^{n+1} \setminus \{(\xi, \tau)\}.$$

(ii) *$H\Gamma(\cdot; \zeta) = \delta_\zeta$ in $\mathcal{D}'(\mathbb{R}^{n+1})$: namely, for any $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$, the function*

$$u(z) = \int_{\mathbb{R}^{n+1}} \Gamma(z; \zeta)\, \varphi(\zeta)\, d\zeta$$

satisfies $Hu = \varphi$ pointwise in $\mathbb{R}^{n+1}$.

(iii) *$\Gamma(x, t; \xi, \tau) = 0$ whenever $t \leq \tau$ and, given any $T > 0$, for every $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$, $0 < t - \tau \leq T$ the following Gaussian estimates hold true for i, $j = 1, 2, ..., m$:*

$$\begin{aligned} c_0^{-1} E(\xi^{-1} \circ x, c_1^{-1}(t-\tau)) \leq \Gamma(x, t; \xi, \tau) &\leq c_0\, E(\xi^{-1} \circ x, c_1(t-\tau)), \\ |X_j^x \Gamma(x, t; \xi, \tau)| &\leq c_0\, (t-\tau)^{-1/2}\, E(\xi^{-1} \circ x, c_1(t-\tau)), \\ |X_i^x X_j^x \Gamma(x, t; \xi, \tau)|,\ |\partial_t \Gamma(x, t; \xi, \tau)| &\leq c_0\, (t-\tau)^{-1}\, E(\xi^{-1} \circ x, c_1(t-\tau)), \end{aligned} \tag{1.9}$$

for constants c_1 depending only on $\Lambda, \delta, \mathbb{G}$, and c_0 depending also on T.

(iv) *The following reproduction formula holds for any x, $\xi \in \mathbb{R}^n$ and $\tau < s < t$:*

$$\Gamma(x, t; \xi, \tau) = \int_{\mathbb{R}^n} \Gamma(x, t; y, s)\, \Gamma(y, s; \xi, \tau)\, dy. \tag{1.10}$$

(v) *Given $g = g(x)$ continuous and $f = f(x, t)$ continuous in (x, t) and locally Dini continuous in x uniformly w.r.t. t, provided f, g satisfy suitable growth estimate (see (4.2) and (4.3)) and $T > 0$ is small enough, then the function*

$$u(z) = \iint_{\mathbb{R}^n \times (0,t)} \Gamma(z; \zeta) f(\zeta)\, d\zeta + \int_{\mathbb{R}^n} \Gamma(z; \xi, 0) g(\xi)\, d\xi$$

is a classical solution to the Cauchy problem

$$\begin{cases} Hu = f \ \ in\, \mathbb{R}^n \times (0, T), \\ u(\cdot, 0) = g \ \ in\, \mathbb{R}^n. \end{cases}$$

This will be proved throughout Theorems 3.14, 4.2, 4.6, 4.7 and 4.10. We point out that (H3) is needed only for Theorem 4.10, namely for the nonhomogeneous Cauchy problem associated to H; everything else does not need that extra assumption.

The rest of the paper is organized as follows: in Section 2 we give some preliminary notions about the control distance in Carnot groups, together with some properties of Dini continuous functions. Section 3 is fully devoted to the development of the parametrix method: first we provide a general explanation of the procedure, then we implement it in our context, culminating with the construction of a fundamental solution for H and the proof of the relative upper Gaussian estimates. Finally, in Section 4 we deduce existence results for the related Cauchy problem, both in the homogeneous and nonhomogeneous case, under Dini-type conditions on the regularity of the right hand side; we also prove uniform Gaussian estimates from below for Γ.

2. Notation and preliminaries

In this section we introduce basic notions and useful lemmas about Carnot groups, as well as convenient results for moduli of continuity satisfying (H2).

2.1. Carnot groups

As a starting point, we recall the "natural" metric structure of Carnot groups. If $\mathbb{G} = (\mathbb{R}^n, \circ, \delta_\lambda)$ is a Carnot group with Lie generators $\mathbf{X} = \{X_1, ..., X_m\}$, then it can be endowed with a distance d_X induced by the latter; this is called *Carnot-Caratheodory distance* (or *control distance*) and is defined as follows:

Definition 2.1. *Given $X_1, ..., X_m$ smooth Hörmander vector fields on $\mathbb{R}^n$, for any $x, y \in \mathbb{R}^n$ and $\delta > 0$ let $C_{x,y}(\delta)$ denote the set of absolutely continuous curves $\gamma : [0, 1] \to \mathbb{R}^n$ with $\gamma(0) = x$, $\gamma(1) = y$ and*

$$\dot{\gamma}(t) = \alpha^i(t)\,(X_i)_{\gamma(t)} \quad a.e.\ in\ (0, 1), \quad \|\alpha_i\|_{L^\infty(0,1)} \le \delta, \quad \forall 1 \le i \le m;$$

we call Carnot-Caratheodory *(or* CC*, in short)* distance *induced by $X_1, ..., X_m$ the function $d_X : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$, $d_X(x, y) = \inf\{\delta > 0 : C_{x,y}(\delta) \neq \emptyset\}$.*

It is well known (see [12, Theorem 1.45]) that d_X is indeed a distance on $\mathbb{R}^n$ (in particular, any couple of points $x, y \in \mathbb{R}^n$ can be joined by a curve γ as above) and it is compatible with translations and dilations of $\mathbb{G}$, namely:

$$d_X(z \circ x, z \circ y) = d_X(x, y), \quad \forall x,\ y,\ z \in \mathbb{R}^n,$$
$$d_X(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d_X(x, y), \quad \forall x,\ y \in \mathbb{R}^n,\ \lambda > 0.$$

The latter implies d_X is induced by an homogeneous norm on $\mathbb{G}$, hence it is equivalent to the homogeneous quasidistance d introduced in Section 1 (see, e.g., [12, Proposition 3.12]).

Let us also define a regularity class of functions induced by the structure of $\mathbb{G}$; to help clarity of the definition below, we recall that the *Lie derivative* along a smooth vector field X of a smooth function u at a point x_0 is defined as

$$Xu(x_0) = \lim_{h\to 0} \frac{u(x_0 \circ \mu_X(h)) - u(x_0)}{h},$$

(if the limit exists), being μ_X the solution of the Cauchy problem

$$\dot{\psi}(t) = X_{\psi(t)}, \quad \psi(0) = 0,$$

(which is well defined, at least locally, by Cauchy theorem). From a geometrical viewpoint, μ_X is the integral curve of the vector field X passing from x_0.

Definition 2.2. *(The class $\mathfrak{C}^2$; see [6, Definition 2.6]) Let $\Omega \subseteq \mathbb{R}^{n+1}$ be an open set. We denote by $\mathfrak{C}^2(\Omega)$ the class of functions $u(x,t)$ defined on Ω which are continuous in Ω w.r.t. the pair (x,t) and such that $u(\cdot,t)$ has continuous Lie derivatives up to second order along the vector fields $X_1, ..., X_m$ (w.r.t. x, for every fixed t) and $u(x,\cdot)$ has continuous derivative (w.r.t. t, for every fixed x), in their respective domains of definition.*

We conclude this overview with the following sort of "Lagrange theorem" in Carnot groups (see [13, Lemma 7.6]):

Lemma 2.3. *Given $X_1, ..., X_m$ smooth Hörmander vector fields on $\mathbb{R}^n$, there exists a constant $c > 0$ such that, for every $x_1,\ x_2 \in \mathbb{R}^n$ and for every function $u \in C^1(\mathbb{R}^n)$, the following holds:*

$$|u(x_1) - u(x_2)| \le c\, d(x_1, x_2) \sup_{d(x_1,y)\le c\, d(x_1,x_2)} (|X_1 u(y)| + ... + |X_m u(y)|).$$

2.2. Dini continuity

Let us now briefly deal with some useful properties of (double) Dini moduli of continuity, starting from the following basic remark:

Remark 2.4. *Let ω be a modulus of continuity which satisfies (1.4). Then also $r \mapsto \omega(\sqrt{r})$ and $r \mapsto \omega(cr)$ (for any $c > 0$) are moduli of continuity satisfying (1.4), and there hold:*

$$\int_0^r \frac{\omega(cx)}{x}\, dx = \int_0^{cr} \frac{\omega(x)}{x}\, dx = \widetilde{\omega}(cr), \qquad \int_0^r \frac{dy}{y} \int_0^y \frac{\omega(cx)}{x}\, dx = \widetilde{\widetilde{\omega}}(cr),$$
$$\int_0^r \frac{\omega(\sqrt{x})}{x}\, dx = \int_0^{\sqrt{r}} \frac{2\omega(\rho)}{\rho}\, d\rho = 2\widetilde{\omega}(\sqrt{r}), \qquad \int_0^r \frac{dy}{y} \int_0^y \frac{\omega(\sqrt{x})}{x}\, dx = 4\widetilde{\widetilde{\omega}}(\sqrt{r}).$$

If, in addition, ω satisfies also (1.6) for some constant $\delta > 0$, then

$$\widetilde{\omega}(r) = \int_0^r \frac{\omega(x)}{x}\, dx \ge C\, r^{-\delta}\, \omega(r) \int_0^r x^{\delta-1}\, dx = C\, \delta^{-1}\, \omega(r), \quad \forall r \ge 0, \tag{2.1}$$

namely ω can be bounded from above by $\widetilde{\omega}$ (up to a constant).

We now turn to two less trivial results:

Lemma 2.5. *Let ω satisfy (1.4)–(1.6), let $c_0 > 0$ be a constant. Then for any $\beta > 0$ there exist constants $c_1 = c_1(\beta) > 0$ and $c_2 > 0$ such that the estimates*

$$\omega^\beta(d(x,\xi))\, E(\xi^{-1} \circ x, c_0(t-\tau)) \le c_1 \omega^\beta(\sqrt{t-\tau})\, E(\xi^{-1} \circ x, 2c_0(t-\tau)), \tag{2.2}$$

$$\omega(d(x,\xi) + \sqrt{t-\tau})\, E(\xi^{-1} \circ x, c_0(t-\tau)) \le c_2 \omega(2\sqrt{t-\tau})\, E(\xi^{-1} \circ x, 2c_0(t-\tau)) \tag{2.3}$$

hold true for every $x, \xi \in \mathbb{R}^n$, $t > \tau$.

Proof. For every $0 < s \le r$ we have $\omega(r) \le C(r/s)^\delta \omega(s)$ by (1.6), therefore

$$\omega^\beta(r) \exp\left\{-\frac{r^2}{c_0 s^2}\right\} \le C^\beta \left(\frac{r}{s}\right)^{\beta\delta} \omega^\beta(s) \exp\left\{-\frac{r^2}{c_0 s^2}\right\}.$$

Since the map $t \mapsto t^\epsilon e^{-kt}$ is bounded in $\mathbb{R}_+$ (for any $\epsilon > 0, k > 0$), we get

$$\left(\frac{r}{s}\right)^{\beta\delta} \exp\left\{-\frac{r^2}{2c_0 s^2}\right\} = \left(\frac{r^2}{s^2}\right)^{\beta\delta/2} \exp\left\{-\frac{1}{2c_0}\frac{r^2}{s^2}\right\} \le c(\beta).$$

Then, multiplying both sides by s^{-Q}, we deduce that for every $0 < s \le r$ there holds

$$\omega^\beta(r) s^{-Q} \exp\left\{-\frac{r^2}{c_0 s^2}\right\} \le c_1(\beta)\, \omega^\beta(s) s^{-Q} \exp\left\{-\frac{r^2}{2c_0 s^2}\right\};$$

by monotonicity of ω, the above inequality trivially holds also for $0 \le r < s$. Choosing $r = d(x,\xi)$ and $s = \sqrt{t-\tau}$, we get (2.2) (see (1.2)). Again by monotonicity of ω, in the case $d(x,\xi) \le \sqrt{t-\tau}$, we get

$$\omega(d(x,\xi) + \sqrt{t-\tau})\, E(\xi^{-1} \circ x, c_0(t-\tau)) \le \omega(2\sqrt{t-\tau})\, E(\xi^{-1} \circ x, c_0(t-\tau)),$$

i.e., (2.3) holds; if instead $d(x,\xi) > \sqrt{t-\tau}$, we find

$$\omega(d(x,\xi) + \sqrt{t-\tau})\, E(\xi^{-1} \circ x, c_0(t-\tau)) \le \omega(2d(x,\xi))\, E(\xi^{-1} \circ x, c_0(t-\tau))$$

and (2.3) is achieved by exploiting (2.2) with $\beta = 1$. □

Lemma 2.6. *Let ω satisfy (1.4). For any $\epsilon \in (0, \frac{1}{2})$ and for any $T > 0$ it holds*

$$\int_0^T \frac{\omega^{\frac{1}{2}+\epsilon}(x)}{x}\, dx < +\infty.$$

For the proof see [14], where the same proof is carried out under the further assumption that ω is subadditive (which, however, does not play a role in this result).

Remark 2.7. *In view of the above result, the extra assumption (H3) is "almost harmless", and it is actually satisfied by the simplest example of modulus of continuity which satisfies (1.4) without being of Hölder type (that is, the one in Remark 1.1), hence this condition does not restrict our framework to the one treated in [6].*

3. Construction of the fundamental solution

3.1. General idea of the parametrix method

As mentioned in Notation 1.2, we may denote points of $\mathbb{R}^{n+1}$ by $z = (x,t)$, $\zeta = (\xi,\tau)$ and $\chi = (y,\eta)$; until not specified otherwise, we assume $\tau < \eta < t$. For fixed $\zeta \in \mathbb{R}^{n+1}$, we consider the constant coefficients' operator

$$H_\zeta := \partial_t - a^{ij}(\zeta)\, X_i X_j$$

obtained from H by "freezing" the coefficients a_{ij} in the point ζ. A deep theory about the fundamental solution of H_ζ (typically called *parametrix* for H) has already been developed and is summarized in the following two theorems:

Theorem 3.1 (The parametrix and its properties). *For any fixed $\zeta \in \mathbb{R}^{n+1}$, the operator H_ζ admits a fundamental solution Γ_ζ given by $\Gamma_\zeta(x,t;y,s) = \gamma_\zeta(y^{-1}\circ x, t-s)$, with $\gamma_\zeta : \mathbb{R}^{n+1}\setminus\{0\} \to \mathbb{R}$ non-negative smooth function such that:*

(i) $\gamma(x,t) = 0$ if and only if $t \le 0$, and $\gamma_\zeta(x,t) \to 0$ as $\|x\| + \sqrt{|t|} \to +\infty$;
(ii)

$$\int_{\mathbb{R}^n} \gamma_\zeta(x,t)\, dx = 1, \qquad \forall t > 0; \tag{3.1}$$

(iii) for any $x \in \mathbb{R}^n$, $t > 0$, $\tau > 0$, we have the reproduction property

$$\gamma_\zeta(x, t+\tau) = \int_{\mathbb{R}^n} \gamma_\zeta(y^{-1}\circ x, t)\, \gamma_\zeta(y,\tau)\, dy; \tag{3.2}$$

(iv) the function $\Gamma_\zeta(\cdot;\zeta)$ is locally integrable in $\mathbb{R}^{n+1}$ and there holds

$$H_\zeta \Gamma_\zeta(\cdot;\zeta) = \delta_\zeta \quad \text{in } D'(\mathbb{R}^{n+1});$$

(v) for any continuous function $g : \mathbb{R}^n \to \mathbb{R}$ with $|g(x)| \le Me^{\nu\|x\|^2}$ in $\mathbb{R}^n$ (for some $\nu > 0$, $M > 0$), there exists $T > 0$ such that the function

$$u(z) := \int_{\mathbb{R}^n} \Gamma_\zeta(z; y, 0)\, g(y)\, dy$$

is well defined in $\mathbb{R}^n \times (0,T)$ and solves the Cauchy problem

$$\begin{cases} H_\zeta\, u = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ u(\cdot,0) = g & \text{in } \mathbb{R}^n. \end{cases}$$

For the proof, see [13, Theorem 2.1].

In the theorem below, for any nonnegative integers h,k we denote by $D_{h,k}$ a differential monomial $X_{i_1}...X_{i_h}\partial_t^k$, with indices $i_1,...,i_h \in \{1,...,m\}$ (with $D_{0,0}$ denoting the identity operator, by convention).

Theorem 3.2 (Gaussian estimates for the parametrix). *For any nonnegative integers h,k and points $(x,t) \in \mathbb{R}^n \times \mathbb{R}^+$ and $\zeta,\ \zeta' \in \mathbb{R}^{n+1}$, the following hold:*

$$\overline{c}_0^{-1}\, E(x, \overline{c}_0^{-1} t) \le \gamma_\zeta(x,t) \le \overline{c}_0\, E(x, \overline{c}_0 t), \tag{3.3}$$

$$|D^{h,k}\gamma_\zeta(x,t)| \le \overline{c}_{h+2k} \cdot t^{-\frac{h}{2}-k}\, E(x, \overline{c}_{h+2k}\cdot t), \tag{3.4}$$

$$|D^{h,k}\gamma_\zeta(x,t) - D^{h,k}\gamma_{\zeta'}(x,t)| \le \overline{c}_{h+2k}\, \omega(d_p(\zeta,\zeta'))\, t^{-\frac{h}{2}-k}\, E(x, \overline{c}_{h+2k}\cdot t), \tag{3.5}$$

where $\{\overline{c}_q\}_{q\ge 0}$ *are positive constants depending only on* Λ, δ *and* $\mathbb{G}$*.*

For the proof of (3.3) and (3.4) see [13, Theorem 2.5], whereas (3.5) follows by assumption (H2) in view of the following result by Bramanti and Fanciullo:

$$|X_{i_1}...X_{i_p}\partial_t^q\, \gamma_\zeta(x,t) - X_{i_1}...X_{i_p}\partial_t^q\, \gamma_{\zeta'}(x,t)| \le c(p,q)\, \|A(\zeta)-A(\zeta')\|_*\, (t-\tau)^{-\frac{p}{2}-q}\, E(x, c't),$$

with $\|\cdot\|_*$ denoting the usual matrix norm (see [15, Proposition 6.2]).

We point out that (3.3) actually means that the "heat kernel" $E(x,t)$ and the parametrix $\gamma_\zeta(x,t)$ are interchangeable (for any fixed ζ), up to fixed constants; thus, denoted by γ_0 the parametrix with pole at $\zeta = 0$ (i.e., the function satisfying $(\partial_t - a^{ij}(0)X_iX_j)\gamma_0 = \delta_0$ in distributional sense), we can express (2.2), (2.3), (3.4), and (3.5) in terms of γ_0 instead of E, namely (for $x,\ \xi \in \mathbb{R}^n$ and $t > \tau$):

$$\omega^\beta(d(x,\xi))\, \gamma_0(\xi^{-1}\circ x, c_0(t-\tau)) \le c_1\omega^\beta(\sqrt{t-\tau})\, \gamma_0(\xi^{-1}\circ x, 2c_0(t-\tau)), \tag{3.6}$$

$$\omega(d_p((x,t),(\xi,\tau)))\, \gamma_0(\xi^{-1}\circ x, c_0(t-\tau)) \le c_2\omega(2\sqrt{t-\tau})\, \gamma_0(\xi^{-1}\circ x, 2c_0(t-\tau)), \tag{3.7}$$

$$|D^{h,k}\gamma_\zeta(x,t)| \le \check{c}_{h+2k}\, t^{-\frac{h}{2}-k}\, \gamma_0(\xi^{-1}\circ x, \check{c}_{h+2k}\cdot t), \tag{3.8}$$

$$|D^{h,k}\gamma_\zeta(x,t) - D^{h,k}\gamma_{\zeta'}(x,t)| \le \check{c}_{h+2k}\, \omega(d_p(\zeta,\zeta'))\, t^{-\frac{h}{2}-k}\, \gamma_0(\xi^{-1}\circ x, \check{c}_{h+2k}\cdot t), \tag{3.9}$$

for constants $c_1 = c_1(\beta)$, $c_2 > 0$ and $\{\check{c}_q\}_q = \overline{c}_0\{\overline{c}_q\}_q$, which depend only on our data (that is Λ, δ and the structure of $\mathbb{G}$). Moreover, since for all $c_2 > c_1 > 0$

$$E(x, c_1t) \le (c_1t)^{-\frac{Q}{2}} \exp\left\{-\frac{\|x\|^2}{c_2t}\right\} = \left(\frac{c_2}{c_1}\right)^{\frac{Q}{2}} E(x, c_2t),$$

then by (3.2)-(3.3), we get that for any $c_2 > c_1 > 0$ there holds

$$\begin{aligned}
\int_{\mathbb{R}^n} \gamma_0(x, c_1t)\gamma_0(y^{-1}\circ x, c_2\tau)\, dy &\le \overline{c}_0^2 \int_{\mathbb{R}^n} E(x, \overline{c}_0c_1t)E(y^{-1}\circ x, \overline{c}_0c_2\tau)\, dy \\
&\le \overline{c}_0^2 \left(\frac{c_2}{c_1}\right)^{\frac{Q}{2}} \int_{\mathbb{R}^n} E(x, \overline{c}_0c_1t)E(y^{-1}\circ x, \overline{c}_0c_1\tau)\, dy \\
&\le \overline{c}_0^4 \left(\frac{c_2}{c_1}\right)^{\frac{Q}{2}} \int_{\mathbb{R}^n} \gamma_0(x, \overline{c}_0^2c_1t)\gamma_0(y^{-1}\circ x, \overline{c_0}^2c_1\tau)\, dy \\
&= \overline{c}_0^4 \left(\frac{c_2}{c_1}\right)^{\frac{Q}{2}} \gamma_0(x, \overline{c}_0^2c_1(t+\tau)),
\end{aligned}$$

namely for all $c_1, c_2 > 0$ there exists $c_3, c_4 > 0$ such that

$$\int_{\mathbb{R}^n} \gamma_0(x, c_1t)\gamma_0(y^{-1}\circ x, c_2\tau)\, dy \le c_3\gamma_0(x, c_4(t+\tau)), \quad \forall x\in\mathbb{R}^n;\ t,\ \tau > 0. \tag{3.10}$$

Here below we summarize some other consequences of Theorems 3.1 and 3.2.

Corollary 3.3. *The map* $(z;\zeta) \mapsto \gamma_\zeta(z;\zeta)$ *is continuous in* $\Omega = \{z \neq \zeta\}$ *and for any* $x \in \mathbb{R}^n$, $t > 0$, $\zeta \in \mathbb{R}^{n+1}$ *the following vanishing property holds:*

$$\int_{\mathbb{R}^n} (X_i X_j \gamma_\zeta)(y \circ x, t)\, dy = 0, \quad \forall 1 \le i, j \le m,$$
$$\int_{\mathbb{R}^n} (\partial_t \gamma_\zeta)(y \circ x, t)\, dy = 0.$$

Proof. Concerning the continuity of $(z,\zeta) \mapsto \gamma_\zeta(z;\zeta)$, note that

$$|\gamma_\zeta(z;\zeta) - \gamma_{\zeta_0}(z_0;\zeta_0)| \le |\gamma_\zeta(z;\zeta) - \gamma_{\zeta_0}(z;\zeta)| + |\gamma_{\zeta_0}(z;\zeta) - \gamma_{\zeta_0}(z_0;\zeta_0)|$$

for any $(z;\zeta)$, $(z_0,\zeta_0) \in \Omega$. As $(z,\zeta) \to (z_0,\zeta_0)$, the two terms vanish by (3.5) and the continuity of γ_{ζ_0} (see Theorem 3.1), respectively; thus $(z;\zeta) \mapsto \gamma_\zeta(z;\zeta)$ is continuous in $(z_0;\zeta_0)$, for whatever $(z_0;\zeta_0) \in \Omega$, as claimed.

As for the vanishing property of the parametrix, left-invariance of the X_i's and dominated convergence (based on the Gaussian estimate (3.5)) yield

$$\int_{\mathbb{R}^n} (X_i X_j \gamma_\zeta)(y \circ x, t)\, dy = \int_{\mathbb{R}^n} X_i X_j \Big(x \mapsto \gamma_\zeta(y \circ x, t)\Big)\, dy = X_i X_j \int_{\mathbb{R}^n} \gamma_\zeta(y \circ x, t)\, dy = 0$$

in force of (3.1); analogously, one gets

$$\int_{\mathbb{R}^n} (\partial_t \gamma_\zeta)(y \circ x, t)\, dy = \partial_t \int_{\mathbb{R}^n} \gamma_\zeta(y \circ x, t)\, dy = 0.$$

□

According to the classical parametrix method, we look for a fundamental solution of H of the form

$$\Gamma(z;\zeta) = \Gamma_\zeta(z;\zeta) + \iint_{\mathbb{R}^n \times (\tau,t)} \Gamma_\chi(z;\chi)\, \mu(\chi,\zeta)\, d\chi \tag{3.11}$$

for an unknown kernel function $\mu = \mu(y,\eta;\xi,\tau)$ defined whenever $\eta > \tau$. To get an ansatz for the latter, let us formally apply H to both sides of (3.11): differentiating the integral term under the integral sign (for spatial derivatives) or via Leibniz' integral rule (for the t-derivative), and denoting

$$Z_1(z;\zeta) := (H\Gamma_\zeta(\cdot;\zeta))(z), \quad \forall z \neq \zeta, \tag{3.12}$$

for every $z = (x,t)$, $\zeta = (\xi,\tau) \in \mathbb{R}^{n+1}$ with $t > \tau$, we formally get

$$0 = Z_1(z;\zeta) + \lim_{t \to \tau^+} \int_{\mathbb{R}^n} \Gamma_\chi(z;y,\eta)\, \mu(y,\eta;\zeta)\, dy + \iint_{\mathbb{R}^n \times (\tau,t)} Z_1(z;\chi)\, \mu(\chi;\zeta)\, d\chi.$$

By point (v) in Theorem 3.1, this formally rewrites as

$$0 = Z_1(z;\zeta) + \mu(z;\zeta) + \iint_{\mathbb{R}^n \times (\tau,t)} Z_1(z;\chi)\, \mu(\chi;\zeta)\, d\chi.$$

Seeing ζ as a fixed parameter in (3.11) and defining the integral operator T as

$$\phi \mapsto (T\phi)(z) := -\iint_{\mathbb{R}^n\times(\tau,t)} Z_1(z;\chi)\,\phi(\chi)\,d\chi,$$

the above equation becomes

$$-Z_1(z;\zeta) = [(I-T)\,\mu(\cdot;\zeta)](z),$$

hence the theory of Neumann series formally gives

$$\mu(z;\zeta) = -[(I-T)^{-1}Z_1(\cdot;\zeta)](z) = -\left[\sum_{j=0}^{\infty} T^j Z_1(\cdot;\zeta)\right](z).$$

Summarizing these (formal) computations, we may expect H to admit a fundamental solution expressed in the form (3.11), with the kernel μ assigned by

$$\mu(z;\zeta) := -\sum_{j=1}^{\infty} Z_j(z;\zeta), \quad \forall z,\, \zeta \in \mathbb{R}^{n+1} \text{ with } t>\tau, \tag{3.13}$$

where, for $j \geq 2$, the sequence $\{Z_j(z;\zeta)\}_j$ is defined recursively as

$$Z_j(\cdot;\zeta) := T^{j-1}Z_1(\cdot;\zeta) = -\iint_{\mathbb{R}^n\times(\tau,t)} Z_1(\cdot;\chi)Z_{j-1}(\chi;\zeta)\,d\chi.$$

The rest of this section is devoted to justify in a rigorous way this intuition.

3.2. *Properties of the kernel μ*

First of all we show the convergence of the series (3.13) in the set $\{t > \tau\}$, namely that μ is well defined therein. To this aim, we introduce the auxiliary sequence

$$\begin{aligned}
&Z_1^\lambda(z;\zeta) := e^{-\lambda(t-\tau)} Z_1^\lambda(z;\zeta),\\
&Z_j^\lambda(z;\zeta) := T^{j-1}Z_1^\lambda(z;\zeta) = -\iint_{\mathbb{R}^n\times(\tau,t)} Z_1^\lambda(z;\chi)Z_{j-1}^\lambda(\chi;\zeta)\,d\chi \quad (j\geq 2),
\end{aligned}$$

for $\lambda > 0$ to be chosen later; this is just a "shrunk" version of our sequence $\{Z_j\}_j$, as it can be easily shown that $Z_j^\lambda(z;\zeta) = e^{-\lambda(t-\tau)}Z_j(z;\zeta)$ for all $j \geq 1$. As already done in the parabolic case [11], the strategy is to find suitable upper bounds for the Z_j^λ's, and then choose λ large enough to ensure convergence of the bounding series. Due to the recursive definition of the Z_j^λ's, this can be done by induction on $j \geq 1$; as for $j = 1$, by Theorem 3.1 we have

$$Z_1(\cdot;\zeta) = (H - H_\zeta)\Gamma_\zeta(\cdot;\zeta) = \sum_{i,j=1}^{m} (a_{ij}(\cdot) - a_{ij}(\zeta))\, X_i^x X_j^x \Gamma_\zeta(\cdot;\zeta), \tag{3.14}$$

and thus (1.5), (3.7) and (3.8) yield (for $x,\, \xi \in \mathbb{R}^n$, $t > \tau$)

$$|Z_1^\lambda(z;\zeta)| \leq \check{c}_2 \frac{\omega(2\sqrt{t-\tau})}{t-\tau}\, \gamma_0(\xi^{-1}\circ x, \check{c}_2(t-\tau))\, e^{-\lambda(t-\tau)}. \tag{3.15}$$

Remark 3.4. *We point out that in the corresponding Hölder case [6] there is no need for this "λ-shrink", as the upper estimates one gets directly for $\{Z_j\}$ are already good enough to ensure convergence of the series. This is not the case for double Dini coefficients, as the lack of regularity weakens the estimates for the Z_j^λ's, thus making necessary a proper shrink of the sequence.*

Through the procedure sketched above, we can prove the following:

Theorem 3.5 (Well-definiteness of μ)**.** *For any $T > 0$, the series (3.13) totally converges over $\{T^{-1} \leq t-\tau \leq T\}$; moreover, for every $z, \zeta \in \mathbb{R}^{n+1}$ with $0 < t-\tau \leq T$ the following estimate holds true:*

$$|\mu(z;\zeta)| \leq 2\check{c}_2 e^{\lambda T/2}\omega(2\sqrt{t-\tau})\,(t-\tau)^{-1}\gamma_0(\xi^{-1}\circ x, \check{c}_2(t-\tau)), \tag{3.16}$$

being $\check{c}_2 > 0$ the constant "of order 2" in (3.8)-(3.9).

Proof. Let us denote $\phi(\lambda,\epsilon) := 8\widetilde{\omega}(2\epsilon) + 16\Lambda\lambda^{-1}\epsilon^{-2}$ for any $\epsilon \in (0,1)$ and $\lambda > 0$. We prove the following estimates by induction on $j \geq 1$:

$$|Z_j^{(\lambda)}(z;\zeta)| \leq \check{c}_2^{\,j}\,[\phi(\lambda,\epsilon)]^{j-1}\,\frac{\omega(2\sqrt{t-\tau})}{t-\tau}\,\gamma_0(\xi^{-1}\circ x, \check{c}_2(t-\tau))\,e^{-\lambda(t-\tau)/2}; \tag{3.17}$$

the base case $j = 1$ trivially follows from (3.15). Assuming now the thesis holds true at step $j-1$ (with $j \geq 2$), for any $\epsilon \in (0,1)$ and $\lambda > 0$ the above yields

$$\begin{aligned}|Z_j^{\lambda}(z;\zeta)| &\leq \check{c}_2^{\,j}\,[\phi(\lambda,\epsilon)]^{j-2}e^{-\lambda(t-\tau)/2}\int_\tau^t \frac{\omega(2\sqrt{t-\eta})\omega(2\sqrt{\eta-\tau})}{(t-\eta)(\eta-\tau)}e^{-\lambda(t-\eta)/2}\left(\int_{\mathbb{R}^n}\gamma_0(y^{-1}\circ x, \check{c}_2(t-\eta)\right.\\ &\quad \left.\times\gamma_0(\xi^{-1}\circ y, \check{c}_2(\eta-\tau))\,dy\right)d\eta\\ &= \check{c}_2^{\,j}\,[\phi(\lambda,\epsilon)]^{j-2}e^{-\lambda(t-\tau)/2}\gamma_0(\xi^{-1}\circ x, \check{c}_2(t-\tau))\cdot I(\tau,t),\end{aligned} \tag{3.18}$$

having set

$$I = I(\tau,t) := \int_\tau^t \frac{\omega(2\sqrt{t-\eta})\,\omega(2\sqrt{\eta-\tau})}{(t-\eta)(\eta-\tau)}e^{-\lambda(t-\eta)/2}\,d\eta.$$

To suitably estimate I, we distinguish two cases: either $(t-\tau)/2 \leq \epsilon^2$ or $(t-\tau)/2 > \epsilon^2$.

(1) If $(t-\tau)/2 \leq \epsilon^2$, we decompose I as:

$$I = \left(\int_\tau^{(t+\tau)/2} + \int_{(t+\tau)/2}^t\right)\{...\}\,d\eta = I_1 + I_2.$$

Since in I_1 we have $t-\tau \geq t-\eta \geq (t-\tau)/2$, from (1.3) we deduce that

$$I_1 \leq 2\,\frac{\omega(2\sqrt{t-\tau})}{t-\tau}\int_\tau^{(t+\tau)/2}\frac{\omega(2\sqrt{\eta-\tau})}{\eta-\tau}\,d\eta \leq 4\,\frac{\omega(2\sqrt{t-\tau})}{t-\tau}\,\widetilde{\omega}(2\epsilon).$$

Similarly, since in I_2 there holds $t-\tau \geq \eta-\tau \geq (t-\tau)/2$, we also get

$$I_2 \leq 4\,\frac{\omega(2\sqrt{t-\tau})}{t-\tau}\,\widetilde{\omega}(2\epsilon).$$

(2) If instead $(t-\tau)/2 > \epsilon^2$, then we set

$$I = \left(\int_{\tau}^{\tau+\epsilon^2} + \int_{\tau+\epsilon^2}^{(t+\tau)/2} + \int_{(t+\tau)/2}^{t-\epsilon^2} + \int_{t-\epsilon^2}^{t} \right) \{...\}\, d\eta = I_1' + I_2' + I_3' + I_4'.$$

I_1' and I_4' can be treated as we did for I_1 and I_2, obtaining

$$I_1' + I_4' \leq 8(t-\tau)^{-1} \omega(2\sqrt{t-\tau})\, \widetilde{\omega}(2\epsilon).$$

Let us pass to I_3'. From (1.8) and the fact that $t - \eta > \epsilon^2$, we have

$$\begin{aligned} I_3' &\leq 2\epsilon^{-2}\, (t-\tau)^{-1}\, \omega(2\sqrt{t-\tau}) \int_{(t+\tau)/2}^{t-\epsilon^2} \omega(2\sqrt{\eta-\tau})\, e^{-\lambda(t-\eta)/2}\, d\eta \\ &\leq 4\Lambda\epsilon^{-2} \frac{\omega(2\sqrt{t-\tau})}{t-\tau} \int_{\tau}^{t} e^{-\lambda(t-\eta)/2}\, d\eta \leq 8\Lambda\lambda^{-1}\epsilon^{-2} \frac{\omega(2\sqrt{t-\tau})}{t-\tau}. \end{aligned}$$

A similar procedure shows the validity of the same estimate also for I_2'.

Combining the estimates for I_1, I_2, or $I_1' + I_4'$, I_2', I_3', we get

$$I \leq \frac{\omega(2\sqrt{t-\tau})}{t-\tau} \left[8\, \widetilde{\omega}(2\epsilon) + 16\Lambda\lambda^{-1}\epsilon^{-2} \right] = \frac{\omega(2\sqrt{t-\tau})}{t-\tau}\, \phi(\lambda, \epsilon)$$

for any $\epsilon \in (0,1)$, $\lambda > 0$. Plugging this back into (3.18) we get claim (3.17).

It is then immediate that Z_j is well defined for any $j \in \mathbb{N}$ and there holds

$$|Z_j(z;\zeta)| \leq \check{c}_2^{\,j}\, [\phi(\lambda,\epsilon)]^{j-1}\, \frac{\omega(2\sqrt{t-\tau})}{t-\tau}\, \gamma_0(\xi^{-1} \circ x, \check{c}_2(t-\tau))\, e^{\lambda(t-\tau)/2} \tag{3.19}$$

for any $\epsilon \in (0,1)$ and $\lambda > 0$. It follows that

$$\sum_{j=1}^{\infty} |Z_j(z;\zeta)| \leq \check{c}_2\, \frac{\omega(2\sqrt{t-\tau})}{t-\tau}\, \gamma_0(\xi^{-1} \circ x, \check{c}_2(t-\tau))\, e^{\lambda(t-\tau)/2} \sum_{j=0}^{\infty} [\check{c}_2 \phi(\lambda,\epsilon)]^j\,.$$

Recalling $\phi(\lambda,\epsilon) = 8\widetilde{\omega}(2\epsilon) + 16\Lambda\lambda^{-1}\epsilon^{-2}$, we eventually fix $\epsilon = \overline{\epsilon}$ small enough to have $\widetilde{\omega}(2\overline{\epsilon}) < (32\check{c}_2)^{-1}$ and then $\lambda = \overline{\lambda} > 64\Lambda(\check{c}_2\overline{\epsilon}^2)^{-1}$, so that $\check{c}_2\phi(\overline{\lambda},\overline{\epsilon}) < 1/2$; these values yield the convergence of the geometric series at the far right hand side (to a value not greater than 2), leading us to

$$\begin{aligned} \sum_{j=1}^{\infty} |Z_j(z;\zeta)| &\leq 2\check{c}_2\, \frac{\omega(2\sqrt{t-\tau})}{t-\tau}\, \gamma_0(\xi^{-1} \circ x, \check{c}_2(t-\tau))\, e^{\overline{\lambda}(t-\tau)/2} \\ &\leq 2\check{c}_2\, \frac{\omega(2\sqrt{t-\tau})}{t-\tau}\, \gamma_0(\xi^{-1} \circ x, \check{c}_2(t-\tau))\, e^{\overline{\lambda}T/2} \end{aligned}$$

for all $z = (x,t)$, $\zeta = (\xi,\tau) \in \mathbb{R}^{n+1}$ with $0 < t - \tau \leq T$. The convergence is total in any subset $\{T^{-1} \leq t - \tau \leq T\}$ in view of the boundedness of the function

$$(x,t;\xi,\tau) \mapsto \frac{\omega(2\sqrt{t-\tau})}{t-\tau}\, \gamma_0(\xi^{-1} \circ x, \check{c}_2(t-\tau))$$

therein. □

To remove the "λ-shrink" introduced above from the estimate (3.19) for Z_j, we fix λ to the value $\overline{\lambda}$ of the previous proof, obtaining

$$|Z_j(z;\zeta)| \le \check{c}_2 e^{\overline{\lambda}T/2}\,(t-\tau)^{-1}\omega(2\sqrt{t-\tau})\,\gamma_0(\xi^{-1}\circ x, \check{c}_2(t-\tau)) \tag{3.20}$$

for every z, $\zeta \in \mathbb{R}^{n+1}$ with $0 < t-\tau \le T$.

Corollary 3.6. *For every z, $\zeta \in \mathbb{R}^{n+1}$ with $t > \tau$, we have*

$$Z_1(z;\zeta) + \mu(z;\zeta) + \iint_{\mathbb{R}^n\times(\tau,t)} Z_1(z;\chi)\,\mu(\chi;\zeta)\,d\chi = 0.$$

Proof. By the total convergence of the series (3.13), we have

$$\begin{aligned}
-\iint_{\mathbb{R}^n\times(\tau,t)} Z_1(z;\chi)\,\mu(\chi;\zeta)\,d\chi &= \iint_{\mathbb{R}^n\times(\tau,t)} Z_1(z;\chi)\,\sum_{j=1}^{\infty} Z_j(\chi;\zeta)\,d\chi \\
&= \sum_{j=1}^{\infty}\iint_{\mathbb{R}^n\times(\tau,t)} Z_1(z;\chi)\,Z_j(\chi;\zeta)\,d\chi = -\sum_{j=1}^{\infty} Z_{j+1}(z;\zeta) \\
&= \mu(z;\zeta) + Z_1(z;\zeta).
\end{aligned}$$

□

In order to analyze the integral term in (3.11), we need to combine the Gaussian estimate (3.16) with suitable regularity results for μ; what we need is $\mu(z;\zeta)$ continuous for $z \ne \zeta$ and Dini continuous w.r.t. $x \in \mathbb{R}^n$ (for any fixed $\xi \in \mathbb{R}^n$, $t > \tau$). This will be the content of the following two results.

Proposition 3.7 (Continuity of μ)**.** *$\mu(x,t;\xi,\tau)$ is continuous in $\{t > \tau\}$.*

Proof. Thanks to the uniform convergence of the series (3.13) (see Proposition 3.5), it is sufficient to prove the continuity of Z_j for every j. If $j = 1$ the claim trivially holds by Theorem 3.1 and the assumption (H2). Let us then show that the continuity of Z_j implies that of Z_{j+1}, so that the thesis follows by induction; to this aim, we prove that the latter is a uniform limit of the functions

$$Z^{\sigma}_{j+1}(z;\zeta) := -\iint_{\mathbb{R}^n\times(\tau+\sigma,t-\sigma)} Z_1(z;\chi)Z_j(\chi;\zeta)\,d\chi, \quad z,\zeta\in\mathbb{R}^{n+1} : t > \tau + 2\sigma$$

as $\sigma \to 0^+$, in every compact subset of $\{t > \tau\}$.

Note that Z^{σ}_{j+1} is continuous by dominated convergence (based on (3.20)), hence the validity of the above claim is sufficient to get the continuity of Z_{j+1} in the set $\{t > \tau\}$. Given $K \Subset \{t > \tau\}$, denoting $I_\sigma := (\tau, \tau+\sigma) \cup (t-\sigma, t)$ and exploiting (3.2) and (3.20), we obtain

$$\begin{aligned}
\sup_K \left|Z_{j+1} - Z^{\sigma}_{j+1}\right| &\le \sup_{(z;\zeta)\in K}\iint_{\mathbb{R}^n\times I_\sigma} \left|Z_1(z;\chi)\,Z_j(\chi;\zeta)\right|\,d\chi \\
&\le c(K)\sup_{(z;\zeta)\in K}\int_{I_\sigma}\frac{\omega(2\sqrt{t-\eta})\,\omega(2\sqrt{\eta-\tau})}{(t-\eta)(\eta-\tau)}\,d\eta\int_{\mathbb{R}^n}\gamma_0(y^{-1}\circ x,\check{c}_2(t-\eta))\gamma_0(\xi^{-1}\circ y,\check{c}_2(\eta-\tau))\,dy \\
&\le c(K)\sup_{(z;\zeta)\in K}\gamma_0(\xi^{-1}\circ x,\check{c}_2(t-\tau))\int_{I_\sigma}\frac{\omega(2\sqrt{t-\eta})\,\omega(2\sqrt{\eta-\tau})}{(t-\eta)(\eta-\tau)}d\eta.
\end{aligned}$$

Setting $\rho = \sqrt{\eta - \tau}$ if $\eta \in (\tau, \tau + \sigma)$ or $\rho = \sqrt{t - \eta}$ if $\eta \in (t - \sigma, t)$, we get

$$\int_{I_\sigma} \frac{\omega(2\sqrt{t-\eta})\,\omega(2\sqrt{\eta-\tau})}{(t-\eta)(\eta-\tau)}\,d\eta \le 8\,\frac{\omega(2\sqrt{t-\tau})}{t-\tau}\int_0^{\sqrt{\sigma}} \frac{\omega(2\rho)}{\rho}\,d\rho.$$

Therefore, by (1.3) and the boundedness of γ_0 in compact subsets of $\{t > \tau\}$:

$$\sup_{(z;\zeta)\in K} \left| Z_{j+1}(z;\zeta) - Z^{\sigma}_{j+1}(z;\zeta)\right| \le c(K)\,\widetilde{\omega}(2\sqrt{\sigma})$$
$$\times \sup_{(z;\zeta)\in K} \left\{ \gamma_0(\xi^{-1}\circ x, \check{c}_2(t-\tau))\omega(2\sqrt{t-\tau})(t-\tau)^{-1} \right\} \longrightarrow 0, \quad \text{as } \sigma \to 0^+.$$

□

Theorem 3.8 (Dini continuity of μ in x)**.** *Given an arbitrary $T > 0$, the following estimate holds for every x, x', $\xi \in \mathbb{R}^n$, $0 < t - \tau \le T$:*

$$\begin{aligned} |\mu(x,t;\zeta) - \mu(x',t;\zeta)| &\le c(T)\left[\gamma_0(\xi^{-1}\circ x, \widehat{c_2}(t-\tau)) + \gamma_0(\xi^{-1}\circ x', \widehat{c_2}(t-\tau))\right] \\ &\quad \times (t-\tau)^{-1}\left[\omega(2d(x,x')) + \omega(2\sqrt{t-\tau})\,\widetilde{\omega}(2d(x,x'))\right] \end{aligned} \tag{3.21}$$

for a constant $\widehat{c_2}$ depending only on $\Lambda, \delta, \mathbb{G}$, and c depending also on T.

The proof exploits the following lemma, which is an immediate consequence of the quasi-triangle inequality of d:

Lemma 3.9. *For every c_1, $c_2 > 0$ there exist c_3, $c_4 > 0$ such that*

$$\sup_{d(x,y)\le c_2\sqrt{t-\tau}} \gamma_0(\xi^{-1}\circ y, c_1(t-\tau)) \le c_3 \cdot \gamma_0(\xi^{-1}\circ x, c_4(t-\tau))$$

for every $(x,t), (\xi,\tau) \in \mathbb{R}^{n+1}$ with $t > \tau$.

Proof. In view of the "equivalence" (3.3), we can prove the statement for $E(x,t)$ in place of $\gamma_0(x,t)$. For every $y \in \mathbb{R}^n$ with $d(x,y) \le c_2\sqrt{t-\tau}$ there holds

$$d(y,\xi) \ge \frac{d(x,\xi)}{\kappa} - d(x,y) \ge \frac{d(x,\xi)}{\kappa} - c_2\sqrt{t-\tau},$$

where $\kappa \ge 1$ is the constant of the quasi-triangle inequality of $d(\cdot,\cdot)$. Therefore, recalling that $(a-b)^2 \ge \frac{a^2}{2} - b^2$ for every $a, b \in \mathbb{R}$, we deduce that

$$\sup_{d(x,y)\le c_2\sqrt{t-\tau}} \exp\left\{-\frac{d(y,\xi)^2}{c_1(t-\tau)}\right\} \le \exp\left\{-\frac{d(x,\xi)^2}{2c_1\kappa^2(t-\tau)}\right\}\exp\left\{\frac{c_2^2}{c_1}\right\}$$

from which the thesis trivially follows, recalling the definition (1.2). □

Proof of Theorem 3.8. If $d(x,x')^2 \ge (t-\tau)/2$ then the thesis immediately follows from (3.16), being ω non-decreasing; hence we assume $d(x,x')^2 < (t-\tau)/2$. For the sake of simplicity, we introduce the following notation for the x-variation of a function f:

$$f(x',t;\zeta) - f(x,t;\zeta) =: \Delta_{(x,x')} f(\cdot,t;\zeta).$$

From Corollary 3.6, it is straightforward that

$$\Delta_{(x,x')}\mu(\cdot,t;\zeta) = \Delta_{(x,x')}Z_1(\cdot,t;\zeta) + \iint_{\mathbb{R}^n\times(\tau,t)} \Delta_{(x,x')}Z_1(\cdot,t;\chi)\,\mu(\chi;\zeta)\,d\chi. \tag{3.22}$$

In view of (3.14), we can decompose $\Delta_{(x,x')}Z_1(\cdot,t;\zeta)$ as follows:

$$\begin{aligned}\Delta_{(x,x')}Z_1(\cdot,t;\zeta) &= \Delta_{(x,x')}a^{ij}(\cdot,t)\,(X_i^xX_j^x\Gamma_\zeta)(x',t;\zeta)\\ &\quad + [a^{ij}(x,t)-a^{ij}(\zeta)]\Delta_{(x,x')}X_iX_j\Gamma_\zeta(\cdot,t;\zeta)\\ &= K_1(x,x',t;\zeta)+K_2(x,x',t;\zeta).\end{aligned} \tag{3.23}$$

As for K_1, by (1.5), (3.8) and Lemma 3.9, we immediately get

$$|K_1(x,x',t;\zeta)| \le \check{c}_2\,\omega(d(x,x'))\,(t-\tau)^{-1}\,\gamma_0(\xi^{-1}\circ x',\check{c}_2(t-\tau)). \tag{3.24}$$

We turn to K_2: in view of Lemma 2.3, we have

$$\begin{aligned}\left|\Delta_{(x,x')}X_iX_j\Gamma_\zeta(\cdot,t;\zeta)\right| &\le c_0 d(x,x') \sup_{d(x,y)\le c_0 d(x,x')} \sum_{k=1}^m \left|(X_k^xX_i^xX_j^x\Gamma_\zeta)(y,t;\zeta)\right| \le c_1 d(x,x')\\ \times (t-\tau)^{-\frac{3}{2}} &\sup_{d(x,y)\le c_0\sqrt{t-\tau}} \gamma_0(\xi^{-1}\circ y,\check{c}_3(t-\tau)) \le c_2 d(x,x')\,(t-\tau)^{-\frac{3}{2}}\,\gamma_0(\xi^{-1}\circ x,c_3(t-\tau))\end{aligned}$$

for any $1\le i,j\le m$, for constants $c_2,c_3>0$. Therefore, by (1.5) and (3.7),

$$|K_2(x,x',t;\zeta)| \le c\omega(2\sqrt{t-\tau})\,d(x,x')\,(t-\tau)^{-\frac{3}{2}}\,\gamma_0(\xi^{-1}\circ x,\widetilde{c}(t-\tau)). \tag{3.25}$$

Finally, from (1.6) there holds

$$\begin{aligned}d(x,x')\,\omega(2\sqrt{t-\tau}) &= d(x,x')^{1-\delta}\left[d(x,x')^{\delta}\,\omega(2\sqrt{t-\tau})\right]\\ &\le (t-\tau)^{\frac{1-\delta}{2}}\left[c(t-\tau)^{\frac{\delta}{2}}\,\omega(2d(x,x'))\right]\\ &= c(t-\tau)^{\frac{1}{2}}\,\omega(2d(x,x')),\end{aligned}$$

thus for positive constant $\widetilde{c}_0$ and $\widetilde{c}_1$, we have

$$|K_2(x,x',t;\zeta)| \le \widetilde{c}_0\,\omega(2d(x,x'))\,(t-\tau)^{-1}\,\gamma_0(\xi^{-1}\circ x,\widetilde{c}_1(t-\tau)).$$

The combination of the latter with (3.24) gives

$$|\Delta_{(x,x')}Z_1(\cdot,t;\zeta)| \le \widetilde{c}_2\omega(2d(x,x'))\,(t-\tau)^{-1}\,\gamma_0(\xi^{-1}\circ x,\widetilde{c}_3(t-\tau)). \tag{3.26}$$

Let us now focus on the integral term in (3.22), which we decompose as:

$$\begin{aligned}\int_\tau^t \left\{\int_{\mathbb{R}^n} \Delta_{(x,x')}Z_1(\cdot,t;y,\eta)\,\mu(y,\eta;\zeta)dy\right\}d\eta &= \left(\int_\tau^{(t+\tau)/2} + \int_{(t+\tau)/2}^{t-d(x,x')^2} + \int_{t-d(x,x')^2}^{t}\right)\{...\}\,d\eta\\ &= F_1+F_2+F_3\end{aligned}$$

and estimate F_1, F_2, F_3 separately. By (1.3), (3.10), (3.16) and (3.26),

$$
\begin{aligned}
|F_1| &\le c(T)\,\omega(2d(x,x'))\int_{\tau}^{(\tau+t)/2}\frac{\omega(2\sqrt{\eta-\tau})}{(t-\eta)(\eta-\tau)}\,d\eta\int_{\mathbb{R}^n}\gamma_0(y^{-1}\circ x,\widetilde{c}_3(t-\eta))\,\gamma_0(\xi^{-1}\circ y,\check{c}_2(\eta-\tau))\,dy\\
&\le c(T)\,\omega(2d(x,x'))\,(t-\tau)^{-1}\,\gamma_0(\xi^{-1}\circ x,c_1(t-\tau))\int_{\tau}^{(\tau+t)/2}\frac{\omega(2\sqrt{\eta-\tau})}{\eta-\tau}\,d\eta\\
&\le c(T)\,\omega(2d(x,x'))\,(t-\tau)^{-1}\,\widetilde{\omega}(2\sqrt{t-\tau})\gamma_0(\xi^{-1}\circ x,c_1(t-\tau)).
\end{aligned}
$$

As for F_3, by (3.20) and Lemma 3.9, we have

$$
\begin{aligned}
|\Delta_{(x,x')}Z_1(\cdot,t;\zeta)| &\le |Z_1(x,t;\zeta)|+|Z_1(x',t;\zeta)|\\
&\le c(T)\,\omega(2\sqrt{t-\tau})(t-\tau)^{-1}\Big[\gamma_0(\xi^{-1}\circ x,\widetilde{c}_3(t-\tau))\\
&\quad+\gamma_0(\xi^{-1}\circ x',\widetilde{c}_3(t-\tau))\Big]\\
&\le c(T)\,\omega(2\sqrt{t-\tau})\,(t-\tau)^{-1}\,\gamma_0(\xi^{-1}\circ x,c_1(t-\tau)).
\end{aligned}
$$

Hence, in view of (H2), (3.10) and (3.16),

$$
\begin{aligned}
|F_3| &\le c(T)\int_{t-d(x,x')^2}^{t}\frac{\omega(2\sqrt{t-\eta})\,\omega(2\sqrt{\eta-\tau})}{(t-\eta)(\eta-\tau)}\,d\eta\int_{\mathbb{R}^n}\gamma_0(\xi^{-1}\circ y,\check{c}_2(\eta-\tau))\gamma_0(y^{-1}\circ x,c_1(t-\eta))\,dy\\
&\le c(T)\,\omega(2\sqrt{t-\tau})\,(t-\tau)^{-1}\,\gamma_0(\xi^{-1}\circ x,c_2(t-\tau))\int_{t-d(x,x')^2}^{t}\frac{\omega(2\sqrt{t-\eta})}{t-\eta}\,d\eta\\
&\le c(T)\,\omega(2\sqrt{t-\tau})\,\widetilde{\omega}(2d(x,x'))\,(t-\tau)^{-1}\,\gamma_0(\xi^{-1}\circ x,c_2(t-\tau)).
\end{aligned}
$$

Finally, to estimate F_2 we resort once again to the decomposition (3.23):

$$
\begin{aligned}
F_2 &= \int_{(\tau+t)/2}^{t-d(x,x')^2}\int_{\mathbb{R}^n}[K_1(x,x',t;\chi)+K_2(x,x',t;\chi)]\,\mu(\chi;\zeta)\,d\chi\\
&= F_2^1+F_2^2.
\end{aligned}
$$

As for F_2^2, by (3.10), (3.16) and (3.25), we obtain

$$
\begin{aligned}
|F_2^2| &\le c(T)\,d(x,x')\int_{(\tau+t)/2}^{t-d(x,x')^2}\frac{\omega(2\sqrt{t-\eta})\,\omega(2\sqrt{\eta-\tau})}{(t-\eta)^{3/2}(\eta-\tau)}\,d\eta\int_{\mathbb{R}^n}\gamma_0(y^{-1}\circ x,\widetilde{c}(t-\eta))\\
&\quad\times\gamma_0(\xi^{-1}\circ y,\check{c}_2(\eta-\tau))\,dy\\
&\le c(T)\,d(x,x')\,\frac{\omega(2\sqrt{t-\tau})}{t-\tau}\,\gamma_0(\xi^{-1}\circ x,c_1(t-\tau))\int_{(\tau+t)/2}^{t-d(x,x')^2}\frac{\omega(2\sqrt{t-\eta})}{(t-\eta)^{3/2}}\,d\eta.
\end{aligned}
$$

For $\eta\in[(\tau+t)/2,t-d(x,x')^2]$, we have $t-\eta\ge d(x,x')^2$, hence by (1.6):

$$
(t-\eta)^{-\delta/2}\omega(2\sqrt{t-\eta})\le c\,d(x,x')^{-\delta}\,\omega(2d(x,x')).
$$

Recalling that $(t-\tau)/2\ge d(x,x')^2$, it follows that

$$\int_{(\tau+t)/2}^{t-d(x,x')^2} \frac{\omega(2\sqrt{t-\eta})}{(t-\eta)^{3/2}}\,d\eta \le c\,\frac{\omega(2d(x,x'))}{d(x,x')^{\delta}} \int_{(\tau+t)/2}^{t-d(x,x')^2} (t-\eta)^{-(3-\delta)/2}\,d\eta$$
$$\le c\,\frac{\omega(2d(x,x'))}{d(x,x')^{\delta}}\left[\left(\frac{t-\tau}{2}\right)^{-(1-\delta)/2} + d(x,x')^{-(1-\delta)}\right]$$
$$\le c\,\frac{\omega(2d(x,x'))}{d(x,x')}.$$

Plugging this inequality in the above estimate of F_2^2 we find

$$|F_2^2| \le c(T)\,\omega(2d(x,x'))\,\omega(2\sqrt{t-\tau})\,(t-\tau)^{-1}\,\gamma(\xi^{-1}\circ x, c_1(t-\tau)).$$

Let us turn to F_2^1. By (3.10), (3.16) and (3.24), we deduce

$$|F_2^1| \le c(T)\,\omega(d(x,x')) \int_{(\tau+t)/2}^{t-d(x,x')^2} \frac{\omega(2\sqrt{\eta-\tau})}{(t-\eta)(\eta-\tau)}\,d\eta \int_{\mathbb{R}^n} \gamma_0(y^{-1}\circ x, \check{c}(t-\eta))\gamma_0(\xi^{-1}\circ y, \check{c}_2(\eta-\tau))\,dy$$
$$\le c(T)\,\omega(d(x,x'))\,\frac{\omega(2\sqrt{t-\tau})}{t-\tau}\gamma_0(\xi^{-1}\circ x, c_1(t-\tau)) \int_{(\tau+t)/2}^{t-d(x,x')^2} \frac{1}{t-\eta}\,d\eta.$$

Since $d(x,x')^2 \le t-\eta$ and ω is non decreasing, then

$$\omega(d(x,x')) \le \omega^{\frac{1}{2}-\alpha}(d(x,x'))\,\omega^{\frac{1}{2}+\alpha}(\sqrt{t-\eta}), \quad \forall \alpha \in \left(0, \frac{1}{4}\right).$$

Therefore, in view of Lemma 2.6, for any $\alpha \in \left(0, \frac{1}{4}\right)$, it holds

$$|F_2^1| \le c(T)\,\omega^{\frac{1}{2}-\alpha}(d(x,x'))\,\frac{\omega(2\sqrt{t-\tau})}{t-\tau}\,\gamma_0(\xi^{-1}\circ x, c_1(t-\tau)) \int_\tau^t \frac{\omega^{\frac{1}{2}+\alpha}(\sqrt{t-\eta})}{t-\eta}\,d\eta$$
$$\le c(T)\,\omega^{\frac{1}{2}-\alpha}(d(x,x'))\frac{\omega(2\sqrt{t-\tau})}{t-\tau}\,\gamma_0(\xi^{-1}\circ x, c_1(t-\tau)).$$

Among all terms of which the x-variation of μ consists, F_2^1 is the only one not bounded from the right-hand side of (3.8); nonetheless, the above estimates yield

$$|\Delta_{(x,x')}\mu(\cdot,t;\xi,\tau)| \le c(T)\,(t-\tau)^{-1}\,\gamma_0(\xi^{-1}\circ x, c_1(t-\tau))\left[\omega^{\frac{1}{2}-\alpha}(2d(x,x')) + \omega(2\sqrt{t-\tau})\,\widetilde{\omega}(2d(x,x'))\right], \tag{3.27}$$

which is a sort of first step towards the thesis; now the strategy is to re-estimate F_2^1 in view of this. We further decompose the latter as follows:

$$F_2^1 = \int_{(\tau+t)/2}^{t-d(x,x')^2} \int_{\mathbb{R}^n} \Delta_{(x,x')}a^{ij}(\cdot,t)X_iX_j\gamma_\chi(y^{-1}\circ x', t-\eta)\,\mu(x',\eta;\zeta)\,d\chi$$
$$+ \int_{(\tau+t)/2}^{t-d(x,x')^2} \int_{\mathbb{R}^n} \Delta_{(x,x')}a_{ij}(\cdot,t)X_iX_j\gamma_\chi(y^{-1}\circ x', t-\eta)\,\Delta_{(x',y)}\mu(\cdot,\eta;\zeta)\,d\chi = V_1 + V_2$$

and use (3.27) to bound V_2; with the aid of (1.5), (3.2), (3.6), (3.8), this gives

$$\begin{aligned}
|V_2| &\le c(T)\,\omega(d(x,x'))\int_{(\tau+t)/2}^{t-d(x,x')^2}\frac{d\eta}{(t-\eta)(\eta-\tau)}\int_{\mathbb{R}^n}\gamma_0(y^{-1}\circ x', c_1(t-\eta))\gamma_0(\xi^{-1}\circ y, c_1(\eta-\tau))\\
&\times\left[\omega^{\frac{1}{2}-\alpha}(2d(x',y))+\omega(2\sqrt{\eta-\tau})\,\widetilde{\omega}(2d(x',y))\right]dy\\
&\le c(T)\,\omega(d(x,x'))\,(t-\tau)^{-1}\int_{(t+\tau)/2}^{t-d(x,x')^2}\frac{\omega^{\frac{1}{2}-\alpha}(2\sqrt{t-\eta})+\omega(2\sqrt{\eta-\tau})\widetilde{\omega}(2\sqrt{t-\eta})}{t-\eta}\,d\eta\\
&\times\int_{\mathbb{R}^n}\gamma_0(y^{-1}\circ x', c_1(t-\eta))\,\gamma_0(\xi^{-1}\circ y, c_1(\eta-\tau))\,dy\\
&\le c(T)\,\omega^{1-2\alpha}(d(x,x'))\,(t-\tau)^{-1}\,\gamma_0(\xi^{-1}\circ x', c_1(t-\tau))\int_0^t\frac{\omega^{\frac{1}{2}+\alpha}(2\sqrt{t-\eta})\,+\widetilde{\omega}(2\sqrt{t-\eta})}{t-\eta}d\eta.
\end{aligned}$$

The last integral converges by (1.4) and Lemma 2.6, thus

$$|V_2|\le c(T)\,\omega^{1-2\alpha}(d(x,x'))\,(t-\tau)^{-1}\,\gamma_0(\xi^{-1}\circ x', c_1(t-\tau)).$$

As for V_1, first of all we exploit Corollary 3.3 to state that

$$V_1=\int_{(\tau+t)/2}^{t-d(x,x')^2}\int_{\mathbb{R}^n}\Delta_{(x,x')}a^{ij}(\cdot,t)\,\mu(x',\eta;\zeta)\left[X_iX_j\gamma_{(y,\eta)}(y^{-1}\circ x',t-\eta)-X_iX_j\gamma_{(x',\eta)}(y^{-1}\circ x',t-\eta)\right]d\chi.$$

Note that, for any $\eta\in\left(\frac{t+\tau}{2},t\right)$, in view of (1.2)–(3.3), the following holds:

$$\begin{aligned}
&E(\xi^{-1}\circ x, c(\eta-\tau))\le 2^{Q/2}\,E(\xi^{-1}\circ x, c(t-\tau))\\
&\implies\ \gamma_0(\xi^{-1}\circ x, c(\eta-\tau))\le c'\gamma_0(\xi^{-1}\circ x, c(t-\tau)).
\end{aligned}\tag{3.28}$$

Together with (H2), (3.1), (3.6), (3.9), (3.16), the above gives

$$\begin{aligned}
|V_1| &\le c(T)\,\omega(d(x,x'))\int_{(\tau+t)/2}^{t-d(x,x')^2}\frac{\omega(2\sqrt{\eta-\tau})}{(t-\eta)(\eta-\tau)}\,\gamma_0(\xi^{-1}\circ x', c_1(\eta-\tau))\,d\eta\int_{\mathbb{R}^n}\omega(d(y,x'))\\
&\times\gamma_0(y^{-1}\circ x',\check{c}_2(t-\eta))\,dy\\
&\le c(T)\,\omega(d(x,x'))\,\omega(2\sqrt{t-\tau})\,(t-\tau)^{-1}\,\gamma_0(\xi^{-1}\circ x', c_1(t-\tau))\\
&\times\int_{(\tau+t)/2}^{t-d(x,x')^2}\frac{\omega(\sqrt{t-\eta})}{t-\eta}\,d\eta\int_{\mathbb{R}^n}\gamma_0(y^{-1}\circ x',\check{c}_2(t-\eta))\,dy\\
&\le c(T)\,\omega(d(x,x'))\,\omega(2\sqrt{t-\tau})\,(t-\tau)^{-1}\,\gamma_0(\xi^{-1}\circ x', c_1(t-\tau)).
\end{aligned}$$

Combining the estimates for V_1, V_2 and using Lemma 3.9, we obtain

$$|F_2^1|\le c(T)\,(t-\tau)^{-1}\,\omega^{1-2\alpha}(d(x,x'))\,\gamma_0(\xi^{-1}\circ x, c_1(t-\tau)),$$

from which it follows that

$$|\Delta_{(x,x')}\mu(\cdot,t;\xi,\tau)|\le c(T)\,(t-\tau)^{-1}\,\gamma_0(\xi^{-1}\circ x, c_1(t-\tau))\left[\omega^{1-2\alpha}(2d(x,x'))+\omega(2\sqrt{t-\tau})\,\widetilde{\omega}(2d(x,x'))\right].\tag{3.29}$$

Iterating the above argument, to conclude we must update only the estimate of V_2. Following the procedure above but using (3.29) in place of (3.27), we find

$$|V_2| \le c(T)\,\omega(d(x,x'))\,(t-\tau)^{-1}\,\gamma_0(\xi^{-1}\circ x', c_1(t-\tau)) \int_0^t \frac{\omega^{1-2\alpha}(2\sqrt{t-\eta}) + \widetilde{\omega}(2\sqrt{t-\eta})}{t-\eta}\,d\eta.$$

Since $\alpha < \frac{1}{4}$ we can directly apply Lemma 2.6, which together with (H2) gives

$$|V_2| \le c(T)\,\omega(d(x,x'))\,(t-\tau)^{-1}\,\gamma_0(\xi^{-1}\circ x', c_1(t-\tau)),$$

thus ending the proof, according to the remark made above. □

To shorten the notation, sometimes it will be convenient to express (3.21) as

$$|\Delta_{(x,x')}\mu(\cdot,t;\zeta)| \le c(T)(t-\tau)^{-1}\widetilde{\omega}(2d(x,x'))[\gamma_0(\xi^{-1}\circ x, \widehat{c_2}(t-\tau)) + \gamma_0(\xi^{-1}\circ x', \widehat{c_2}(t-\tau))] \tag{3.30}$$

for all $x, \xi \in \mathbb{R}^n$ and $0 < t-\tau \le T$ (it follows from (3.21) by (2.1)).

Remark 3.10. *The above proof could not work if the coefficients $a_{ij}(x,t)$ of H were only Dini continuous: indeed we have exploited multiple times Lemma 2.6, which in turn requires the double Dini condition to hold for ω.*

3.3. Differentiability of the integral term

Let us exploit estimates and regularity of the kernel μ to study well-definiteness and differentiability (along H) of the integral term

$$J(z;\zeta) := \iint_{\mathbb{R}^n\times(\tau,t)} \Gamma_\chi(z;\chi)\,\mu(\chi;\zeta)\,d\chi, \quad x\in\mathbb{R}^n,\ \xi\in\mathbb{R}^n,\ t>\tau.$$

Note that (1.3), (3.8), (3.10) and (3.16) give

$$\begin{aligned}
&\iint_{\mathbb{R}^n\times(\tau,t)} \left|\Gamma_\chi(z;\chi)\,\mu(\chi;\zeta)\right|\,d\chi \\
&\le c(T)\int_\tau^t \frac{\omega(2\sqrt{\eta-\tau})}{\eta-\tau}\,d\eta \int_{\mathbb{R}^n} \gamma_0(y^{-1}\circ x, \check{c}_0(t-\eta))\gamma_0(\xi^{-1}\circ y, \check{c}_2(\eta-\tau))\,dy \\
&\le c(T)\,\widetilde{\omega}(2\sqrt{t-\tau})\,\gamma_0(\xi^{-1}\circ x, c_1(t-\tau)),
\end{aligned}$$

namely J is well defined and the following estimate holds:

$$|J(z;\zeta)| \le c(T)\,\widetilde{\omega}(2\sqrt{t-\tau})\,\gamma_0(\xi^{-1}\circ x, \widehat{c_3}(t-\tau)) \tag{3.31}$$

for any $x\in\mathbb{R}^n$, $\xi\in\mathbb{R}^n$, $0 < t-\tau \le T$, for a positive constant $\widehat{c_3}$ depending only on Λ, δ and $\mathbb{G}$. Moreover:

Proposition 3.11. *$J(x,t;\xi,\tau)$ is a continuous function over $\{t > \tau\}$.*

Proof. As for the proof of Proposition 3.7, it suffices to show that J is the uniform limit (in every $K \Subset \{t > \tau\}$) of the family of continuous functions

$$J^\sigma(z;\zeta) := \iint_{\mathbb{R}^n \times (\tau+\sigma, t-\sigma)} \Gamma_\chi(z;\chi)\,\mu(\chi;\zeta)\,d\chi$$

as $\sigma \to 0^+$. Given $K \Subset \{t > \tau\}$, by (3.8), (3.10) and (3.16), we have

$$\begin{aligned}
\sup_{(z;\zeta)\in K} |J^\sigma(z;\zeta) - J(z;\zeta)| &\le \sup_{(z;\zeta)\in K} \left(\int_\tau^{\tau+\sigma} + \int_{t-\sigma}^t \right) \int_{\mathbb{R}^n} \left| \Gamma_\chi(z;\chi)\,\mu(\chi,\zeta) \right| d\chi \\
&\le c(K) \sup_{(z;\zeta)\in K} \left(\int_\tau^{\tau+\sigma} + \int_{t-\sigma}^t \right) \frac{\omega(2\sqrt{\eta-\tau})}{\eta-\tau} d\eta \int_{\mathbb{R}^n} \\
&\times \gamma_0(y^{-1}\circ x, t-\eta)\gamma_0(\xi^{-1}\circ y, \check{c}_2(\eta-\tau))\,dy \\
&\le c(K) \sup_{(z;\zeta)\in K} \left[\gamma_0(\xi^{-1}\circ x, c_1(t-\tau)) \left(\int_\tau^{\tau+\sigma} + \int_{t-\sigma}^t \right) \frac{\omega(2\sqrt{\eta-\tau})}{\eta-\tau} d\eta \right].
\end{aligned}$$

In $K \Subset \{t > \tau\}$ we have $t-\tau$ far from zero, hence γ_0 is bounded; for $\eta \in (t-\sigma, t)$ we use $\frac{t-\tau}{2} \le \eta-\tau \le t-\tau$, whereas for $\eta \in (\tau, \tau+\sigma)$ we use (H2), to get

$$\begin{aligned}
\sup_{(z;\zeta)\in K} |J^\sigma(z;\zeta) - J(z;\zeta)| &\le c(K) \sup_{(z;\zeta)\in K} \left\{ \int_\tau^{\tau+\sigma} \frac{\omega(2\sqrt{\eta-\tau})}{\eta-\tau} d\eta + \int_{t-\sigma}^t \frac{\omega(2\sqrt{\eta-\tau})}{\eta-\tau} d\eta \right\} \\
&\le c(K) \sup_{(z;\zeta)\in K} \left\{ 2\widetilde{\omega}(2\sqrt{\sigma}) + \frac{\omega(2\sqrt{t-\tau})}{(t-\tau)/2} \cdot \sigma \right\} \longrightarrow 0
\end{aligned}$$

as $\sigma \to 0^+$, which concludes the proof. □

Then, according to Definition 2.2, we have:

Theorem 3.12 (Derivatives of J)**.** *For every fixed $\zeta = (\xi,\tau) \in \mathbb{R}^{n+1}$, the function $J(\cdot;\zeta)$ belongs to $\mathfrak{C}^2(\{z = (x,t) \in \mathbb{R}^{n+1} : t > \tau\})$ and we have*

$$X_j^x J(z;\zeta) = \iint_{\mathbb{R}^n \times (\tau,t)} X_j^x \Gamma_\chi(z;\chi)\,\mu(\chi;\zeta)\,d\chi, \tag{3.32}$$

$$X_i^x X_j^x J(z;\zeta) = \lim_{\epsilon\to 0} \iint_{\mathbb{R}^n \times (\tau,t-\epsilon)} X_i^x X_j^x \Gamma_\chi(z;\chi)\,\mu(\chi;\zeta)\,d\chi, \tag{3.33}$$

$$\partial_t J(z;\zeta) = \mu(z;\zeta) + \lim_{\epsilon\to 0} \iint_{\mathbb{R}^n \times (\tau,t-\epsilon)} \partial_t \Gamma_\chi(z;\chi)\,\mu(\chi;\zeta)\,d\chi. \tag{3.34}$$

Moreover, the following estimates hold for $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$, $0 < t-\tau \le T$:

$$|X_j^x J(z;\zeta)| \le c(T)\,\widetilde{\omega}(2\sqrt{t-\tau})\,(t-\tau)^{-\frac{1}{2}}\,E(\xi^{-1}\circ x, \widehat{c_4}(t-\tau)), \tag{3.35}$$

$$|X_i^x X_j^x J(z;\zeta)| \le c(T)\,\widetilde{\omega}(2\sqrt{t-\tau})\,(t-\tau)^{-1}\,E(\xi^{-1}\circ x, \widehat{c_4}(t-\tau)), \tag{3.36}$$

$$|\partial_t J(z;\zeta)| \le c(T)\,\widetilde{\omega}(2\sqrt{t-\tau})\,(t-\tau)^{-1}\,E(\xi^{-1}\circ x, \widehat{c_4}(t-\tau)) \tag{3.37}$$

for a constant $\widehat{c_4}$ depending only on $\Lambda, \delta, \mathbb{G}$, and c depending also on T.

The proof exploits the following technical lemma:

Lemma 3.13. *[6, Lemma 2.8] Let $X \in \mathfrak{g}$ and let $\{u_j\}_j$ be a sequence of continuous functions, defined on an open subset $A \subseteq \mathbb{R}^n$, with continuous Lie derivative along X. Suppose that u_j converges pointwise in A to some function u and that Xu_j converges to some function w uniformly on the compact subsets of A. Then there exists the Lie derivative of u along X, $Xu(x) = w(x)$, for every $x \in A$.*

Proof of Theorem 3.12. As for (3.32)–(3.34), it is enough to prove them for $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$, $0 < t-\tau \leq T$, for arbitrary $T > 0$. For any $\epsilon > 0$ (small enough) we introduce the function

$$J_\epsilon(z;\zeta) := \iint_{\mathbb{R}^n \times (\tau, t-\epsilon)} \Gamma_\chi(z;\chi)\, \mu(\chi;\zeta)\, d\chi.$$

It is immediate that $J_\epsilon(\cdot, t; \xi, \tau) \longrightarrow J(\cdot, t; \xi, \tau)$ pointwise in $\mathbb{R}^n$ as $\epsilon \to 0^+$.

By Theorem 3.1 and dominated convergence (based on (3.4) and (3.16)), we have

$$X_j^x J_\epsilon(z;\zeta) = \iint_{\mathbb{R}^n \times (\tau, t-\epsilon)} X_j^x \Gamma_\chi(z;\chi)\, \mu(\chi;\zeta)\, d\chi,$$
$$X_i^x X_j^x J_\epsilon(z;\zeta) = \iint_{\mathbb{R}^n \times (\tau, t-\epsilon)} X_i^x X_j^x \Gamma_\chi(z;\chi)\, \mu(\chi;\zeta)\, d\chi$$

for any $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$, $t > \tau$. Now we introduce the function

$$w(z;\zeta) := \iint_{\mathbb{R}^n \times (\tau, t)} X_j^x \Gamma_\chi(z;\chi)\, \mu(\chi;\zeta)\, d\chi \quad z,\ \zeta \in \mathbb{R}^{n+1};\ t > \tau,$$

which, by (3.8) and (3.16), is well defined and satisfy

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^n} |X_j^x J_\epsilon(x,t;\xi,\tau) - w(x,t;\xi,\tau)| \\
&\leq c(T) \sup_{x\in\mathbb{R}^n} \int_{t-\epsilon}^{t} \frac{\omega(2\sqrt{\eta-\tau})}{\sqrt{t-\eta}\,(\eta-\tau)} d\eta \\
&\quad \times \int_{\mathbb{R}^n} \gamma_0(y^{-1} \circ x, \check{c}_1(t-\eta))\, \gamma_0(\xi^{-1} \circ y, \check{c}_2(\eta-\tau))\, dy.
\end{aligned}$$

Let us assume that $\epsilon < (t-\tau)/2$, so that $t - \epsilon > (t+\tau)/2$. By (3.3) and (3.10), for some $c_1 > 0$ and $c_2 = \overline{c}_0 c_1$, we have

$$\begin{aligned}
\sup_{x\in\mathbb{R}^n} \left| X_j^x J_\epsilon(x,t;\zeta) - w(x,t;\zeta) \right| &\leq c(T) \frac{\omega(2\sqrt{t-\tau})}{t-\tau} \int_{t-\epsilon}^{t} \frac{1}{\sqrt{t-\eta}}\, d\eta \cdot \sup_{x\in\mathbb{R}^n} \gamma_0(\xi^{-1} \circ x, c_1(t-\tau)) \\
&\leq c(T)\, \frac{\omega(2\sqrt{t-\tau})}{t-\tau}\, \sqrt{\epsilon} \cdot \sup_{x\in\mathbb{R}^n} E(\xi^{-1} \circ x, c_2(t-\tau)) \\
&= c(T)(t-\tau)^{-\frac{Q}{2}-1}\, \omega(2\sqrt{t-\tau})\, \sqrt{\epsilon} \longrightarrow 0
\end{aligned}$$

as $\epsilon \to 0^+$, thus (3.32) holds in view of Lemma 3.13. Furthermore, proceeding as for the last chain of inequalities, it is immediate to get

$$|X_j^x J(z;\zeta)| \leq c(T) E(\xi^{-1} \circ x, c_2(t-\tau)) \int_\tau^t \frac{\omega(2\sqrt{\eta-\tau})}{(t-\eta)^{1/2}(\eta-\tau)}\, d\eta.$$

Decomposing $(\tau, t) = (\tau, (t+\tau)/2] \cup ((t+\tau)/2, t)$ and using (1.3), we obtain

$$\int_{\tau}^{\frac{t+\tau}{2}} \frac{\omega(2\sqrt{\eta-\tau})}{(t-\eta)^{\frac{1}{2}}(\eta-\tau)}\, d\eta \leq \frac{\sqrt{2}}{(t-\tau)^{\frac{1}{2}}} \int_{\tau}^{\frac{t+\tau}{2}} \frac{\omega(2\sqrt{\eta-\tau})}{\eta-\tau} d\eta \leq \sqrt{2}\, \frac{\widetilde{\omega}(2\sqrt{t-\tau})}{(t-\tau)^{\frac{1}{2}}},$$
$$\int_{\frac{t+\tau}{2}}^{t} \frac{\omega(2\sqrt{\eta-\tau})}{(t-\eta)^{\frac{1}{2}}(\eta-\tau)}\, d\eta \leq 2\, \frac{\omega(2\sqrt{t-\tau})}{t-\tau} \int_{\frac{t+\tau}{2}}^{t} (t-\eta)^{-1/2} d\eta \leq C\, \frac{\widetilde{\omega}(2\sqrt{t-\tau})}{(t-\tau)^{\frac{1}{2}}},$$

which plugged in the inequality above give (3.35).

As for (3.33), in view of Lemma 3.13, it suffices to prove that

$$\sup_{x\in\mathbb{R}^n} \left| \int_{t-\epsilon}^{t} d\eta \int_{\mathbb{R}^n} X_i^x X_j^x \Gamma_{(y,\eta)}(z; y, \eta)\, \mu(y, \eta; \zeta)\, dy \right| \longrightarrow 0, \quad \text{as } \epsilon \to 0^+. \tag{3.38}$$

To this purpose, we consider the integral

$$\widehat{I} = \int_{\mathbb{R}^n} X_i^x X_j^x \Gamma_{(y,\eta)}(z; y, \eta)\, \mu(y, \eta; \zeta)\, dy; \tag{3.39}$$

fixed $y_0 \in \mathbb{R}^n$, we split $\widehat{I}$ as $\widehat{I} = \widehat{I_1} + \widehat{I_2} + \widehat{I_3}$, with

$$\begin{aligned}
\widehat{I_1} &= \int_{\mathbb{R}^n} X_i^x X_j^x \Gamma_{(y,\eta)}(z; y, \eta)\, [\mu(y, \eta; \zeta) - \mu(y_0, \eta; \zeta)]\, dy, \\
\widehat{I_2} &= \mu(y_0, \eta; \zeta) \int_{\mathbb{R}^n} X_i^x X_j^x [\Gamma_{(y,\eta)}(z; y, \eta) - \Gamma_{(y_0,\eta)}(z; y, \eta)]\, dy, \\
\widehat{I_3} &= \mu(y_0, \eta; \zeta) \int_{\mathbb{R}^n} X_i^x X_j^x \Gamma_{(y_0,\eta)}(z; y, \eta)\, dy.
\end{aligned}$$

Corollary 3.3 yields $\widehat{I_3} = 0$. To estimate $\widehat{I_1}$ and $\widehat{I_2}$ we fix $y_0 = x$; from (3.1), (3.6), (3.8), (3.10), (3.21) and (3.28), we get

$$\begin{aligned}
|\widehat{I_1}| \leq\, & c(T)(t-\eta)^{-1}(\eta-\tau)^{-1} \int_{\mathbb{R}^n} [\omega(2d(x,y)) \\
& + \omega(2\sqrt{\eta-\tau})\widetilde{\omega}(2d(x,y))]\, \gamma_0(y^{-1}\circ x, \check{c}_2(t-\eta)) \\
& \times \left[\gamma_0(\xi^{-1}\circ y, \widehat{c_2}(\eta-\tau)) + \gamma_0(\xi^{-1}\circ x, \widehat{c_2}(\eta-\tau))\right] dy \\
\leq\, & c(T)\, \frac{\omega(2\sqrt{t-\eta}) + \omega(2\sqrt{\eta-\tau})\widetilde{\omega}(2\sqrt{t-\eta})}{(t-\eta)(\eta-\tau)} \\
& \times \left[\int_{\mathbb{R}^n} \gamma_0(y^{-1}\circ x, c_1(t-\eta))\gamma_0(\xi^{-1}\circ y, c_1(\eta-\tau))\, dy \right.\\
& \left. + \gamma_0(\xi^{-1}\circ x, c_1(\eta-\tau)) \int_{\mathbb{R}^n} \gamma_0(y^{-1}\circ x, c_1(t-\eta))\, dy \right] \\
\leq\, & c(T)\, \frac{\omega(2\sqrt{t-\eta}) + \omega(2\sqrt{\eta-\tau})\, \widetilde{\omega}(2\sqrt{t-\eta})}{(t-\eta)(\eta-\tau)}\, \gamma_0(\xi^{-1}\circ x, c_1(t-\tau)).
\end{aligned}$$

Hence, by using (1.2)–(3.3) and the fact that $\eta - \tau \geq (t-\tau)/2$, we find

$$\sup_{x\in\mathbb{R}^n}\int_{t-\epsilon}^{t}|\widehat{I_1}|\,d\eta \le c(T)\,(t-\tau)^{-1-\frac{Q}{2}}\left[\int_{t-\epsilon}^{t}\frac{\omega(2\sqrt{t-\eta})+\widetilde{\omega}(2\sqrt{t-\eta})}{t-\eta}\,d\eta\right]$$
$$= c(T)\,(t-\tau)^{-1-\frac{Q}{2}}\left[\widetilde{\omega}(2\sqrt{\epsilon})+\widetilde{\widetilde{\omega}}(2\sqrt{\epsilon})\right]\longrightarrow 0,\quad \text{as } \epsilon\to 0^+,$$

thanks to (H2), i.e., the double Dini continuity of the coefficients a_{ij}.

As for $\widehat{I_2}$, from (3.1), (3.6), (3.9), (3.16) and (3.28), we get

$$|\widehat{I_2}| \le c(T)\,(t-\eta)^{-1}\,(\eta-\tau)^{-1}\,\omega(2\sqrt{\eta-\tau})\,\gamma_0(\xi^{-1}\circ x,\check{c}_2(\eta-\tau))\int_{\mathbb{R}^n}\omega(d(x,y))\,\gamma_0(y^{-1}\circ x,\check{c}_2(t-\eta))\,dy$$
$$\le c(T)\frac{\omega(\sqrt{t-\eta})}{t-\eta}\,\frac{\omega(2\sqrt{t-\eta})}{\eta-\tau}\,\gamma_0(\xi^{-1}\circ x,\check{c}_2(t-\tau))\int_{\mathbb{R}^n}\gamma_0(y^{-1}\circ x,\check{c}_2(t-\eta))\,dy$$
$$= c(T)\,(t-\eta)^{-1}\,(\eta-\tau)^{-1}\,\omega(2\sqrt{\eta-\tau})\,\omega(\sqrt{t-\eta})\,\gamma_0(\xi^{-1}\circ x,\check{c}_2(t-\tau))$$

and so by (H2), (1.2), (3.3) and the fact that $t-\eta\ge(t-\tau)/2$, we deduce

$$\sup_{x\in\mathbb{R}^n}\int_{t-\epsilon}^{t}|\widehat{I_2}|\,d\eta \le c(T)\,(t-\tau)^{-\frac{Q}{2}-1}\,\omega(2\sqrt{t-\tau})\int_{t-\epsilon}^{t}\frac{\omega(\sqrt{t-\eta})}{t-\eta}\,d\eta$$
$$= c(T)\,(t-\tau)^{-\frac{Q}{2}-1}\,\widetilde{\omega}(\sqrt{\epsilon})\longrightarrow 0,\quad \text{as } \epsilon\to 0^+,$$

namely (3.38) holds, thus yielding (3.33). For what concerns (3.36), we have

$$|X_i^x X_j^x J(z;\zeta)| \le \int_{\tau}^{\frac{t+\tau}{2}}|\widehat{I}|\,d\eta+\lim_{\epsilon\to 0}\int_{\frac{t+\tau}{2}}^{t-\epsilon}(|\widehat{I_1}|+|\widehat{I_2}|)\,d\eta=\widehat{J_1}+\widehat{J_2}.$$

From the definition (3.39), it is immediate that $\widehat{I}$ can be bounded as follows:

$$|\widehat{I}| \le c(T)\,\frac{\omega(2\sqrt{\eta-\tau})}{(t-\eta)(\eta-\tau)}\,\gamma_0(\xi^{-1}\circ x,\check{c}_2(t-\tau))$$

and consequently

$$\widehat{J_1} \le c(T)\,(t-\tau)^{-1}\,\gamma_0(\xi^{-1}\circ x,\check{c}_2(t-\tau))\int_{\tau}^{\frac{t+\tau}{2}}\frac{\omega(2\sqrt{\eta-\tau})}{\eta-\tau}\,d\eta$$
$$= c(T)\,\widetilde{\omega}(2\sqrt{t-\tau})\,(t-\tau)^{-1}\,\gamma_0(\xi^{-1}\circ x,\check{c}_2(t-\tau)).$$

Consider now the above estimates for $\widehat{I_1}$ and $\widehat{I_2}$. From the monotonicity of ω it is immediate that, for any $\eta\in(\tau,\frac{t+\tau}{2})$, it holds

$$|\widehat{I_1}|+|\widehat{I_2}| \le c(T)\,\frac{\omega(2\sqrt{t-\eta})+\omega(2\sqrt{\eta-\tau})\,\widetilde{\omega}(2\sqrt{t-\eta})}{(t-\eta)(\eta-\tau)}\,\gamma_0(\xi^{-1}\circ x,c_1(t-\tau))$$

and therefore, exploiting also the property (2.1), it follows that

$$\begin{aligned}
\widehat{J_2} &\le c(T)(t-\tau)^{-1}\,\gamma_0(\xi^{-1}\circ x, c_1(t-\tau))\left[\int_{\frac{t+\tau}{2}}^{t}\frac{\omega(2\sqrt{t-\eta})}{t-\eta}\,d\eta\ +\omega(2\sqrt{t-\tau})\int_{\frac{t+\tau}{2}}^{t}\frac{\widetilde{\omega}(2\sqrt{t-\eta})}{t-\eta}\,d\eta\right]\\
&\le c(T)\,\frac{\widetilde{\omega}(2\sqrt{t-\tau})+\omega(2\sqrt{t-\tau})\,\widetilde{\widetilde{\omega}}(2\sqrt{t-\tau})}{t-\tau}\,\gamma_0(\xi^{-1}\circ x, c_1(t-\tau))\\
&\le c(T)\,(t-\tau)^{-1}\,\widetilde{\omega}(c_2\sqrt{t-\tau})\,\gamma_0(\xi^{-1}\circ x, c_1(t-\tau)).
\end{aligned}$$

Combining the estimates for $\widehat{J_1}$ and $\widehat{J_2}$ we get (3.36) (up to using (3.3)).

Finally, let us pass to the proof of (3.34), which consists of three major steps.

We start by claiming that $t\mapsto J_\epsilon(x,t;\xi,\tau)$ has continuous derivative given by

$$\partial_t J_\epsilon(z;\zeta)=\int_{\mathbb{R}^n}\gamma_{(y,\,t-\epsilon)}(y^{-1}\circ x,\epsilon)\mu(y,t-\epsilon;\zeta)\,dy+\iint_{\mathbb{R}^n\times(\tau,t-\epsilon)}\partial_t\Gamma_\chi(z;\chi)\mu(\chi;\zeta)\,d\chi. \tag{3.40}$$

Let us prove (3.40). Given $h\in\mathbb{R}$ (with $|h|$ small enough), we have

$$\begin{aligned}
\frac{1}{h}\,[J_\epsilon(x,t+h;\zeta)-J_\epsilon(x,t;\zeta)] &= \int_{t-\epsilon}^{t+h-\epsilon}\int_{\mathbb{R}^n}\frac{1}{h}\,\Gamma_\chi(x,t+h;\chi)\,\mu(\chi;\zeta)\,d\chi\\
&\quad+\int_{\tau}^{t-\epsilon}\int_{\mathbb{R}^n}\frac{1}{h}\left[\Gamma_\chi(x,t+h;\chi)-\Gamma_\chi(x,t;\chi)\right]\mu(\chi;\zeta)\,d\chi\\
&= A_1(h)+A_2(h).
\end{aligned}\tag{3.41}$$

Using the Lagrange mean value theorem, we obtain

$$A_2(h)=\int_{\tau}^{t-\epsilon}\int_{\mathbb{R}^n}\partial_t\Gamma_\chi(x,t^*;\chi)\,\mu(\chi;\zeta)\,d\chi$$

for some $t^*\in(t,t+h)$ if $h>0$, for some $t^*\in(t+h,t)$ if $h<0$. Then, since $(\partial_t\Gamma_\chi)(x,\cdot;\chi)$ is continuous from Theorem 3.1, by dominated convergence (based on (3.4) and (3.16)), we deduce that

$$\lim_{h\to0}A_2(h)=\int_{\tau}^{t-\epsilon}\int_{\mathbb{R}^n}\partial_t\Gamma_\chi(x,t;\chi)\,\mu(\chi;\zeta)\,d\chi. \tag{3.42}$$

Concerning A_1, recall that $\chi=(y,\eta)$ and $\Gamma_\chi(x,t;\chi)=\gamma_{(y,\eta)}(y^{-1}\circ x,t-\eta)$; through the change of variable $\eta\mapsto r=\frac{\eta-(t-\epsilon)}{h}$, we get

$$A_1(h)=\int_0^1 dr\int_{\mathbb{R}^n}\gamma_{(y,\,t-\epsilon+rh)}(y^{-1}\circ x,\epsilon+h-rh)\,\mu(y,t-\epsilon+rh;\xi,\tau)\,dy.$$

Since $\gamma_{(\xi,\tau)}(\xi^{-1}\circ x,t-\tau)$ and $\mu(x,t;\xi,\tau)$ are continuous w.r.t. $x,\ \xi\in\mathbb{R}^n$, $t>\tau$ (by Corollary 3.3 and Proposition 3.7), dominated convergence gives

$$\lim_{h\to0}A_1(h)=\int_{\mathbb{R}^n}\gamma_{(y,\,t-\epsilon)}(y^{-1}\circ x,\epsilon)\,\mu(y,t-\epsilon;\xi,\tau)\,dy, \tag{3.43}$$

thus (3.40) holds true as a consequence of (3.41)–(3.43). As a second step we claim that, given an arbitrary $K \Subset (\tau, +\infty)$, there holds

$$\lim_{\epsilon\to 0^+} \sup_{t\in K} \left|\mu(x,t;\xi,\tau) - \int_{\mathbb{R}^n} \gamma_{(y,t-\epsilon)}(y^{-1}\circ x,\epsilon)\,\mu(y,t-\epsilon;\xi,\tau)\,dy\right| = 0. \tag{3.44}$$

Let us prove the claim; note that for any $0 < \epsilon < \inf_K((t-\tau)/2)$, we have

$$\begin{aligned}
&\sup_{t\in K} \left|\mu(x,t;\xi,\tau) - \int_{\mathbb{R}^n} \gamma_{(y,t-\epsilon)}(y^{-1}\circ x,\epsilon)\,\mu(y,t-\epsilon;\xi,\tau)\,dy\right| \\
&\le \sup_{t\in K} \int_{\mathbb{R}^n} \left|\left[\gamma_{(y,t-\epsilon)}(y^{-1}\circ x,\epsilon) - \gamma_{(x,t-\epsilon)}(y^{-1}\circ x,\epsilon)\right]\,\mu(y,t-\epsilon;\xi,\tau)\right|\,dy \\
&+ \sup_{t\in K} \int_{\mathbb{R}^n} \gamma_{(x,t-\epsilon)}(y^{-1}\circ x,\epsilon)\ |\mu(y,t-\epsilon;\xi,\tau) - \mu(x,t-\epsilon;\xi,\tau)|\,dy \\
&+ \sup_{t\in K} \left|\mu(x,t-\epsilon;\xi,\tau) \int_{\mathbb{R}^n} \gamma_{(x,t-\epsilon)}(y^{-1}\circ x,\epsilon)\,dy\ - \mu(x,t;\xi,\tau)\right| = \overline{S}_1 + \overline{S}_2 + \overline{S}_3.
\end{aligned}$$

By (3.1) and the uniform continuity of $\mu(x,\cdot;\xi,\tau)$ over K (for any fixed $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$), it is straightforward that

$$\overline{S}_3 = \sup_{t\in K} |\mu(x,t-\epsilon;\xi,\tau) - \mu(x,t;\xi,\tau)| \longrightarrow 0 \quad \text{as } \epsilon \to 0^+.$$

As for $\overline{S}_1$, by (3.6), (3.9) and (3.10), we get

$$\begin{aligned}
\overline{S}_1 &\le c(K) \sup_{t\in K} \left[\frac{\omega(2\sqrt{t-\tau})}{t-\epsilon-\tau} \int_{\mathbb{R}^n} \omega(d(x,y))\,\gamma_0(\xi^{-1}\circ y, \check{c}_2(t-\epsilon-\tau))\gamma_0(y^{-1}\circ x, \check{c}_0\epsilon)\,dy\right] \\
&\le c(K)\,\omega(\sqrt{\epsilon})\ \sup_{t\in K} (t-\tau)^{-1} \int_{\mathbb{R}^n} \gamma_0(y^{-1}\circ x, \check{c}_0\epsilon)\gamma_0(\xi^{-1}\circ y, \check{c}_2(t-\epsilon-\tau))\,dy \\
&\le c(K)\,\omega(\sqrt{\epsilon})\ \sup_{t\in K} \frac{\gamma_0(\xi^{-1}\circ x, c_1(t-\tau))}{t-\tau};
\end{aligned}$$

similarly, with the aid of (3.1), (3.6), (3.8), (3.10) and (3.30),

$$\begin{aligned}
\overline{S}_2 &\le c(K) \sup_{t\in K} (t-\epsilon-\tau)^{-1} \int_{\mathbb{R}^n} [\gamma_0(\xi^{-1}\circ y, \check{c}_1(t-\epsilon-\tau)) + \gamma_0(\xi^{-1}\circ x, \widehat{c}_2(t-\epsilon-\tau))] \\
&\times \gamma_0(y^{-1}\circ x,\epsilon)\,\widetilde{\omega}(2d(x,y))\,dy \le c(K)\,\widetilde{\omega}(2\sqrt{\epsilon})\ \sup_{t\in K} (t-\tau)^{-1} \left[\int_{\mathbb{R}^n} \gamma_0(\xi^{-1}\circ y, \widehat{c}_2(t-\epsilon-\tau))\right. \\
&\left.\times \gamma_0(y^{-1}\circ x,\epsilon)\,dy + \gamma_0(\xi^{-1}\circ x, \widehat{c}_2(t-\epsilon-\tau)) \int_{\mathbb{R}^n} \gamma_0(y^{-1}\circ x,\epsilon)\,dy\right] \\
&\le c(K)\,\widetilde{\omega}(2\sqrt{\epsilon}) \sup_{t\in K} \frac{\gamma_0(\xi^{-1}\circ x, c_1(t-\tau)) + \gamma_0(\xi^{-1}\circ x, \widehat{c}_2(t-\epsilon-\tau))}{t-\tau}.
\end{aligned}$$

Then, from (3.28) and the usual compactness argument, we conclude that

$$\overline{S}_1 + \overline{S}_2 \le c(K)[\omega(\sqrt{\epsilon}) + \widetilde{\omega}(2\sqrt{\epsilon})] \longrightarrow 0, \quad \text{as } \epsilon \to 0^+,$$

thus (3.44) holds true. The final step consists in proving the following:

$$\lim_{\epsilon\to 0^+}\sup_{t\in K}\left|\int_{t-\epsilon}^{t} d\eta\int_{\mathbb{R}^n}\partial_t\gamma_{(y,\eta)}(y^{-1}\circ x, t-\eta)\,\mu(y,\eta;\xi,\tau)\right| dy = 0; \tag{3.45}$$

since the proof follows exactly the lines of that of (3.38), we omit the details.

Combining (3.40), (3.44) and (3.45), we finally get (3.34). As for (3.37), thanks to the fact that $\partial_t\Gamma_\zeta(z;\zeta)$ and $X_i^x X_j^x\Gamma_\zeta(z;\zeta)$ enjoy the same estimate up to constants (see Theorem 3.2), reasoning as for the proof of (3.36) one easily gets

$$\left|\lim_{\epsilon\to 0}\int_{\tau}^{t-\epsilon} d\eta\int_{\mathbb{R}^n}\partial_t\gamma_{(y,\eta)}(y^{-1}\circ x, t-\eta)\,\mu(y,\eta;\xi,\tau)\,dy\right| \le c(T)\,\widetilde{\omega}(2\sqrt{t-\tau})\,(t-\tau)^{-1}\,\gamma_0(\xi^{-1}\circ x, c_1(t-\tau)).$$

Furthermore, combining the estimate (3.16) with (2.1) one easily deduces that also $\mu(z;\zeta)$ enjoys the same estimate, thus yielding (3.37). □

Note that, according to definitions (3.11)–(3.13), in principle $\Gamma(x,t;\xi,\tau)$ is defined only for $t>\tau$. In what follows, we agree to extend Γ to the whole $\mathbb{R}^{n+1}$ by setting $\Gamma(x,t;\xi,\tau)=0$ for $t\le\tau$. Then, by Theorem 3.12, one gets:

Theorem 3.14 (Upper Gaussian estimates for Γ)**.** *Γ is a continuous function outside the diagonal of $\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$, and for any $\zeta\in\mathbb{R}^{n+1}$ we have that*

$$\Gamma(\cdot;\zeta)\in\mathfrak{C}^2(\mathbb{R}^{n+1}\setminus\{\zeta\}), \tag{3.46}$$

$$H(\,\Gamma(\cdot;\zeta)\,)=0 \ \ \textit{in}\ \mathbb{R}^{n+1}\setminus\{\zeta\}. \tag{3.47}$$

Fixed any $T>0$, the following estimates hold for $x\in\mathbb{R}^n$, $\xi\in\mathbb{R}^n$, $0<t-\tau\le T$:

$$|\Gamma(z;\zeta)|\le c(T)\,E(\xi^{-1}\circ x,\widehat{c}_5(t-\tau)), \tag{3.48}$$

$$|X_j^x\Gamma(z;\zeta)|\le c(T)\,(t-\tau)^{-\frac{1}{2}}\,E(\xi^{-1}\circ x,\widehat{c}_5(t-\tau)), \tag{3.49}$$

$$|X_i^x X_j^x\Gamma(z;\zeta)|,\ |\partial_t\Gamma(z;\zeta)|\le c(T)\,(t-\tau)^{-1}\,E(\xi^{-1}\circ x,\widehat{c}_5(t-\tau)) \tag{3.50}$$

with $\widehat{c}_5$ depending only on Λ,δ, $\mathbb{G}$, and c depending also on T.

Proof. Everything except (3.47) is straightforward from the properties of the parametrix Γ_χ and the integral term J (see Theorems 3.1, 3.2, 3.12, Corollary 3.3 and Proposition 3.11). As for (3.47), for $t\le\tau$ the thesis is immediate; for $t>\tau$, combining (3.32)–(3.34) with (3.12) and the fact that

$$\int_{t-\epsilon}^{t} d\eta\int_{\mathbb{R}^n}\left|X_j\gamma_{(y,\eta)}(y^{-1}\circ x, t-\eta)\,\mu(y,\eta;\xi,\tau)\right| dy\to 0,\quad \text{as}\ \epsilon\to 0^+$$

(which has been shown in the proof of Theorem 3.12), we immediately get

$$\begin{aligned}(H\Gamma(\cdot;\zeta))\,(z) &= \Big(H\Gamma_\zeta(\cdot;\zeta)\Big)(z)+\mu(z;\zeta)\,+\lim_{\epsilon\to 0^+}\int_{\tau}^{t-\epsilon}\int_{\mathbb{R}^n}\Big(H\Gamma_\chi(\cdot;\chi)\Big)(z)\,\mu(\chi;\zeta)\,d\chi\\ &= Z_1(z;\zeta)+\mu(z;\zeta)\,+\lim_{\epsilon\to 0^+}\int_{\tau}^{t-\epsilon}\int_{\mathbb{R}^n}Z_1(z;\chi)\,\mu(\chi;\zeta)\,d\chi.\end{aligned}$$

Note that, given any $z, \zeta \in \mathbb{R}^{n+1}$ with $t > \tau$, the map $Z_1(z;\cdot)\,\mu(\cdot;\zeta)$ belongs to $L^1(\mathbb{R}^n \times (\tau, t))$ in view of (3.16), (3.20), thus by Corollary 3.6

$$(H\Gamma(\cdot;\zeta))\,(z) = Z_1(z;\zeta) + \mu(z;\zeta) + \iint_{\mathbb{R}^n\times(\tau,t)} Z_1(z;\chi)\,\mu(\chi;\zeta)\,d\chi = 0.$$

□

4. Cauchy problem for H and properties of Γ

In this section we use the fundamental solution Γ previously constructed to build a solution for the Cauchy problem

$$\begin{cases} Hu = f \ \text{ in}\,\mathbb{R}^n \times (0,T), \\ u(\cdot,0) = g \ \text{ in}\,\mathbb{R}^n, \end{cases} \tag{4.1}$$

under suitable assumptions about the regularity and growth of both f and g and with $T > 0$ small enough. We also exploit the result concerning the homogeneous version of (4.1) (see Theorem 4.2 below) to prove lower Gaussian estimates for Γ, thus concluding the proof of Theorem 1.3.

Definition 4.1. *By* solution *of the problem (4.1) we mean a function $u : \mathbb{R}^n \times (0,T) \to \mathbb{R}$ with the following properties:*

(i) $u \in \mathfrak{C}^2(\mathbb{R}^n \times (0,T))$ (see Definition 2.2);

(ii) u satisfies the equation $Hu = f$ pointwise, that is

$$Hu(x,t) = f(x,t), \quad \forall (x,t) \in \mathbb{R}^n \times (0,T);$$

(iii) The initial datum g is assumed as a pointwise limit, namely for every $x_0 \in \mathbb{R}^n$ it holds that

$$u(x,t) \longrightarrow g(x_0), \quad \text{as } d_p((x,t),(x_0,0)) = d(x,x_0) + \sqrt{|t|} \to 0.$$

4.1. Homogeneous Cauchy problem

Let us start with the solution of the homogeneous version of (4.1):

Theorem 4.2 (Solution of the homogeneous Cauchy problem)**.** *Let $\nu \geq 0$ and let $g : \mathbb{R}^n \to \mathbb{R}$ be a continuous function subject to the growth condition*

$$|g(x)| \leq M e^{\nu \|x\|^2}, \quad \forall\, x \in \mathbb{R}^n \tag{4.2}$$

for some constant $M > 0$. Then there exists $\delta > 0$ such that, for any $T > 0$ satisfying $T\nu < \delta$, the function $u : \mathbb{R}^n \times (0,T] \to \mathbb{R}$ given by

$$u(x,t) := \int_{\mathbb{R}^n} \Gamma(x,t;y,0)\,g(y)\,dy$$

belongs to $\mathfrak{C}^2(\mathbb{R}^n \times (0,T)) \cap C(\mathbb{R}^n \times [0,T])$ and solves the Cauchy problem

$$\begin{cases} Hu = 0 \ \ \textit{in}\,\mathbb{R}^n \times (0,T), \\ u(\cdot,0) = g \ \ \textit{in}\,\mathbb{R}^n. \end{cases}$$

In particular, if g is bounded, all the above holds for any $T > 0$.

Proof. By (3.49), (3.50) and (4.2), dominated convergence gives

$$\begin{aligned}
X_j u(x,t) &= \int_{\mathbb{R}^n} (X_j^x \Gamma)(x,t;y,0)\, g(y)\, dy \quad (\forall 1 \le j \le m),\\
X_i^x X_j^x u(x,t) &= \int_{\mathbb{R}^n} (X_i^x X_j^x \Gamma)(x,t;y,0)\, g(y)\, dy \quad (\forall 1 \le i,j \le m),\\
\partial_t u(x,t) &= \int_{\mathbb{R}^n} (\partial_t \Gamma)(x,t;y,0)\, g(y)\, dy,
\end{aligned}$$

for any $x \in \mathbb{R}^n$, $t > 0$, hence $u \in \mathfrak{C}^2(\mathbb{R}^n \times (0,T)) \cap C(\mathbb{R}^n \times (0,T])$ by (3.46); moreover, the above and (3.47) yield

$$Hu(x,t) = \int_{\mathbb{R}^n} (H\Gamma(\cdot,\cdot;y,0))\,(x,t)\, g(y)\, dy = 0 \quad \text{in } \mathbb{R}^n \times (0,T).$$

We are left to discuss the initial condition. Denoting again

$$J(z;\zeta) := \iint_{\mathbb{R}^n \times (\tau,t)} \Gamma_\chi(z;\chi)\, \mu(\chi;\zeta)\, d\chi$$

for any $z = (x,t)$, $\zeta = (\xi,\tau) \in \mathbb{R}^{n+1}$ with $t > \tau$, we have

$$\begin{aligned}
|u(x,t) - g(x_0)| \le{}& \int_{\mathbb{R}^n} |J(x,t;y,0)\, g(y)|\, dy + \left| \int_{\mathbb{R}^n} \gamma_{(x_0,0)}(y^{-1} \circ x, t)\, g(y)\, dy - g(x_0) \right|\\
&+ \int_{\mathbb{R}^n} \left| \left[\gamma_{(y,0)}(\xi^{-1} \circ x, t) - \gamma_{(x_0,0)}(y^{-1} \circ x, t) \right] g(y) \right| dy = W_1 + W_2 + W_3.
\end{aligned}$$

From (3.3), (3.31) and (4.2), we have

$$\begin{aligned}
W_1 &\le c(M,T)\, \widetilde{\omega}(2\sqrt{t}) \int_{\mathbb{R}^n} \gamma_0(y^{-1} \circ x, c_1 t)\, e^{\nu \|y\|^2}\, dy\\
&\le c(M,T)\, \widetilde{\omega}(2\sqrt{t})\, t^{-\frac{Q}{2}} \int_{\mathbb{R}^n} \exp\left\{ -\frac{d(x,y)^2}{c_2 t} + \nu \|y\|^2 \right\} dy.
\end{aligned}$$

By quasi-triangle inequality $\|y\| \le \kappa\,(\|y^{-1} \circ x\| + \|x\|)$, hence

$$\|y\|^2 \le 2\kappa^2\,(\|y^{-1} \circ x\|^2 + \|x\|^2) = 2\kappa^2 (d(x,y)^2 + \|x\|^2);$$

assuming $T\nu < \delta := (4c\kappa^2)^{-1}$, one easily gets

$$-\frac{d(x,y)^2}{c_2 t} + \nu \|y\|^2 \le -\frac{d(x,y)^2}{c_2 t} \left[1 - 2c_2 t \nu \kappa^2 \right] + 2\nu \|x\|^2 \le -\frac{d(x,y)^2}{2c_2 t} + 2\nu\kappa^2 \|x\|^2$$

for all $x, y \in \mathbb{R}^n$ and $0 < t \le T$, and so by (3.1):

$$W_1 \le c(M,T)\, \widetilde{\omega}(2\sqrt{t})\, e^{2\nu\kappa^2 \|x\|^2} \int_{\mathbb{R}^n} t^{-\frac{Q}{2}} \exp\left\{ -\frac{d(x,y)^2}{2c_2 t} \right\} dy \le c(M,T)\, \widetilde{\omega}(2\sqrt{t})\, e^{2\nu\kappa^2 \|x\|^2} \longrightarrow 0$$

as $(x,t) \to (x_0,0)$. As for W_2, from Theorem 3.1(v), there holds

$$\int_{\mathbb{R}^n} \gamma_{(x_0,0)}(y^{-1} \circ x, t)\, g(y)\, dy \longrightarrow g(x), \quad \text{as } t \to 0^+,\ \forall x \in \mathbb{R}^n,$$

hence it is immediate to deduce that, as $(x,t) \to (x_0,0)$, we have

$$W_2 \leq \left| \int_{\mathbb{R}^n} \gamma_{(x_0,0)}(y^{-1} \circ x, t)\, g(y)\, dy - g(x) \right| + |g(x) - g(x_0)| \longrightarrow 0.$$

As for W_3, let $\sigma \in (0,1)$ to be chosen later; then we split

$$W_3 = \left(\int_{d(x,y)\leq\sigma} + \int_{d(x,y)>\sigma} \right) \left| \left[\gamma_{(y,0)}(y^{-1} \circ x, t) - \gamma_{(x_0,0)}(y^{-1} \circ x, t)) \right] g(y) \right| dy = W_3^1 + W_3^2.$$

The integration domain of W_3^1 is contained in the closed ball $\overline{B_r(x_0)}$ with radius $r = \kappa[d(x,x_0)+1]$ by quasi-triangle inequality, hence g is bounded (uniformly w.r.t. σ) therein. Then, by (3.1) and (3.9),

$$W_3^1 \leq \check{c}_0 \int_{d(x,y)<\sigma} \omega(d(x_0,y))\, \gamma_0(y^{-1} \circ x, \check{c}_0 t))\, dy \leq \check{c}_0 \omega(\kappa[d(x,x_0)+\sigma])$$

for all $\sigma \in (0,1)$. Instead, using (3.8) and then proceeding as for W_1,

$$\begin{aligned} W_3^2 &\leq 2M\check{c}_0 \int_{d(x,y)>\sigma} \gamma_0(y^{-1} \circ x, \check{c}_0 t)\, e^{\nu\|y\|^2}\, dy \\ &\leq c(M,\nu,x_0)\, e^{2\nu\kappa^2} \int_{d(x,y)>\sigma} t^{-\frac{Q}{2}} \exp\left\{ -\frac{d(x,y)^2}{2c_1 t} \right\} dy. \end{aligned}$$

Setting $y \mapsto \xi = \delta_{1/\sqrt{t}}(y^{-1} \circ x)$, which yields $d\xi = t^{-Q/2}\, dy$, we get

$$W_3^2 \leq c(M,\nu,x_0) \int_{\|\xi\|>\frac{\sigma}{\sqrt{t}}} \exp\left\{ -\frac{\|\xi\|^2}{2c_1} \right\} d\xi,$$

which gives us the following:

$$W_3 \leq C(M,\nu,x_0) \left[\omega(\kappa[d(x,x_0)+\sigma]) + \int_{\|\xi\|>\frac{\sigma}{\sqrt{t}}} \exp\left\{ -\frac{\|\xi\|^2}{2c_1} \right\} d\xi \right].$$

Given $\epsilon > 0$, by continuity of ω_g we can choose $\delta_x > 0$ such that $\omega_g(\kappa\delta_x) < \epsilon/4$ and then $0 < \sigma < 1$ such that $\omega_g(\kappa\,[\delta_x + \sigma]) < \epsilon/2$. Moreover, by Lebesgue's theorem there exists $\delta_t > 0$ such that

$$\int_{\|\xi\|>\sigma/\sqrt{\delta_t}} \exp\left\{ -\frac{\|\xi\|^2}{2c_1} \right\} d\xi < \frac{\epsilon}{2},$$

thus for any $(x,t) \in B(x_0,\delta_x) \times (0,\delta_t)$ it holds $W_3 \leq C(M,\nu,x_0)\,\epsilon$, which by arbitrariness of $\epsilon > 0$ gives $W_3 \longrightarrow 0$ as $(x,t) \to (x_0,0)$. □

4.2. Maximum principles and lower Gaussian estimates

Let us introduce some maximum principles for the operator H, which will be useful both to prove the nonnegativity of the fundamental solution Γ and to get uniqueness of solutions for the associated Cauchy problem (in suitable spaces); the latter, in turn, will reveal crucial to obtain the reproduction property (1.10) of Γ and, consequently, also the lower Gaussian estimate in (1.9).

Theorem 4.3 (Weak maximum principle in bounded sets)**.** *Let Ω be an open bounded subset of $\mathbb{R}^{n+1}$ and let $t_0 \in \mathbb{R}$. Then*

$$\begin{cases} u \in \mathfrak{C}^2(\Omega), \\ Hu \le 0 \ \ in\, \Omega \cap \{t < t_0\}, \\ \limsup u \le 0 \ \ in\, \partial\Omega \cap \{t \le t_0\}, \end{cases} \qquad \Longrightarrow \qquad u \le 0 \ \ in \ \ \Omega \cap \{t < t_0\}.$$

(by "$\limsup u \le 0$ in $\partial\Omega \cap \{t \le t_0\}$" we mean that, for any $(\overline{x}, \overline{t}) \in \partial\Omega \cap \{t \le t_0\}$ and any sequence $\{(x_n, t_n)\}_{n\ge1} \subset \Omega \cap \{t \le t_0\}$ converging to $(\overline{x}, \overline{t})$, there holds $\limsup_{n\to\infty} u(x_n, t_n) \le 0$).

For the proof see [6, Theorem 4.3] (keeping in mind that, in the latter, the opposite sign convention for H is used). Denoting by $C_*(\Omega)$ the space of continuous functions in Ω which vanish at infinity, an immediate corollary is the following:

Corollary 4.4 (Weak maximum principle in infinite strips)**.** *Given $T_1 < T_2$, there holds*

$$\begin{cases} u \in \mathfrak{C}^2(\mathbb{R}^n \times (T_1, T_2)) \cap C_*(\mathbb{R}^n \times [T_1, T_2]), \\ Hu = 0 \ \ in\, \mathbb{R}^n \times (T_1, T_2), \\ u = 0 \ \ on\, \mathbb{R}^n \times \{T_1\}, \end{cases} \qquad \Longrightarrow \ u = 0 \ \ in \ \ \mathbb{R}^n \times (T_1, T_2).$$

Proof. Since u vanishes at infinity, for any $\epsilon > 0$ there exists $M_\epsilon > 0$ such that

$$\sup_{\|x\|>M_\epsilon,\, t\in(T_1,T_2)} u(x,t) < \epsilon.$$

Let $M > M_\epsilon$, $\Omega := \{\|x\| < M\} \times (T_1, T_2)$ and $t_0 \in (T_1, T_2)$. Consider the function $w_\epsilon(z) := u(z) - \epsilon$, $z \in \mathbb{R}^n \times (T_1, T_2)$:

$$\begin{cases} w_\epsilon \in \mathfrak{C}^2(\mathbb{R}^n \times (T_1, T_2)), \\ Hw_\epsilon = 0 \quad \text{in}\, \mathbb{R}^n \times (T_1, T_2), \\ w_\epsilon \le 0 \qquad \text{in}\, \partial\Omega \cap \{t \le t_0\} = \{\|x\| = M\} \times (T_1, t_0), \end{cases}$$

thus $w_\epsilon \le 0$ in $\{\|x\| < M\} \times (T_1, t_0)$ by Theorem 4.3. By arbitrariness of $M > M_\epsilon$ and $t_0 \in (T_1, T_2)$, it immediately follows that $w_\epsilon \le 0$ in $\mathbb{R}^n \times (T_1, T_2)$. Then, since this holds for any $\epsilon > 0$ and the sequence $\{w_\epsilon\}_{\epsilon>0}$ converges uniformly to u in $\mathbb{R}^n \times (T_1, T_2)$ as $\epsilon \to 0^+$, we get that $u \le 0$ in $\mathbb{R}^n \times (T_1, T_2)$.

Note now that also $-u$ vanishes at infinity, hence all the above procedure can be analogously applied to $-u$, yielding $-u \le 0$ in $\mathbb{R}^n \times (T_1, T_2)$, namely $u \ge 0$ in $\mathbb{R}^n \times (T_1, T_2)$. Combining the two results the thesis is achieved. □

It is immediate to deduce the following uniqueness result for solutions of (4.1) vanishing at infinity:

Corollary 4.5. *Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ be continuous functions. Then, for any $T_1 < T_2$, the solution of the Cauchy problem*

$$\begin{cases} Hu = f \ \ \text{in}\, \mathbb{R}^n \times (T_1, T_2), \\ u(\cdot, T_1) = g \ \ \text{in}\, \mathbb{R}^n, \end{cases}$$

if it exists, is unique in the space $\mathfrak{C}^2(\mathbb{R}^n \times (T_1, T_2)) \cap C_*(\mathbb{R}^n \times [T_1, T_2])$.

Proof. Let and v, $w \in \mathfrak{C}^2(\mathbb{R}^n \times (T_1, T_2)) \cap C_*(\mathbb{R}^n \times [T_1, T_2])$ be both solutions of (4.1). Then the function $\psi := v - w$ clearly belongs to the same regularity class and satisfy $H\psi = 0$ in $\mathbb{R}^n \times (T_1, T_2)$ and $\psi(\cdot, T_1) \equiv 0$; hence by Corollary 4.4 there holds $\psi \equiv 0$ in $\mathbb{R}^n \times (T_1, T_2)$, namely $v = w$ in $\mathbb{R}^n \times (T_1, T_2)$. □

Coming back to the fundamental solution Γ, we are now ready to show that:

Theorem 4.6. *The fundamental solution Γ is nonnegative in $(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}) \setminus D$ and the reproduction property (1.10) holds true.*

Proof. Assume by contradiction that $\Gamma(x_0, t_0; \xi_0, \tau_0) < 0$ for some $(x_0, t_0; \xi_0, \tau_0) \notin D$; then $\Gamma(x_0, t_0; \cdot, \tau_0) < 0$ in $B_\epsilon(\xi_0)$ for some $\epsilon > 0$, by continuity of Γ. Let $T > 0$ be small enough and $g \neq 0$ be a nonnegative smooth function in $\mathbb{R}^n$ with support in $B_\epsilon(\xi_0)$; then, by Theorem 4.2, the function

$$u(x, t) = \int_{\mathbb{R}^n} \Gamma\left(x, t; \xi, \tau_0 - \frac{T}{2}\right) g(\xi)\, d\xi, \quad (x, t) \in \mathbb{R}^{n+1}$$

is well defined and solves

$$\begin{cases} Hu = 0 & \text{in}\, \mathbb{R}^n \times (\tau_0 - \frac{T}{2}, \tau_0 + \frac{T}{2}), \\ u(\cdot, \tau_0 - \frac{T}{2}) = g & \text{in}\, \mathbb{R}^n. \end{cases}$$

Moreover, by dominated convergence and Theorem 3.1,

$$\lim_{\|x\| \to +\infty} u(x, t) = \int_{\mathbb{R}^n} \lim_{\|x\| \to +\infty} \Gamma\left(x, t; \xi, \tau_0 - \frac{T}{2}\right) g(\xi)\, d\xi = 0.$$

Hence, $Hu = 0$ in $\mathbb{R}^n \times (\tau_0 - T/2, \tau_0 + T/2)$ and $\liminf u \geq 0$ on $\mathbb{R}^n \times \{\tau_0 - T/2\}$ and at infinity, which (by Corollary 4.4) yields $u \geq 0$ in $\mathbb{R}^n \times (\tau_0 - T/2, \tau_0 + T/2)$.

However, since $supp(g) \subset B_\epsilon(\xi_0)$ but g does not vanish everywhere, we have

$$u(x_0, t_0) = \int_{B_\epsilon(\xi_0)} \Gamma\left(x_0, t_0; \xi, \tau_0 - \frac{T}{2}\right) g(\xi)\, d\xi < 0,$$

which is a contradiction. Thus Γ is nonnegative outside D. As for the reproduction property (1.10), let us fix $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$, $\tau < s$; consider the Cauchy problem

$$\begin{cases} Hu = 0 & \text{in}\, \mathbb{R}^n \times (s, s + T), \\ u(\cdot, s) = g & \text{in}\, \mathbb{R}^n, \end{cases}$$

with $g = \Gamma(\cdot, s; \xi, \tau)$. Note that, since $s > \tau$, then g is well defined and continuous on $\mathbb{R}^n$; moreover, in view of (3.3) and the fact that s and τ are fixed and distinct, g is also bounded. By Theorem 4.2 it follows that the function

$$\psi(x, t; s, \xi, \tau) := \int_{\mathbb{R}^n} \Gamma(x, t; y, s)\, g(y)\, dy = \int_{\mathbb{R}^n} \Gamma(x, t; y, s)\, \Gamma(y, s; \xi, \tau)\, dy$$

is well defined for all $x, \xi \in \mathbb{R}^n$, $t > \tau$, belongs to $\mathfrak{C}^2(\mathbb{R}^n \times (s, s+T))$ and solves the Cauchy problem above. Furthermore, by (3.48) and dominated convergence,

$$\lim_{\|x\|\to+\infty} \psi(x,t;s,\xi,\tau) = \int_{\mathbb{R}^n} \lim_{\|x\|\to+\infty} \Gamma(x,t;y,s)\,\Gamma(y,s;\xi,\tau)\,dy = 0.$$

However also $\Gamma(x,t;\xi,\tau) \in \mathfrak{C}^2(\mathbb{R}^n \times (s, s+T))$, solves the Cauchy problem and vanishes at infinity (being the fundamental solution of H); so by Theorem 4.5

$$\Gamma(x,t;\xi,\tau) = \psi(x,t;s,\xi,\tau) = \int_{\mathbb{R}^n} \Gamma(x,t;y,s)\,\Gamma(y,s;\xi,\tau)\,dy$$

for all $x, \xi \in \mathbb{R}^n$, $\tau < s < t < s+T$. By arbitrariness of $T > 0$, we get (1.10). □

Let us now focus on proving a Gaussian estimate from below for the fundamental solution Γ: first, we prove it in local form, namely for points $(z;\zeta)$ "sufficiently close" to the diagonal D; then, we extend the result to the whole space by (1.10), following a geometric idea used in [16, Theorem 2.2].

Theorem 4.7 (Lower Gaussian estimate for the fundamental solution)**.** *For any $T > 0$ there exists a constant $\overline{\overline{c}} = \overline{\overline{c}}(T) > 0$ such that, for every $(x,t), (\xi,\tau) \in \mathbb{R}^{n+1}$ with $0 < t-\tau \le T$, the following uniform Gaussian estimate holds true:*

$$\Gamma(x,t;\xi,\tau) \ge \overline{\overline{c}}^{\,-1} (t-\tau)^{-Q/2} \exp\left\{-\overline{\overline{c}}\,\frac{d(x,\xi)^2}{t-\tau}\right\}.$$

Proof. Recall that $\Gamma(x,t;\xi,\tau) = \Gamma_{(\xi,\tau)}(x,t;\xi,\tau) + J(x,t;\xi,\tau)$, with the former term at right hand side bounded below by (3.3) and the latter bounded above by (3.31); by equivalence between d and the CC distance d_X, namely

$$\beta^{-1} d_X(x,\xi)^2 \le d(x,\xi)^2 \le \beta d_X(x,\xi)^2, \quad \forall x, \xi \in \mathbb{R}^n$$

for a constant $\beta \ge 1$, the mentioned estimates can be written as

$$\Gamma_{(\xi,\tau)}(x,t;\xi,\tau) \ge \overline{c}_0^{\,-1} (t-\tau)^{-Q/2} \exp\left\{-\overline{c}_0 \beta \frac{d_X(x,\xi)^2}{t-\tau}\right\},$$
$$|J(x,t;\xi,\tau)| \le c(t-\tau)^{-Q/2}\, \widetilde{\omega}(2\sqrt{t-\tau}) \exp\left\{-\frac{d_X(x,\xi)^2}{\widehat{c}_3 \beta (t-\tau)}\right\}$$

(with the advantage that d_X is a true distance). These yield

$$\begin{aligned}\Gamma(x,t;\xi,\tau) &\ge \overline{c}_0^{\,-1}(t-\tau)^{-Q/2} \exp\left\{-\overline{c}_0 \frac{d_X(x,\xi)^2}{t-\tau}\right\}\left[1 - \overline{c}_0 c\,\widetilde{\omega}(2\sqrt{t-\tau}) \exp\left\{\left(\overline{c}_0 - \frac{1}{\widehat{c}_3}\right)\frac{d_X(x,\xi)^2}{t-\tau}\right\}\right] \\ &=: \overline{c}_0^{\,-1}(t-\tau)^{-Q/2} \exp\left\{-\overline{c}_0 \frac{d_X(x,\xi)^2}{t-\tau}\right\}\left[1 - c_2 \widetilde{\omega}(2\sqrt{t-\tau}) \exp\left\{c_3 \frac{d_X(x,\xi)^2}{t-\tau}\right\}\right].\end{aligned}$$

We choose $0 < \delta < \frac{1}{4}$ such that $c_3\delta < 1$ and $\widetilde{\omega}(2\sqrt{\delta}) \le \frac{1}{4c_2}$, which is clearly possible in view of the properties of a modulus of continuity. Then for any x, t, ξ, τ with $0 < t-\tau \le \delta$ and $d_X(x,\xi)^2 \le \delta(t-\tau)$ there holds

$$1 - c_2 \widetilde{\omega}(2\sqrt{t-\tau}) \exp\left\{c_3 \frac{d_X(x,\xi)^2}{t-\tau}\right\} \ge 1 - c_2\, \widetilde{\omega}(2\sqrt{\delta})\, e^{c_3\delta} > \frac{1}{4},$$

thus

$$\Gamma(x,t;\xi,\tau) \geq (4\overline{c}_0)^{-1}(t-\tau)^{-Q/2}\exp\left\{-4\overline{c}_0\frac{d_X(x,\xi)^2}{t-\tau}\right\}.$$

Now, fixed any $T>0$, we show an analogous bound for all points $(x,t),(\xi,\tau)\in\mathbb{R}^{n+1}$ with $0<t-\tau\leq T$; it is not restrictive to assume $T\geq 1$. Let

$$k := \left\lfloor \max\left\{\frac{T}{\delta},\ \frac{4d_X(x,\xi)^2}{\delta(t-\tau)}\right\}\right\rfloor + 1, \quad \sigma := \frac{1}{4}\sqrt{\delta\frac{t-\tau}{k+1}}.$$

Given $\epsilon>0$, take an absolutely continuous curve $\gamma:[0,1]\to\mathbb{R}^n$ joining x to ξ with length smaller than $d_X(x,\xi)+\epsilon$, a chain of points $x\equiv x_0,\ x_1,\ ...,\ x_{k+1}\equiv\xi$ on γ and a chain of times $t\equiv t_0>t_1>...>t_{k+1}\equiv\tau$ with

$$d_X(x_j,x_{j+1}) = \frac{d(x,\xi)+\epsilon}{k+1}, \quad t_j-t_{j+1} = \frac{t-\tau}{k+1} \quad (\forall 0\leq j\leq k).$$

Then, for any $0\leq j\leq k$ and any $y\in B_{d_X}(x_j,\sigma),\ y'\in B_{d_X}(x_{j+1},\sigma)$, there holds

$$\begin{aligned}
d_X(y,y') &\leq d_X(y,x_j)+d_X(x_j,x_{j+1})+d_X(x_{j+1},y') \leq \frac{d_X(x,\xi)+\epsilon}{k+1}+2\sigma \\
&< \frac{\frac{1}{2}\sqrt{k\delta(t-\tau)}}{k+1}+\frac{1}{2}\sqrt{\delta\frac{t-\tau}{k+1}} < \sqrt{\delta\frac{t-\tau}{k+1}} = \sqrt{\delta(t_j-t_{j+1})}
\end{aligned}$$

(provided ϵ is small enough); since we also have $t_j-t_{j+1}\leq T/(k+1)<\delta$, then the lower Gaussian estimate holds for points $(y,t_j),(y',t_{j+1})$, i.e.,

$$\Gamma(y,t_j;y',t_{j+1}) \geq c^{-1}(t_j-t_{j+1})^{-Q/2}\exp\left\{-c\frac{d_X(y,y')^2}{t_j-t_{j+1}}\right\}$$

for any constant $c\geq 4\overline{c}_0$. Setting $y_0=x$ and $y_{k+1}=\xi$ for homogeneity of notation, we iterate (1.10) and exploit the nonnegativity of Γ:

$$\begin{aligned}
\Gamma(x,t;\xi,\tau) &= \int_{(\mathbb{R}^n)^k}\Gamma(y_0,t_0;y_1,t_1)\Gamma(y_1,t_1;y_2,t_2)...\Gamma(y_k,t_k;y_{k+1},t_{k+1})\,dy_1...dy_k \\
&\geq \int_{\Omega_\sigma:=B_{d_X}(x_1,\sigma)\times...\times B_{d_X}(x_k,\sigma)}\Gamma(x_0,t_0;y_1,t_1)...\Gamma(y_k,t_k;x_{k+1},t_{k+1})\,dy_1...dy_k \\
&\geq c^{-k-1}\left(\frac{t-\tau}{k+1}\right)^{-\frac{Q}{2}(k+1)}\int_{\Omega_\sigma}\exp\left\{-c\sum_{j=0}^{k}\frac{d_X(y_j,y_{j+1})^2}{t_j-t_{j+1}}\right\}dy_1...dy_k \\
&\geq c^{-k-1}\left(\frac{t-\tau}{k+1}\right)^{-\frac{Q}{2}(k+1)}\exp\{-c\delta(k+1)\}\,\left[\left|B_{d_X}(0,1)\right|\,\sigma^Q\right]^k \\
&= c^{-k-1}(t-\tau)^{-Q/2}(k+1)^{Q/2}|B_{d_X}(0,1)|^k4^{-Qk}\,\exp\{-c\delta(k+1)\} \\
&\geq c^{-1}(t-\tau)^{-Q/2}\exp\{-ck\}\exp\left\{-c\delta(k+1)+ck-k\log\left(\frac{c\cdot 4^Q}{|B_{d_X}(0,1)|}\right)\right\}.
\end{aligned}$$

We fix c to a value $\overline{c}$ large enough for the argument of the last exponential to be positive whenever $k\geq 1$, so that $\Gamma(x,t;\xi,\tau)\geq\overline{c}^{-1}(t-\tau)^{-Q/2}\exp\{-\overline{c}k\}$; finally, we split in two cases depending on which one among $\frac{T}{\delta}$ and $\frac{4d_X(x,\xi)^2}{\delta(t-\tau)}$ is bigger:

(i) If $\frac{4d_X(x,\xi)^2}{t-\tau} \le T$, then $k = \left\lfloor \frac{T}{\delta} \right\rfloor + 1 < 2\frac{T}{\delta}$, so

$$\Gamma(x,t;\xi,\tau) \ge \overline{c}^{-1}(t-\tau)^{-Q/2}\exp\left\{-2\overline{c}T\delta^{-1}\right\} = [\overline{c}e^{-2\overline{c}T/\delta}]^{-1}(t-\tau)^{-Q/2}$$
$$\ge [\overline{c}e^{-2\overline{c}T/\delta}]^{-1}(t-\tau)^{-Q/2}\exp\left\{-[\overline{c}e^{-2\overline{c}T/\delta}]\frac{d(x,\xi)^2}{t-\tau}\right\};$$

(ii) Otherwise $\frac{4d_X(x,\xi)^2}{t-\tau} > T \ge 1$ and $k = \left\lfloor \frac{4d_X(x,\xi)^2}{\delta(t-\tau)} \right\rfloor + 1 < \frac{8d_X(x,\xi)^2}{\delta(t-\tau)}$; by equivalence between d and d_X:

$$\Gamma(x,t;\xi,\tau) \ge \overline{c}^{-1}(t-\tau)^{-Q/2}\exp\left\{-\overline{c}\frac{8\beta d(x,\xi)^2}{\delta(t-\tau)}\right\}$$
$$\ge [8\overline{c}\beta\delta^{-1}]^{-1}(t-\tau)^{-Q/2}\exp\left\{-[8\overline{c}\beta\delta^{-1}]\frac{d(x,\xi)^2}{t-\tau}\right\},$$

hence the thesis holds with

$$\overline{\overline{c}} = \overline{\overline{c}}(T) = \max\{\overline{c}e^{-2\overline{c}T/\delta},\ 8\overline{c}\beta\delta^{-1}\}.$$

□

4.3. Nonhomogeneous Cauchy problem

Lemma 4.8. *Let us assume that ω satisfies also the condition (H3). Let $\nu \ge 0$, let $T > 0$ be small enough. Let $f \in C(\mathbb{R}^n \times [0,T])$ be locally Dini continuous in x uniformly w.r.t. t, namely for any $K \Subset \mathbb{R}^N$ the modulus of continuity*

$$\omega_{f,K}(r) := \sup_{x,y\in K:\, d(x,y)<r} |f(x) - f(y)|$$

satisfies

$$\int_0^r \frac{\omega_{f,K}(x)}{x}\,dx < +\infty, \quad \forall r > 0.$$

Assume moreover that f is subject to the growth condition

$$|f(x,t)| \le Me^{\nu\|x\|^2}, \quad \forall\, x \in \mathbb{R}^n,\ t \in [0,T] \tag{4.3}$$

for some constant $M > 0$. Then the function $V_f : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ given by

$$V_f(x,t) = \iint_{\mathbb{R}^n\times(0,t)} \Gamma_\zeta(x,t;\zeta)f(\zeta)\,d\zeta, \quad \mathit{if}\, t > 0, \quad V_f(x,0) = 0$$

is well defined, belongs to $\mathfrak{C}^2(\mathbb{R}^n \times (0,T)) \cap C(\mathbb{R}^n \times [0,T])$ and

$$HV_f(x,t) = f(x,t) + \iint_{\mathbb{R}^n\times(0,t)} Z_1(x,t;\zeta)f(\zeta)\,d\zeta, \quad \forall (x,t) \in \mathbb{R}^n \times (0,T). \tag{4.4}$$

Proof. Note that, except for the continuity of V_f at $t = 0$, the statement is the same as that of Theorem 3.12 (with $\tau = 0$ and the integral map $\mu(\cdot;\zeta)$ replaced by f); thus, up to substituting the inequality (3.16) with (4.3) and the Dini-type estimate (3.21) with the local Dini continuity of $f(\cdot,t)$, the same arguments can be repeated, yielding $V_f \in \mathfrak{C}^2(\mathbb{R}^n \times (0,T)) \cap C(\mathbb{R}^n \times (0,T])$ with

$$HV_f(x,t) = f(x,t) + \iint_{\mathbb{R}^n\times(0,t)} [H\Gamma(\cdot;\zeta)](x,t)\, f(\zeta)\, d\zeta = f(x,t).$$

We are left to prove that $V_f(\cdot,t) \to 0$ as $t \to 0^+$; by quasi-triangle inequality

$$|f(\xi,\tau)| \le Me^{\nu\|\xi\|^2} \le Me^{2\nu\kappa^2(\|x\|^2+d(x,\xi)^2)}$$

for all $x, \xi \in \mathbb{R}^n$ and $0 < t-\tau \le T$; so, in view also of (3.1) and (3.8), we have

$$0 \le |V_f(x,t)| \le \check{c}_0 M e^{2\nu\kappa^2\|x\|^2} \int_0^t ds \int_{\mathbb{R}^n} \frac{\exp\left\{-\frac{d(x,\xi)^2}{\check{c}_0(t-\tau)}\left[1-2\nu\kappa^2\check{c}_0 T\right]\right\}}{(t-\tau)^{Q/2}} d\xi.$$

Thus, assuming $4\nu\kappa^2\check{c}_0 T < 1$ (i.e., T small enough), we get

$$\begin{aligned} 0 \le |V_f(x,t)| &\le \check{c}_0 M e^{2\nu\kappa^2\|x\|^2} \int_0^t \int_{\mathbb{R}^n} (t-\tau)^{-\frac{Q}{2}} \exp\left\{-\frac{d(x,\xi)^2}{2\check{c}_0(t-\tau)}\right\} d\xi \\ &\le \check{c}_0 M e^{2\nu\kappa^2\|x\|^2} \int_0^t ds \int_{\mathbb{R}^n} \gamma_0(\xi^{-1}\circ x, 2\check{c}_0(t-\tau))\, d\xi = c(M)e^{2\nu\kappa^2\|x\|^2} t \to 0, \end{aligned}$$

as $t \to 0^+$ (for every $x \in \mathbb{R}^n$), which concludes the proof. □

Lemma 4.9. *Let $f : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ be a continuous function subject to the growth condition (4.3) for some constants $\nu > 0$, $M > 0$. Then the function*

$$\widetilde{f}(z) := \iint_{\mathbb{R}^n\times(0,t)} \mu(z;\zeta) f(\zeta)\, d\zeta, \quad z \in \mathbb{R}^n \times [0,T]$$

is well-defined, continuous w.r.t. (x,t) and locally Dini continuous in x, uniformly w.r.t. $t \in [0,T]$. Furthermore, $\widetilde{f}$ is subject to the growth condition

$$|\widetilde{f}(x,t)| \le c\,\widetilde{\omega}(2\sqrt{T})e^{2\nu\kappa^2\|x\|^2} \quad \forall (x,t) \in \mathbb{R}^n \times [0,T]$$

for a constant $c = c(M,T) > 0$.

Proof. Given $T > 0$, $K \Subset \mathbb{R}^n$ (compact) and fixed $x, x' \in K$ and $t, t+h \in (0,T)$, we study the oscillation of $\widetilde{f}$ in space and time separately. As for the former

$$|\widetilde{f}(x,t) - \widetilde{f}(x',t)| \le \iint_{\mathbb{R}^n\times(0,t)} |\mu(x,t;\zeta) - \mu(x',t;\zeta)| \cdot |f(\zeta)|\, d\zeta.$$

The use of quasi-triangle inequality as in Lemma 4.8 for both x, x' gives

$$|f(\xi,\tau)| \le Me^{\nu\|\xi\|^2} \le Me^{\nu\kappa^2\cdot\min\{\|x\|^2+d(x,\xi)^2,\, \|x'\|^2+d(x',\xi)^2\}};$$

moreover, starting from either (3.21) if $t-\tau \geq d(x,x')^2$ or (3.16) if $t-\tau < d(x,x')^2$, by monotonicity of ω and (3.3) one gets

$$|\mu(x',t;\xi,\tau)-\mu(x,t;\xi,\tau)| \leq c(T)\,(t-\tau)^{-1-\frac{Q}{2}}\,\Big[\omega^{1/2}(2d(x,x'))\,\omega^{1/2}(2\sqrt{t-\tau})$$
$$+\widetilde{\omega}(2\sqrt{t-\tau})\,\widetilde{\omega}(2d(x,x'))\Big]\cdot\left[\exp\left\{-\frac{d(x,\xi)^2}{\widetilde{c}(t-\tau)}\right\}+\exp\left\{-\frac{d(x',\xi)^2}{\widetilde{c}(t-\tau)}\right\}\right]$$

provided $0<t-\tau\leq T$, for a constant $\widetilde{c}>0$. In compact notation

$$|\mu(x,t;\zeta)-\mu(x',t;\zeta)| \leq c(T)(t-\tau)^{-1-\frac{Q}{2}}\omega_1(d(x,x'))\,\omega_1(\sqrt{t-\tau})[\Phi(x)+\Phi(x')],$$
$$\text{where } \omega_1(r) := \omega^{1/2}(2r)+\widetilde{\omega}(2r), \quad \Phi := \exp\left\{-\frac{d(\cdot,\xi)^2}{\widetilde{c}(t-\tau)}\right\}.$$

Note that by (H2)-(H3) ω_1 satisfies the Dini condition, and provided T is small enough (i.e., $T < (4\widetilde{c}\nu\kappa^2)^{-1}$, as for the previous proof), we have

$$\Phi(\cdot)\,|f(\xi,\tau)| \leq M\exp\left\{-\frac{d(\cdot,\xi)^2}{\widetilde{c}(t-\tau)}(1-\widetilde{c}\nu\kappa^2 T)\right\} \leq M\exp\left\{-\frac{d(\cdot,\xi)^2}{2\widetilde{c}(t-\tau)}\right\}$$

in both x, x'; all the above combined with (3.1), (3.3) yield

$$|\widetilde{f}(x',t)-\widetilde{f}(x,t)| \leq c_K\omega_1(d(x,x'))\int_0^t \frac{\omega_1(\sqrt{t-\tau})}{t-\tau}\int_{\mathbb{R}^n}(t-\tau)^{-\frac{Q}{2}}\sum_{y\in\{x,x'\}}\exp\left\{-\frac{d(y,\xi)^2}{2\widetilde{c}(t-\tau)}\right\}d\xi$$
$$\leq c_K\omega_1(d(x,x'))\times\int_0^t\frac{\omega_1(\sqrt{t-\tau})}{t-\tau}\,d\tau\int_{\mathbb{R}^n}\sum_{y\in\{x,x'\}}\gamma_0(\xi^{-1}\circ y,\overline{c}_0\widetilde{c}(t-\tau))\,d\xi \leq c_K\omega_1(d(x,x')) \tag{4.5}$$

for a constant $c_K = c(T,M,\nu,K)$, that is the t-uniform local Dini continuity of $\widetilde{f}(\cdot,t)$. We deduce the global continuity studying also the time oscillation:

$$\widetilde{f}(x',t+h)-\widetilde{f}(x',t) = \iint_{\mathbb{R}^n\times(t,t+h)}\mu(x',t+h;\zeta)\,f(\zeta)\,d\zeta$$
$$+\iint_{\mathbb{R}^n\times(0,t)}(\mu(x',t+h;\zeta)-\mu(x',t;\zeta))\,f(\zeta)\,d\zeta = B_1(h)+B_2(h).$$

Since $\mu(x',\cdot;\zeta)$ is continuous, as $h\to 0$ $B_2(h)$ vanishes by dominated convergence; moreover, by (3.1), (3.16), (4.3) and treating again the exponential $e^{\nu\|\xi\|^2}$ via quasi-triangle inequality, one gets

$$|B_1(h)| \leq c(T,M)e^{2\nu\kappa^2\|x'\|^2}\int_t^{t+h}\frac{\omega(2\sqrt{t+h-\tau})}{t+h-\tau}\,d\tau\int_{\mathbb{R}^n}\gamma_0(\xi^{-1}\circ x',\overline{c}_0\widetilde{c}(t+h-\tau))\,d\xi$$
$$= c(T,M)e^{2\nu\kappa^2\|x'\|^2}\widetilde{\omega}(2\sqrt{h}) \longrightarrow 0$$

as $h\to 0$, which combined with (4.5) gives the continuity of $\widetilde{f}$ over $\mathbb{R}^n\times[0,T]$. Finally, repeating the last integral computation but with τ ranging in $(0,t)$, we get

$$|\widetilde{f}(x,t)| \leq c(T,M)\,e^{2\nu\kappa^2\|x\|^2}\int_0^t\frac{\omega(2\sqrt{t-\tau})}{t-\tau}\,d\tau \leq c(T,M)\,\widetilde{\omega}(2\sqrt{T})\,e^{2\nu\kappa^2\|x\|^2}$$

for all $(x,t)\in\mathbb{R}^n\times(0,T)$. □

Theorem 4.10. *Under the assumptions of Lemma 4.8, the function*

$$u(z) := \iint_{\mathbb{R}^n\times(0,t)} \Gamma(z;\zeta)\, f(\zeta)\, d\zeta, \quad \forall z \in \mathbb{R}^n \times [0,T]$$

belongs to $\mathfrak{C}^2(\mathbb{R}^n \times (0,T)) \cap C(\mathbb{R}^n \times [0,T])$ *and solves the Cauchy problem*

$$\begin{cases} Hu = f \ \ in\, \mathbb{R}^n \times (0,T), \\ u(\cdot,0) = 0 \ \ in\, \mathbb{R}^n. \end{cases}$$

Proof. First of all, exploiting the decomposition (3.11), we get

$$\begin{aligned} u(z) &= \iint_{\mathbb{R}^n\times(0,t)} \Gamma(z;\zeta)\, f(\zeta)\, d\zeta = \iint_{\mathbb{R}^n\times(0,t)} (\Gamma_\zeta(z;\zeta) + J(z;\zeta))\, f(\zeta)\, d\zeta \\ &= V_f(z) + \iint_{\mathbb{R}^n\times(0,t)} \left[\iint_{\mathbb{R}^n\times(\tau,t)} \Gamma_\chi(z;\chi)\, \mu(\chi;\zeta)\, d\chi\right] f(\zeta)\, d\zeta \\ &= V_f(z) + \iint_{\mathbb{R}^n\times(0,t)} \Gamma_\chi(z;\chi) \left[\iint_{\mathbb{R}^n\times(0,\eta)} \mu(\chi;\zeta)\, f(\zeta)\, d\zeta\right] d\chi \\ &= V_f(z) + \iint_{\mathbb{R}^n\times(0,t)} \Gamma_\chi(z;\chi)\, \widetilde{f}(\chi)\, d\chi = V_f(z) + V_{\widetilde{f}}(z). \end{aligned}$$

Applying Lemma 4.8 on $\widetilde{f}$ (in view of Lemma 4.9) and exploiting Corollary 3.6,

$$\begin{aligned} HV_{\widetilde{f}}(z) &= \widetilde{f}(z) + \iint_{\mathbb{R}^n\times(0,t)} Z_1(z;\zeta)\, \widetilde{f}(\zeta)\, d\zeta \\ &= \iint_{\mathbb{R}^n\times(0,t)} f(\chi)\mu(z;\chi)\, d\chi + \iint_{\mathbb{R}^n\times(0,t)} Z_1(z;\zeta)\left(\iint_{\mathbb{R}^n\times(0,\tau)} f(\chi)\mu(\zeta;\chi)\, d\chi\right) d\zeta \\ &= \iint_{\mathbb{R}^n\times(0,t)} f(\chi)\left[\mu(z;\chi) + \iint_{\mathbb{R}^n\times(\eta,t)} Z_1(z;\zeta)\mu(\zeta;\chi)\, d\zeta\right] d\chi \\ &= -\iint_{\mathbb{R}^n\times(0,t)} f(\chi) Z_1(z;\chi)\, d\chi. \end{aligned}$$

Combining this last result with (4.4) we get $Hu = f$ in $\mathbb{R}^n\times(0,T)$; the attainment of the initial condition follows directly from definition of V_f and $V_{\widetilde{f}}$. □

Theorem 4.11. *Under the assumptions of Theorem 4.2 and Lemma 4.8, the function*

$$u(z) := \iint_{\mathbb{R}^n\times(0,t)} \Gamma(z;\zeta) f(\zeta)\, d\zeta + \int_{\mathbb{R}^n} \Gamma(z;\xi,0) g(\xi)\, d\xi, \quad z \in \mathbb{R}^n \times [0,T]$$

belongs to $\mathfrak{C}^2(\mathbb{R}^n \times (0,T)) \cap C(\mathbb{R}^n \times [0,T])$ *and solves the Cauchy problem (4.1).*

Proof. The thesis is a direct consequence of Theorems 4.2 and 4.10, thanks to the linearity of the Cauchy problem (4.1). □

5. Conclusions

We built a fundamental solution that emulates standard heat kernels in the context of degenerate parabolic operators $H = \partial_t - a^{ij}(x,t)X_iX_j$, under the assumptions that the coefficients $\{a_{ij}\}$ are double Dini continuous and the vector fields $\{X_i\}$ generate a Carnot group; this work extended previous results [3, 6, 11] by requiring weaker conditions on the considered operators.

The assumption on the coefficients' regularity is somehow natural for the implemented method, as it justifies that the fundamental solution of H should be "similar" to that of the associated constant coefficients' operator; more precisely, in [17] it has been explicitly proved that the assumption of Dini continuity for the $\{a_{ij}\}$ is sharp, meaning that H might not have a "good" fundamental solution under weaker regularity of its coefficients. The regularity we actually required in this work - i.e., *double* Dini continuity - is slightly higher, but we believe that this stronger assumption is mainly technical and might be relaxed to Dini continuity (as done in [17] for uniformly parabolic operators in the Euclidean space) with further technical complications. Concerning the structure of the operator, the validity of Hörmander condition for the X_i's is essential, as existence of smooth fundamental solutions can not occur for truly degenerate operators; the geometric assumptions of translation-invariance and homogeneity w.r.t. dilations, instead, may be slightly relaxed: in view of the global lifting [18] for homogeneous Hörmander vector fields, we expect analogous results to hold without the need of the X_i's to be left-invariant w.r.t. some group structure, as long as homogeneity is assumed (otherwise, it is our opinion that only local results might be obtained). Nevertheless, since the parametrix method in this degenerate framework is intrinsically delicate and technical, here we chose to avoid these more general assumptions.

The results in the current paper may be a starting point for the study of further properties of solutions to the associated global equation and Cauchy problem, such as Harnack inequalities (as discussed in [16] for Hölder continuous coefficients), regularity of solutions in the Dini spaces, asymptotic behavior of solutions (as done in the correspondent Hölder case [6]).

Use of Generative-AI tools declaration

The author declares he/she has not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This paper is an extract of the namesake Master Thesis, conduct by the author at Politecnico di Milano under the supervision of professor Marco Bramanti. A sincere acknowledgment goes to the latter for all the guidance and support that lead to the development of the present work. The author wishes to thank also the referees for having read carefully the paper, providing useful suggestions for its improvement.

Conflict of interest

The author declares no conflict of interest.

References

1. E. E. Levi, Sulle equazioni lineari totalmente ellittiche alle derivate parziali, *Rend. Circ. Matem. Palermo*, **24** (1907), 275–317. https://doi.org/10.1007/BF03015067
2. E. E. Levi, I problemi dei valori al contorno per le equazioni lineari totalmente ellittiche alle derivate parziali, *Met. Soc. It. dei Sc. XL*, 1909.
3. A. Friedman, *Partial differential equations of parabolic type*, Prentice Hall, 1964.
4. S. Polidoro, On a class of ultraparabolic operators of Kolmogorov-Fokker-Planck type, *Le Matematiche*, **49** (1994), 53–105.
5. E. Lanconelli, S. Polidoro, On a class of hypoelliptic evolution operators, *Rend. Sem. Mat. Univ. Politec. Torino*, **52** (1994), 29–63.
6. A. Bonfiglioli, E. Lanconelli, F. Uguzzoni, Fundamental solutions for non-divergence form operators on stratified groups, *Trans. Amer. Math. Soc.*, **356** (2003), 2709–2737. https://doi.org/10.1090/S0002-9947-03-03332-4
7. M. Bramanti, L. Brandolini, E. Lanconelli, F. Uguzzoni, *Non-divergence equations structured on Hörmander vector fields: heat kernels and Harnack inequalities*, American Mathematical Society, 2010.
8. S. Biagi, M. Bramanti, Non-divergence operators structured on homogeneous Hörmander vector fields: heat kernels and global Gaussian bounds, *Adv. Differential Equations*, **26** (2021), 621–658. https://doi.org/10.57262/ade026-1112-621
9. G. Lucertini, G. S. Pagliarani, A. Pascucci, Optimal regularity for degenerate Kolmogorov equations in non-divergence form with rough-in-time coefficients, *J. Evol. Equ.*, **23** (2023), 1–37. https://doi.org/10.1007/s00028-023-00916-9
10. M. Bramanti, S. Polidoro, Fundamental solutions for Kolmogorov-Fokker-Planck operators with time-depending measurable coefficients, *Math. Eng.*, **2** (2020), 734–771. https://doi.org/10.3934/mine.2020035
11. I. V. Zhenyakova, M. F. Cherepova, The Cauchy problem for a multidimensional parabolic equation with Dini-continuous coefficients, *J. Math. Sci.*, **264** (2022), 581–602. https://doi.org/10.1007/s10958-022-06018-0
12. M. Bramanti, L. Brandolini, *Hörmander operators*, World Scientific, 2023.
13. A. Bonfiglioli, E. Lanconelli, F. Uguzzoni, Uniform Gaussian estimates for the fundamental solutions for heat operators on Carnot groups, *Adv. Differential Equations*, **7** (2002), 1153–1192. https://doi.org/10.57262/ade/1356651633
14. E. A. Baderko, Potential for $2p$-parabolic equations, *Differ. Uravn.*, **19** (1983), 9–18.
15. M. Bramanti, M. S. Fanciullo, BMO estimates for nonvariational operators with discontinuous coefficients structured on Hörmander's vector fields on Carnot groups, *Adv. Differential Equations*, **18** (2013), 955–1004. https://doi.org/10.57262/ade/1372777765

16. A. Bonfiglioli, F. Uguzzoni, Harnack inequality for non-divergence form operators on stratified groups, *Trans. Amer. Math. Soc.*, **359** (2007), 2463–2481. https://doi.org/10.1090/S0002-9947-07-04273-0

17. E. A. Baderko, K. V. Semenov, Regular fundamental solution to parabolic equation with Dini continuous coefficients in many spatial variables, *J. Math. Sci.*, **274** (2023), 441–459. https://doi.org/10.1007/s10958-023-06612-w

18. S. Biagi, A. Bonfiglioli, The existence of a global fundamental solution for homogeneous Hörmander operators via a global lifting method, *Proc. Lond. Math. Soc.*, **114** (2017), 855–889. https://doi.org/10.1112/plms.12024