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Research article

Comparison results for the p-torsional rigidity on convex domains

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Abstract: For each open, bounded and convex domain $\Omega \subset \mathbb{R}^D$, $D \geq 2$, and each real number $p > 1$, we denote by u_p the *p-torsion function* on Ω, i.e., the solution of the *torsional creep problem* $\Delta_p u = -1$ in Ω, $u = 0$ on $\partial\Omega$, where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian. Let $T_p(\Omega)$ be the *p-torsional rigidity* on Ω, defined as $T_p(\Omega) := \int_\Omega u_p dx$. Define $T(p;\Omega) := |\Omega|^{p-1} T_p(\Omega)^{1-p}$, where $|\Omega|$ stands for the Lebesgue measure of Ω. The main purpose of this paper is to compare the values of $T(p;\Omega)$ for bounded convex domains having different inradii. We prove that for any $0 < a < b$ there exists an explicit constant $\gamma_{D,p} \in [1/D, 1)$, depending only on the dimension D and the parameter p, such that $T(p;\Omega_b) \leq T(p;\Omega_a)$, for all $\Omega_a \in \mathbb{P}^D(a)$, and $\Omega_b \in \mathbb{P}^D(b)$, if and only if $\gamma_{D,p} b \geq a$, where $\mathbb{P}^D(r)$ denotes the family of convex bounded domains in $\mathbb{R}^D$ of inradius r. In addition, we discuss the asymptotic equality case, the limiting regimes $p \to 1^+$ and $p \to \infty$, and the behaviour of the scale-invariant functional on model families such as rectangles, orthotopes, ellipses, and triangles. We also derive a fixed-volume monotonicity consequence for the unnormalised torsional rigidity, parameterized by the inradius.

Keywords: inradius; open, bounded, convex domain; p-torsional rigidity; p-Laplacian

1. Introduction and statement of main results

1.1. Definitions and notation

For each integer $D \geq 1$ we denote by $\mathbb{R}^D$ the D-dimensional Euclidean space. For a bounded convex set $\Omega \subset \mathbb{R}^D$, denote by R_Ω its *inradius*, i.e., the radius of the largest ball inscribed in Ω, and by $\partial\Omega$ its boundary.

Given $r > 0$, we write B_r^D for the open ball of radius r in $\mathbb{R}^D$, and define the following two classes of sets

$$\mathbb{P}^D := \{\Omega \subset \mathbb{R}^D \mid \Omega \text{ is open, bounded and convex}\},$$

and

$$\mathbb{P}^D(r) := \{\Omega \in \mathbb{P}^D \mid R_\Omega = r\}.$$

For a given $\Omega \in \mathbb{P}^D$ and $p \in (1, \infty)$, we consider the *p-torsion problem*

$$\begin{cases} -\Delta_p u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ stands for the *p-Laplacian* operator. A simple application of the *Direct Method of the Calculus of Variations* shows that for each $p \in (1, \infty)$ problem (1.1) possesses a unique solution, $u_p \in W_0^{1,p}(\Omega)$, called the *p-torsion function* on Ω. The associated *p-torsional rigidity* on Ω is defined as follows

$$T_p(\Omega) := \int_\Omega u_p \, dx.$$

The following variational characterization will, however, be more useful for our subsequent analysis:

$$T_p(\Omega)^{p-1} = \sup_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\left(\displaystyle\int_\Omega |u| \, dx\right)^p}{\displaystyle\int_\Omega |\nabla u|^p \, dx},$$

(see, e.g., [1, relations (18) and (19)]). Next, for convenience we introduce the *normalized p-torsional rigidity*

$$T(p; \Omega) := |\Omega|^{p-1} T_p(\Omega)^{1-p}, \tag{1.2}$$

where $|\Omega|$ is the Lebesgue measure of Ω, which can also be written as

$$T(p; \Omega) := \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\displaystyle\frac{1}{|\Omega|} \int_\Omega |\nabla u|^p \, dx}{\left(\displaystyle\frac{1}{|\Omega|} \int_\Omega |u| \, dx\right)^p}. \tag{1.3}$$

Note that, for general domains $\Omega \in \mathbb{P}^D$, neither u_p nor $T(p; \Omega)$ can be computed explicitly.

In the engineering literature, torsional rigidity is a classical measure of the resistance of a prismatic bar to twisting. When $D = 2$ and $p = 2$, Ω represents the cross-section of the bar and the problem (1.1) reduces to the classical linear torsion problem, in which the torsional rigidity depends strongly on the geometry of the cross-section. The nonlinear case $p \neq 2$ is also relevant in mechanics: It is related to torsional creep and to power-law type material response, in particular for bars subject to torsion over long time intervals at high temperature (see [2]; see also [3,4] for the underlying theory of creep and plasticity). In this setting, the parameter p encodes the nonlinear constitutive behaviour of the material. Thus, comparison principles for $T_p(\Omega)$ and $T(p; \Omega)$ under geometric constraints such as prescribed inradius or fixed volume provide quantitative information on how the shape of the cross-section influences the torsional response, and are therefore relevant to shape optimization and engineering design.

1.2. Formulation of the problem

For the ball B_R^D, the p-torsion function is explicitly known (see [2]):

$$u_p(x) = \frac{D(p-1)}{p}\left[\left(\frac{R}{D}\right)^{\frac{p}{p-1}} - \left|\frac{x}{D}\right|^{\frac{p}{p-1}}\right], \quad \forall\, x \in B_R^D.$$

Further computations (see, e.g., [5, relation (2.7)]) yield

$$T\left(p; B_R^D\right) = D\left(D + \frac{p}{p-1}\right)^{p-1} \frac{1}{R^p}, \quad \forall\, p > 1.$$

Hence, for two different radii $0 < R_1 < R_2$, we have

$$T\left(p; B_{R_2}^D\right) < T\left(p; B_{R_1}^D\right), \quad \forall\, p > 1.$$

This observation naturally leads to the following question: *given $\Omega_a \in \mathbb{P}^D(a)$ and $\Omega_b \in \mathbb{P}^D(b)$, with $0 < a < b$ fixed, is it true that we have*

$$T(p; \Omega_b) < T(p; \Omega_a), \qquad \forall p > 1? \tag{1.4}$$

1.3. Motivation

For comparison, recall that the *first Dirichlet eigenvalue of the p-Laplacian* on $\Omega \in \mathbb{P}^D$ is defined by

$$\lambda_1(p; \Omega) := \min_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^p \, dx}{\displaystyle\int_\Omega |u|^p \, dx}. \tag{1.5}$$

In [6, Theorem 1], the last two authors established the following sharp comparison principle.

Theorem 1. *Let $D \geq 1$ be a fixed integer. Given real numbers $p > 1$ and $0 < a < b$, we define the constant*

$$C(D; p) := \left[\frac{\lambda_1\left(p; B_1^D\right)}{\lambda_1\left(p; B_1^1\right)}\right]^{1/p} > 0. \tag{1.6}$$

We then have

$$\lambda_1(p; \Omega_b) < \lambda_1(p; \Omega_a), \quad \forall\, \Omega_a \in \mathbb{P}^D(a), \text{ and } \Omega_b \in \mathbb{P}^D(b), \tag{1.7}$$

if and only if $C(D; p)^{-1} b \geq a$.

Moreover, when $C(D; p)^{-1} b = a$ then there exists a sequence of domains $\{\Omega_n\}_n \subset \mathbb{P}^D(a)$ for which the equality in (1.7) holds asymptotically if $\Omega_b = B_b^D$, in the sense that

$$\lim_{n\to\infty} \lambda_1(p; \Omega_n) = \lambda_1\left(p; B_b^D\right). \tag{1.8}$$

Since $\lambda_1(p; \Omega)$ and $T(p; \Omega)$ are defined through closely related variational principles, it is natural to ask whether an analogue of Theorem 1 holds for $T(p; \Omega)$. This expectation is further supported by the case $p = 1$, where both constants $\lambda_1(p; \Omega)$ and $T(p; \Omega)$ degenerate to the *Cheeger's constant* of the domain Ω, denoted by $h(\Omega)$. In such a case, the following result is known, being also established in [6, Corollary 1]:

Corollary 1. *Let $D \geq 1$ be a fixed integer and $0 < a < b$ be two given real numbers. Then*

$$h(\Omega_b) \leq h(\Omega_a), \quad \forall\ \Omega_a \in \mathbb{P}^D(a), \quad \Omega_b \in \mathbb{P}^D(b), \tag{1.9}$$

if and only if $D^{-1}b \geq a$ (with strict inequality in (1.9) when $D^{-1}b > a$).

Moreover, when $D^{-1}b = a$ then there exists a sequence of domains $\{\Omega_n\}_n \subset \mathbb{P}^D(a)$ for which the equality in (1.9) holds asymptotically if $\Omega_b = B_b^D$, in the sense that

$$\lim_{n\to\infty} h(\Omega_n) = h\left(B_b^D\right).$$

1.4. Main results

We now state the main theorem of this paper, which provides the analogue of Theorem 1 for the *normalized p-torsional rigidity*.

Theorem 2. *Let $D \geq 2$ be a fixed integer. Given real numbers $p > 1$ and $0 < a < b$, we define the constant*

$$\gamma_{D,p} := \left[\frac{T\left(p; B_1^D\right)}{T\left(p; B_1^1\right)}\right]^{-1/p} > 0. \tag{1.10}$$

We then have

$$T\left(p; \Omega_b\right) \leq T\left(p; \Omega_a\right), \quad \forall\ \Omega_a \in \mathbb{P}^D(a), \text{ and } \Omega_b \in \mathbb{P}^D(b), \tag{1.11}$$

if and only if $\gamma_{D,p}b \geq a$, with strict inequality in (1.11) when $\gamma_{D,p}b > a$.

Moreover, when $\gamma_{D,p}b = a$ then the equality in (1.11) occurs asymptotically if $\Omega_b = B_b^D$, in the sense that there exists a sequence of domains $\{\omega_n\}_n \subset \mathbb{P}^D(a)$ for which we have

$$\lim_{n\to\infty} T\left(p; \omega_n\right) = T\left(p; B_b^D\right). \tag{1.12}$$

As an immediate consequence, we obtain an answer to the problem we addressed in Subsection 1.2.

Corollary 2. *Let $D \geq 2$ be a fixed integer and $0 < a < b$ be two given real numbers. Then the following assertions hold.*

If $D^{-1}b < a$ then there exists a real number $p_D \in (1, \infty)$ and two sets $\Omega_a \in \mathbb{P}^D(a)$ and $\Omega_b \in \mathbb{P}^D(b)$ such that

$$T\left(p_D; \Omega_a\right) < T\left(p_D; \Omega_b\right). \tag{1.13}$$

Conversely, if $D^{-1}b \geq a$ then for each set $\Omega_a \in \mathbb{P}^D(a)$ and $\Omega_b \in \mathbb{P}^D(b)$ we have

$$T\left(p; \Omega_b\right) < T\left(p; \Omega_a\right), \quad \forall\ p \in (1, \infty). \tag{1.14}$$

Remark 1. *Theorem 2 and Corollary 2 solve the problem (1.4) proposed for investigation in this paper. The elementary fact that, for two balls in the Euclidean space of dimension D with distinct radii, the normalized p-torsional rigidity of the ball with larger radius is smaller than that of the ball with smaller radius does not extend trivially to two arbitrary bounded convex domains with different inradii. More precisely, for fixed $p > 1$, Theorem 2 shows that the general comparison holds if and only if a suitable balance between the inradii holds, expressed by the fact that the ratio b/a between* *the larger and the smaller inradius is bounded from below by the explicit constant $\gamma_{D,p}^{-1}$, depending on both p and D. Corollary 2 gives the corresponding dimension-only condition when the comparison is required for all $p > 1$.*

As we will see in the next two sections, the constant $\gamma_{D,p}$ is closely related to two sharp estimates, from below and from above, for the normalized p-torsional rigidity on bounded convex domains, namely the Hersch–Protter type inequality and the Buser type inequality, respectively. These estimates are due to [1, 7, 8], and are key ingredients in the proofs of our main results. However, Theorem 2 and Corollary 2 are not merely formal consequences of these estimates: They show how the sharp lower and upper bounds combine to yield an explicit comparison threshold for convex domains with prescribed inradii.

1.5. Structure of the paper.

The rest of the paper is organized as follows. In Section 2 we collect several basic properties of the normalized p-torsional rigidity. Section 3 is devoted to the proofs of the main results. In Section 4 we reformulate the comparison principle in terms of the average integral of the distance function to the boundary. In Section 5 we evaluate the scale-invariant functional $Q_p(\Omega)$ defined in (3.1) on several model families of convex domains, including rectangles, higher-dimensional orthotopes, planar ellipses, and triangles. In Section 6 we discuss the limiting regimes as $p \to 1^+$ and $p \to \infty$. Finally, Section 7 contains concluding remarks and a fixed-volume monotonicity consequence for the unnormalised torsional rigidity.

2. Properties of $T(p;\Omega)$

In this section we recall several known properties of $T(p;\Omega)$, when $D \geq 1$, $p \in (1,\infty)$ and $\Omega \in \mathbb{P}^D$. These inequalities will be repeatedly used in the proofs of our main results. These include sharp lower and upper bounds of *Hersch–Protter* and *Buser type* inequalities, as well as a simple scaling rule.

2.1. A lower bound of $T(p;\Omega)$

According to [1, Theorem 4.3] (see also [8, Corollary 1.2] applied with $q = 1$) one has the following sharp *Hersch–Protter type inequality*

$$T(p;\Omega) \geq \left(\frac{2p-1}{p-1}\right)^{p-1} \frac{1}{R_\Omega^p}, \quad \forall\, \Omega \in \mathbb{P}^D,\ \forall\, p \in (1,\infty). \tag{2.1}$$

Note that $\left(\frac{2p-1}{p-1}\right)^{p-1} = T\left(p; B_1^1\right)$. Moreover, we recall that by the proof of [8, Corollary 1.2] (see also Subsection 5.1 below), we know that inequality (2.1) is *asymptotically sharp* by using, for example, the slab-type sequences $\Omega_L := (-L, L)^{D-1} \times (0, 2R)$ and observing that $\Omega_L \in \mathbb{P}^D(R)$ (when $L > R$) and

$$\lim_{L\to\infty} T\left(p;\Omega_L\right) = \left(\frac{2p-1}{p-1}\right)^{p-1} \frac{1}{R^p}, \quad \forall\, p \in (1,\infty).$$

2.2. An upper bound of $T(p;\Omega)$

According to [1, Theorem 1.1(ii)] (see also [7] for a similar result) one has the following sharp *Buser-type inequality*

$$T(p;\Omega) \le D\Big(D + \tfrac{p}{p-1}\Big)^{p-1} \frac{1}{R_\Omega^p}, \qquad \forall \Omega \in \mathbb{P}^D,\ \forall\, p \in (1,\infty). \tag{2.2}$$

Moreover, equality holds, up to translations and dilations, for balls. More precisely,

$$D\Big(D + \tfrac{p}{p-1}\Big)^{p-1} = T\left(p; B_1^D\right).$$

2.3. Rescaled domains

Finally, here we note that if $t > 0$ and $\Omega \in \mathbb{P}^D$, then the rescaled domain $t\Omega := \{tx : x \in \Omega\}$ satisfies (see, e.g., [5, relation (2.5)]) $R_{t\Omega} = tR_\Omega$, that is $t\Omega \in \mathbb{P}^D(tR_\Omega)$, and

$$T(p;t\Omega) = t^{-p}T(p;\Omega), \quad \forall\, p \in (1,\infty). \tag{2.3}$$

This scaling property will play a key role in the construction of extremal sequences in the proof of Theorem 2.

3. Proof of the main results

3.1. Proof of Theorem 2

Let $\Omega \in \mathbb{P}^D$. We introduce the following scale-invariant quantity

$$Q_p(\Omega) := \left(\frac{T(p;\Omega)\, R_\Omega^p}{(\frac{2p-1}{p-1})^{p-1}}\right)^{1/p} \tag{3.1}$$

and divide the proof into four steps.

Step 1 (Choosing the constant $\gamma_{D,p}$).

We first present a geometric corridor that will be used in Section 5. By [7, Main theorem] (applied with $q = 1$) one has the Buser-type estimate

$$T(p;\Omega) < \left(\frac{2p-1}{p-1}\right)^{p-1} \left(\frac{P(\Omega)}{|\Omega|}\right)^{p}, \qquad \forall \Omega \in \mathbb{P}^D,\ \forall\, p > 1,$$

where $P(\Omega)$ denotes the perimeter of Ω, together with the geometric inequality (see [7, Lemma B.1])

$$\frac{R_\Omega}{D} \le \frac{|\Omega|}{P(\Omega)} < R_\Omega, \qquad \forall\, \Omega \in \mathbb{P}^D.$$

Combining these with the Hersch–Protter inequality (2.1) and recalling the definition (3.1) of Q_p, we obtain the geometric corridor

$$1 \ \le\ Q_p(\Omega)\ <\ R_\Omega \frac{P(\Omega)}{|\Omega|}\ \le\ D, \qquad \forall\, \Omega \in \mathbb{P}^D,\ \forall\, p > 1. \tag{3.2}$$

Combining instead Hersch–Protter inequality (2.1), Buser-type bound (2.2), and the facts that $\left(\frac{2p-1}{p-1}\right)^{p-1} = T\left(p; B_1^1\right)$ and $D\Big(D + \frac{p}{p-1}\Big)^{p-1} = T\left(p; B_1^D\right)$, we obtain the sharp global corridor

$$1 \;\le\; Q_p(\Omega) \;\le\; \gamma_{D,p}^{-1}, \qquad \forall\, \Omega \in \mathbb{P}^D,\ \forall\, p > 1, \tag{3.3}$$

where $\gamma_{D,p}$ is defined by relation (1.10) and the upper endpoint is attained, up to translations and dilations, by balls.

This motivates the definition of the following constants:

$$\alpha(p; D) := \inf_{\Omega\in\mathbb{P}^D} Q_p(\Omega), \qquad \beta(p; D) := \sup_{\Omega\in\mathbb{P}^D} Q_p(\Omega). \tag{3.4}$$

The lower endpoint of the corridor (3.2) is sharp. Indeed, the Hersch–Protter type inequality (2.1) is sharp, as recalled in Subsection 2.1 (see [1, Theorem 4.3 and Proposition 4.4] and [8, Corollary 1.2]), so that

$$\alpha(p; D) = 1, \quad \forall\, p > 1, \quad \forall\, D \ge 2. \tag{3.5}$$

The upper endpoint, by contrast, is *attained*. The result [1, Theorem 1.1(ii)] provides the corresponding sharp lower bound for the scale-invariant torsional functional $T_p(\Omega)/(|\Omega|\, R_\Omega^{p/(p-1)})$, the minimum being realized by the ball; via the definition (1.2) of $T(p; \Omega)$, this is equivalent to the sharp upper bound for $T(p; \Omega)\, R_\Omega^p$. Combining the explicit value

$$T(p; B_R^D) = D(D + \tfrac{p}{p-1})^{p-1} R^{-p}$$

recalled in Subsection 1.2 with [1, Theorem 1.1(ii)], one obtains

$$T(p; \Omega)\, R_\Omega^p \le D\Big(D + \tfrac{p}{p-1}\Big)^{p-1}, \qquad \forall\, \Omega \in \mathbb{P}^D,$$

with equality, up to translations and dilations, for balls. Recalling the (3.1) of Q_p and (1.10) of $\gamma_{D,p}$, this gives

$$\beta(p; D) = Q_p(B_R^D) = \left(\frac{p-1}{2p-1}\right)^{(p-1)/p} \left[D\Big(D + \tfrac{p}{p-1}\Big)^{p-1}\right]^{1/p} = \gamma_{D,p}^{-1}, \tag{3.6}$$

the supremum being attained, up to translations and dilations, for balls. In particular, for $D = 2$, $p = 2$ one obtains $\beta(2; 2) = \sqrt{8/3}$. Note that the use of infima and suprema reflects the fact that the infimum is not attained within the class $\mathbb{P}^D$, but only approached by suitable degenerating sequences of convex domains, whereas the supremum is attained by balls. We then have

$$1 = \alpha(p; D) < \beta(p; D) = \gamma_{D,p}^{-1}. \tag{3.7}$$

In particular we note that

$$\gamma_{D,p} = \frac{\alpha(p; D)}{\beta(p; D)} < 1.$$

Step 2 (Direct implication).

Assume $\gamma_{D,p} b \ge a$. We want to show that inequality (1.11) is valid. Suppose, for contradiction, that there exist domains $\Omega_a \in \mathbb{P}^D(a)$ and $\Omega_b \in \mathbb{P}^D(b)$ such that

$$T(p; \Omega_a) < T(p; \Omega_b).$$

Recalling the definition of $Q_p(\Omega)$ and combining this inequality with relations (2.1), (2.2) and (3.6), we get

$$\left(\frac{2p-1}{p-1}\right)^{p-1}\frac{1}{a^p} \leq T(p;\Omega_a) < T(p;\Omega_b) \leq \left(\frac{2p-1}{p-1}\right)^{p-1}\frac{\gamma_{D,p}^{-p}}{b^p}.$$

This leads to the contradiction $\gamma_{D,p}b < a$. Hence, $T(p;\Omega_b) \leq T(p;\Omega_a)$ must hold.

Note that the above argument shows that for any two real numbers $a, b \in (0,\infty)$ satisfying $\gamma_{D,p}b \geq a$, inequality (1.11) holds true. In particular, the assumption that a and b are fixed at the beginning of the proof of Theorem 2 can be relaxed for this part of Step 2.

Next, let us assume $\gamma_{D,p}b > a$. Let $\Omega_a \in \mathbb{P}^D(a)$ and $\Omega_b \in \mathbb{P}^D(b)$. Then, considering the rescaled domain $\frac{\gamma_{D,p}b}{a}\Omega_a \in \mathbb{P}^D(\gamma_{D,p}b)$, by the scaling property (2.3) we have

$$T\left(p;\frac{\gamma_{D,p}b}{a}\Omega_a\right) = \left(\frac{\gamma_{D,p}b}{a}\right)^{-p} T(p;\Omega_a). \tag{3.8}$$

Finally, since $\gamma_{D,p} \in (0,1)$ we have $a' := \gamma_{D,p}b < b$. By the remark above, relation (1.11) holds for any pair of inradii satisfying $\gamma_{D,p}b \geq a'$, hence in particular for (a', b). Rescale Ω_a to $\Omega'_a := \frac{a'}{a}\,\Omega_a \in \mathbb{P}^D(a')$. Applying (1.11) to (Ω'_a, Ω_b) and using (2.3) yields

$$T(p;\Omega_b) \;\leq\; T(p;\Omega'_a) = \left(\frac{a'}{a}\right)^{-p} T(p;\Omega_a) = \left(\frac{\gamma_{D,p}b}{a}\right)^{-p} T(p;\Omega_a).$$

Since $\gamma_{D,p}b/a > 1$, we have $(\gamma_{D,p}b/a)^{-p} < 1$, and therefore the right-hand side is strictly smaller than $T(p;\Omega_a)$, whence

$$T(p;\Omega_b) \;<\; T(p;\Omega_a).$$

Therefore, inequality (1.11) holds strictly.

Step 3 (Converse implication).

We assume that relation (1.11) holds true, that is $T(p;\Omega_b) \leq T(p;\Omega_a)$ for all $\Omega_a \in \mathbb{P}^D(a)$ and $\Omega_b \in \mathbb{P}^D(b)$, and we show that $\gamma_{D,p}b \geq a$.

Indeed, by the definition of $\alpha(p;D)$, there exists an extremal sequence of sets, $(\omega_n)_n \subset \mathbb{P}^D$ such that

$$\lim_{n\to\infty} Q_p(\omega_n) = \alpha(p;D) = 1. \tag{3.9}$$

Moreover, by the scaling invariance property (2.3), we recall that we have

$$T(p;t\Omega)R^p_{t\Omega} = T(p;\Omega)R^p_{\Omega}, \qquad \forall\,\Omega \in \mathbb{P}^D,\ \forall\, t > 0.$$

Now, the limit in (3.9) remains valid if we replace ω_n by the rescaled domain $\frac{a}{R_{\omega_n}}\omega_n \in \mathbb{P}^D(a)$. Hence we may assume $(\omega_n)_n \subset \mathbb{P}^D(a)$ (since $R_{\frac{a}{R_{\omega_n}}\omega_n} = a$). In this case, relation (3.9) becomes

$$\lim_{n\to\infty} T(p;\omega_n)^{1/p} = \left(\frac{2p-1}{p-1}\right)^{(p-1)/p}\frac{1}{a} = T(p;B^1_1)^{1/p}\frac{1}{a}. \tag{3.10}$$

On the other hand, since for each integer $n \geq 1$ we have $\omega_n \in \mathbb{P}^D(a)$ and $B^D_b \in \mathbb{P}^D(b)$, relation (1.11) gives

$$T(p;B^D_b) \leq T(p;\omega_n), \quad \forall\, n \geq 1, \tag{3.11}$$

or, equivalently,

$$T(p; B_b^D)^{1/p} \le T(p; \omega_n)^{1/p}, \quad \forall\, n \ge 1. \tag{3.12}$$

Letting $n \to \infty$ in (3.12) and using (3.10), we obtain

$$T(p; B_b^D)^{1/p} \ \le \ T(p; B_1^1)^{1/p} \frac{1}{a},$$

which in view of (1.10) and (2.3) gives, $a \le \gamma_{D,p} b$.

Step 4 (Equality case).

We assume that $\gamma_{D,p} b = a$ and we show that the equality in (1.11) occurs asymptotically.

Indeed, as in the proof of Step 3, there exists a sequence of sets $(\omega_n)_n \subset \mathbb{P}^D(a)$ such that relation (3.10) holds true. Since $\gamma_{D,p} b = a$ relation (3.10) reads, equivalently, as follows

$$\lim_{n\to\infty} T(p; \omega_n)^{1/p} = \left(\frac{2p-1}{p-1}\right)^{(p-1)/p} \frac{1}{a} = T(p; B_1^1)^{1/p} \frac{1}{\gamma_{D,p} b}.$$

Now, recalling that by relation (1.10) we have

$$\gamma_{D,p} := \left[\frac{T\left(p; B_1^D\right)}{T\left(p; B_1^1\right)}\right]^{-1/p},$$

the last two relations and (2.3) imply

$$\lim_{n\to\infty} T(p; \omega_n)^{1/p} = T(p; B_1^D)^{1/p} \frac{1}{b} = T(p; B_b^D)^{1/p},$$

so equality in (1.11) holds asymptotically, as claimed. The proof of Theorem 2 is thus complete.

3.2. *Proof of Corollary 2*

For each $p \in (1, \infty)$ let $\gamma_{D,p}$ be defined by relation (1.10). Simple computations show that

$$\gamma_{D,p} = \left[\frac{1}{D}\left(\frac{2p-1}{(D+1)p-D}\right)^{p-1}\right]^{1/p} > \frac{1}{D}, \quad \forall\, p > 1, \tag{3.13}$$

and the lower bound D^{-1} is optimal, since $\lim_{p\to 1^+} \gamma_{D,p} = D^{-1}$. Thus, we have

$$\inf_{p\in(1,\infty)} \gamma_{D,p} = D^{-1}.$$

Assume that $D^{-1} b < a$. Then there exists $\varepsilon_0 > 0$ such that $(D^{-1} + \varepsilon_0) b < a$ (for instance, one may take $\varepsilon_0 = \frac{1}{2}(\frac{a}{b} - \frac{1}{D})$). Since $D^{-1} = \inf_{p\in(1,\infty)} \gamma_{D,p}$, it follows that there exists $p_{\varepsilon_0} \in (1, \infty)$ such that

$$\gamma_{D,p_{\varepsilon_0}} < D^{-1} + \varepsilon_0.$$

Consequently $\gamma_{D,p_{\varepsilon_0}} b < a$. Applying Theorem 2 with $p = p_{\varepsilon_0}$, it follows that there exist two sets $\Omega_a \in \mathbb{P}^D(a)$ and $\Omega_b \in \mathbb{P}^D(b)$ such that

$$T\left(p_{\epsilon_0}; \Omega_a\right) < T\left(p_{\epsilon_0}; \Omega_b\right),$$

i.e., inequality (1.13) holds true with $p_D = p_{\epsilon_0}$.

On the other hand, if $D^{-1} b \ge a$, from relation (3.13), we have $\gamma_{D,p} b > D^{-1} b \ge a$, for all $p \in (1, \infty)$. By Theorem 2 we deduce that (1.14) holds. The proof of Corollary 2 is now complete.

4. An extension in terms of the average distance function

A key argument in the analysis of the problem considered in [6] and in the formulation of Theorem 1 therein in terms of the inradius of the domain, is the following asymptotic formula due to [9, Lemma 1.5] (see also [10, Theorem 3.1]):

$$\lim_{p\to\infty} \sqrt[p]{\lambda_1(p;\Omega)} = R_\Omega^{-1}, \quad \forall\, \Omega \in \mathbb{P}^D. \tag{4.1}$$

Denote by δ_Ω the distance function to the boundary of $\Omega \in \mathbb{P}^D$, i.e.,

$$\delta_\Omega(x) := \inf_{y\in\partial\Omega} |x-y|, \qquad \forall x \in \Omega,$$

and define *the average integral of the distance function* by

$$\delta(\Omega) := \frac{1}{|\Omega|}\int_\Omega \delta_\Omega(x)\,dx, \qquad \Omega \in \mathbb{P}^D.$$

We recall that by [5, relation (2.4)], we know that we have

$$\lim_{p\to\infty} \sqrt[p]{T(p;\Omega)} = \delta(\Omega)^{-1}, \quad \forall\, \Omega \in \mathbb{P}^D. \tag{4.2}$$

For each real number $r > 0$, we define

$$\mathbb{P}^D[r] := \{\Omega \in \mathbb{P}^D \mid \delta(\Omega) = r\}.$$

Let now $0 < a < b$ be two real numbers. Using (4.2), we observe that for any *fixed* domains $\Omega_a \in \mathbb{P}^D[a]$ and $\Omega_b \in \mathbb{P}^D[b]$, there exists a constant $p_0 = p_0(\Omega_a, \Omega_b)$ such that

$$T(p;\Omega_b) < T(p;\Omega_a), \quad \forall\, p > p_0. \tag{4.3}$$

This naturally leads to a reformulation of the question raised at the end of Subsection 1.2, namely: *Is it true that, given $\Omega_a \in \mathbb{P}^D[a]$ and $\Omega_b \in \mathbb{P}^D[b]$, with $0 < a < b$ fixed, inequality (4.3) holds for every $p \in (1,\infty)$?*

To investigate this question, we recall the following sharp inequality from [11, Proposition 6.1]

$$\frac{1}{D+1} \le \frac{\delta(\Omega)}{R_\Omega} \le \frac{1}{2}, \quad \forall\, \Omega \in \mathbb{P}^D. \tag{4.4}$$

Note that the value $(D+1)^{-1}$ from the left-hand side in the above inequality is achieved if Ω is a ball while the value 2^{-1} from the right-hand side in the above inequality is asymptotically attained by considering (for example) a sequence of slab domains $\Omega_\varepsilon := (0,1)^{D-1} \times (0,\varepsilon)$ for $\varepsilon > 0$, and letting $\varepsilon \to 0^+$ (see, [11, proof of Proposition 6.1]).

Combining now (4.4) with (2.1) and (2.2), we obtain

$$\left(\frac{2p-1}{p-1}\right)^{p-1}\left(\frac{1}{(D+1)\delta(\Omega)}\right)^p \;\le\; T(p;\Omega) \;\le\; D\left(D+\frac{p}{p-1}\right)^{p-1}\left(\frac{1}{2\,\delta(\Omega)}\right)^p \tag{4.5}$$

for all $\Omega \in \mathbb{P}^D$ and all $p \in (1, \infty)$.

Moreover, since $\delta(t\Omega) = t\,\delta(\Omega)$ for all $t > 0$, it follows that $t\Omega \in \mathbb{P}^D[t\,\delta(\Omega)]$, and hence scale-invariance remains valid when the functional is expressed in terms of $\delta(\Omega)$ rather than R_Ω. Consequently, for each $D \geq 2$ and $p \in (1, \infty)$ we introduce the scale-invariant quantity

$$\overline{Q}_p(\Omega) := \left[\frac{T(p;\Omega)\,\delta(\Omega)^p}{(\frac{2p-1}{p-1})^{p-1}}\right]^{1/p}, \qquad \Omega \in \mathbb{P}^D. \tag{4.6}$$

From (4.5) and (4.6), we immediately obtain the scale-free corridor

$$\frac{1}{D+1} \;\leq\; \overline{Q}_p(\Omega) \;\leq\; \frac{1}{2\gamma_{D,p}}, \qquad \forall\,\Omega \in \mathbb{P}^D,\ \forall\,p \in (1, \infty), \tag{4.7}$$

where $\gamma_{D,p}$ is defined in relation (1.10). This motivates the definition of

$$\overline{\alpha}(p;D) := \inf_{\Omega \in \mathbb{P}^D} \overline{Q}_p(\Omega), \qquad \overline{\beta}(p;D) := \sup_{\Omega \in \mathbb{P}^D} \overline{Q}_p(\Omega). \tag{4.8}$$

We then set

$$\overline{\gamma}_{D,p} := \frac{\overline{\alpha}(p;D)}{\overline{\beta}(p;D)} \tag{4.9}$$

and deduce by (4.7) that $\overline{\gamma}_{D,p} \in \left[\frac{2\gamma_{D,p}}{D+1}, 1\right)$. All these arguments allow us to repeat the proof of Theorem 2, with R_Ω replaced by $\delta(\Omega)$, and obtain the following extended result:

Theorem 3. *Let $D \geq 2$ be a fixed integer. Let $p > 1$ and $0 < a < b$ be three given real numbers and define $\gamma_{D,p}$ as in relation (1.10). Then there exists a constant $\overline{\gamma}_{D,p} \in [\,\frac{2\gamma_{D,p}}{D+1}, 1)$, depending only on D and p, such that we have*

$$T(p;\Omega_b) \leq T(p;\Omega_a), \quad \forall \quad \Omega_a \in \mathbb{P}^D[a], \quad \text{and} \quad \Omega_b \in \mathbb{P}^D[b], \tag{4.10}$$

if and only if $\overline{\gamma}_{D,p} b \geq a$, with strict inequality in (4.10) whenever $\overline{\gamma}_{D,p} b > a$.

Moreover, when $\overline{\gamma}_{D,p} b = a$ then the equality in (4.10) occurs asymptotically, in the sense that there exist two sequences of domains $\{\widetilde{\omega}_n\}_n \subset \mathbb{P}^D[a]$ and $\{\widetilde{\Omega}_n\}_n \subset \mathbb{P}^D[b]$ for which we have

$$\lim_{n\to\infty} T(p;\widetilde{\omega}_n) = \lim_{n\to\infty} T\left(p;\widetilde{\Omega}_n\right). \tag{4.11}$$

Next, we set

$$\overline{\gamma}_D := \inf_{p \in (1,\infty)} \overline{\gamma}_{D,p}.$$

Since for any $p > 1$ we have $\overline{\gamma}_{D,p} \geq \frac{2\gamma_{D,p}}{D+1}$ and by (3.13) we know that $\gamma_{D,p} > D^{-1}$ it follows that $\overline{\gamma}_{D,p} > \frac{2}{D(D+1)}$, for all $p > 1$. Thus, $\overline{\gamma}_D \geq \frac{2}{D(D+1)}$. Taking into account that fact, as a direct consequence of Theorem 3, we obtain the following corollary, which extends Corollary 2 to the average-distance framework:

Corollary 3. *Let $D \geq 2$ be a fixed integer and $0 < a < b$ be two given real numbers. Then there exists a constant $\overline{\gamma}_D \in [\,\frac{2}{D(D+1)}, 1)$, depending only on D, such that the following assertions hold.*

If $\overline{\gamma}_D b < a$ then there exists a real number $q_D \in (1, \infty)$ and two sets $\Omega_a \in \mathbb{P}^D[a]$ and $\Omega_b \in \mathbb{P}^D[b]$ such that

$$T(q_D; \Omega_a) < T(q_D; \Omega_b). \tag{4.12}$$

Conversely, if $\overline{\gamma}_D b \geq a$ then for each set $\Omega_a \in \mathbb{P}^D[a]$ and $\Omega_b \in \mathbb{P}^D[b]$, we have

$$T(p; \Omega_b) \leq T(p; \Omega_a), \quad \forall\, p \in (1, \infty), \tag{4.13}$$

with strict inequality in (4.13) when $\overline{\gamma}_D b > a$.

Remark 2. *The comparison constants associated with $\overline{Q}_p$ are also defined via infima and suprema over $\mathbb{P}^D$, and therefore the corresponding bounds should be understood in a limiting sense. However, unlike the case of Q_p, the available estimates for $\overline{\gamma}_{D,p}$ are not expected to be sharp.*

In contrast, the explicit examples discussed in Section 5 illustrate the behaviour of Q_p within several families of convex domains. For Q_p, the lower endpoint $\alpha(p; D) = 1$ is approached through degenerating families, whereas the upper extremum $\beta(p; D)$ is attained by balls, see (3.6).

5. Examples and behaviour on some families of convex sets

In this section we discuss the scale-invariant functional $Q_p(\Omega)$ on several model families of convex domains. The purpose of this section is illustrative: We present some consequences of the geometric corridor (3.2) for familiar model families, without attempting to determine the sharp constants for each family. These examples illustrate the behavior of Q_p within each family and the role that geometric degeneration may play, in some families, in approaching the lower endpoint $\alpha(p; D) = 1$ of the corridor (3.2). They should not, however, be interpreted as saturating the global upper endpoint in general: The global supremum $\beta(p; D)$ of Q_p over $\mathbb{P}^D$ is attained, up to translations and dilations, by balls, see (3.6). Thus, in what follows, the family-wise estimates are based mainly on the geometric corridor (3.2), while the sharp global corridor identifies the universal upper endpoint, which is attained, up to translations and dilations, by balls. Accordingly, the family bounds below are lower estimates for $\gamma_{\mathcal{F}}(p)$ rather than evaluations of the global constant $\gamma_{D,p}$. Specifically, by highlighting these phenomena, we describe the behavior within each family $\mathcal{F} \subset \mathbb{P}^D$, of the ratio

$$\gamma_{\mathcal{F}}(p) := \frac{\alpha_{\mathcal{F}}(p)}{\beta_{\mathcal{F}}(p)}, \qquad \alpha_{\mathcal{F}}(p) := \inf_{\Omega \in \mathcal{F}} Q_p(\Omega), \qquad \beta_{\mathcal{F}}(p) := \sup_{\Omega \in \mathcal{F}} Q_p(\Omega).$$

In parallel with the average-distance framework of Section 4, we also comment on the qualitative behavior of the alternative scale-invariant quantity $\overline{Q}_p$ defined in (4.6). While Q_p yields explicit formulas or effective lower estimates for $\gamma_{\mathcal{F}}(p)$ in the model settings below, for $\overline{Q}_p$ we only have universal (dimensional) bounds, and no comparable closed-form expressions for the associated constants are available.

Unless otherwise stated, all domains are open, bounded, and convex, and R denotes the inradius, so that $\Omega \in \mathbb{P}^D(R)$.

5.1. Rectangles and higher-dimensional orthotopes

For $\kappa \geq 2$, consider the family of axis-aligned orthotopes

$$O_\kappa := \Big\{\Omega = \prod_{i=1}^{D}(0, \ell_i) :\ \ell_1 = 2R,\ \ell_j \geq \kappa R \text{ for } j = 2, \ldots, D\Big\}.$$

Each element of O_κ is a D-dimensional axis-aligned box with inradius R and aspect ratio at least κ between the shortest side and the remaining ones. For $D = 2$ this family reduces to the planar rectangles $(0, 2R) \times (0, \ell_2)$ with $\ell_2 \geq \kappa R$, and the estimate below specializes, in that case, to $\gamma_{O_\kappa}(p) \geq \kappa/(\kappa + 2)$.

Proposition 1. *For every $p > 1$ and $\kappa \geq 2$, we have*

$$\gamma_{O_\kappa}(p) \;\geq\; \frac{\kappa}{\kappa + 2(D-1)}.$$

Proof. We have

$$\frac{P(\Omega)}{|\Omega|} = 2\sum_{i=1}^{D} \frac{1}{\ell_i}.$$

With $\ell_1 = 2R$ and $\ell_j \geq \kappa R$ for $j \geq 2$, it follows that

$$R_\Omega \frac{P(\Omega)}{|\Omega|} \;\leq\; 2R_\Omega\left(\frac{1}{2R_\Omega} + \sum_{j=2}^{D} \frac{1}{\kappa R_\Omega}\right) = 1 + \frac{2(D-1)}{\kappa}.$$

Using (3.2) we deduce

$$\gamma_{O_\kappa}(p) \;\geq\; \frac{1}{1 + \frac{2(D-1)}{\kappa}} = \frac{\kappa}{\kappa + 2(D-1)}.$$

□

Remark 3. *For $\kappa = 2$ (which includes the hypercube) one obtains $\gamma_{O_\kappa}(p) \geq 1/D$, in agreement with the universal barrier $[1/D, 1)$. Moreover, as $\kappa \to \infty$, the geometric corridor in (3.2) shrinks to the singleton $\{1\}$. Consequently, $Q_p(\Omega) \to 1$ and $\gamma_{O_\kappa}(p) \nearrow 1$. This conclusion is independent of whether the limit $\kappa \to \infty$ is achieved by letting the sidelengths ℓ_j ($j \geq 2$) tend to infinity with R fixed, or by letting $R \to 0$ with the other sidelengths fixed.*

Remark 4 (Analogue in terms of $\overline{Q}_p$)**.** *For any $\Omega \in O_\kappa$, the bounds in (4.7) yield*

$$\frac{1}{D+1} \leq \overline{Q}_p(\Omega) \leq \frac{D}{2},$$

independently of the aspect ratio κ. As a consequence, $\overline{\gamma}_{O_\kappa}(p)$ admits no sharper explicit expression than the universal estimate $\overline{\gamma}_{D,p} \geq 2/(D(D+1))$.

The behavior of $\overline{Q}_p(\Omega)$ as $\kappa \to \infty$ depends on the mechanism by which this limit is achieved. If $\kappa \to \infty$ through the regime where the sidelengths $\ell_j \to \infty$ for $j \geq 2$ with R fixed, the geometry converges locally to $\mathbb{R}^{D-1} \times (0, 2R)$ and the p-torsion problem becomes asymptotically independent of the longitudinal variables. In this case, $\overline{Q}_p(\Omega) \to 1/2$.

If instead $\kappa \to \infty$ through the thin-domain regime $R \to 0$ with the other sidelengths fixed, the inradius R_Ω tends to zero and the domain collapses to a lower-dimensional set. Consequently, $\delta(\Omega) \to 0$, and no specific limit is claimed for $\overline{Q}_p$ in this regime, although the same universal bounds still apply.

5.2. Planar ellipses for $p = 2$

For $\kappa \geq 1$, we consider the family of ellipses

$$\mathcal{E}_\kappa := \{\, E = \{(x, y) : x^2/a^2 + y^2/b^2 < 1\},\ 1 \leq a/b \leq \kappa \,\}.$$

Note that for $E \in \mathcal{E}_\kappa$ with $a \geq b$, the inradius is given by $R_E = b$.

Proposition 2. *For every $\kappa \geq 1$, we have*

$$\gamma_{\mathcal{E}_\kappa}(2) = \frac{1}{\sqrt{2}} \sqrt{1 + \frac{1}{\kappa^2}}.$$

Proof. First we note that for $E \in \mathcal{E}_\kappa$, the torsion function is explicit (see, e.g., [12]):

$$u(x, y) = \frac{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}{2(\frac{1}{a^2} + \frac{1}{b^2})}.$$

A direct computation yields

$$T_2(E) = \frac{\pi a^3 b^3}{4(a^2 + b^2)}, \qquad |E| = \pi ab,$$

hence

$$Q_2(E) = \frac{2}{\sqrt{3}} \sqrt{1 + \frac{1}{(a/b)^2}}.$$

Since $1 \leq a/b \leq \kappa$, we have

$$\alpha_{\mathcal{E}_\kappa}(2) = \frac{2}{\sqrt{3}} \sqrt{1 + \frac{1}{\kappa^2}}, \qquad \beta_{\mathcal{E}_\kappa}(2) = \sqrt{\tfrac{8}{3}},$$

and the conclusion follows. □

Remark 5. *The function $\gamma_{\mathcal{E}_\kappa}(2)$ is strictly decreasing in κ and varies from 1 (the disk $\kappa = 1$) down to $1/\sqrt{2}$ as $\kappa \to \infty$.*

Remark 6 (Analogue in terms of $\overline{Q}_2$). *For a planar ellipse $E \in \mathcal{E}_\kappa$ one has $R_E = b$ and*

$$\frac{b}{3} \leq \delta(E) \leq \frac{b}{2}.$$

As a consequence,

$$\frac{1}{3} \leq \overline{Q}_2(E) \leq \frac{2}{3}, \qquad E \in \mathcal{E}_\kappa,$$

and therefore $\overline{\gamma}_{\mathcal{E}_\kappa}(2) = 1/2$. Even in this completely explicit case, $\overline{Q}_2$ remains confined to a universal interval and does not distinguish between the extremal shapes within the family, in contrast with the sharp evaluation provided by Q_2.

In the thin-domain limit $b \to 0$ with a fixed, the inradius $R_E = b$ tends to zero. By the geometric bounds relating $\delta(E)$ and R_E, the average distance to the boundary also converges to zero. The ellipse degenerates to a line segment, and in this regime $\overline{Q}_2$ no longer distinguishes between different shapes within the family, beyond the universal bounds.

5.3. Triangles

We finally consider the family of all open, bounded triangles in the plane, denoted by $\mathcal{T} \subset \mathbb{P}^2$.

Proposition 3. *For every $p > 1$, we have*

$$\gamma_{\mathcal{T}}(p) \ \geq \ \frac{1}{2}.$$

Proof. For $T \in \mathcal{T}$, with inradius R_T, perimeter $P(T)$, and area $|T|$, the well-known identity

$$|T| = R_T \, \frac{P(T)}{2}$$

implies

$$R_T \, \frac{P(T)}{|T|} = 2.$$

Inserted into the general corridor (3.2), this yields

$$1 \leq Q_p(T) < 2, \qquad \forall T \in \mathcal{T}, \ \forall p > 1.$$

Therefore,

$$\gamma_{\mathcal{T}}(p) = \frac{\inf_{\mathcal{T}} Q_p}{\sup_{\mathcal{T}} Q_p} \geq \frac{1}{2}.$$

□

Remark 7. *The upper bound $Q_p(T) < 2$ used above comes from the geometric corridor (3.2) and is not sharp on $\mathcal{T}$: By (3.6) the global supremum of Q_p over all planar convex sets is $\beta(p; 2) < 2$, attained, up to translations and dilations, by disks, so that $\sup_{T \in \mathcal{T}} Q_p(T) \leq \beta(p; 2) < 2$. Hence the value $1/2$ in Proposition 3 is only a lower estimate and is not claimed to be optimal. The exact determination of $\sup_{T \in \mathcal{T}} Q_p(T)$ is a separate shape-optimization problem, which lies outside the scope of the present paper.*

Remark 8 (Analogue in terms of $\overline{Q}_p$)**.** *For every planar triangle T one has the exact identity*

$$\delta(T) = \frac{R_T}{3}.$$

Consequently,

$$\frac{1}{3} \leq \overline{Q}_p(T) \leq \frac{2}{3}, \qquad T \in \mathcal{T}.$$

Thus, even for triangles, where explicit geometric formulas are available, $\overline{Q}_p$ remains confined to a fixed universal interval and does not distinguish between extremal shapes within the family.

Taken together, these examples clarify the geometric meaning of $\gamma_{D,p}$. The lower bound $Q_p(\Omega) \geq 1$ is attained in dimension $D = 1$; for $D \geq 2$ it is the infimum of Q_p over convex domains, approached in some families, for instance as $\kappa \to \infty$ for rectangles and orthotopes, through collapse to one-dimensional configurations. Within a fixed family, larger values of Q_p tend to occur for more balanced geometries of smaller aspect ratio. These family-wise values, however, do not in general reach the global supremum $\beta(p;D) = \gamma_{D,p}^{-1}$, which over $\mathbb{P}^D$ is attained by balls; for $D = 2, p = 2$, $\beta(2;2) = \sqrt{8/3} \approx 1.6329$. Thus $\gamma_{D,p}$ is governed by the ball, while $\alpha(p;D) = 1$ is approached only through degenerating families.

In contrast, the scale-invariant quantity $\overline{Q}_p$ remains confined to universal dimensional intervals in all these examples. In particular, when degenerating families collapse to lower-dimensional sets, the inradius tends to zero and, by the geometric inequalities linking $\delta(\Omega)$ and R_Ω, the average distance to the boundary vanishes as well. As a consequence, $\overline{Q}_p$ does not capture the extremal geometric behavior responsible for the sharpness of the comparison constants, even in settings where explicit formulas are available.

6. Limiting regimes: the cases $p \to 1^+$ and $p \to \infty$

The scale-invariant functionals $Q_p(\Omega)$ and $\overline{Q}_p(\Omega)$ provide a natural bridge between the two extremal regimes of the p-torsion problem, namely the Cheeger-type limit $p \to 1^+$ and the distance-function-dominated limit $p \to \infty$. In this section we recall the precise asymptotic behavior of the normalized p-torsional rigidity $T(p;\Omega)$ and describe the corresponding limits of $Q_p(\Omega)$ and $\overline{Q}_p(\Omega)$ within the framework introduced in Sections 1–4.

For each $p > 1$, let u_p denote the solution of the p-torsion problem in Ω, and recall that

$$T_p(\Omega) := \int_\Omega u_p\, dx, \qquad T(p;\Omega) := |\Omega|^{p-1}\, T_p(\Omega)^{1-p}.$$

Recall also the definitions of the two scale-invariant functionals,

$$Q_p(\Omega)^p = \frac{T(p;\Omega)\, R_\Omega^p}{\left(\frac{2p-1}{p-1}\right)^{p-1}}, \qquad \overline{Q}_p(\Omega)^p = \frac{T(p;\Omega)\, \delta(\Omega)^p}{\left(\frac{2p-1}{p-1}\right)^{p-1}}.$$

(i) The limit $p \to 1^+$

It is known (see, e.g., [13, Theorem 2]) that

$$\lim_{p\to 1^+} T_p(\Omega)^{1-p} = h(\Omega),$$

where

$$h(\Omega) := \inf_{A\subset\Omega} \frac{P(A)}{|A|}$$

denotes the Cheeger constant of Ω. Consequently,

$$\lim_{p\to 1^+} T(p;\Omega) = h(\Omega).$$

Observe also that

$$\lim_{p\to 1^+} \left(\frac{2p-1}{p-1}\right)^{p-1} = 1.$$

Recalling the definitions above, we obtain

$$\lim_{p\to 1^+} Q_p(\Omega) = R_\Omega\, h(\Omega) =: Q_1(\Omega), \qquad \lim_{p\to 1^+} \overline{Q}_p(\Omega) = \delta(\Omega)\, h(\Omega) =: \overline{Q}_1(\Omega).$$

Thus, in the limit $p \to 1^+$, the functional Q_p reduces to the scale-invariant product $R_\Omega\, h(\Omega)$, while the functional $\overline{Q}_p$ reduces to the product $\delta(\Omega)\, h(\Omega)$, where $\delta(\Omega)$ denotes the average distance to the boundary. For convex domains $\Omega \in \mathbb{P}^D$, the functional Q_1 satisfies the corridor

$$1 \le Q_1(\Omega) \le D, \qquad \forall\, \Omega \in \mathbb{P}^D,$$

obtained by letting $p \to 1^+$ in the geometric corridor (3.2). The lower endpoint is approached by slab-type boxes $\Omega_L = (-L, L)^{D-1} \times (-R, R)$, as $L/R \to \infty$; in this regime $h(\Omega_L)R_{\Omega_L} \to 1$, and hence $Q_1(\Omega_L) \to 1$, while the upper endpoint is attained by balls, for which $h(B_R^D) = D/R$ and hence $Q_1(B_R^D) = D$. For $\overline{Q}_1$, the same estimates yield the dimensional interval

$$\frac{1}{D+1} \le \overline{Q}_1(\Omega) \le \frac{D}{2}, \qquad \forall\, \Omega \in \mathbb{P}^D,$$

obtained by combining the bounds $\frac{1}{D+1} \le \delta(\Omega)/R_\Omega \le \frac{1}{2}$ and $1 \le R_\Omega h(\Omega) \le D$; we do not claim sharpness of these endpoints.

(ii) The limit $p \to \infty$

Recall from (4.2) that the normalized p-torsional rigidity satisfies

$$\lim_{p\to\infty} T(p;\Omega)^{1/p} = \delta(\Omega)^{-1}.$$

Observe also that

$$\lim_{p\to\infty} \left(\tfrac{2p-1}{p-1}\right)^{(p-1)/p} = 2.$$

Recalling the definitions above, we obtain

$$\lim_{p\to\infty} Q_p(\Omega) = \frac{R_\Omega}{2\delta(\Omega)} =: Q_\infty(\Omega), \qquad \lim_{p\to\infty} \overline{Q}_p(\Omega) = \frac{1}{2} =: \overline{Q}_\infty(\Omega).$$

Thus, in the limit $p \to \infty$, the functional Q_p reduces to the scale-invariant ratio $R_\Omega/(2\delta(\Omega))$, while the functional $\overline{Q}_p$ reduces to the constant value $1/2$, for every $\Omega \in \mathbb{P}^D$. By [11, Proposition 6.1], the inradius R_Ω and the average distance to the boundary $\delta(\Omega)$ are comparable through purely dimensional constants, and the functional Q_∞ satisfies the corridor

$$1 \le Q_\infty(\Omega) \le \frac{D+1}{2}, \qquad \forall\, \Omega \in \mathbb{P}^D.$$

The lower endpoint is approached by slab-type boxes $\Omega_L = (-L, L)^{D-1} \times (-R, R)$, as $L/R \to \infty$; indeed, $R_{\Omega_L} = R$, $\delta(\Omega_L)/R_{\Omega_L} \to 1/2$, and hence $Q_\infty(\Omega_L) \to 1$, in agreement with the asymptotic saturation observed in Section 5. On the other hand, the upper endpoint is attained by balls, for which $\delta(B_R^D) = R/(D+1)$ and hence $Q_\infty(B_R^D) = (D+1)/2$.

Remark 9. *We emphasize that the present limiting discussion concerns both functionals $Q_p(\Omega)$ and $\overline{Q}_p(\Omega)$, and that the two regimes are symmetric in structure. In the regime $p \to 1^+$, $Q_p(\Omega)$ converges to the product $R_\Omega\, h(\Omega)$, thereby encoding simultaneously the inradius and the Cheeger constant of the domain, while $\overline{Q}_p(\Omega)$ converges to the product $\delta(\Omega)\, h(\Omega)$, encoding simultaneously the average distance to the boundary and the Cheeger constant of the domain. In the opposite regime $p \to \infty$, the functional $Q_p(\Omega)$ converges to the ratio $R_\Omega/(2\delta(\Omega))$, linking the comparison principle to the average-distance geometry discussed earlier, while the functional $\overline{Q}_p(\Omega)$ converges to the constant value $\frac{1}{2}$, for every $\Omega \in \mathbb{P}^D$.*

7. Conclusions

In this paper, we have established a unified comparison criterion for the normalized p-torsional rigidity $T(p;\Omega)$ on convex domains, valid for all $p \in (1,\infty)$. More precisely, for any fixed dimension $D \geq 2$ and any pair of real numbers $0 < a < b$, we proved that there exists a constant $\gamma_{D,p}$ given in relation (1.10), depending only on D and p, such that

$$T(p;\Omega_b) \leq T(p;\Omega_a), \quad \text{for all } \Omega_a \in \mathbb{P}^D(a),\ \Omega_b \in \mathbb{P}^D(b),$$

if and only if $\gamma_{D,p} b \geq a$. This result provides the natural analogue, in the framework of p-torsional rigidity, of classical comparison inequalities for the first Dirichlet eigenvalue of the p-Laplacian (see [6]).

The introduction of the scale-invariant functional $Q_p(\Omega)$ allows for a precise geometric interpretation of the comparison principle established in this paper. The upper extremum $\beta(p;D)$ of Q_p over $\mathbb{P}^D$ is attained, up to translations and dilations, by balls, see (3.6), and the comparison constant $\gamma_{D,p} = 1/\beta(p;D)$ is therefore governed by the ball; the lower endpoint $\alpha(p;D) = 1$ is not attained in dimension $D \geq 2$ and is approached only through degenerating domains. The explicit evaluation of $Q_p(\Omega)$ on several model families, including rectangles, higher-dimensional orthotopes, planar ellipses, and triangles, illustrates this behaviour of Q_p within these classes rather than the sharpness of the global upper endpoint: In several families, larger values of Q_p are often associated with more balanced geometries, for instance squares among rectangles and hypercubes among orthotopes, whereas thin-domain regimes lead to values of Q_p close to the lower endpoint 1, reflecting collapse toward one-dimensional configurations.

Throughout this work, the analysis has been carried out for the normalized p-torsional rigidity $T(p;\Omega)$, rather than for the classical torsional rigidity

$$T_p(\Omega) := \int_\Omega u_p \, dx.$$

This choice is dictated by the favorable scaling properties of $T(p;\Omega)$, which are essential for the formulation and proof of the comparison principle developed here. Nevertheless, several explicit examples for convex domains suggest that analogous comparison phenomena should also hold at the level of the unnormalized functional $T_p(\Omega)$. At present, however, the lack of an appropriate scale-invariant framework for $T_p(\Omega)$ prevents a direct extension of the present approach, and the identification of alternative normalization strategies or comparison mechanisms leading to similar results remains an interesting direction for future research. Within this perspective, we point out that Theorem 2 yields a fixed-volume monotonicity statement for T_p, naturally related to but distinct from the classical Saint-Venant inequality. More precisely, the *Saint-Venant inequality* (see, e.g., [14, Theorem 2.5] for its general version, [15, relation (4)] and [16, Chapter V] for its classical formulation obtained in the case where $p = 2$, and the recent historical discussion in [17]) states that for any open bounded set $\Omega \subset \mathbb{R}^D$ and any $p > 1$, we have

$$T_p(\Omega) \leq T_p(\Omega^\star), \tag{7.1}$$

where $\Omega^\star$ denotes a ball with $|\Omega^\star| = |\Omega|$. Next, for any real number $c > 0$ let us define the set

$$\mathbb{P}^D_c := \{\Omega \in \mathbb{P}^D : \ |\Omega| = c\}.$$

Since the balls belonging to $\mathbb{P}_c^D$ have the largest inradii (equal to $(c/v_D)^{1/D}$, where $v_D := |B_1^D|$) among all the sets belonging to $\mathbb{P}_c^D$, and since they maximize T_p within $\mathbb{P}_c^D$ by the Saint-Venant inequality (7.1), it is natural to ask whether T_p is monotone with respect to the prescribed inradius inside the fixed-volume class, namely: *Given $c > 0$, $p > 1$ and $0 < a < b$ four real numbers, is it true that we have*

$$T_p(\Omega_a) < T_p(\Omega_b), \quad \forall\, \Omega_a \in \mathbb{P}_c^D \cap \mathbb{P}^D(a), \quad \Omega_b \in \mathbb{P}_c^D \cap \mathbb{P}^D(b)?$$

By Theorem 2 and relation (1.2) we get the following answer to the above question:

Proposition 4. *Let $D \geq 2$ be a fixed integer. Let $p > 1$, $c > 0$ and $0 < a < b$ be four given real numbers. Assume that $b < (c/v_D)^{1/D}$, and let $\gamma_{D,p}$ be the constant given in relation (1.10) from Theorem 2. If $\gamma_{D,p} b \geq a$, then*

$$T_p(\Omega_a) \leq T_p(\Omega_b), \quad \forall\, \Omega_a \in \mathbb{P}_c^D \cap \mathbb{P}^D(a), \quad \Omega_b \in \mathbb{P}_c^D \cap \mathbb{P}^D(b), \tag{7.2}$$

with strict inequality in (7.2) whenever $\gamma_{D,p} b > a$.

Remark 10. *We state Proposition 4 as a sufficient condition only. Indeed, Theorem 2 gives a necessary and sufficient condition on the full classes $\mathbb{P}^D(a)$ and $\mathbb{P}^D(b)$, whereas Proposition 4 concerns the restricted fixed-volume classes $\mathbb{P}_c^D \cap \mathbb{P}^D(a)$ and $\mathbb{P}_c^D \cap \mathbb{P}^D(b)$. The extremal sequences used in the converse part of Theorem 2 are rescaled to have prescribed inradii, and such rescalings do not generally preserve the constraint $|\Omega| = c$. Therefore, an intrinsic necessary and sufficient condition in the fixed-volume setting would require constants defined directly on the restricted classes $\mathbb{P}_c^D \cap \mathbb{P}^D(a)$ and $\mathbb{P}_c^D \cap \mathbb{P}^D(b)$, rather than the universal constant $\gamma_{D,p}$. More explicitly, one could introduce the restricted constants*

$$\alpha_{c,a}(p; D) := \inf_{\Omega \in \mathbb{P}_c^D \cap \mathbb{P}^D(a)} Q_p(\Omega), \qquad \beta_{c,b}(p; D) := \sup_{\Omega \in \mathbb{P}_c^D \cap \mathbb{P}^D(b)} Q_p(\Omega).$$

These need not coincide with the global constants $\alpha(p; D)$ and $\beta(p; D)$ entering $\gamma_{D,p} = \alpha(p; D)/\beta(p; D)$, since the fixed-volume constraint $|\Omega| = c$ restricts the admissible domains. Consequently, the condition $\gamma_{D,p} b \geq a$ is sufficient for (7.2), but it should not be interpreted as necessary for the fixed-volume classes $\mathbb{P}_c^D \cap \mathbb{P}^D(a)$ and $\mathbb{P}_c^D \cap \mathbb{P}^D(b)$.

Finally, we analyzed the limiting regimes $p \to 1^+$ and $p \to \infty$. In the former case, the comparison problem naturally connects to the Cheeger constant and the 1-Laplacian, while in the latter it is governed by the ratio between the inradius and the average distance to the boundary, in the spirit of ∞-Laplacian-type problems. Possible directions for further research include the application of this scale-invariant framework to other variational functional with similar homogeneity properties. While extensions to non-convex geometries present significant challenges due to the loss of sharp inradius-based estimates, the methodology developed here appears robust for studying anisotropic operators or for establishing quantitative stability estimates for the comparison inequalities.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. F. Della Pietra, N. Gavitone, S. G. L. Bianco, On functionals involving the torsional rigidity related to some classes of nonlinear operators, *J. Differ. Equations*, **265** (2018), 6424–6442. https://doi.org/10.1016/j.jde.2018.07.030
2. B. Kawohl, On a familiy of torsional creep problems, *J. Reine Angew. Math.*, **410** (1990), 1–22. https://doi.org/10.1515/crll.1990.410.1
3. L. M. Kachanov, *The theory of creep*, National Lending Library for Science and Technology, Boston Spa, Yorkshire, 1967.
4. L. M. Kachanov, *Foundations of the theory of plasticity*, Applied Mathematics and Mechanics, Vol. 12, North-Holland, Amsterdam, 1971.
5. C. Enache, M. Mihăilescu, A monotonicity property of the p-torsional rigidity, *Nonlinear Anal.*, **208** (2021), 112326. https://doi.org/10.1016/j.na.2021.112326
6. M. Mihăilescu, D. Stancu-Dumitru, Comparing the first eigenvalues of the p-Laplace operator on distinct bounded and convex domains, *Tohoku Math. J.*, submitted for publication.
7. L. Brasco, On principal frequencies and isoperimetric ratios in convex sets, *Ann. Fac. Sci. Toulouse: Math.*, **29** (2020), 977–1005. https://doi.org/10.5802/afst.1653
8. F. Prinari, A. C. Zagati, On the sharp Makai inequality, *arXiv*, 2023. https://doi.org/10.48550/arXiv.2307.06086
9. P. Juutinen, P. Lindqvist, J. J. Manfredi, The ∞-eigenvalue problem, *Arch. Rational Mech. Anal.*, **148** (1999), 89–105. https://doi.org/10.1007/s002050050157
10. N. Fukagai, M. Ito, K. Narukawa, Limit as $p \to \infty$ of p-Laplace eigenvalue problems and L^∞-inequality of Poincaré type, *Differ. Integral Equ.*, **12** (1999), 183–206. https://doi.org/10.57262/die/1367265629
11. L. Briani, G. Buttazzo, F. Prinari, Inequalities between torsional rigidity and principal eigenvalue of the p-Laplacian, *Calc. Var.*, **61** (2022), 78. https://doi.org/10.1007/s00526-021-02129-9
12. G. Keady, A. McNabb, The elastic torsion problem: solutions in convex domains, *New Zealand J. Math.*, **22** (1993), 43–64.
13. H. Bueno, G. Ercole, Solutions of the Cheeger problem via torsion functions, *J. Math. Anal. Appl.*, **381** (2011), 263–279. https://doi.org/10.1016/j.jmaa.2011.03.002

14. F. Della Pietra, N. Gavitone, Sharp bounds for the first eigenvalue and the torsional rigidity related to some anisotropic operators, *Math. Nachr.*, **287** (2014), 194–209. https://doi.org/10.1002/mana.201200296

15. M. van den Berg, G. Buttazzo, On capacity and torsional rigidity, *Bull. London Math. Soc.*, **53** (2021), 347–359. https://doi.org/10.1112/blms.12422

16. G. Pólya, G. Szegö, Isoperimetric inequalities in mathematical physics, *Annals of Mathematics Studies*, Princeton: Princeton University Press, 1951.

17. A. Berger, A Beautiful Inequality by Saint-Venant and Pólya Revisited, *Am. Math. Mon.*, **130** (2023), 239–250. https://doi.org/10.1080/00029890.2022.2156241

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