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Research article

Weak Harnack inequality for the nonhomogeneous nonlocal equations with general growth†

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Abstract: In this paper, we aim to establish a new class of weak Harnack inequalities for weak supersolutions to the nonhomogeneous nonlocal equations with general growth. Our approach mainly relies on the expansion of positivity in the spirit of De Giorgi classes, along with a refined energy estimate.

Keywords: nonlocal equations; weak Harnack estimates; general growth

1. Introduction

Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be a bounded domain. In this paper, we consider the following nonlinear nonlocal equations with general growth

$$\mathcal{L}u = f(x, u) \quad \text{in } \Omega \tag{1.1}$$

with

$$\mathcal{L}u(x) := \text{P.V.} \int_{\mathbb{R}^N} g\left(\frac{|u(x) - u(y)|}{|x - y|^s}\right) \frac{u(x) - u(y)}{|u(x) - u(y)|} \frac{K(x, y)}{|x - y|^s} \, \mathrm{d}y,$$

where P.V. stands for "in the principal value sense", $s \in (0, 1)$ and the kernel $K\colon \mathbb{R}^N \times \mathbb{R}^N \to (0, \infty]$ is a measurable function fulfilling

$$\frac{\lambda}{|x - y|^N} \leq K(x, y) = K(y, x) \leq \frac{\Lambda}{|x - y|^N} \tag{1.2}$$

for some $\Lambda \geq \lambda > 0$. Moreover, the continuous function $g\colon [0,\infty) \to [0,\infty)$ is strictly increasing satisfying $g(0) = 0$, $\lim_{t\to\infty} g(t) = \infty$ and

$$1 < p \leq \frac{tg(t)}{G(t)} \leq q < \infty \quad \text{with } G(t) = \int_0^t g(\tau)\mathrm{d}\tau, \tag{1.3}$$

where the conditions on G will be introduced in Section 2 in detail. The nonhomogeneous term f satisfies

$$|f(x,u)| \leq d_1 + d_2 g(|u|) \quad \text{for a.a. } x \in \Omega \text{ and } u \in \mathbb{R},$$

with some fixed positive constants d_1 and d_2.

Equation (1.1) is regarded as the nonlocal nonhomogeneous analogue of the G-Laplace equation, which is typically defined as

$$-\operatorname{div}\left(g(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right) = 0.$$

Such equations have been widely studied over the past years, and we refer the readers to [1,13,14,22,24,25] and references therein for related results. In the special case $g(t) = t^{p-1}$ and $K = |x-y|^{-N}$, the operator $\mathcal{L}$ in (1.1) can also be represented by $(-\Delta)_p^s$, which is called fractional p-Laplacian, see [6,18] for instance. Regarding the regularity theory for this class of problems, Di Castro et al. [12] investigated local boundedness and Hölder continuity using the De Giorgi-Nash-Moser iteration; see also [10] for similar results derived with the aid of fractional De Giorgi classes. Recently, Liao [23] proved weak Harnack estimates for the nonlocal p-Laplace equation, which exhibit strong nonlocality. For nonlocal parabolic equations, Kassmann and Weidner [21] established the Hölder regularity and a Harnack inequality with tail terms appearing on both sides. The nonlocal operators characterized by nonsymmetric forms can be found in [19,20].

In terms of the higher regularity theory, Brasco, Lindgren and Schikorra [5] provided an explicit Hölder exponent for solutions of the fractional p-Laplacian equation in the superquadratic case, while the subquadratic case was studied by Garain-Lindgren in [15]. In [17], C^α-regularity up to the boundary for weak solutions of the fractional p-Laplacian equation was proved by Iannizzotto, Mosconi and Squassina using barrier arguments. Very recently, the authors in [2,3] showed that the (s,p)-harmonic functions have fractional differentiability of the gradient with the restriction of order s.

When $g(\cdot)$ has a general structure, Fang and the third author [14] employed the expansion of positivity and energy estimates to establish a Harnack inequality. Later, Byun et al. [8] provided a more simplified proof to obtain the Harnack inequality, under no assumptions on G other than (1.3); see also [4,7,22] for further regularity results. Coming to the nonlocal double phase structure, De Filippis and Palatucci [11] proved the Hölder regularity for the viscosity solutions and Byun, Ok and Song [9] showed that weak solutions are locally bounded and Hölder continuous.

This paper aims to establish weak Harnack inequalities for the nonhomogeneous nonlocal equations with general growth. Inspired by the results mentioned above, we use the expansion of positivity, which describes the propagation of pointwise positivity based on measure information. Our approach mainly relies on an energy estimate that accounts for the influence of the nonhomogeneous term. Unlike typical weak Harnack inequalities that display the nonlocal tail only on one side, our result takes into account the feature of nonlocal terms on the positive and negative parts of the weak solutions at both sides of the inequality.

We introduce the following tail space

$$L^g_s(\mathbb{R}^N) = \left\{u \text{ is a measurable function in } \mathbb{R}^N\colon \int_{\mathbb{R}^N} g\left(\frac{|u(x)|}{(1+|x|)^s}\right)\frac{\mathrm{d}x}{(1+|x|)^{N+s}} < \infty\right\}.$$

The nonlocal tail of u is defined as

$$\mathrm{Tail}(u; B_R(x_0)) = \int_{\mathbb{R}^N\setminus B_R(x_0)} g\left(\frac{|u(x)|}{|x-x_0|^s}\right)\frac{\mathrm{d}x}{|x-x_0|^{N+s}}.$$

It is known from [7] that $u \in L^g_s(\mathbb{R}^N)$ if and only if $\mathrm{Tail}(u; B_R(x_0))$ is finite for any $x_0 \in \mathbb{R}^N$ and $R > 0$. Set

$$u_+ = \max\{u, 0\} \quad \text{and} \quad u_- = \max\{-u, 0\}.$$

Now, we proceed to state our main results, which highlight the positivity contribution arising from the long-range characteristics of the supersolution to Eq (1.1).

Theorem 1.1. *Let $R \in (0,1]$ and $B_{2r}(x_0) \subset B_R(x_0)$. Suppose that $u \in \mathbb{W}^{s,G}(\Omega)$, satisfying $u \geq 0$ in $B_R(x_0) \subset \Omega$, is a weak supersolution to Eq (1.1). Then there exists a constant $\eta \in (0,1)$ depending only on $s, N, p, q, \lambda, \Lambda, d_2$, such that*

$$\begin{aligned}&\operatorname*{ess\,inf}_{B_{\frac{r}{2}}(x_0)} u + r^s g^{-1}\left(r^s\,\mathrm{Tail}\left(u_-; B_R\left(x_0\right)\right)\right) + r^s G^{-1}G^*\left(d_1 r^s\right)\\ &\geq \eta r^s g^{-1}\left(r^s\,\mathrm{Tail}\left(u_+; B_r\left(x_0\right)\right)\right).\end{aligned}$$

We also derive a weak Harnack estimate similar to the result in [22, Theorem 3.4]. However, our approach primarily relies on energy estimates rather than the Moser iteration employed in [22].

Theorem 1.2. *Let $R \in (0,1]$ and $B_{2r}(x_0) \subset B_R(x_0)$. Suppose that $u \in \mathbb{W}^{s,G}(\Omega)$, satisfying $u \geq 0$ in $B_R(x_0) \subset \Omega$, is a weak supersolution to Eq (1.1). Then there exists a constant $\eta \in (0,1)$ depending only on $s, N, p, q, \lambda, \Lambda, d_2$, such that*

$$\operatorname*{ess\,inf}_{B_{\frac{r}{2}}(x_0)} u + r^s g^{-1}\left(r^s\,\mathrm{Tail}(u_-; B_R(x_0))\right) + r^s G^{-1}G^*(d_1 r^s) \geq \eta r^s g^{-1}\left(\fint_{B_r(x_0)} g\left(\frac{u(x)}{r^s}\right)\mathrm{d}x\right).$$

The article is organized as follows. In Section 2, several basic concepts, inequalities, and function spaces are introduced. In Section 3, we establish an energy estimate that makes an effort to obtain the expansion of positivity and density lemmas in Section 4. Finally, we prove the weak Harnack inequalities in Section 5.

2. Preliminaries

In this section, we shall introduce the definition of weak solutions and function spaces related to solutions and give some basic inequalities to be used later.

Let $B_r(x_0) = \{x \in \mathbb{R}^N\colon |x - x_0| < r\}$ be an open ball with center x_0 and radius $r > 0$, and center x_0 will be omitted when there is no ambiguity. A measurable function $G\colon [0,\infty) \to [0,\infty)$ is said to be an N-function if it is convex and increasing, and satisfies

$$G(0) = 0, \quad \lim_{t\to 0^+}\frac{G(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty}\frac{G(t)}{t} = \infty.$$

For an N-function $G\colon [0,\infty)\to[0,\infty)$, whose conjugate function $G^*\colon [0,\infty)\to[0,\infty)$ is defined by

$$G^*(t)=\sup_{\tau\geq 0}\{\tau t-G(\tau)\}.$$

From the relation (1.3), we immediately deduce several inequalities to be utilized later.

(a) For $t\in[0,\infty)$, $a\in(0,1)$ and $c=p^{1/(q-1)}/q^{1/(p-1)}$,

$$\begin{cases} a^qG(t)\leq G(at)\leq a^pG(t), & a^{\frac{1}{p}}G^{-1}(t)\leq G^{-1}(at)\leq a^{\frac{1}{q}}G^{-1}(t),\\ \dfrac{p}{q}a^{q-1}g(t)\leq g(at)\leq \dfrac{q}{p}a^{p-1}g(t), & ca^{\frac{1}{p-1}}g^{-1}(t)\leq g^{-1}(at)\leq ca^{\frac{1}{q-1}}g^{-1}(t),\end{cases}\tag{2.1}$$

and for $t\in[0,\infty)$, $a\in(1,\infty)$,

$$\begin{cases} a^pG(t)\leq G(at)\leq a^qG(t), & a^{\frac{1}{q}}G^{-1}(t)\leq G^{-1}(at)\leq a^{\frac{1}{p}}G^{-1}(t),\\ \dfrac{q}{p}a^{p-1}g(t)\leq g(at)\leq \dfrac{p}{q}a^{q-1}g(t), & ca^{\frac{1}{q-1}}g^{-1}(t)\leq g^{-1}(at)\leq ca^{\frac{1}{p-1}}g^{-1}(t).\end{cases}\tag{2.2}$$

(b) The Young inequality with $\epsilon\in(0,1]$:

$$t\tau\leq \epsilon^{1-q}G(t)+\epsilon G^*(\tau),\ \ t,\tau\geq 0.\tag{2.3}$$

(c) For $t,\tau\geq 0$,

$$G^*(g(t))\leq (q-1)G(t),\tag{2.4}$$

and

$$2^{-1}(G(t)+G(\tau))\leq G(t+\tau)\leq 2^{q-1}(G(t)+G(\tau)).\tag{2.5}$$

Before giving the definition of weak solutions, we shall introduce the notion of Orlicz-Sobolev space. Providing that N-function G satisfies the condition (1.3), the Orlicz space $L^G(\Omega)$ is described as

$$L^G(\Omega)=\left\{u \text{ is measurable function in } \Omega\colon\ \int_\Omega G(|u(x)|)\,\mathrm{d}x<\infty\right\}.$$

The norm of the above space is

$$\|u\|_{L^G(\Omega)}=\inf\Big\{\lambda>0\colon\ \int_\Omega G\Big(\frac{u(x)}{\lambda}\Big)\,\mathrm{d}x\leq 1\Big\}.$$

We next introduce the fractional Orlicz-Sobolev space $W^{s,G}(\Omega)$

$$W^{s,G}(\Omega)=\left\{u\in L^G(\Omega)\colon\ \int_\Omega\int_\Omega G\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N}<\infty\right\},$$

equipped with the norm

$$\|u\|_{W^{s,G}(\Omega)}=\|u\|_{L^G(\Omega)}+[u]_{W^{s,G}(\Omega)},$$

where the semi-norm is defined as

$$[u]_{W^{s,G}(\Omega)} = \inf\left\{\lambda > 0\colon \int_\Omega \int_\Omega G\Big(\frac{|u(x)-u(y)|}{\lambda|x-y|^s}\Big)\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N} \le 1\right\}.$$

For measurable function u in $\mathbb{R}^N$, we define

$$\mathbb{W}^{s,G}(\Omega) = \left\{u|_\Omega \in L^G(\Omega)\colon \iint_{C_\Omega} G\Big(\frac{|u(x)-u(y)|}{|x-y|^s}\Big)\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N} < \infty\right\},$$

where $C_\Omega :- (\Omega \times \mathbb{R}^N) \cup (\mathbb{R}^N \times \Omega)$.

The definition of weak solutions to Eq (1.1) is provided as follows.

Definition 2.1. *A measurable function $u \in \mathbb{W}^{s,G}(\Omega)$ is a weak supersolution(subsolution) of Eq* (1.1)*, if*

$$\iint_{C_\Omega} g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|u(x)-u(y)|}(\varphi(x)-\varphi(y))\frac{K(x,y)}{|x-y|^s}\,\mathrm{d}x\,\mathrm{d}y \ge (\le) \int_\Omega f(x,u(x))\varphi(x)\,\mathrm{d}x, \quad (2.6)$$

for all non-negative functions $\varphi \in \mathbb{W}^{s,G}(\Omega)$ with compact support in Ω. And u is said to be a weak solution if and only if it is a weak supersolution and a weak subsolution.

3. Energy estimate

We will establish an energy estimate that accounts for the impact of the inhomogeneous term.

Proposition 3.1. *Assume that u is a weak supersoution to* (1.1)*, then there exists a constant $\gamma_* > 0$ depending only on $s, p, q, N, d_2, \lambda, \Lambda$, such that for any $B_r(x_0) \subset B_R(x_0) \subset \Omega$ $(R < 1)$ and any $k \in \mathbb{R}$,*

$$\begin{aligned}
&\int_{B_r}\int_{B_r} G\left(\frac{|\omega_-(x)-\omega_-(y)|}{|x-y|^s}\right)\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N} + \int_{B_r}\omega_-(x)\int_{\mathbb{R}^N} g\left(\frac{\omega_+(y)}{|x-y|^s}\right)\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^{N+s}}\\
&\le \gamma_*\left[G^*(d_1R^s) + G\left(\frac{k}{R^s}\right)\right]|A_-(k,R)| + \gamma_*\frac{R^q}{(R-r)^q}\int_{B_r} G\left(\frac{\omega_-(x)}{R^s}\right)\mathrm{d}x\\
&\quad + \gamma_*\frac{R^{N+sq}}{(R-r)^{N+sq}}\|\omega_-\|_{L^1(B_R)}\,\mathrm{Tail}(\omega_-; B_R(x_0)),
\end{aligned}$$

where $\omega_- = (u-k)_-$ and $A_-(k,R) = \{x \in B_R\colon u(x) \le k\}$.

Proof. Assume $x_0 = 0$ for simplicity. Let $\omega_\pm = (u-k)_\pm$. Now we take a cutoff function $\eta \in C_0^\infty(B_R)$ with $0 \le \eta \le 1$, vanishing outside $B_{\frac{r+R}{2}}$ and equal to 1 in B_r, such that $|\nabla\eta| \le \frac{4}{R-r}$. Now we select $\varphi := \eta^q\omega_-$ as a test function in the weak formulation (2.6) to obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^N} f(x,u(x))\varphi(x)\,\mathrm{d}x\\
&\le \int_{B_R}\int_{B_R} g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|u(x)-u(y)|}\frac{\varphi(x)-\varphi(y)}{|x-y|^s}K(x,y)\,\mathrm{d}x\,\mathrm{d}y\\
&\quad + 2\int_{\mathbb{R}^N\setminus B_R}\int_{B_R} g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|u(x)-u(y)|}\frac{\varphi(x)}{|x-y|^s}K(x,y)\,\mathrm{d}x\,\mathrm{d}y := I_1 + I_2. \quad (3.1)
\end{aligned}$$

Estimate of I_1. If $x, y \notin A_-(k,R)$, then using the fact that $\operatorname{supp}\varphi \subset\subset A_-(k,R)$ we know

$$g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|u(x)-u(y)|}\frac{\varphi(x)-\varphi(y)}{|x-y|^s} = 0. \tag{3.2}$$

If $x \in A_-(k,R)$, $y \notin A_-(k,R)$, it holds that

$$g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|u(x)-u(y)|}(\varphi(x)-\varphi(y)) = -\eta^q(x)\omega_-(x)g\left(\frac{\omega_-(x)+\omega_+(y)}{|x-y|^s}\right).$$

Due to the strictly increasing monotonicity of g, it follows that

$$g\left(\frac{\omega_-(x)+\omega_+(y)}{|x-y|^s}\right) \geq \frac{1}{2}\left[g\left(\frac{\omega_-(x)}{|x-y|^s}\right)+g\left(\frac{\omega_+(y)}{|x-y|^s}\right)\right].$$

By (1.3), we get

$$\begin{aligned}
&g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|u(x)-u(y)|}\frac{\varphi(x)-\varphi(y)}{|x-y|^s}\\
&\leq -\frac{p}{2}\eta^q(x)G\left(\frac{\omega_-(x)}{|x-y|^s}\right)-\frac{1}{2}\eta^q(x)\frac{\omega_-(x)}{|x-y|^s}g\left(\frac{\omega_+(y)}{|x-y|^s}\right)\\
&\leq -\frac{p}{2}\min\{\eta^q(x),\eta^q(y)\}G\left(\frac{|\omega_-(x)-\omega_-(y)|}{|x-y|^s}\right)\\
&\quad -\frac{1}{2}\min\{\eta^q(x),\eta^q(y)\}\frac{\omega_-(x)}{|x-y|^s}g\left(\frac{\omega_+(y)}{|x-y|^s}\right).
\end{aligned} \tag{3.3}$$

If $x, y \in A_-(k,R)$, we assume without loss of generality that $k \geq u(x) \geq u(y)$. Therefore

$$\begin{aligned}
&g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|u(x)-u(y)|}\frac{\varphi(x)-\varphi(y)}{|x-y|^s}\\
&= g\left(\frac{|\omega_-(x)-\omega_-(y)|}{|x-y|^s}\right)\frac{\eta^q(x)\omega_-(x)-\eta^q(y)\omega_-(y)}{|x-y|^s} := K.
\end{aligned}$$

Notice that, if $\eta(x) \leq \eta(y)$, we have

$$\begin{aligned}
K &\leq -g\left(\frac{|\omega_-(x)-\omega_-(y)|}{|x-y|^s}\right)\frac{|\omega_-(x)-\omega_-(y)|}{|x-y|^s}\eta^q(y)\\
&\leq -p\eta^q(y)G\left(\frac{|\omega_-(x)-\omega_-(y)|}{|x-y|^s}\right)\\
&\leq -pG\left(\frac{|\omega_-(x)-\omega_-(y)|}{|x-y|^s}\right)\min\{\eta^q(x),\eta^q(y)\}.
\end{aligned}$$

If $\eta(x) > \eta(y)$, then

$$\begin{aligned}
K &= -g\left(\frac{|\omega_-(x)-\omega_-(y)|}{|x-y|^s}\right)\frac{|\omega_-(x)-\omega_-(y)|\eta^q(x)}{|x-y|^s}\\
&\quad + g\left(\frac{|\omega_-(x)-\omega_-(y)|}{|x-y|^s}\right)\frac{\omega_-(y)(\eta^q(x)-\eta^q(y))}{|x-y|^s} := K_1+K_2.
\end{aligned}$$

Using (1.3), we have

$$K_1 \le -p\eta^q(x)G\left(\frac{|\omega_-(x)-\omega_-(y)|}{|x-y|^s}\right). \tag{3.4}$$

Considering the fact that $\eta^q(x)-\eta^q(y)\le q\eta^{q-1}(x)(\eta(x)-\eta(y))$, we further compute

$$K_2 \le qg\left(\frac{|\omega_-(x)-\omega_-(y)|}{|x-y|^s}\right)\eta^{q-1}(x)\frac{\eta(x)-\eta(y)}{|x-y|^s}\omega_-(y). \tag{3.5}$$

Then, we apply (2.3) and (2.4) with $\epsilon=\min\left\{\frac{p}{2q(q-1)},\frac{1}{2q}\right\}$ to obtain that

$$\begin{aligned}
&g\left(\frac{|\omega_-(x)-\omega_-(y)|}{|x-y|^s}\right)\eta^{q-1}(x)\frac{\eta(x)-\eta(y)}{|x-y|^s}\omega_-(y)\\
&\le \epsilon G^*\left(g\left(\frac{|\omega_-(x)-\omega_-(y)|}{|x-y|^s}\right)\eta^{q-1}(x)\right)+\gamma(\epsilon)G\left(\frac{\eta(x)-\eta(y)}{|x-y|^s}\omega_-(y)\right)\\
&\le \frac{p}{2q}G\left(\frac{|\omega_-(x)-\omega_-(y)|}{|x-y|^s}\right)\eta^q(x)+\gamma(p,q)G\left(\frac{\eta(x)-\eta(y)}{|x-y|^s}\omega_-(y)\right).
\end{aligned} \tag{3.6}$$

By means of (3.4)–(3.6), we finally derive

$$\begin{aligned}
&g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|u(x)-u(y)|}\frac{\varphi(x)-\varphi(y)}{|x-y|^s}\\
&\le -\frac{p}{2}G\left(\frac{|\omega_-(x)-\omega_-(y)|}{|x-y|^s}\right)\min\{\eta^q(x),\eta^q(y)\}\\
&\quad+\gamma(p,q)G\left(\frac{|\eta(x)-\eta(y)|}{|x-y|^s}\max\{\omega_-(x),\omega_-(y)\}\right).
\end{aligned} \tag{3.7}$$

Utilizing (1.2), (3.2), (3.3) and (3.7) and considering the fact that $\eta=1$ in B_r, we derive

$$\begin{aligned}
&\iint_{B_R\times B_R} g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|u(x)-u(y)|}\frac{\varphi(x)-\varphi(y)}{|x-y|^s}K(x,y)\,\mathrm{d}x\,\mathrm{d}y\\
&\le \gamma(p,q,\Lambda)\iint_{B_R\times B_R} G\left(\frac{|\eta(x)-\eta(y)|}{|x-y|^s}\max\{\omega_-(x),\omega_-(y)\}\right)\frac{1}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad-\frac{\lambda p}{2}\iint_{B_r\times B_r} G\left(\frac{|\omega_-(x)-\omega_-(y)|}{|x-y|^s}\right)\frac{1}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad-\frac{\lambda}{2}\iint_{B_r\times B_r}\frac{\omega_-(x)}{|x-y|^{N+s}}g\left(\frac{\omega_+(y)}{|x-y|^s}\right)\mathrm{d}x\,\mathrm{d}y.
\end{aligned} \tag{3.8}$$

Recalling that $|\nabla\eta|\le\frac{4}{R-r}$, we may apply (2.2) and Mean Value Theorem to yield

$$\begin{aligned}
&\int_{B_R}\int_{B_R} G\left(\frac{|\eta(x)-\eta(y)|}{|x-y|^s}\max\{\omega_-(x),\omega_-(y)\}\right)\frac{1}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\\
&\le 2\int_{B_R}\int_{B_R} G\left(\frac{|\eta(x)-\eta(y)|}{|x-y|^s}\omega_-(x)\right)\frac{1}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&\leq 2\int_{B_R}\int_{B_R} G\left(\frac{|\nabla\eta||x-y|}{|x-y|^s}\omega_-(x)\right)\frac{1}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\\
&\leq 2\int_{B_R}\int_{B_R} G\left(\frac{8R}{R-r}\frac{(2R)^s}{|x-y|^s}\frac{\omega_-(x)}{(2R)^s}\right)\frac{1}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\\
&\leq \gamma(N,s,p,q)\frac{R^{sq+q}}{(R-r)^q}\int_{B_R} G\left(\frac{\omega_-(x)}{R^s}\right)\int_{B_R}\frac{1}{|x-y|^{N+sq}}\,\mathrm{d}y\,\mathrm{d}x\\
&\leq \gamma(N,s,p,q)\frac{R^q}{(R-r)^q}\int_{B_R} G\left(\frac{\omega_-(x)}{R^s}\right)\mathrm{d}x,
\end{aligned}
$$

in which the penultimate inequality arises from the facts $\frac{R}{R-r} > 1$ and $\frac{2R}{|x-y|} > 1$. Substituting the above inequality into (3.8) shows that

$$
\begin{aligned}
&\int_{B_R}\int_{B_R} g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|u(x)-u(y)|}\frac{\varphi(x)-\varphi(y)}{|x-y|^s}K(x,y)\,\mathrm{d}x\,\mathrm{d}y\\
\leq& -\frac{\lambda p}{2}\int_{B_r}\int_{B_r} G\left(\frac{|\omega_-(x)-\omega_-(y)|}{|x-y|^s}\right)\frac{1}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\\
&-\frac{\lambda}{2}\int_{B_r}\int_{B_r}\frac{\omega_-(x)}{|x-y|^{N+s}} g\left(\frac{\omega_+(y)}{|x-y|^s}\right)\mathrm{d}x\,\mathrm{d}y\\
&+\gamma(N,s,p,q,\Lambda)\frac{R^q}{(R-r)^q}\int_{B_R} G\left(\frac{\omega_-(x)}{R^s}\right)\mathrm{d}x. \qquad (3.9)
\end{aligned}
$$

Next, we will deal with the second term on the right-hand side of (3.1).

Estimate of I_2. We now turn our attention to the term integrated on $B_R \times (\mathbb{R}^N \setminus B_R)$.

$$
\begin{aligned}
&2\int_{B_R}\eta^q(x)\omega_-(x)\int_{\mathbb{R}^N\setminus B_R} g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|u(x)-u(y)|}\frac{K(x,y)}{|x-y|^s}\,\mathrm{d}y\,\mathrm{d}x\\
\leq& 2\Lambda\int_{A_-(k,\frac{r+R}{2})}\omega_-(x)\int_{\mathbb{R}^N\cap\{u(y)<u(x)\}\setminus B_R} g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{1}{|x-y|^{N+s}}\,\mathrm{d}y\,\mathrm{d}x\\
&-2\lambda\int_{A_-(k,r)}\omega_-(x)\int_{\mathbb{R}^N\cap\{u(y)\geq u(x)\}\setminus B_R} g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{1}{|x-y|^{N+s}}\,\mathrm{d}y\,\mathrm{d}x\\
:=& J_1 - J_2.
\end{aligned}
$$

Given $x \in B_{\frac{r+R}{2}}$ and $y \in \mathbb{R}^N \setminus B_R$, we have $|x-y| \geq \frac{R-r}{2R}|y|$. This together with (2.2) results in

$$
\begin{aligned}
J_1 &\leq 2\Lambda\int_{B_{\frac{r+R}{2}}}\omega_-(x)\int_{\mathbb{R}^N\setminus B_R} g\left(\frac{\omega_-(y)}{|y|^s}\left(\frac{2R}{R-r}\right)^s\right)\frac{1}{|y|^{N+s}}\left(\frac{2R}{R-r}\right)^{N+s}\mathrm{d}y\,\mathrm{d}x\\
&\leq \gamma(N,s,p,q,\Lambda)\frac{R^{N+sq}}{(R-r)^{N+sq}}\int_{B_{\frac{r+R}{2}}}\omega_-(x)\int_{\mathbb{R}^N\setminus B_R} g\left(\frac{\omega_-(y)}{|y|^s}\right)\frac{1}{|y|^{N+s}}\,\mathrm{d}y\,\mathrm{d}x\\
&\leq \gamma(N,s,p,q,\Lambda)\frac{R^{N+sq}}{(R-r)^{N+sq}}\|\omega_-(x)\|_{L^1(B_R)}\,\mathrm{Tail}(\omega_-;B_R). \qquad (3.10)
\end{aligned}
$$

It's easy to check that

$$
\{u(y) \geq k\} \subset \{u(y) \geq u(x)\},
$$

which leads to

$$
\begin{aligned}
J_2 &\geq 2\lambda \int_{B_r} \omega_-(x) \int_{\mathbb{R}^N \cap A_+(k)\setminus B_R} g\left(\frac{\omega_+(y)+\omega_-(x)}{|x-y|^s}\right) \frac{1}{|x-y|^{N+s}}\,\mathrm{d}y\,\mathrm{d}x \\
&\geq 2\lambda \int_{B_r} \omega_-(x) \int_{\mathbb{R}^N \cap A_+(k)\setminus B_R} g\left(\frac{\omega_+(y)}{|x-y|^s}\right) \frac{1}{|x-y|^{N+s}}\,\mathrm{d}y\,\mathrm{d}x \\
&= 2\lambda \int_{B_r} \omega_-(x) \int_{\mathbb{R}^N \setminus B_R} g\left(\frac{\omega_+(y)}{|x-y|^s}\right) \frac{1}{|x-y|^{N+s}}\,\mathrm{d}y\,\mathrm{d}x, \qquad (3.11)
\end{aligned}
$$

where $A_+(k) = \{x \in \mathbb{R}^N \colon u(x) \geq k\}$. Here, to obtain the last line we use the fact that $\omega_+(y) = 0$, while $y \notin \mathbb{R}^N \cap A_+(k)$. It follows from (3.10) and (3.11) that

$$
\begin{aligned}
&\int_{B_R}\int_{\mathbb{R}^N\setminus B_R} g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|u(x)-u(y)|}\frac{\varphi(x)-\varphi(y)}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y \\
&\leq \gamma(N,s,p,q,\Lambda)\frac{R^{N+sq}}{(R-r)^{N+sq}}\|\omega_-(x)\|_{L^1(B_R)}\operatorname{Tail}(\omega_-;B_R) \\
&\quad - \gamma(N,s,p,q,\lambda)\int_{B_r}\omega_-(x)\int_{\mathbb{R}^N\setminus B_R} g\left(\frac{\omega_+(y)}{|x-y|^s}\right)\frac{1}{|x-y|^{N+s}}\,\mathrm{d}y\,\mathrm{d}x. \qquad (3.12)
\end{aligned}
$$

To finish the proof, we shall estimate the term on the left-hand side of (3.1). Recalling the property of $f(x,u(x))$, we get

$$
\int_{\mathbb{R}^N} |f(x,u(x))|\varphi(x)\,\mathrm{d}x \leq \int_{A_-(k,\frac{r+R}{2})} (d_1 + d_2 g(|u(x)|))\eta^q(x)\omega_-(x)\,\mathrm{d}x. \qquad (3.13)
$$

For the first term of the inequality above, we may apply the Young inequality (2.3) with

$$
\epsilon = \left(\frac{R}{R-r}\right)^{\frac{q}{1-q}} < 1,
$$

to obtain

$$
\begin{aligned}
&\int_{A_-(k,\frac{r+R}{2})} d_1\eta^q(x)\omega_-(x)\,\mathrm{d}x \\
&\leq \left(\frac{R}{R-r}\right)^{\frac{q}{1-q}} G^*(d_1R^s)|A_-(k,R)| + \left(\frac{R}{R-r}\right)^q \int_{B_R} G\left(\frac{\omega_-(x)}{R^s}\right)\mathrm{d}x. \qquad (3.14)
\end{aligned}
$$

For the second term of the inequality (3.13), we notice that if $x \in A_-(k)$

$$
g(|u(x)|) = g(|\omega_-(x)-k|) \leq g(\omega_-(x)+k) \leq \gamma(p,q)g(\omega_-(x)) + \gamma(p,q)g(k),
$$

then

$$
\begin{aligned}
&\int_{A_-(k,\frac{r+R}{2})} d_2 g(|u(x)|)\eta^q(x)\omega_-(x)\,\mathrm{d}x \\
&\leq \gamma(N,s,p,q)\int_{A_-(k,\frac{r+R}{2})} \eta^q d_2[g(\omega_-(x))\omega_-(x) + g(k)\omega_-(x)]\,\mathrm{d}x \\
&\leq \gamma(N,s,p,q,d_2)\left[\left(\frac{R}{R-r}\right)^q\int_{B_R} G\left(\frac{\omega_-(x)}{R^s}\right)\mathrm{d}x + G\left(\frac{k}{R^s}\right)|A_-(k,R)|\right], \qquad (3.15)
\end{aligned}
$$

where we use (2.3), (2.4) and the fact that $R < 1$. By putting together (3.9), (3.12), (3.14) and (3.15) and adjusting the constants, we finally complete the proof. □

4. Density lemma

The following lemma shows the spread of pointwise positivity in space.

Lemma 4.1. *Let $k \geq 0$ and $R \leq 1$ be parameters. Assume that u is a supersolution of* (1.1)*, nonnegative in $B_R(x_0) \subset \Omega$. Then there exists a constant $\nu \in (0,1)$ depending only on the data $s, p, q, N, d_1, d_2, \lambda, \Lambda$, such that if*

$$|\{u \leq k\} \cap B_r(x_0)| \leq \nu |B_r(x_0)|,$$

then either

$$r^s g^{-1}\left(r^s \operatorname{Tail}(u_-; B_R(x_0))\right) + r^s G^{-1} G^*(d_1 r^s) > k,$$

or

$$u \geq \frac{1}{2}k, \quad \forall x \in B_{\frac{1}{2}r}(x_0),$$

where $B_r(x_0) \subset B_R(x_0)$.

Proof. Without loss of generality, we suppose $x_0 = 0$ and

$$r^s g^{-1}\left(r^s \operatorname{Tail}(u_-; B_R)\right) + r^s G^{-1} G^*(d_1 r^s) \leq k. \tag{4.1}$$

For all $n \in \mathbb{N}_0$, set

$$\begin{cases} k_n = \dfrac{k}{2} + \dfrac{k}{2^{n+1}}, \\ r_n = \dfrac{r}{2} + \dfrac{r}{2^{n+1}}, \ \tilde{r}_n = \dfrac{r_n + r_{n+1}}{2}, \\ \hat{r}_n = \dfrac{3r_n + r_{n+1}}{4}, \ \overline{r}_n = \dfrac{r_n + 3r_{n+1}}{4}, \\ B_n = B_{r_n}, \ \tilde{B}_n = B_{\tilde{r}_n}, \ \hat{B}_n = B_{\hat{r}_n}, \ \overline{B}_n = B_{\overline{r}_n}. \end{cases}$$

Observe that $B_{n+1} \subset \overline{B}_n \subset \tilde{B}_n \subset \hat{B}_n \subset B_n$. Now we take a cutoff function ϕ in B_n, vanishing outside $\hat{B}_n$, and equal to the identity in $\tilde{B}_n$, such that

$$|D\phi| \leq \frac{2^n}{r}.$$

Selecting $k = k_n$, $B_r = \tilde{B}_n$ and $B_R = B_n$, we can make use of the energy estimate of Proposition 3.1. As a result, we have

$$\begin{aligned} &\int_{B_{n+1}} \int_{B_{n+1}} G\left(\frac{|\omega_-(x) - \omega_-(y)|}{|x-y|^s}\right) \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^N} \\ &\leq \gamma_* \Big[G^*(d_1 r^s) + G\Big(\frac{k_n}{r^s}\Big)\Big] |B_n \cap \{u < k_n\}| + \gamma_* \frac{r_n^q}{(r_n - \tilde{r}_n)^q} \int_{B_n} G\Big(\frac{\omega_-(x)}{r_n^s}\Big) \mathrm{d}x \\ &\quad + \gamma_* \frac{r_n^{N+sq}}{(r_n - \tilde{r}_n)^{N+sq}} \|\omega_-\|_{L^1(B_n)} \operatorname{Tail}(\omega_-; B_n), \end{aligned} \tag{4.2}$$

where $\omega_- = (u - k_n)_-$. Firstly, we shall focus on estimating the first term on the right-hand side of (4.2). Recalling (4.1) that $r^s G^{-1} G^*(d_1 r^s) \leq k$, we arrive that

$$\left[G^*(d_1 r^s) + G\left(\frac{k_n}{r^s}\right)\right] |B_n \cap \{u < k_n\}| \leq \gamma(p,q) G\left(\frac{k_n}{r^s}\right) |B_n \cap \{u < k_n\}|. \tag{4.3}$$

From the definitions of r_n and $\tilde{r}_n$, there holds

$$\frac{r_n}{r_n - \tilde{r}_n} = 2^{n+2} + 4.$$

Consequently, we estimate the second term as

$$\begin{aligned}\frac{r_n^q}{(r_n - \tilde{r}_n)^q} \int_{B_n} G\Big(\frac{\omega_-(x)}{r_n^s}\Big)\,\mathrm{d}x &\le \gamma(p,q)2^{nq} \int_{B_n} G\left(\frac{\omega_-(x)}{r^s}\right)\mathrm{d}x \\ &\le \gamma(p,q)2^{nq} G\left(\frac{k_n}{r^s}\right)|B_n \cap \{u < k_n\}|. \end{aligned} \tag{4.4}$$

The last term in (4.2) shall be estimated as follows. An application of (2.5) and (1.3) gives that

$$g\left(\frac{k_n + u_-}{|x_0 - y|^s}\right) \le \frac{q2^{q-1}}{p}\left[g\left(\frac{k_n}{|x_0 - y|^s}\right) + g\left(\frac{u_-(y)}{|x_0 - y|^s}\right)\right].$$

We further get

$$\begin{aligned}&\frac{r_n^{N+sq}}{(r_n - \tilde{r}_n)^{N+sq}}\|\omega_-\|_{L^1(B_n)}\,\mathrm{Tail}(\omega_-; B_n) \\ &\le \gamma(p,q)2^{n(N+sq)}k_n|B_n \cap \{u < k_n\}| \int_{\mathbb{R}^N\setminus B_n} g\left(\frac{k_n + u_-(y)}{|x_0 - y|^s}\right)\frac{\mathrm{d}y}{|x_0 - y|^{N+s}} \\ &\le \gamma(p,q)2^{n(N+sq)}k_n|B_n \cap \{u < k_n\}| \int_{\mathbb{R}^N\setminus B_n} \left[g\left(\frac{k_n}{r_n^s}\right) + g\left(\frac{u_-(y)}{|x_0 - y|^s}\right)\right]\frac{\mathrm{d}y}{|x_0 - y|^{N+s}} \\ &\le \gamma(p,q)2^{n(N+sq)}k_n|B_n \cap \{u < k_n\}| \left[r^{-s}g\left(\frac{k_n}{r^s}\right) + \mathrm{Tail}(u_-; B_R)\right].\end{aligned}$$

By means of (1.3) and $r^s g^{-1}(r^s\,\mathrm{Tail}(u_-; B_R)) \le k$ in (4.1), we arrive at

$$\frac{r_n^{N+sq}}{(r_n - \tilde{r}_n)^{N+sq}}\|\omega_-\|_{L^1(B_n)}\,\mathrm{Tail}(\omega_-; B_n) \le \gamma 2^{n(N+sq)}k_n\,|B_n \cap \{u < k_n\}|\, r^{-s}g\Big(\frac{k_n}{r^s}\Big). \tag{4.5}$$

Combining (4.2)–(4.5), we can get

$$\int_{B_{n+1}}\int_{B_{n+1}} G\left(\frac{|\omega_-(x) - \omega_-(y)|}{|x - y|^s}\right)\frac{\mathrm{d}x\,\mathrm{d}y}{|x - y|^N} \le \gamma 2^{n(N+sq+q)}G\left(\frac{k_n}{r^s}\right)|B_n \cap \{u < k_n\}|.$$

According to Lemma 4.1 in [7], there exists a constant $\theta > 1$ depending only on N, s, such that

$$\begin{aligned}&\left[\fint_{B_{n+1}} G^\theta\left(\frac{|\omega_- - (\omega_-)_{B_{n+1}}|}{r_{n+1}^s}\right)\mathrm{d}x\right]^{\frac{1}{\theta}} \\ &\le \gamma \fint_{B_{n+1}}\int_{B_{n+1}} G\left(\frac{|\omega_-(x) - \omega_-(y)|}{|x - y|^s}\right)\frac{\mathrm{d}x\,\mathrm{d}y}{|x - y|^N} \\ &\le \gamma 2^{n(N+sq+q)}G\left(\frac{k_n}{r^s}\right)\frac{|B_n \cap \{u < k_n\}|}{|B_n|}.\end{aligned}$$

We further employ the Jensen inequality to get the following display

$$\left[\fint_{B_{n+1}} G^{\theta}\left(\frac{\omega_-}{r_{n+1}^s}\right)\mathrm{d}x\right]^{\frac{1}{\theta}} \le \gamma\fint_{B_{n+1}} G\left(\frac{\omega_-}{r_{n+1}^s}\right)\mathrm{d}x + \gamma\left[\fint_{B_{n+1}} G^{\theta}\left(\frac{|\omega_- - (\omega_-)_{B_{n+1}}|}{r_{n+1}^s}\right)\mathrm{d}x\right]^{\frac{1}{\theta}}.$$

Recall the definition of r_{n+1} with $\frac{r}{2} \le r_{n+1} \le r$, then

$$G^{\theta}\left(\frac{\omega_-}{r_{n+1}^s}\right) \ge G^{\theta}\left(\frac{k_n - k_{n+1}}{r_{n+1}^s}\right)\chi_{\{u<k_{n+1}\}} \ge \gamma' 2^{-nq\theta} G^{\theta}\left(\frac{k_n}{r^s}\right)\chi_{\{u<k_{n+1}\}}.$$

It can be deduced that

$$\gamma 2^{-nq} G\left(\frac{k_n}{r^s}\right)\left(\fint_{B_{n+1}} \chi_{\{u<k_{n+1}\}}\,\mathrm{d}x\right)^{\frac{1}{\theta}} \le \gamma 2^{n(N+sq+q)} G\left(\frac{k_n}{r^s}\right)\frac{|B_n \cap \{u < k_n\}|}{|B_n|},$$

which means

$$\left(\frac{|B_{n+1} \cap \{u < k_{n+1}\}|}{|B_{n+1}|}\right)^{\frac{1}{\theta}} \le \gamma 2^{n(N+sq+2q)}\frac{|B_n \cap \{u < k_n\}|}{|B_n|}.$$

Denote

$$A_n := \frac{|B_n \cap \{u < k_n\}|}{B_n}.$$

Then we have

$$A_{n+1} \le \gamma 2^{n(N+sq+2q)\theta} A_n^{\theta}.$$

According to [16, Lemma 7.1], we have to make sure that

$$A_0 = \frac{|B_r \cap \{u < k\}|}{|B_r|} \le \gamma^{\frac{-1}{\theta-1}} 2^{-(N+sq+2q)\frac{\theta}{(\theta-1)^2}}$$

to obtain $A_n \to 0$ as $n \to \infty$. Notice that

$$\gamma^{\frac{-1}{\theta-1}} 2^{-(N+sq+2q)\frac{\theta}{(\theta-1)^2}} < 1$$

and γ depending only on $N, s, p, q, d_2, \lambda, \Lambda$, so there exists a constant $\nu \in (0,1)$ depending on $N, s, p, q, d_2, \lambda, \Lambda$ to justify the desired result

$$\lim_{n\to\infty} A_n = 0.$$

In other words, we draw a conclusion

$$u \ge \frac{k}{2} \quad \text{in } B_{\frac{r}{2}}.$$

We now complete the proof. □

5. Proof of main results

The following two lemmas apply the energy estimate to obtain two different measure density inequalities, which play a key role in proving our main results, Theorems 1.1 and 1.2. The first lemma shows that a measure shrinking inequality depends on a local integral associated with g.

Lemma 5.1. *Let $k \geq 0$ and $R \leq 1$. Let $u \in \mathbb{W}^{s,G}(\Omega)$ be a non-negative supersolution of* (1.1) *in $B_R(x_0) \subset \Omega$. There exists a constant $\gamma_1 > 1$ depending only on $s, p, q, N, d_2, \lambda, \Lambda$, such that either*

$$r^s g^{-1}\left(r^s \operatorname{Tail}(u_-; B_R(x_0))\right) + r^s G^{-1} G^*(d_1 r^s) > k,$$

or

$$|\{u \leq k\} \cap B_r(x_0)| \leq \frac{\gamma_1\, g\left(\frac{k}{r^s}\right)}{\fint_{B_r} g(\frac{u(x)}{r^s})\,\mathrm{d}x} |B_r|,$$

where $B_{2r}(x_0) \subset B_R(x_0)$.

Proof. Without loss of generality, we suppose $x_0 = 0$, and

$$r^s g^{-1}(r^s \operatorname{Tail}(u_-; B_R)) + r^s G^{-1} G^*(d_1 r^s) \leq k.$$

Apply the energy estimate Proposition 3.1 in $B_r \subset B_{2r}$ to get

$$\begin{aligned}
&\int_{B_r} \omega_-(x) \int_{\mathbb{R}^N} g\left(\frac{\omega_+(y)}{|x-y|^s}\right) \frac{1}{|x-y|^{N+s}}\,\mathrm{d}y\,\mathrm{d}x \\
&\leq \gamma_* \int_{B_{2r}} G\left(\frac{\omega_-(x)}{r^s}\right)\mathrm{d}x + \gamma_* \|\omega_-\|_{L^1(B_{2r})} \operatorname{Tail}(\omega_-; B_{2r}) \\
&\quad + \gamma_* \left[G^*(d_1 r^s) + G\left(\frac{k}{r^s}\right)\right] |A_-(k, 2r)| \\
&\leq \gamma G\left(\frac{k}{r^s}\right)|B_r| + \gamma k |B_r| \operatorname{Tail}(\omega_-; B_{2r}) + \gamma G^*(d_1 r^s)|B_r|.
\end{aligned} \tag{5.1}$$

According to the assumption $r^s g^{-1}(r^s \operatorname{Tail}(u_-; B_R)) \leq k$, we can estimate the tail term as follows

$$\begin{aligned}
\operatorname{Tail}(\omega_-; B_{2r}) &= \int_{\mathbb{R}^N \setminus B_{2r}} g\left(\frac{\omega_-(x)}{|x|^s}\right) \frac{1}{|x|^{N+s}}\,\mathrm{d}x \\
&= \int_{B_R \setminus B_{2r}} g\left(\frac{\omega_-(x)}{|x|^s}\right) \frac{1}{|x|^{N+s}}\,\mathrm{d}x + \int_{\mathbb{R}^N \setminus B_R} g\left(\frac{\omega_-(x)}{|x|^s}\right) \frac{1}{|x|^{N+s}}\,\mathrm{d}x \\
&\leq \gamma r^{-s} g\left(\frac{k}{r^s}\right),
\end{aligned} \tag{5.2}$$

where in the last inequality we note that, by (2.1),

$$g\left(\frac{\omega_-(x)}{|x|^s}\right) \leq \frac{q}{p} 2^{(1-p)s} g\left(\frac{k}{r^s}\right) \quad \text{for all } x \in B_{2r} \subset B_R.$$

Due to (1.3), (5.2), and the assumption that $r^sG^{-1}G^*(d_1r^s) \le k$, (5.1) turns into

$$\int_{B_r} \omega_-(x) \int_{\mathbb{R}^N} g\left(\frac{\omega_+(y)}{|x-y|^s}\right) \frac{1}{|x-y|^{N+s}}\, \mathrm{d}y\, \mathrm{d}x \le \gamma \frac{k}{r^s} g\left(\frac{k}{r^s}\right)|B_r|. \tag{5.3}$$

In the following, we will focus on the left part of (5.3). From (1.3) and (2.5), we know

$$g\left(\frac{u_+(y)}{|x-y|^s}\right) \le g\left(\frac{(u(y)-k)_+ + k}{|x-y|^s}\right) \le \gamma(p,q)\left[g\left(\frac{\omega_+(y)}{|x-y|^s}\right) + g\left(\frac{k}{|x-y|^s}\right)\right].$$

Based on the above inequality, we can see

$$\begin{aligned}
&\int_{B_r} \omega_-(x) \int_{\mathbb{R}^N} g\left(\frac{\omega_+(y)}{|x-y|^s}\right) \frac{1}{|x-y|^{N+s}}\, \mathrm{d}y\, \mathrm{d}x \\
\ge &\int_{B_r}\int_{B_r} \omega_-(x) g\left(\frac{\omega_+(y)}{|x-y|^s}\right) \frac{1}{|x-y|^{N+s}}\, \mathrm{d}y\, \mathrm{d}x \\
\ge &\, C\int_{B_r}\int_{B_r} \frac{\omega_-(x)}{|x-y|^{N+s}}\left[g\left(\frac{u(y)}{|x-y|^s}\right) - g\left(\frac{k}{|x-y|^s}\right)\right] \mathrm{d}y\, \mathrm{d}x \\
\ge &\, Cr^{-(N+s)}\int_{B_r}\int_{B_r} \omega_-(x) g\left(\frac{u(y)}{r^s}\right) \mathrm{d}y\, \mathrm{d}x - \gamma\frac{k}{r^s} g\left(\frac{k}{r^s}\right)|B_r|.
\end{aligned}$$

Combine this with (5.3) to reach

$$\int_{B_r} \omega_-(x) ⨍_{B_r} g\left(\frac{u(y)}{r^s}\right) \mathrm{d}y\, \mathrm{d}x \le \gamma k g\left(\frac{k}{r^s}\right)|B_r|.$$

Noticing that

$$\int_{B_r} \omega_-(x)\, \mathrm{d}x \ge \frac{1}{2}k|\{u < \frac{1}{2}k\} \cap B_r|,$$

we arrive that

$$|\{u < \frac{1}{2}k\} \cap B_r| \le \frac{\gamma g\left(\frac{k}{r^s}\right)}{⨍_{B_r} g\left(\frac{u(x)}{r^s}\right) \mathrm{d}x}|B_r|.$$

In this way, we complete our proof. □

The next lemma measures the shrinking set with a nonlocal integral.

Lemma 5.2. *Let* $k \ge 0$ *and* $0 < R \le 1$. *Let* $u \in \mathbb{W}^{s,G}(\Omega)$ *be a non-negative supersolution of* (1.1) *in* $B_R(x_0) \subset \Omega$. *There exists a constant* $\gamma_2 > 1$ *depending only on* $s, p, q, N, d_2, \lambda, \Lambda$, *such that either*

$$r^s g^{-1}\left(r^s \operatorname{Tail}(u_-; B_R(x_0))\right) + r^sG^{-1}G^*(d_1r^s) > k,$$

or

$$|\{u \le k\} \cap B_r(x_0)| \le \frac{\gamma_2\, g\left(\frac{k}{r^s}\right)}{r^s \operatorname{Tail}(u_+; B_r(x_0))}|B_r|,$$

where $B_{2r}(x_0) \subset B_R(x_0)$.

Proof. Without sacrificing generality, we also assume that $x_0 = 0$, and

$$r^s g^{-1}(r^s \operatorname{Tail}(u_-; B_R)) + r^s G^{-1}G^*(d_1 r^s) \le k.$$

Using the similar proceed as in Lemma 5.1, we can get

$$\int_{B_r} \omega_-(x) \int_{\mathbb{R}^N} g\left(\frac{\omega_+(y)}{|x-y|^s}\right) \frac{1}{|x-y|^{N+s}} \,\mathrm{d}y\,\mathrm{d}x \le \gamma \frac{k}{r^s} g\left(\frac{k}{r^s}\right)|B_r|, \tag{5.4}$$

and

$$g\left(\frac{u_+(y)}{|x-y|^s}\right) \le g\left(\frac{(u(y)-k)_+ + k}{|x-y|^s}\right) \le C\left[g\left(\frac{\omega_+(y)}{|x-y|^s}\right) + g\left(\frac{k}{|x-y|^s}\right)\right].$$

Then, we know

$$\begin{aligned}
&\int_{B_r} \omega_-(x) \int_{\mathbb{R}^N} g\left(\frac{\omega_+(y)}{|x-y|^s}\right) \frac{1}{|x-y|^{N+s}} \,\mathrm{d}y\,\mathrm{d}x \\
\ge &\int_{B_r}\int_{\mathbb{R}^N\setminus B_r} \omega_-(x) g\left(\frac{\omega_+(y)}{|x-y|^s}\right) \frac{1}{|x-y|^{N+s}} \,\mathrm{d}y\,\mathrm{d}x \\
\ge C &\int_{B_r}\int_{\mathbb{R}^N\setminus B_r} \frac{\omega_-(x)}{(2|y|)^{N+s}}\left[g\left(\frac{u_+(y)}{(2|y|)^s}\right) - g\left(\frac{k}{(2|y|)^s}\right)\right]\mathrm{d}y\,\mathrm{d}x \\
\ge C &\int_{B_r}\int_{\mathbb{R}^N\setminus B_r} \frac{\omega_-(x)}{|y|^{N+s}} g\left(\frac{u_+(y)}{|y|^s}\right)\mathrm{d}y\,\mathrm{d}x - \gamma\frac{k}{r^s} g\left(\frac{k}{r^s}\right)|B_r|.
\end{aligned}$$

Combine the above inequality and (5.4) to get

$$\int_{B_r}\int_{\mathbb{R}^N\setminus B_r} \frac{\omega_-(x)}{|y|^{N+s}} g\left(\frac{u_+(y)}{|y|^s}\right)\mathrm{d}y\,\mathrm{d}x \le \gamma\frac{k}{r^s} g\left(\frac{k}{r^s}\right)|B_r|.$$

Besides, we can obtain the following result by the method of Lemma 5.2:

$$\left|\left\{u < \frac{1}{2}k\right\} \cap B_r\right| \le \frac{\gamma g\left(\frac{k}{r^s}\right)}{r^s \operatorname{Tail}(u_+; B_r)}|B_r|.$$

Then the proof is complete. □

Finally, we present the proof of Theorems 1.1 and 1.2.

Proof of Theorems 1.1 and 1.2. For simplicity, we choose to omit the symbol x_0. From Lemma 4.1, we know there exists a constant $\nu \in (0,1)$ depending only on $s, N, p, q, \lambda, \Lambda$, which means we shall choose k_1 and k_2 to satisfy the following conditions

$$\frac{\gamma_1 g\left(\frac{k_1}{r^s}\right)}{\fint_{B_r} g\left(\frac{u(x)}{r^s}\right)\mathrm{d}x} \le \nu \qquad \text{and} \qquad \frac{\gamma_2 g\left(\frac{k_2}{r^s}\right)}{r^s \operatorname{Tail}(u_+; B_r)} \le \nu,$$

where γ_1 and γ_2 are determined by Lemmas 5.1 and 5.2. Utilizing the inequality (2.1), we can choose

$$k_1 = r^s g^{-1}\left[\frac{\nu}{\gamma_1}\fint_{B_r} g\left(\frac{u(x)}{r^s}\right)\mathrm{d}x\right] \ge \eta_1 r^s g^{-1}\left[\fint_{B_r} g\left(\frac{u(x)}{r^s}\right)\right],$$

and

$$k_2 = r^s g^{-1}\left(\frac{\nu}{\gamma_2} r^s \operatorname{Tail}(u_+; B_r)\right) \geq \eta_2 r^s g^{-1}\left(r^s \operatorname{Tail}(u_+; B_r)\right),$$

where η_1 and η_2 can be described respectively by

$$\eta_1 = p^{1/(q-1)}/q^{1/(p-1)}\left(\frac{\nu}{\gamma_1}\right)^{\frac{1}{p-1}} \qquad \text{and} \qquad \eta_2 = p^{1/(q-1)}/q^{1/(p-1)}\left(\frac{\nu}{\gamma_2}\right)^{\frac{1}{p-1}}.$$

It comes to us from Lemma 4.1 that for $i = 1, 2$, either

$$r^s g^{-1}(r^s \operatorname{Tail}(u_-; B_R)) + r^s G^{-1} G^*(d_1 r^s) > k_i,$$

or

$$u \geq \frac{1}{2} k_i.$$

Therefore, the proof is finished. □

6. Conclusions

This paper establishes two weak Harnack inequalities for weak supersolutions to nonhomogeneous nonlocal equations with general growth. Using the expansion of positivity and a refined energy estimate, our results incorporate nonlocal tail terms for both the positive and negative parts of solutions on both sides of the inequalities addressing strong nonlocality more comprehensively. These findings advance the regularity theory of such nonlocal equations, and the preliminary energy and density lemmas offer foundational tools for future related research.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest. We also declare that this manuscript has no associated data.

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