http://www.aimspress.com/journal/mine

Mathematics in Engineering, 8(1): 140–149.
DOI:10.3934/mine.2026005
Received: 12 October 2025
Accepted: 04 February 2026
Published: 10 February 2026

Research article

Lump waves in a generalized Bogoyavlensky-Konopelchenko model with spatially balanced derivatives

Wen-Xiu Ma[1,2,3,4,*]

[1] Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China
[2] Research Center of Astrophysics and Cosmology, Khazar University, 41 Mehseti Street, Baku 1096, Azerbaijan
[3] Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA
[4] Material Science Innovation and Modelling, North-West University, Mafikeng Campus, Mmabatho 2735, South Africa

* **Correspondence:** Email: wma3@usf.edu.

Abstract: This paper investigates lump waves in a generalized (2+1)-dimensional Bogoyavlensky–Konopelchenko model with spatially balanced derivatives. Using a sum-of-squares ansatz, symbolic computation in Maple is employed to construct lump wave solutions of the nonlinear model from positive quadratic functions. The interplay of four sets of nonlinear terms and five dispersion terms gives rise to the resulting lump waves. The critical points of these quadratic functions are determined, and they travel at constant velocities along a straight line in the spatial plane. Along this characteristic line, the constructed lump waves remain invariant. Concluding remarks are provided in the final section.

Keywords: lump wave; Hirota bilinear form; soliton; symbolic computation; nonlinearity; dispersion

1. Introduction

The determination of nonlinear waves in optical, fluid, and oceanological models is often a challenging task [1, 2]. In rare cases, closed-form solutions can be obtained. However, soliton theory provides systematic trial-and-error approaches to construct such solutions. One important class, known as soliton solutions, can be presented explicitly. In formulating these nonlinear dispersive wave solutions, the interplay between nonlinearity and dispersion plays a crucial role.

The inverse scattering transform [3] and the Hirota direct method [4] are two efficient techniques to both soliton and lump wave solutions. The inverse scattering transform was developed for solving

Cauchy problems of nonlinear evolution equations generated from Lax pairs [5], through which one can also analyze long-time asymptoticis of solitonless solutions [6]. It is a nonlinear generalization analogous to to the Fourier transform. The Hirota direct method is a straightforward but powerful technique to determine both soliton and lump waves, particularly for nonlinear dispersive wave equations in (2+1)-dimensions [7, 8].

We consider a (2+1)-dimensional situation. Assume that R is a given polynomial in time t and two space variables x, y. Hirota bilinear derivatives are defined as follows [4]:

$$D_t^m D_x^n D_y^k f \cdot f = (\frac{\partial}{\partial t} - \frac{\partial}{\partial t'})^m (\frac{\partial}{\partial x} - \frac{\partial}{\partial x'})^n (\frac{\partial}{\partial y} - \frac{\partial}{\partial y'})^k f(t,x,y) f(t',x',y')\Big|_{t'=t,x'=x,y'=y}, \tag{1.1}$$

where m, n, k are nonnegative integers. Then, we can introduce a (2+1)-dimensional Hirota bilinear differential equation

$$R(D_t, D_x, D_y) f \cdot f = 0. \tag{1.2}$$

From such a Hirota bilinear equation, a nonlinear partial differential equation with a dependent variable u can be presented often by one of the logarithmic derivative transformations

$$u = 2(\ln f)_{xx},\ u = 2(\ln f)_{yy},\ u = 2(\ln f)_{xy},\ u = 2(\ln f)_x,\ u = 2(\ln f)_y. \tag{1.3}$$

Within the Hirota bilinear theory, an N-soliton solution is formulated (see, e.g., [7, 9]) as

$$f = \sum_{\nu=0,1} \exp(\sum_{i=1}^{N} \nu_i \zeta_i + \sum_{i<j} \nu_i \nu_j c_{ij}). \tag{1.4}$$

Here $\sum_{\nu=0,1}$ denotes a sum with all possibilities when $\nu_1, \nu_2, \cdots, \nu_N$ take either zero or one, and the phase shifts c_{ij} and the wave variables ζ_i are defined via

$$\exp(c_{ij}) = -\frac{R(\omega_j - \omega_i, m_i - m_j, n_i - n_j)}{R(\omega_j + \omega_i, m_i + m_j, n_i + n_j)},\ 1 \le i < j \le N, \tag{1.5}$$

and

$$\zeta_i = m_i x + n_i y - \omega_i t + \zeta_{i,0},\ 1 \le i \le N, \tag{1.6}$$

where the constant phase shifts $\zeta_{i,0}$ are arbitrary. To construct an N-soliton solution in a nonlinear model equation, the wave numbers m_i, n_i and the frequencies ω_i need to satisfy

$$R(-\omega_i, m_i, n_i) = 0,\ 1 \le i \le N, \tag{1.7}$$

which are called the dispersion relations in the theory of nonlinear waves. Based on these dispersion relations (1.7), an algorithm to show if a function f in (1.4) solves a Hirota bilinear equation (1.2) is proposed and illustrative examples are made in [9].

Recent studies have shown that abundant lump waves (and rogue waves) exist in nonlinear integrable models, remarkably similar to soliton waves [10]. Lump waves, describing important nonlinear phenomena, are expressed in terms of rational functions, and they are localized in all directions in space (see, e.g., [10, 11]). It is known that the KPI equation possesses various lump waves (see, e.g., [8]), and taking long wave limits of soliton solutions can yield particular lump wave solutions [12]. There also exist lump waves in nonlinear nonintegrable models, and such examples include generalized (2+1)-dimensional KP, BKP and KP-Boussinesq equations [13, 14]. Moreover, linear models in higher dimensions have been shown to possess lump waves (see, e.g., [15]).

An initial step to find lump waves is to construct quadratic function solutions to bilinear equations [8, 10]. Lump waves to nonlinear model equations are then determined from positive quadratic function solutions through the logarithmic derivative transformations. In this paper, we would like to search for lump waves in a generalized (2+1)-dimensional spatially balanced-derivative Bogoyavlensky-Konopelchenko model via such a process starting from quadratic functions. The starting point is the Hirota bilinear form with spatially balanced-derivative terms. The considered generalized (2+1)-dimensional Bogoyavlensky-Konopelchenko model has four sets of nonlinear terms and five second-order linear dispersion terms. The nonlinear terms and the dispersion terms play together to form lump waves. Symbolic computation with Maple will be arried out to get lump waves. A few of characteristic properties will be analyzed for the resultant lump waves. The conclusion is the last section.

2. A generalized Bogoyavlensky-Konopelchenko model with spatially balanced derivatives

Let $\alpha_1, \alpha_2, \beta_1, \beta_2$ and γ_i, $1 \le i \le 5$, be arbitrary real constants. To explore lump wave solutions created jointly by dispersion and nonlinearity, let us consider a generalized (2+1)-dimensional Bogoyavlensky-Konopelchenko model equation with spatially balanced derivatives:

$$\begin{aligned} K(u,v,w,r,s) := {} & \alpha_1(6v_xv_y + 6vv_{xy} + v_{xxxy}) + \beta_1(3u_{xx}r_y + 3u_xr_{xy} + 3u_{xy}v + 3u_yv_x + u_{xxxy}) \\ & + \alpha_2(6w_xw_y + 6ww_{xy} + w_{xxxy}) + \beta_2(3u_{yy}s_x + 3u_ys_{xy} + 3u_{xy}w + 3u_xw_y + u_{xyyy}) \\ & + \gamma_1u_{tx} + \gamma_2u_{ty} + \gamma_3u_{xx} + \gamma_4u_{yy} + \gamma_5u_{xy} = 0, \end{aligned} \tag{2.1}$$

subject to the constraints $v_y = u_x, w_x = u_y, r_x = v, s_y = w$.

In the special case $\alpha_2 = \beta_2 = \gamma_2 = 0$, this model reduces to a generalized (2+1)-dimensional Bogoyavlensky-Konopelchenko equation:

$$\begin{aligned} & \alpha_1(6v_xv_y + 6vv_{xy} + v_{xxxy}) + \beta_1(3u_{xx}r_y + 3u_xr_{xy} + 3u_{xy}v + 3u_yv_x + u_{xxxy}) \\ & + \gamma_1u_{tx} + \gamma_3u_{xx} + \gamma_4u_{yy} + \gamma_5u_{xy} = 0, \end{aligned} \tag{2.2}$$

with the same constraints $v_y = u_x, w_x = u_y, r_x = v, s_y = w$. Additional reductions of this type have also been investigated in the literature (see, e.g., [16–19]).

A straightforward computation tells that if we take the logarithmic derivative transformations

$$u = 2(\ln f)_{xy},\ v = 2(\ln f)_{xx},\ w = 2(\ln f)_{yy},\ r = 2(\ln f)_x,\ s = 2(\ln f)_y, \tag{2.3}$$

then the above generalized (2+1)-dimensional Bogoyavlensky-Konopelchenko model equation with spatially balanced derivatives (2.1) is changed into the Hirota bilinear form:

$$\begin{aligned} R(f) := {} & (\alpha_1D_x^4 + \beta_1D_x^3D_y + \alpha_2D_y^4 + \beta_2D_y^3D_x \\ & + \gamma_1D_xD_t + \gamma_2D_yD_t + \gamma_3D_x^2 + \gamma_4D_y^2 + \gamma_5D_xD_y)f \cdot f \\ = {} & 2[\alpha_1(f_{xxxx}f - 4f_{xxx}f_x + 3f_{xx}^2) + \beta_1(f_{xxxy}f - 3f_{xxy}f_x + 3f_{xy}f_{xx} - f_yf_{xxx}) \end{aligned}$$

$$+\alpha_2(f_{yyyy}f-4f_{yyy}f_y+3f_{yy}^2)+\beta_2(f_{xyyy}f-3f_{xyy}f_y+3f_{xy}f_{yy}-f_xf_{yyy})$$
$$+\gamma_1(f_{tx}f-f_tf_x)+\gamma_2(f_{ty}f-f_tf_y)+\gamma_3(f_{xx}f-f_x^2)$$
$$+\gamma_4(f_{yy}f-f_y^2)+\gamma_5(f_{xy}f-f_xf_y)]=0, \quad (2.4)$$

where D_t, D_x and D_y are the standard Hirota bilinear derivatives [4] (see also (1.1)). Based on a symbolic computation, one can explore a precise relation between the nonlinear model equation and the bilinear model equation:

$$K(u,v,w,r,s)=[\frac{R(f)}{f^2}]_{xy}, \quad (2.5)$$

where u, v, w, r, s are defined in terms of f in (2.3). Such connections also exist in a spatial symmetric KP model [20] and a spatial symmetric HSI model [21]. In light of (2.5), if f solves the bilinear model equation (2.4), then u, v, w, r, s determined by (2.3) present a solution to the generalized (2+1)-dimensional Bogoyavlensky-Konopelchenko model equation (2.1) with spatially balanced derivatives. We would like to search for a class of lump wave solutions in this nonlinear model equation in the following section.

3. Lump waves formed by dispersion and nonlinearity

Let us now construct lump waves in the generalized (2+1)-dimensional Bogoyavlensky-Konopelchenko model equation (2.1) with spatially balanced derivatives, through conducting symbolic computations. It is easy to check that the above general nonlinear model equation does not pass the three-soliton test (see, e.g., [9] for the three-soliton test).

We apply a general ansatz on lump waves in (2+1)-dimensions [8], and begin with positive quadratic function solutions

$$f=\zeta_1^2+\zeta_2^2+a_9,\ \zeta_1=a_1x+a_2y+a_3t+a_4,\ \zeta_2=a_5x+a_6y+a_7t+a_8, \quad (3.1)$$

to the corresponding Hirota bilinear equation (2.4), in which the nine parameters a_i's are real constants to be found (see, e.g., [8, 10] for illustrative examples). It is known that this is a general form for lump wave solutions of lower order in (2+1)-dimensions [10]. The next step is to conduct symbolic computations to find the involved constant parameters a_i's.

By a straightforward symbolic computation with Maple, one can find a set of solutions for the parameters:

$$\begin{cases} a_3=-\dfrac{[(a_1^2+a_5^2)a_1\gamma_3+(a_1a_2^2+2a_2a_5a_6-a_1^2a_6)\gamma_4+(a_1^2+a_5^2)a_2\gamma_5]\gamma_1}{(a_1\gamma_1+a_2\gamma_2)^2+(a_5\gamma_1+a_6\gamma_2)^2} \\ \qquad -\dfrac{[(a_1^2a_2+2a_1a_5a_6-a_2a_5^2)\gamma_3+(a_2^2+a_6^2)a_2\gamma_4+(a_2^2+a_6^2)a_1\gamma_5]\gamma_2}{(a_1\gamma_1+a_2\gamma_2)^2+(a_5\gamma_1+a_6\gamma_2)^2}, \\ a_7=-\dfrac{[(a_1^2+a_5^2)a_5\gamma_3+(a_5a_6^2+2a_1a_2a_6-a_2^2a_5)\gamma_4+(a_1^2+a_5^2)a_6\gamma_5]\gamma_1}{((a_1\gamma_1+a_2\gamma_2)^2+(a_5\gamma_1+a_6\gamma_2)^2} \\ \qquad -\dfrac{[(a_5^2a_6+2a_1a_2a_5-a_1^2a_6)\gamma_3+(a_2^2+a_6^2)a_6\gamma_4+(a_2^2+a_6^2)a_5\gamma_5]\gamma_2}{(a_1\gamma_1+a_2\gamma_2)^2+(a_5\gamma_1+a_6\gamma_2)^2}, \\ a_9=-\dfrac{3[(a_1\gamma_1+a_2\gamma_2)^2+(a_5\gamma_1+a_6\gamma_2)^2]a_{10}}{(a_1a_6-a_2a_5)^2(\gamma_1^2\gamma_4-\gamma_1\gamma_2\gamma_5+\gamma_2^2\gamma_3)}, \end{cases} \quad (3.2)$$

where a_{10} is defined by

$$a_{10} = (a_1^2 + a_5^2)[(a_1^2 + a_5^2)\alpha_1 + (a_1a_2 + a_5a_6)\beta_1] + (a_2^2 + a_6^2)[(a_2^2 + a_6^2)\alpha_2 + (a_1a_2 + a_5a_6)\beta_2], \tag{3.3}$$

and all other parameters can be arbitrarily chosen.

The above expressions of the two frequency parameters, a_3 and a_7, show a class of dispersion relations in (2+1)-dimensional nonlinear dispersive wave models, and the expression of the constant term parameter, a_9, tells a complicated solution in terms of the wave numbers, which is crucial in forming lump waves, within the Hirota bilinear formulation. We point out that a sort of higher-order dispersion relations appearing in the formulation of lump waves have also been presented in the second model equation of the integrable KP hierarchy [22].

Under the help of Maple, we have simplified all the above expressions for the wave frequencies and the constant term in (3.2). Note that if

$$a_1\gamma_1 + a_2\gamma_2 = a_5\gamma_1 + a_6\gamma_2 = 0, \tag{3.4}$$

then

$$(\gamma_1^2 + \gamma_2^2)(a_1a_6 - a_2a_5) = 0. \tag{3.5}$$

Accordingly, to determine lump wave solutions through the logarithmic derivative transformations, we need to satisfy two essential conditions:

$$a_1a_6 - a_2a_5 \neq 0, \tag{3.6}$$

and

$$\frac{a_{10}}{\gamma_1^2\gamma_4 - \gamma_1\gamma_2\gamma_5 + \gamma_2^2\gamma_3} < 0. \tag{3.7}$$

The above conditions ensure the fundamental properties of lump waves. The condition (3.6) guarantees that the resulting solutions of u, v, w, r, s are localized in all spatial directions, and the two conditions, (3.6) and (3.7), work together to assure that $a_9 > 0$ so that u, v, w, r, s are analytic in the whole spatial and temporal space. We will show later that the condition of $a_9 > 0$ is also necessary for u, v, w to be analytic.

Note that the second condition defined by (3.7) involves the coefficients, $\alpha_1, \alpha_2, \beta_1, \beta_2$, of the four nonlinear terms and the coefficients, γ_i, $1 \leq i \leq 5$, of the five dispersion terms. Obviously, if $a_1a_2 + a_5a_6 > 0$, then (3.7) becomes

$$\gamma_1^2\gamma_4 - \gamma_1\gamma_2\gamma_5 + \gamma_2^2\gamma_3 < 0 \ (> 0),$$

when $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ (< 0). If $a_1a_2 + a_5a_6 < 0$, then (3.7) becomes

$$\gamma_1^2\gamma_4 - \gamma_1\gamma_2\gamma_5 + \gamma_2^2\gamma_3 > 0 \ (< 0),$$

when $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ (< 0). Therefore, the dispersion and the nonlinearity in the model equation (2.1) play together to form lump waves. However, the nonlinear terms do not affect the speeds of the two single waves in the lumps, in view of (3.2). The above analyses also imply that we always need the condition on the dispersion terms

$$\gamma_1^2\gamma_4 - \gamma_1\gamma_2\gamma_5 + \gamma_2^2\gamma_3 \neq 0, \tag{3.8}$$

in the formulation of the lump waves.

Two special reduced important cases can be computed. If we take $\alpha_2 = \beta_2 = \gamma_2 = 0$, we obtain

$$\begin{cases} a_3 = -\dfrac{(a_1^2 + a_5^2)a_1\gamma_3 + (a_1a_2^2 + 2a_2a_5a_6 - a_1^2a_6)\gamma_4 + (a_1^2 + a_5^2)a_2\gamma_5}{(a_1^2 + a_5^2)\gamma_1}, \\ a_7 = -\dfrac{(a_1^2 + a_5^2)a_5\gamma_3 + (a_5a_6^2 + 2a_1a_2a_6 - a_2^2a_5)\gamma_4 + (a_1^2 + a_5^2)a_6\gamma_5}{(a_1^2 + a_5^2)\gamma_1}, \\ a_9 = -\dfrac{3(a_1^2 + a_5^2)^2[(a_1^2 + a_5^2)\alpha_1 + (a_1a_2 + a_5a_6)\beta_1]}{(a_1a_6 - a_2a_5)^2\gamma_4}. \end{cases} \quad (3.9)$$

The necessary and sufficient conditions for the existence of lump waves in this first reduced case become

$$(a_1a_6 - a_2a_5)\gamma_1 \neq 0, \ [(a_1^2 + a_5^2)\alpha_1 + (a_1a_2 + a_5a_6)\beta_1]\gamma_4 < 0. \quad (3.10)$$

Consequently, we see that the linear dispersion terms, u_{xt} and u_{yy}, are crucial for a lump wave to exist, and their coefficients can be either positive or negative.

Similarly, if we take $\alpha_1 = \beta_1 = \gamma_1 = 0$, then we arrive at

$$\begin{cases} a_3 = -\dfrac{(a_1^2a_2 + 2a_1a_5a_6 - a_2a_5^2)\gamma_3 + (a_2^2 + a_6^2)a_2\gamma_4 + (a_2^2 + a_6^2)a_1\gamma_5}{(a_2^2 + a_6^2)\gamma_2}, \\ a_7 = -\dfrac{(a_5^2a_6 + 2a_1a_2a_5 - a_1^2a_6)\gamma_3 + (a_2^2 + a_6^2)a_6\gamma_4 + (a_2^2 + a_6^2)a_5\gamma_5}{(a_2^2 + a_6^2)\gamma_2}, \\ a_9 = -\dfrac{3(a_2^2 + a_6^2)^2[(a_2^2 + a_6^2)\alpha_2 + (a_1a_2 + a_5a_6)\beta_2]}{(a_1a_6 - a_2a_5)^2\gamma_3}. \end{cases} \quad (3.11)$$

The necessary and sufficient conditions for the existence of lump waves in this second reduced case become

$$(a_1a_6 - a_2a_5)\gamma_2 \neq 0, \ [(a_2^2 + a_6^2)\alpha_2 + (a_1a_2 + a_5a_6)\beta_2]\gamma_3 < 0. \quad (3.12)$$

Therefore, we see that the linear dispersion terms, u_{yt} and u_{xx}, are crucial in the presentation of lump waves, and the two corresponding coefficients can be either positive and negative.

4. Characteristic properties

Let us now explore the characteristic properties of the resultant lump waves presented previously.

4.1. Line of critical points

We view f defined by (3.1) as a function of the two spatial variables x and y. Let us compute its critical points to determine extreme values of the lump waves. To the end, we consider the system

$$f_x(x(t), y(t), t) = 0, \ f_y(x(t), y(t), t) = 0. \quad (4.1)$$

Because f is quadratic, this precisely yields

$$a_1\zeta_1 + a_5\zeta_2 = 0, \ a_2\zeta_1 + a_6\zeta_2 = 0. \quad (4.2)$$

Then, in light of the condition by (3.6), we arrive at

$$\zeta_1 = a_1 x + a_2 y + a_3 t + a_4 = 0,\ \zeta_2 = a_5 x + a_6 y + a_7 t + a_8 = 0. \tag{4.3}$$

Again because of (3.6), we can solve this linear system for x and y, and obtain all critical points of the quadratic function f:

$$\begin{cases} x(t) = \dfrac{[(a_1^2+a_5^2)\gamma_3 - (a_2^2+a_6^2)\gamma_4]\gamma_1 + [2(a_1a_2+a_5a_6)\gamma_3 + (a_2^2+a_6^2)\gamma_5]\gamma_2}{(a_1\gamma_2+a_2\gamma_2)^2 + (a_5\gamma_1+a_6\gamma_2)^2} t + \dfrac{a_2a_8 - a_4a_6}{a_1a_6 - a_2a_5}, \\ y(t) = \dfrac{[2(a_1a_2+a_5a_6)\gamma_4 + (a_1^2+a_5^2)\gamma_5]\gamma_1 - [(a_1^2+a_5^2)\gamma_3 - (a_2^2+a_6^2)\gamma_4]\gamma_2}{(a_1\gamma_2+a_2\gamma_2)^2 + (a_5\gamma_1+a_6\gamma_2)^2} t - \dfrac{a_1a_8 - a_4a_5}{a_1a_6 - a_2a_5}, \end{cases} \tag{4.4}$$

where time t is arbitrary.

All critical points are aligned on a characteristic straight line, where the two spatial coordinates propagate at constant velocities. Along this line, the lump waves u, v, w, r, s given by (2.3) remain invariant (see [20, 21] for other examples).

4.2. *Analyticity condition*

Note that all critical points satisfy $\zeta_1 = \zeta_2 = 0$, i.e., (4.3), and the sum of two squares, i.e., the function $f - a_9$ vanishes at all critical points defined by (4.4). Therefore, we see that $f > 0$ in the whole spatial and temporal space if and only if the constant term $a_9 > 0$. The sufficiency is obvious, as discussed earlier. The necessity is due to that $f = 0$ at the critical points if $a_9 = 0$, and $f = 0$ at any point on the circle $\zeta_1^2 + \zeta_2^2 = -a_9$ if $a_9 < 0$.

It then follows that the three functions u, v, w defined by (2.3) are analytic in $\mathbb{R}^3$ if and only if the parameter a_9 must be positive. Now, in view of the result on the positiveness of a_9 in Section 3, the necessary and sufficient conditions for u, v, w, r, s to be analytic are the two conditions in (3.6) and (3.7).

In addition, we can observe that when $a_1a_6 - a_2a_5$ goes to zero, namely, the two spatial directions (a_1, a_2) and (a_5, a_6) tend to be parallel to each other, the lump waves of u, v, w, r, s do not decay in all cases of wave numbers.

5. Conclusions

Based on symbolic computations with Maple, we constructed a class of lump wave solutions to a generalized (2+1)-dimensional Bogoyavlensky-Konopelchenko model with spatially balanced derivatives. The interplay of dispersion and nonlinearity gives rise to these lump waves, which are presented explicitly and exactly. The frequencies a_3, a_7 and the constant term a_9 were determined in terms of the wave numbers in the quadratic function f. Characteristic properties of the lump waves were analyzed, and the influences of both dispersion and nonlinear terms were discussed.

It is well known that fascinating lump waves exist in linear wave models [15], as well as in nonlinear (2+1)-dimensional models (see, e.g., [23–27]) and (3+1)-dimensional models (see, e.g., [28, 29]). The Hirota bilinear and generalized bilinear forms play a key role in constructing lump waves [10, 30]. Moreover, interaction solutions between lump waves and other intriguing structures, including homoclinic and heteroclinic waves, have been explored in (2+1)-dimensional integrable models (see, e.g., [13, 31]).

Recently, multi-component integrable models have been systematically studied by groups reductions (see, e.g., [32–34]), and generated from higher-order matrix spectral problems (see, e.g., [35, 36]), including 4×4 cases (see, e.g., [37, 38]). It is of particular interest to determine whether N-soliton solutions and lump waves occur in these reduced and multi-component integrable equations, and whether they can be constructed by Riemann-Hilbert methods or the Hirota direct method. Further exploration of soliton and lump structures will enhance our understanding of nonlinear wave phenomena.

Use of Generative-AI tools declaration

The author declares that no artificial intelligence tools were used to create the results reported in this article.

Acknowledgments

The work was supported in part by National Natural Science Foundation of China (12271488 and 11975145), and the Ministry of Science and Technology of China (G2021016032L and G2023016011L).

Conflict of interest

The author declares that there is no conflict of interest regarding the publication of this paper.

References

1. J. S. Russell, Report of the committee on waves, *Report of the 7th Meeting of British Association for the Advancement of Science*, London: John Murray, 1838, 417–496.
2. J. S. Russell, Report on waves, *Report of the 14th meeting of the British Association for the Advancement of Science*, London: John Murray, 1844, 311–390.
3. M. J. Ablowitz, H. Segur, *Solitons and the inverse scattering transform*, SIAM, Philadelphia, 1981. https://doi.org/10.1137/1.9781611970883
4. R. Hirota, *The direct method in soliton theory*, New York: Cambridge University Press, 2004. https://doi.org/10.1017/CBO9780511543043
5. V. E. Zakharov, A. B. Shabat, A scheme for integrating the nonlinear equations of mathematical physics by the method of the inverse scattering problem. I, *Funct. Anal. Appl.*, **8** (1974), 226–235. https://doi.org/10.1007/BF01075696
6. M. J. Ablowitz, A. C. Newell, The decay of the continuous spectrum for solutions of the Korteweg-de Vries equation, *J. Math. Phys.*, **14** (1973), 1277–1284. https://doi.org/10.1063/1.1666479
7. R. Hirota, Exact solution of the Korteweg-de Vries equation for multiple collisions of solitons, *Phys. Rev. Lett.*, **27** (1971), 1192–1194. https://doi.org/10.1103/PhysRevLett.27.1192

8. W. X. Ma, Lump solutions to the Kadomtsev-Petviashvili equation, *Phys. Lett. A*, **379** (2015), 1975–1978. https://doi.org/10.1016/j.physleta.2015.06.061
9. W. X. Ma, *N*-soliton solutions and the Hirota conditions in (2+1)-dimensions, *Opt. Quant. Electron.*, **52** (2020), 511. https://doi.org/10.1007/s11082-020-02628-7
10. W. X. Ma, Y. Zhou, Lump solutions to nonlinear partial differential equations via Hirota bilinear forms, *J. Differ. Equ.*, **264** (2018), 2633–2659. https://doi.org/10.1016/j.jde.2017.10.033
11. W. Tan, H. P. Dai, Z. D. Dai, W. Y. Zhong, Emergence and space-time structure of lump solution to the (2+1)-dimensional generalized KP equation, *Pramana–J. Phys.*, **89** (2017), 77. https://doi.org/10.1007/s12043-017-1474-0
12. S. V. Manakov, V. E. Zakharov, L. A. Bordag, A. R. Its, V. B. Matveev, Two-dimensional solitons of the Kadomtsev-Petviashvili equation and their interaction, *Phys. Lett. A*, **63** (1977), 205–206. https://doi.org/10.1016/0375-9601(77)90875-1
13. W. X. Ma, X. L. Yong, H. Q. Zhang, Diversity of interaction solutions to the (2+1)-dimensional Ito equation, *Comput. Math. Appl.*, **75** (2018), 289–295. https://doi.org/10.1016/j.camwa.2017.09.013
14. W. X. Ma, Lump waves and their dynamics in a generalized Kadomtsev–Petviashvili-like model, *Mod. Phys. Lett. A*, **41** (2026), 2550215. https://doi.org/10.1142/S0217732325502153
15. W. X. Ma, Lump and interaction solutions to linear PDEs in (2+1)-dimensions via symbolic computation, *Mod. Phys. Lett. B*, **33** (2019), 1950457. https://doi.org/10.1142/S0217984919504578
16. S. S. Ray, On conservation laws by Lie symmetry analysis for (2+1)-dimensional Bogoyavlensky-Konopelchenko equation in wave propagation, *Comput. Math. Appl.*, **74** (2017), 1158–1165. https://doi.org/10.1016/j.camwa.2017.06.007
17. H. Triki, Z. Jovanoski, A. Biswas, Shock wave solutions to the Bogoyavlensky-Konopelchenko equation, *Indian J. Phys.*, **88** (2014), 71–74. https://doi.org/10.1007/s12648-013-0380-7
18. L. Cheng, W. X. Ma, Lump structures and their dynamics in a generalized Calogero–Bogoyavlenskii–Schiff-like wave model, *Int. J. Theoret. Phys.*, **65** (2026), 20. https://doi.org/10.1007/s10773-025-06235-8
19. C. M. Khalique, O. D. Adeyemo, K. Maefo, Symmetry solutions and conservation laws of a new generalized 2D Bogoyavlensky-Konopelchenko equation of plasma physics, *AIMS Math.*, **7** (2022), 9767–9788. https://doi.org/10.3934/math.2022544
20. W. X. Ma, S. Batwa, S. Manukure, Dispersion-managed lump waves in a spatial symmetric KP model, *East Asian J. Appl. Math.*, **13** (2023), 246–256. https://doi.org/10.4208/eajam.2022-038.180922
21. W. X. Ma, Y. S. Bai, A. Adjiri, Nonlinearity-managed lump waves in a spatial symmetric HSI model, *Eur. Phys. J. Plus*, **136** (2021), 240. https://doi.org/10.1140/epjp/s13360-021-01212-6
22. W. X. Ma, L. Q. Zhang, Lump solutions with higher-order rational dispersion relations, *Pramana–J. Phys.*, **94** (2020), 43. https://doi.org/10.1007/s12043-020-1918-9
23. S. Manukure, Y. Zhou, W. X. Ma, Lump solutions to a (2+1)-dimensional extended KP equation, *Comput. Math. Appl.*, **75** (2018), 2414–2419. https://doi.org/10.1016/j.camwa.2017.12.030
24. B. Ren, W. X. Ma, J. Yu, Characteristics and interactions of solitary and lump waves of a (2+1)-dimensional coupled nonlinear partial differential equation, *Nonlinear Dyn.*, **96** (2019), 717–727. https://doi.org/10.1007/s11071-019-04816-x

25. J. P. Yu, Y. L. Sun, Study of lump solutions to dimensionally reduced generalized KP equations, *Nonlinear Dyn.*, **87** (2017), 2755–2763. https://doi.org/10.1007/s11071-016-3225-z
26. Z. L. Zhao, L. C. He, Y. B. Gao, Rogue wave and multiple lump solutions of the (2+1)-dimensional Benjamin-Ono equation in fluid mechanics, *Complexity*, **2019** (2019), 8249635. https://doi.org/10.1155/2019/8249635
27. L. Cheng, W. X. Ma, Soliton and lump solutions to a fourth-order nonlinear wave equation in (2+1)-dimensions, *Qual. Theory Dyn. Syst.*, **24** (2025), 237. https://doi.org/10.1007/s12346-025-01384-x
28. W. X. Ma, Lump-type solutions to the (3+1)-dimensional Jimbo-Miwa equation, *Int. J. Nonlinear Sci. Numer. Simul.*, **17** (2016), 355–359. https://doi.org/10.1515/ijnsns-2015-0050
29. Y. Sun, B. Tian, X. Y. Xie, J. Chai, H. M. Yin, Rogue waves and lump solitons for a (3+1)-dimensional B-type Kadomtsev-Petviashvili equation in fluid dynamics, *Waves Random Complex Media*, **28** (2018), 544–552. https://doi.org/10.1080/17455030.2017.1367866
30. W. X. Ma, Dispersion-governed lump waves in a generalized Calogero–Bogoyavlenskii–Schiff-like model with spatially symmetric nonlinearity, *Axioms*, **14** (2025), 869. https://doi.org/10.3390/axioms14120869
31. T. C. Kofane, M. Fokou, A. Mohamadou, E. Yomba, Lump solutions and interaction phenomenon to the third-order nonlinear evolution equation, *Eur. Phys. J. Plus*, **132** (2017), 465. https://doi.org/10.1140/epjp/i2017-11747-6
32. W. X. Ma, An application of dual group reductions to the AKNS integrable mKdV model, *Mod. Phys. Lett. B*, **39** (2025), 2550233. https://doi.org/10.1142/S0217984925502331
33. W. X. Ma, Integrable matrix Modified Korteweg-de Vries equations derived from reducing AKNS Lax pairs, *Rom. J. Phys.*, **70** (2025), 110. https://doi.org/10.59277/RomJPhys.2025.70.110
34. W. X. Ma, Reduced matrix integrable hierarchies via group reduction involving off-diagonal block matrices, *Commun. Theor. Phys.*, **78** (2026), 015001. https://doi.org/10.1088/1572-9494/adf42a
35. X. X. Xu, Solving an integrable coupling system of Merola-Ragnisco-Tu lattice equation by Darboux transformation of Lax pair, *Commun. Nonlinear Sci. Numer. Simul.*, **23** (2015), 192–201. https://doi.org/10.1016/j.cnsns.2014.11.002
36. X. Y. Li, Q. L. Zhao, Q. Q. Yang, Integrable asymmetric AKNS model with multi-component, *Commun. Nonlinear Sci. Numer. Simul.*, **91** (2020), 105434. https://doi.org/10.1016/j.cnsns.2020.105434
37. W. X. Ma, A combined integrable hierarchy with four potentials and its recursion operator and bi-Hamiltonian structure, *Indian J. Phys.*, **99** (2025), 1063–1069. https://doi.org/10.1007/s12648-024-03364-4
38. W. X. Ma, A soliton hierarchy derived from a fourth-order matrix spectral problem possessing four fields, *Chaos Solitons Fract.*, **195** (2025), 116309. https://doi.org/10.1016/j.chaos.2025.116309

AIMS Press