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Research article

Nonlocal Harnack inequality in a disconnected region†

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Abstract: We establish a Harnack inequality for weak solutions of nonlocal nonlinear equations in a disconnected region. The inequality compares the value of a solution on one connected component with that on another, capturing a purely nonlocal phenomenon with no local analogue. We provide three different approaches based on a new weak Harnack inequality, the Poisson formula and the localized maximum principle.

Keywords: Harnack inequality; nonlocal equation; disconnected region; Poisson kernel; maximum principle

1. Introduction

The classical Harnack inequality for harmonic functions can be formulated as follows.

Theorem 1.1. *Let $u : B_{2r}(x_0) \subset \mathbb{R}^n \to \mathbb{R}$ be a nonnegative harmonic function. Then there exists a constant $C = C(n)$ such that*

$$\sup_{B_r(x_0)} u \le C \inf_{B_r(x_0)} u.$$

Although the proof of this inequality for harmonic functions is quite immediate from the Poisson integral formula, it implies significant consequences in the regularity theory; Liouville theorem, removable singularity theorem and Hölder regularity of harmonic functions, just to name a few. Moreover, the Harnack inequality remains valid for solutions of a wide class of partial differential equations, not only linear equations with measurable uniformly elliptic coefficients, but also quasilinear equations involving p-Laplacian for $p > 1$. For divergence form operators, the De Giorgi–Nash–Moser theory [16, 42–44] provides a very flexible tool for studying the local behaviors of solutions such as local boundedness, weak Harnack inequality and Harnack inequality. In short, this theory is based on the iteration of Caccioppoli inequalities derived from choosing suitable test functions. We also refer to Serrin [45] and Trudinger [46] for quasilinear equations involving p-Laplace equations. On the other hand, Krylov–Safonov [36, 37] developed a priori estimates (the so-called expansion of positivity) for both elliptic and parabolic operators in nondivergence form; see also the paper by Caffarelli [10] for a similar result in the fully nonlinear setting. We finally refer to the monograph by Kassmann [23] for further introduction and historical background of the Harnack inequality.

We now move our attention to the nonlocal Harnack inequality. For the fractional Laplacian operator $(-\Delta)^s$, the nonlocal counterpart of Theorem 1.1 can be found in the book by Landkof [38] under the assumption that u is nonnegative in the whole Euclidean space $\mathbb{R}^n$. The necessity of the global nonnegative condition on u was pointed out by Kassmann [22]. In fact, in order to replace the global nonnegative condition with the nonnegative assumption only in a ball $B_{2r}(x_0)$, one needs to introduce the nonlocal tail term $\mathrm{Tail}(u_-; x_0, r)$ that encodes the long-range interaction of u; see Theorem 2.3 for the precise statement. The nonlocal De Giorgi–Nash–Moser theory has been extensively studied by many authors to establish the nonlocal Harnack inequality and the local Hölder regularity. We refer to Bass–Kassmann [1, 2] and Kassmann [24, 25] for linear nonlocal operators in divergence form, Di Castro–Kuusi–Palatucci [17, 18] and Cozzi [15] for nonlinear nonlocal operators in divergence form, and Caffarelli–Silvestre [8, 9] for fully nonlinear nonlocal operators in nondivergence form. See also [7, 11, 21, 27, 28, 40] for Harnack inequality and related regularity results in different settings.

The goal of this paper is to establish the nonlocal Harnack inequality in a *disconnected region*, which becomes possible due to the nonlocal nature of equations. To illustrate the issue, we first consider the corresponding situation in the local case. Let u be harmonic in $B_{2r}(x_1) \cup B_{2r}(x_2)$, where $B_{2r}(x_1)$ and $B_{2r}(x_2)$ are two disjoint balls in $\mathbb{R}^n$. Moreover, suppose that u is nonnegative in a large ball containing both $B_{2r}(x_1)$ and $B_{2r}(x_2)$. The classical Harnack inequality (Theorem 1.1) says that the ratio $u(y_1)/u(y_2)$ is bounded from above and below by positive universal constants, whenever two points y_1 and y_2 are contained in the same ball (either $B_r(x_1)$ or $B_r(x_2)$). However, this cannot happen when one chooses $y_1 \in B_r(x_1)$ and $y_2 \in B_r(x_2)$; just consider $u \equiv M$ in $B_{2r}(x_1)$ and $u \equiv 1$ in $B_{2r}(x_2)$ for any $M \geq 0$. In other words, the information on one connected component cannot be transported to another in the local setting.

Surprisingly, the ratio between two values of a solution u in disjoint balls admits a universal bound when it comes to the nonlocal equation. To be precise, let u be a weak solution of nonlocal nonlinear equations in divergence form given by

$$0 = \mathcal{L}u(x) := 2\,\mathrm{p.v.} \int_{\mathbb{R}^n} |u(x) - u(y)|^{p-2}(u(x) - u(y))k(x, y)\,\mathrm{d}y, \tag{1.1}$$

where the kernel $k : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty]$ satisfies a symmetry and uniform ellipticity condition of the form

$$\Lambda^{-1}|x - y|^{-n-sp} \leq k(x, y) = k(y, x) \leq \Lambda|x - y|^{-n-sp}$$

for some $s \in (0, 1)$, $p > 1$ and $\Lambda \geq 1$. In particular, the choice $k(x, y) = |x-y|^{-n-sp}$ (up to a normalization constant) corresponds to the fractional p-Laplacian operator $(-\Delta_p)^s$. Moreover, we fix notation on a disconnected region consisting of two disjoint balls throughout the paper. Let $x_1, x_2 \in \mathbb{R}^n$ be points such that $4r \leq |x_1 - x_2| \leq 8r$. Suppose that $B_{2r}(x_1) \cup B_{2r}(x_2) \subset B_{R/2} = B_{R/2}(0)$. In particular, these assumptions imply that $B_{2r}(x_1) \cap B_{2r}(x_2) = \varnothing$ and that $|x_1|, |x_2| \leq R/2$.

The following Harnack inequality in a disconnected region is the main theorem of this paper. See Section 2 for the definition of the nonlocal tail, $\mathrm{Tail}(\cdot)$.

Theorem 1.2 (Harnack inequality in a disconnected region)**.** *Let u be a weak solution of* (1.1) *in* $B_{2r}(x_1) \cup B_{2r}(x_2)$ *and u is nonnegative in B_R. Then there exists a constant $C = C(n, s, p, \Lambda) > 0$ such that*

$$\sup_{B_r(x_2)} u \le C \inf_{B_r(x_1)} u + C \left(\frac{r}{R}\right)^{\frac{sp}{p-1}} \mathrm{Tail}(u_-; 0, R),$$

where $u_- := \max\{-u, 0\}$ denotes the negative part of u.

As we observed in the earlier example under the local situation, Theorem 1.2 reveals a purely nonlocal phenomenon: The nonlocal nature of operators ensures the transfer of information despite the region being disconnected. In particular, we show that Theorem 1.2 is non-robust in the sense that the universal constant C in Theorem 1.2 blows up when $s \to 1^-$; see Remark 4.1 and Remark 5.5 for details. Moreover, it would be interesting to develop a parabolic counterpart of Theorem 1.2; we leave this for future work.

In Section 3, we provide the proof of Theorem 1.2 for general nonlocal nonlinear operators $\mathcal{L}$. In fact, this proof was suggested by Marvin Weidner after an earlier version of the manuscript had been completed. We apply a new weak Harnack inequality that was recently developed by Kassmann–Weidner [27, Theorem 1.9] for the linear case and Liao [39, Theorem 7.2] for the nonlinear case; see Theorem 3.1 for the precise statement. Theorem 3.1 says that the tail can be bounded from above in terms of the (interior) infimum of a supersolution u. As mentioned in [39], this estimate measures the positivity contribution from the long-range behavior of u.

In Sections 4 and 5, we present two alternative approaches for proving Theorem 1.2 under either of the following hypotheses:

(i) $\mathcal{L} = (-\Delta)^s$ (or generally, $p = 2$ and $s \in (0, 1)$ with $2s < n$); or

(ii) $p = 2$, $s \in (0, 1)$ and k is translation invariant, i.e., there exists a function $K : \mathbb{R}^n \to [0, \infty]$ such that
$$k(x, y) = K(x - y) \quad \text{for any } x, y \in \mathbb{R}^n.$$

Although these methods require additional assumptions on $\mathcal{L}$, they have the advantage of being relatively elementary and are of independent interest.

In the first approach, we employ a representation formula for a solution u of the Dirichlet problem. In particular, the solution u can be represented as the Poisson integral of its exterior data with the associated Poisson kernel P. Then we can capture the long-range interaction of a solution u between the two disjoint balls. While the fractional Laplacian case $(-\Delta)^s$ can be rather easily treated due to the explicit formula for the Poisson kernel, an additional effort is necessary to obtain the Poisson kernel estimates for general operators $\mathcal{L}$; see Remark 4.2 for details. We also refer to [6, 12–14, 20, 33] for the Green function and the Poisson kernel estimates in different settings.

The second approach relies on the construction of barriers and the application of the localized maximum principle. We suggest two simple barrier functions w_1 and w_2 in Lemmas 5.1 and 5.2 so that a certain linear combination of w_1 and w_2 becomes a weak subsolution of (1.1) with an appropriate profile. The translation invariant condition in (ii) is necessary only in Lemma 5.2, where the finiteness of a certain integral term follows from the control of $\delta(w_2; x, y) \coloneqq 2w_2(x) - w_2(x+y) - w_2(x-y)$; see also [8]. At this stage, the standard comparison principle (Korvenpää–Kuusi–Palatucci [35, Lemma 6] for instance) between u and barrier functions is not directly applicable, because we do not impose the nonnegativity of u in the exterior domain $\mathbb{R}^n \setminus B_R$. Instead, we utilize the localized maximum principle (Lemma 5.4) that is similar to the one by Kim–Lee [31, Theorem 1.7].

Furthermore, we develop the weak Harnack inequality for weak supersolutions u of (1.1). In contrast to Theorem 1.2, Theorem 1.3 below requires u to be a weak supersolution only in $B_{2r}(x_1)$ and imposes no condition in $B_{2r}(x_2)$. We also remark that, in Sections 3 and 4, we first establish Theorem 1.3 and then deduce Theorem 1.2 with the aid of the Harnack inequality in a ball (Theorem 2.3).

Theorem 1.3 (Weak Harnack inequality in a disconnected region)**.** *Let u be a weak supersolution of* (1.1) *in $B_{2r}(x_1)$ and u is nonnegative in B_R. Then there exists a constant $C = C(n, s, p, \Lambda) > 0$ such that*

$$\left(\fint_{B_r(x_2)} u^{p-1}(z)\,\mathrm{d}z \right)^{\frac{1}{p-1}} \leq C \inf_{B_r(x_1)} u + C \left(\frac{r}{R} \right)^{\frac{sp}{p-1}} \mathrm{Tail}(u_-; 0, R). \tag{1.2}$$

Let us end with an immediate corollary of Theorem 1.2: The Harnack inequality in an annulus, i.e., two disjoint intervals, when $n = 1$. Note that for $n \geq 2$, one can iterate the Harnack inequality in a ball (Theorem 2.3), together with the standard covering argument, to obtain the Harnack inequality in an annulus $B_R \setminus \{0\}$; see Theorem 2.4. Nevertheless, the covering argument does not work in the special case $n = 1$, since $(-R, R) \setminus \{0\}$ is no longer connected.

Corollary 1.4 (Harnack inequality in disjoint intervals)**.** *Let $n = 1$ and let u be a weak solution of* (1.1) *in $(-R, R) \setminus \{0\}$ such that $u \geq 0$ in $(-R, R)$. Then for any $r \leq R/8$, we have*

$$\sup_{(-3r,-r)} u \leq C \inf_{(r,3r)} u + C \left(\frac{r}{R} \right)^{\frac{sp}{p-1}} \mathrm{Tail}(u_-; 0, R),$$

where $C = C(s, p, \Lambda) > 0$.

Corollary 1.4 can be applied to extend the Liouville and Bôcher theorems presented in [31] for $n = 1$. Furthermore, it can be used to simplify the analysis of singular solutions developed in [30]: That work required a separate treatment of the one-dimensional case, which becomes unnecessary in view of Corollary 1.4.

The paper is organized as follows. In Section 2, we collect several definitions and well-known results concerning function spaces and weak solutions of nonlocal equations. Sections 3–5 are devoted to the three different proofs of Theorem 1.2 via a new weak Harnack inequality, the representation formula with the Poisson kernel and the localized maximum principle with barriers, respectively.

2. Preliminaries

We summarize several definitions of function spaces and weak solutions of nonlocal equations, and recall the standard Harnack inequality, which will be used throughout the paper.

For Ω being a bounded $C^{1,1}$ domain in $\mathbb{R}^n$, the *fractional Sobolev space* $W^{s,p}(\Omega)$ consists of measurable functions $u : \Omega \to [-\infty, \infty]$ whose fractional Sobolev norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\|u\|^p_{L^p(\Omega)} + [u]^p_{W^{s,p}(\Omega)}\right)^{1/p} := \left(\int_\Omega |u(x)|^p \,\mathrm{d}x + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}y \,\mathrm{d}x\right)^{1/p}$$

is finite. By $W^{s,p}_{\mathrm{loc}}(\Omega)$ we denote the space of functions u such that $u \in W^{s,p}(G)$ for every open $G \Subset \Omega$. We refer the reader to Di Nezza–Palatucci–Valdinoci [19] for further properties of these spaces.

Since we are concerned with nonlocal equations, we also need the *tail space*

$$L^{p-1}_{sp}(\mathbb{R}^n) := \left\{u \text{ measurable} : \int_{\mathbb{R}^n} \frac{|u(y)|^{p-1}}{(1+|y|)^{n+sp}} \,\mathrm{d}y < \infty\right\};$$

see Kassmann [25] and Di Castro–Kuusi–Palatucci [18]. Note that $u \in L^{p-1}_{sp}(\mathbb{R}^n)$ if and only if the *nonlocal tail* (or *tail* for short)

$$\mathrm{Tail}(u; x_0, r) := \left(r^{sp} \int_{\mathbb{R}^n \setminus B_r(x_0)} \frac{|u(y)|^{p-1}}{|y - x_0|^{n+sp}} \,\mathrm{d}y\right)^{1/(p-1)} \tag{2.1}$$

is finite for any $x_0 \in \mathbb{R}^n$ and $r > 0$.

We next recall the definitions of a weak solution of (1.1) and of an $\mathcal{L}$-harmonic function. To this end, we define, for measurable functions $u, v : \mathbb{R}^n \to [-\infty, \infty]$, the bilinear form

$$\mathcal{E}(u, v) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))k(x, y) \,\mathrm{d}y \,\mathrm{d}x,$$

provided that it is finite.

Definition 2.1. A function $u \in W^{s,p}_{\mathrm{loc}}(\Omega) \cap L^{p-1}_{sp}(\mathbb{R}^n)$ is a *weak supersolution* (resp. *weak subsolution*) of (1.1) in Ω if

$$\mathcal{E}(u, \varphi) \geq 0 \ \text{(resp.} \leq 0) \tag{2.2}$$

for all nonnegative $\varphi \in C^\infty_c(\Omega)$. A function $u \in W^{s,p}_{\mathrm{loc}}(\Omega) \cap L^{p-1}_{sp}(\mathbb{R}^n)$ is a *weak solution* of (1.1) in Ω if (2.2) holds for all $\varphi \in C^\infty_c(\Omega)$. A function u is *$\mathcal{L}$-harmonic* in Ω if it is a weak solution of (1.1) in Ω and $u \in C(\Omega)$. In particular, if u is a continuous weak solution of $(-\Delta)^s u = 0$, then we say that u is *s-harmonic*.

It is useful to deal with a larger class of test functions than $C^\infty_c(\Omega)$ as follows; see also [31, Section 2] and [3, Section 2] for more comments on the test function classes.

Proposition 2.2 ([30, Proposition 2.6]). *A function $u \in W^{s,p}_{\mathrm{loc}}(\Omega) \cap L^{p-1}_{sp}(\mathbb{R}^n)$ is a weak supersolution (resp. weak subsolution) of* (1.1) *in Ω if and only if* (2.2) *holds for all nonnegative $\varphi \in W^{s,p}_{\mathrm{loc}}(\Omega)$ with* $\operatorname{supp} \varphi \Subset \Omega$.

We end this section with Harnack inequalities for weak solutions of (1.1) in a ball when $n \geq 1$ or in an annulus when $n \geq 2$. We again point out that both domains are clearly *connected* and that Theorem 2.4 follows from the iteration of Theorem 2.3 together with the standard covering argument.

Theorem 2.3 ([15, Theorem 6.9]). *Let $n \geq 1$. Let u be a weak solution of* (1.1) *in $B_R(x_0)$ such that $u \geq 0$ in $B_R(x_0)$. Then for any $r \leq R/2$,*

$$\sup_{B_r(x_0)} u \leq C \inf_{B_r(x_0)} u + C\mathrm{Tail}(u_-; x_0, r),$$

where $C = C(n, s, p, \Lambda) > 0$.

Theorem 2.4 ([30, Theorem 3.9]). *Let $n \geq 2$. Let u be a weak solution of* (1.1) *in $B_R \setminus \{0\}$ such that $u \geq 0$ in $B_R \setminus \{0\}$. Then for any $r \leq R/2$,*

$$\sup_{B_r \setminus B_{r/2}} u \leq C \inf_{B_r \setminus B_{r/2}} u + C\left(\frac{r}{R}\right)^{\frac{sp}{p-1}} \mathrm{Tail}(u_-; 0, R),$$

where $C = C(n, s, p, \Lambda) > 0$.

3. Nonlinear case: a new weak Harnack inequality

In this section, we present the proof of Theorem 1.2 for general nonlocal nonlinear operators $\mathcal{L}$. This approach relies on a new weak Harnack inequality that bounds the tail in terms of the infimum of a supersolution u. The following *tail-infimum estimate* was recently developed in both elliptic and parabolic settings by Kassmann–Weidner [27, Theorem 1.9] for $p = 2$ and Liao [39, Theorem 7.2] for $p > 1$.

Theorem 3.1 ([39, Theorem 7.2 and Remark 7.1]). *Let u be a weak supersolution of* (1.1) *in B_R such that $u \geq 0$ in B_R. Then for any $r \leq R/2$,*

$$\eta\mathrm{Tail}(u_+; x_0, r) \leq \inf_{B_r(x_0)} u + \left(\frac{r}{R}\right)^{\frac{sp}{p-1}} \mathrm{Tail}(u_-; x_0, R),$$

where $\eta = \eta(n, s, p, \Lambda) \in (0, 1)$.

Roughly speaking, this estimate serves as the nonlinear counterpart of the Poisson integral formula (4.1) in the linear case. We also remark that Theorem 3.1 indeed holds for more general nonlocal nonlinear equations with lower-order terms, since Liao established this estimate for the fractional De Giorgi class; see [15, 39] for details.

Proof of Theorem 1.3. We apply Theorem 3.1 for a weak supersolution u in $B_{2r}(x_1)$ to find that

$$\eta\mathrm{Tail}(u_+; x_1, r) \leq \inf_{B_r(x_1)} u + 2^{-\frac{sp}{p-1}}\mathrm{Tail}(u_-; x_1, 2r).$$

We estimate two tail terms as follows.

(i) For $y \in B_r(x_2)$, we have

$$|y - x_1| \leq |y - x_2| + |x_2 - x_1| \leq 9r$$

and so

$$\begin{aligned}\mathrm{Tail}^{p-1}(u_+; x_1, r) &= r^{sp} \int_{\mathbb{R}^n \setminus B_r(x_1)} \frac{u_+^{p-1}(y)}{|y - x_1|^{n+sp}}\,\mathrm{d}y \\ &\geq r^{sp} \int_{B_r(x_2)} \frac{u^{p-1}(y)}{|y - x_1|^{n+sp}}\,\mathrm{d}y \\ &\geq C \fint_{B_r(x_2)} u^{p-1}(y)\,\mathrm{d}y.\end{aligned}$$

(ii) For $y \in \mathbb{R}^n \setminus B_R$, we have

$$|y - x_1| \geq |y| - |x_1| \geq \frac{|y|}{2}$$

and so

$$\begin{aligned}
\mathrm{Tail}^{p-1}(u_-; x_1, 2r) &= (2r)^{sp} \int_{\mathbb{R}^n \setminus B_{2r}(x_1)} \frac{u_-^{p-1}(y)}{|y - x_1|^{n+sp}} \, \mathrm{d}y \\
&= (2r)^{sp} \int_{\mathbb{R}^n \setminus B_R} \frac{u_-^{p-1}(y)}{|y - x_1|^{n+sp}} \, \mathrm{d}y \\
&\leq C r^{sp} \int_{\mathbb{R}^n \setminus B_R} \frac{u_-^{p-1}(y)}{|y|^{n+sp}} \, \mathrm{d}y \\
&= C \left(\frac{r}{R}\right)^{sp} \mathrm{Tail}^{p-1}(u_-; 0, R).
\end{aligned}$$

The desired weak Harnack inequality follows from a combination of these estimates. □

Proof of Theorem 1.2. In view of Theorem 1.3, we focus on the estimate of the left-hand side of (1.2) by employing the standard Harnack inequality in a ball. In fact, we apply the Harnack inequality (Theorem 2.3) in a ball $B_{2r}(x_2)$ to find that

$$\sup_{B_r(x_2)} u \leq C \inf_{B_r(x_2)} u + C\mathrm{Tail}(u_-; x_2, r).$$

Here we observe that

$$\mathrm{Tail}(u_-; x_2, r) = \left(r^{sp} \int_{\mathbb{R}^n \setminus B_R} \frac{u_-^{p-1}(y)}{|y - x_2|^{n+sp}} \, \mathrm{d}y \right)^{1/(p-1)} \leq C \left(\frac{r}{R}\right)^{\frac{sp}{p-1}} \mathrm{Tail}(u_-; 0, R),$$

since

$$|y - x_2| \geq |y| - |x_2| \geq \frac{|y|}{2} \quad \text{for all } y \in \mathbb{R}^n \setminus B_R.$$

Therefore, we conclude that

$$\begin{aligned}
\sup_{B_r(x_2)} u &\leq C \inf_{B_r(x_2)} u + C \left(\frac{r}{R}\right)^{\frac{sp}{p-1}} \mathrm{Tail}(u_-; 0, R) \\
&\leq C \left(\fint_{B_r(x_2)} u^{p-1}(z) \, \mathrm{d}z \right)^{\frac{1}{p-1}} + C \left(\frac{r}{R}\right)^{\frac{sp}{p-1}} \mathrm{Tail}(u_-; 0, R).
\end{aligned} \tag{3.1}$$

The desired Harnack inequality follows from the combination of (1.2) and (3.1). □

Remark 3.2. If the aim is to prove Theorem 1.2 without obtaining Theorem 1.3, then one may utilize [17, Lemma 4.2] instead of Theorem 3.1. Although the tail-supremum estimate [17, Lemma 4.2] is weaker than Theorem 3.1, it suffices to prove Theorem 1.2.

4. Linear case 1: Poisson formula

In this section, we provide the first alternative proof, based on the Poisson formula, of Theorems 1.2 and 1.3 for the fractional Laplacian $\mathcal{L} = (-\Delta)^s$ and other linear nonlocal operators.

Given a bounded $C^{1,1}$ domain $\Omega \subset \mathbb{R}^n$ and exterior data $g \in L^1_{2s}(\mathbb{R}^n \setminus \Omega)$, a unique solution u of the Dirichlet problem

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \Omega, \\ u = g & \text{on } \mathbb{R}^n \setminus \Omega, \end{cases}$$

can be written by the representation formula

$$u(x) = \int_{\mathbb{R}^n \setminus \Omega} g(z) P(x,z) \, \mathrm{d}z \quad \text{for all } x \in \Omega, \tag{4.1}$$

where $P = P_{\mathcal{L},\Omega} : \Omega \times (\mathbb{R}^n \setminus \Omega) \to [0,\infty]$ denotes the *Poisson kernel* associated with $\mathcal{L}$ and Ω. We note that this representation formula is only available for the linear case $p = 2$.

In particular, if we choose $\mathcal{L} = (-\Delta)^s$ and $\Omega = B_r(0)$, then the corresponding Poisson kernel P can be written as

$$P(x,y) := c_{n,s} \frac{(r^2 - |x|^2)^s}{(|y|^2 - r^2)^s} \frac{1}{|y-x|^n}, \tag{4.2}$$

where $s \in (0,1)$ and $c_{n,s} = \Gamma(n/2)\pi^{-n/2-1} \sin(s\pi)$; see [4, 5] for details.

Let us now prove Theorem 1.3 for $\mathcal{L} = (-\Delta)^s$.

Proof of Theorem 1.3 for $\mathcal{L} = (-\Delta)^s$. We choose a point $\bar{x} \in \overline{B}_r(x_1)$ such that

$$u(\bar{x}) = \inf_{B_r(x_1)} u.$$

We let $P : B_{3r/2}(x_1) \times (\mathbb{R}^n \setminus B_{3r/2}(x_1)) \to [0,\infty]$ be the Poisson kernel associated with $(-\Delta)^s$ and $B_{3r/2}(x_1)$. If we let v be the unique solution of the Dirichlet problem

$$\begin{cases} (-\Delta)^s v = 0 & \text{in } B_{3r/2}(x_1), \\ v = u & \text{on } \mathbb{R}^n \setminus B_{3r/2}(x_1), \end{cases}$$

then the following representation formula holds:

$$v(x) = \int_{\mathbb{R}^n \setminus B_{3r/2}(x_1)} u(z) P(x,z) \, \mathrm{d}z \quad \text{for all } x \in B_{3r/2}(x_1).$$

On the other hand, the comparison principle between u and v in $B_{3r/2}(x_1)$ yields that $u \geq v$ in $B_{3r/2}(x_1)$; see [34, Lemma 6] for instance. In particular, the nonnegativity of u in B_R gives us that

$$\begin{aligned} u(\bar{x}) \geq v(\bar{x}) &= \int_{\mathbb{R}^n \setminus B_{3r/2}(x_1)} u(z) P(\bar{x},z) \, \mathrm{d}z \\ &\geq \int_{B_r(x_2)} u(z) P(\bar{x},z) \, \mathrm{d}z + \int_{\mathbb{R}^n \setminus B_R} u(z) P(\bar{x},z) \, \mathrm{d}z \\ &\geq C^{-1} \int_{B_r(x_2)} \frac{r^{2s}}{|\bar{x}-z|^{n+2s}} u(z) \, \mathrm{d}z - C \int_{\mathbb{R}^n \setminus B_R} \frac{r^{2s}}{|\bar{x}-z|^{n+2s}} u_-(z) \, \mathrm{d}z \\ &=: I_1 + I_2, \end{aligned}$$

where we used the formula (4.2)

$$C^{-1}\frac{r^{2s}}{|\bar{x}-z|^{n+2s}} \le P(\bar{x},z) \le C\frac{r^{2s}}{|\bar{x}-z|^{n+2s}}$$

for $\bar{x}\in B_r(x_1)$ and $z\in B_r(x_2)\cup(\mathbb{R}^n\setminus B_R)$.

We now estimate two integral terms I_1 and I_2 as follows:

(i) For $z\in B_r(x_2)$, we have

$$|z-\bar{x}| \le |z-x_2|+|x_2-x_1|+|x_1-\bar{x}| \le 10r$$

and so

$$I_1 \ge C \fint_{B_r(x_2)} u(z)\,\mathrm{d}z.$$

(ii) For $z\in\mathbb{R}^n\setminus B_R$, we have

$$|z-\bar{x}| \ge |z|-|\bar{x}| \ge \frac{|z|}{2}$$

and so

$$I_2 \ge -Cr^{2s}\int_{\mathbb{R}^n\setminus B_R}\frac{u_-(z)}{|z|^{n+2s}}\,\mathrm{d}z = -C\left(\frac{r}{R}\right)^{2s}\mathrm{Tail}(u_-;0,R).$$

By combining these estimates above, we conclude that

$$\fint_{B_r(x_2)} u(z)\,\mathrm{d}z \le C\inf_{B_r(x_1)} u + C\left(\frac{r}{R}\right)^{2s}\mathrm{Tail}(u_-;0,R).$$

□

Proof of Theorem 1.2 for $\mathcal{L}=(-\Delta)^s$. As in Section 3, we use the standard Harnack inequality (Theorem 2.3) in the ball $B_{2r}(x_2)$ to estimate the left-hand side of (1.2). □

Remark 4.1 (Non-robustness 1)**.** It is natural to ask whether the universal constant $C>0$ appearing in Theorem 1.2 is robust or not. More precisely, we would like to find the explicit dependence of C on the parameter $s\in(0,1)$ and send $s\to 1^-$. We expect that C must blow up when we let $s\to 1^-$, in view of the simple observation in Section 1 for the local setting.

In fact, such a non-robustness arises from the fact that $c_{n,s}\sim\sin(s\pi)\to 0$ as $s\to 1^-$, where $c_{n,s}$ is the constant appearing in the Poisson kernel P. By tracking down the dependence C on $c_{n,s}$ together with the representation formula, we conclude that the constant $C>0$ in Theorem 1.2 must blow up when $s\to 1^-$.

Remark 4.2 (General linear operators)**.** The Poisson kernel approach in this section can be extended to a class of nonlocal linear operators when $2s<n$. Although we do not have an explicit formula for $P=P_{\mathcal{L},B_r(x_1)}$ in general, we can prove the following two-sided Poisson kernel estimates: for any $x\in B_{r/2}(x_1)$ and $z\in\mathbb{R}^n\setminus B_{2r}(x_1)$,

$$C^{-1}\frac{r^{2s}}{|x-z|^{n+2s}} \le P(x,z) \le C\frac{r^{2s}}{|x-z|^{n+2s}}$$

for some constant $C>0$ depending only on n, Λ and s, but not on r and x_1. To be precise, the Poisson kernel estimates can be derived from

(i) the relation between the Poisson kernel P and the Green function G:

$$P(x,z) = \int_{B_r(x_1)} G(x,y)k(z,y)\,\mathrm{d}y \quad \text{for any } x \in B_r(x_1) \text{ and } z \in \mathbb{R}^n \setminus B_r(x_1),$$

and

(ii) the bounds near the diagonal for the Green function G:

$$G(x,y) \le C|x-y|^{2s-n} \quad \text{for all } x,y \in B_r(x_1)$$

and

$$G(x,y) \ge C^{-1}|x-y|^{2s-n} \quad \text{for all } x,y \in B_r(x_1) \text{ with } |x-y| \le \min\{d(x),d(y)\}$$

for some constant $C > 0$ depending only on n, s and Λ, but not on r and x_1. Here $d(x) := \mathrm{dist}(x, \partial B_r(x_1))$.

We note that these bounds are simplified versions of [32, Corollary 9.6], since we are only concerned with the interior estimates of P when the domain is a ball. We refer to Bucur [6], Chen–Song [13], Kassmann–Kim–Lee [26] and Kim–Weidner [32] for related results.

Once we have the two-sided estimates for P, then we can repeat the proofs of Theorem 1.2 and Theorem 1.3 in this section. It is noteworthy that the assumption $2s < n$ arises from the fact that the Poisson kernel estimates strongly rely on the associated Green function estimates in the context of [26].

5. Linear case 2: localized maximum principle

In this section, we construct barrier functions and employ the localized maximum principle introduced in [31, Theorem 1.7] to prove Theorem 1.2 when $p = 2$ and k is translation invariant. Throughout this section, we suppose that $\mathcal{L}$ is linear, i.e., $p = 2$. However, it is noteworthy that the translation invariant condition is necessary only in the construction of the second barrier function in Lemma 5.2.

We begin with the construction of two appropriate subsolutions in $B_r(x_1)$.

Lemma 5.1 (Barrier 1). *Let $w_1 = \chi_{B_r(x_2)}$. Then there exists a constant $C = C(n,s,\Lambda) > 0$ such that*

$$\mathcal{L}w_1 \le -Cr^{-2s} \quad \text{in } B_r(x_1).$$

Proof. For $x \in B_r(x_1)$, the pointwise value $\mathcal{L}w_1(x)$ can be computed as follows:

$$\begin{aligned}\mathcal{L}w_1(x) &= 2\,\mathrm{p.v.}\int_{\mathbb{R}^n} (w_1(x) - w_1(y))k(x,y)\,\mathrm{d}y \\ &= -2\int_{B_r(x_2)} k(x,y)\,\mathrm{d}y \\ &\le -2\Lambda^{-1}\int_{B_r(x_2)} |x-y|^{-n-2s}\,\mathrm{d}y.\end{aligned}$$

It follows from

$$|x-y| \le |x-x_1| + |x_1-x_2| + |x_2-y| \le 10r \quad \text{for } x \in B_r(x_1) \text{ and } y \in B_r(x_2)$$

that

$$\mathcal{L}w_1(x) \le -Cr^{-2s}$$

as desired. □

Lemma 5.2 (Barrier 2)**.** *Let $w_2 \in C_c^\infty(B_r(x_1))$ be a cut-off function such that $w_2 \equiv 1$ on $B_{r/2}(x_1)$, $0 \le w_2 \le 1$ in $\mathbb{R}^n$ and $\|D^2 w_2\|_\infty \le c(n)r^{-2}$. Then there exists a constant $C = C(n, s, \Lambda) > 0$ such that*

$$\mathcal{L}w_2 \le Cr^{-2s} \quad \text{in } B_r(x_1).$$

Proof. The proof is similar to the one of Lemma 5.1. For $x \in B_r(x_1)$, we use the translation invariant property of k as follows.

$$\begin{aligned}
\mathcal{L}w_2(x) &= 2\,\mathrm{p.v.}\int_{\mathbb{R}^n}(w_2(x)-w_2(y))k(x,y)\,\mathrm{d}y\\
&= \int_{\mathbb{R}^n}(2w_2(x)-w_2(x+y)-w_2(x-y))K(y)\,\mathrm{d}y\\
&\le \Lambda\int_{|y|<r}|2w_2(x)-w_2(x-y)-w_2(x+y)||y|^{-n-2s}\,\mathrm{d}y + 4\Lambda\int_{|y|\ge r}|y|^{-n-2s}\,\mathrm{d}y\\
&\le C\|D^2w_2\|_\infty\int_{|y|<r}|y|^{-n-2s+2}\,\mathrm{d}y + Cr^{-2s}\\
&\le Cr^{-2s}.
\end{aligned}$$

□

In order to compare a weak solution u with barriers, we need the localized maximum principle that is a modified version of [31, Theorem 1.7]. For this purpose, we show the following lemma, which captures the effect of a non-homogeneous term in the maximum principle for nonlocal operators.

Lemma 5.3. *Let C_0 be a nonnegative constant. Let u be a weak supersolution of $\mathcal{L}u = -C_0$ in $B_r(x_1)$ such that $u \ge 0$ a.e. in $\mathbb{R}^n \setminus B_r(x_1)$. Then there exists a constant $C = C(n, s, \Lambda) > 0$ such that*

$$u \ge -CC_0r^{2s} \quad \text{a.e. in } B_r(x_1).$$

Proof. It is enough to prove that the set

$$G := \{u < -CC_0r^{2s}\}\,(\subset B_r(x_1))$$

has measure zero for a sufficiently large constant $C > 0$. We define the function

$$\varphi = (u + CC_0r^{2s})_-,$$

which is admissible for a test function in the weak formulation of $\mathcal{L}u = -C_0$; see Proposition 2.2 or [31, Proposition 2.6] for instance. We observe that

(i) $G^c \supset \mathbb{R}^n \setminus B_r(x_1) \supset \mathbb{R}^n \setminus B_{2r}(x)$ for any $x \in B_r(x_1)$;

(ii) $u(x) - u(y) \le 0$ for $x \in G$ and $y \in G^c$.

Thus, we have

$$\begin{aligned}-C_0\int_G\varphi\,\mathrm{d}x &\le \mathcal{E}(u,\varphi)\\ &\le -\Lambda^{-1}\int_G\int_G\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}y\,\mathrm{d}x+2\Lambda^{-1}\int_G\int_{G^c}\frac{(u(x)-u(y))\varphi(x)}{|x-y|^{n+2s}}\,\mathrm{d}y\,\mathrm{d}x\\ &\le 2\Lambda^{-1}\int_G\int_{\mathbb{R}^n\setminus B_{2r}(x)}\frac{(u(x)-u(y))\varphi(x)}{|x-y|^{n+2s}}\,\mathrm{d}y\,\mathrm{d}x\\ &\le -2CC_0\Lambda^{-1}r^{2s}\int_G\varphi(x)\int_{\mathbb{R}^n\setminus B_{2r}(x)}\frac{1}{|x-y|^{n+2s}}\,\mathrm{d}y\,\mathrm{d}x\\ &= -\frac{CC_0|\mathbb{S}^{n-1}|}{s\Lambda 2^{2s}}\int_G\varphi\,\mathrm{d}x.\end{aligned}$$

Taking $C > 2^{2s}s\Lambda/|\mathbb{S}^{n-1}|$ yields that $|G| = 0$, which finishes the proof. □

We next prove the localized maximum principle that is suitable for our purpose. As pointed out in [31], see Lindgren–Lindqvist [41, Lemma 9], Korvenpää–Kuusi–Palatucci [35, Lemma 6] or Kim–Lee [29, Lemma 5.2] for the standard maximum principle with respect to nonlocal operators.

Lemma 5.4 (Localized maximum principle)**.** *Let u be a weak supersolution of* (1.1) *in $B_r(x_1)$ such that $u \ge 0$ in a.e. $B_R \setminus B_r(x_1)$. Then there exists a constant $C = C(n, s, \Lambda) > 0$ such that*

$$u \ge -C\left(\frac{r}{R}\right)^{2s}\mathrm{Tail}(u_-;0,R) \quad \textit{a.e. in } B_r(x_1).$$

Proof. Let $\eta \in C_c^\infty(B_R)$ be a cut-off function such that $0 \le \eta \le 1$ in $\mathbb{R}^n$ and $\eta \equiv 1$ on $B_{R/2}$. Then for any nonnegative function $\varphi \in C_c^\infty(B_r(x_1))$, we have

$$\begin{aligned}\mathcal{E}(u\eta,\varphi) &\ge -\mathcal{E}(u(1-\eta),\varphi)\\ &= -\int_{B_r(x_1)}\int_{B_r(x_1)}(u(x)(1-\eta(x))-u(y)(1-\eta(y)))(\varphi(x)-\varphi(y))k(x,y)\,\mathrm{d}y\,\mathrm{d}x\\ &\quad -2\int_{B_r(x_1)}\int_{\mathbb{R}^n\setminus B_r(x_1)}(u(x)(1-\eta(x))-u(y)(1-\eta(y)))\varphi(x)k(x,y)\,\mathrm{d}y\,\mathrm{d}x\\ &= 2\int_{B_r(x_1)}\int_{\mathbb{R}^n\setminus B_{R/2}}u(y)(1-\eta(y))\varphi(x)k(x,y)\,\mathrm{d}y\,\mathrm{d}x\\ &\ge -2\int_{B_r(x_1)}\int_{\mathbb{R}^n\setminus B_R}u_-(y)\varphi(x)k(x,y)\,\mathrm{d}y\,\mathrm{d}x\\ &\ge -2\Lambda C_0\int_{B_r(x_1)}\varphi\,\mathrm{d}x,\end{aligned}$$

where

$$C_0 := \sup_{x\in B_r(x_1)}\int_{\mathbb{R}^n\setminus B_R}\frac{u_-(y)}{|x-y|^{n+2s}}\,\mathrm{d}y.$$

Since

$$|x-y| \ge |y|-|x-x_1|-|x_1| \ge |y|/2 \quad \text{for } x\in B_r(x_1) \text{ and } y\in\mathbb{R}^n\setminus B_R,$$

we have

$$0 \le C_0 \le 2^{n+2s} \int_{\mathbb{R}^n \setminus B_R} \frac{u_-(y)}{|y|^{n+2s}}\,\mathrm{d}y = 2^{n+2s} R^{-2s} \mathrm{Tail}(u_-; 0, R).$$

Hence, we conclude that $u\eta$ is a weak supersolution of $\mathcal{L}(u\eta) = -2\Lambda C_0$ in $B_r(x_1)$.

On the other hand, since $u\eta \ge 0$ in $\mathbb{R}^n \setminus B_r(x_1)$, we are able to apply Lemma 5.3 to find that

$$u = u\eta \ge -C\left(\frac{r}{R}\right)^{2s} \mathrm{Tail}(u_-; 0, R) \quad \text{a.e. in } B_r(x_1).$$

for some constant $C = C(n, s, \Lambda) > 0$. □

We are now ready to prove Theorem 1.2 via the localized maximum principle with the barrier functions, provided that $p = 2$ and k is translation invariant.

Proof of Theorem 1.2 when $\mathcal{L}$ is linear and translation invariant. We set $v := w_1 + c_0 w_2$ for barrier functions w_1 and w_2 constructed in Lemmas 5.1 and 5.2, respectively, where c_0 is a universal positive constant that will be determined soon. Indeed, due to Lemmas 5.1 and 5.2, there exists a positive constant c_0 depending only on n, s and Λ such that

$$\begin{cases} \mathcal{L}v \le 0 & \text{in } B_r(x_1), \\ v = c_0 & \text{on } B_{r/2}(x_1), \\ v = 1 & \text{on } B_r(x_2), \\ v = 0 & \text{on } B_R \setminus (B_r(x_1) \cup B_r(x_2)), \\ v = 0 & \text{on } \mathbb{R}^n \setminus B_R. \end{cases}$$

We now apply the localized maximum principle (Lemma 5.4) for $u - mv$ with $m := \inf_{B_r(x_2)} u$ to obtain that

$$\begin{aligned} u &\ge mv - C\left(\frac{r}{R}\right)^{2s} \mathrm{Tail}((u - mv)_-; 0, R) \\ &\ge c_0 m - C\left(\frac{r}{R}\right)^{2s} \mathrm{Tail}(u_-; 0, R) \quad \text{in } B_{r/2}(x_1). \end{aligned}$$

In particular, we arrive at

$$\inf_{B_r(x_2)} u \le C u(x_1) + C\left(\frac{r}{R}\right)^{2s} \mathrm{Tail}(u_-; 0, R).$$

We finish the proof by applying the standard Harnack inequality (Theorem 2.3) in each ball $B_{2r}(x_1)$ and $B_{2r}(x_2)$, together with the fact that $u \ge 0$ in B_R. □

Remark 5.5 (Non-robustness 2)**.** To verify the non-robustness in this approach again, we first have to specify the dependence of the kernel k on s as follows:

$$\Lambda^{-1}(1-s)|x-y|^{-n-2s} \le k(x,y) = k(y,x) \le \Lambda(1-s)|x-y|^{-n-2s}.$$

For simplicity, we may assume that $u \ge 0$ in $\mathbb{R}^n$. Then it is easy to check that two barriers w_1 and w_2 satisfy

$$\mathcal{L}w_1 \le -C(1-s)r^{-2s} \quad \text{and} \quad \mathcal{L}w_2 \le Cr^{-2s} \quad \text{in } B_r(x_1),$$

where $C > 0$ depends only on n and Λ but not on s. In fact, in the proof of Lemma 5.2, the estimate of the integral $\int_{|y|<r} |y|^{-n-2s+2}\,dy$ requires us to multiply an additional $1/(1-s)$. Since we assume that u is nonnegative in $\mathbb{R}^n$, we can just apply the global maximum principle to obtain that the constant $c_0 = c_0(n, \Lambda)$ is comparable to $1-s$. Hence, we conclude that the inequality gives no information when we let $s \to 1^-$, as we already checked in Remark 4.1.

6. Conclusions

In this paper, we have established a nonlocal Harnack inequality that holds in any open set, not necessarily connected. Since no analogous phenomenon occurs in the local setting, our main result is purely nonlocal. We have presented three different approaches for the inequality: a new weak Harnack inequality, a Poisson representation formula and a localized maximum principle. A new weak Harnack inequality (or a tail-infimum estimate), recently developed in [27, 39], serves as a powerful tool for dealing with general nonlocal nonlinear operators. Although the other two approaches are only available for nonlocal linear operators, they have the advantage of being relatively elementary and are of independent interest. Furthermore, the nonlocal Harnack inequality in a disconnected region yields an immediate corollary: Harnack inequality holds in disjoint intervals when $n = 1$.

Use of Generative-AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest in this paper.

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