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Research article

Well-posedness of contact discontinuity solutions and vanishing pressure limit for the Aw–Rascle traffic flow model†

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Abstract: This paper investigates the well-posedness of contact discontinuity solutions and the vanishing pressure limit for the Aw–Rascle traffic flow model with general pressure functions. The well-posedness problem is formulated as a free boundary problem, where initial discontinuities propagate along linearly degenerate characteristics. To address vacuum degeneracy, a condition at density jump points is introduced, ensuring a uniform lower bound for density. The Lagrangian coordinate transformation is applied to fix the contact discontinuity.The well-posedness of contact discontinuity solutions is established, showing that compressive initial data leads to finite-time blow-up of the velocity gradient, while rarefactive initial data ensures global existence. For the vanishing pressure limit, uniform estimates of velocity gradients and density are derived via level set argument. The contact discontinuity solutions of the Aw–Rascle system are shown to converge to those of the pressureless Euler equations, with matched convergence rates for characteristic triangles and discontinuity lines. Furthermore, under the conditions of pressure, enhanced regularity in non-discontinuous regions yields convergence of blow-up times.

Keywords: Aw–Rascle traffic flow model; well-posdness; blow-up; lower bound of density; vanishing pressure limit; convergence rate

1. Introduction

In this paper, we consider the Aw–Rascle traffic flow model proposed by Aw and Rascle [1]:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho (u + P))_t + (\rho u (u + P))_x = 0, \end{cases} \tag{1.1}$$

where $t > 0$ and x represent time and space, ρ, u, P are density, velocity, and pressure respectively. The pressure P is the function of density ρ and the small parameter $\varepsilon > 0$, satisfying $\lim_{\varepsilon\to 0} P(\rho, \varepsilon) = 0$. Here we consider the general pressure

$$P(\rho, \varepsilon) = \varepsilon^2 p(\rho). \tag{1.2}$$

In the paper, for $\rho > 0$, $p \in C^2(\mathbb{R}^+)$ satisfies the following conditions:

$$p'(\rho) > 0, \quad 2p'(\rho) + \rho p''(\rho) > 0, \quad \lim_{\rho\to+\infty} p(\rho) = +\infty, \quad \lim_{\rho\to 0} p(\rho) = k. \tag{1.3}$$

And for different k we have the following two cases:

Case 1: If k is a finite constant, by $(1.1)_2 - \varepsilon^2 k (1.1)_1$, (1.1) are equivalent to

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ \left(\rho\left(u + \varepsilon^2 (p(\rho) - k)\right)\right)_t + \left(\rho u\left(u + \varepsilon^2 (p(\rho) - k)\right)\right)_x = 0. \end{cases}$$

Therefore, we can take $k = 0$, otherwise we can let $\tilde{p}(\rho) = p(\rho) - k$, then the form of (1.1) is unchanged. In this case, we have $\lim_{\rho\to 0} p(\rho) = 0$, which includes γ-law: $p(\rho) = \rho^\gamma$ with $\gamma \geq 1$.

Case 2: If $k = -\infty$, then $\lim_{\rho\to 0} p(\rho) = -\infty$, which includes $p(\rho) = \ln\rho$. To define the inital date, we need to introduce $Lip_s(\mathbb{R})$ to denote the set of piecewise Lipschitz functions:

Definition 1.1. *For each f is belongs to $Lip_s(\mathbb{R})$, there exists a finite set $\{x_i\}_{i=1}^n$ of the first kind of discontinuity points, in which $x_i < x_{i+1}$ for any $i = 1, \cdots, n-1$, such that f is Lipschitz function on (x_i, x_{i+1}). Respectively, f is belongs to C_s^1, if f is C^1 function on each (x_i, x_{i+1}).*

Here, $\{x_i\}_{i=1}^n$ is called partition points set. By the Lipschitz continuous at each subinterval, at each partition point x_i, there exists the left limit $\lim_{x\to x_i^-} f(x) = f\left(x_i^-\right)$ and the right limit $\lim_{x\to x_i^+} f(x) = f\left(x_i^+\right)$, but may not be equal. And, $[f](x_i) = f\left(x_i^+\right) - f\left(x_i^-\right)$ is the jump of f at x_i.

The initial data of (1.1) can be given as

$$(\rho, u)|_{t=0} = (\rho_0, u_0), \tag{1.4}$$

where both ρ_0 and u_0 are bounded, $\rho_0 \in Lip_S(\mathbb{R})$, $u_0 \in Lip(\mathbb{R})$ and $\rho_0 \geq \underline{\rho_0} > 0$. At each partition point x_i of density ρ_0, following conditions are introduce for the low bounded of density:

- ε-condition: If $\rho_0(x_i^-) < \rho_0(x_i^+)$ at x_i, for $x > x_i$,

$$u_0(x_i) + \varepsilon^2 p(\rho_0)(x_i^-) > u_0(x). \tag{1.5}$$

- 0-condition: If $\rho_0(x_i^-) < \rho_0(x_i^+)$ at x_i, for $x > x_i$,

$$u_0(x_i) > u_0(x). \tag{1.6}$$

The necessary and sufficient of above conditions will be discussed in Sections 3 and 4. And, 0-condition is the joint of ε-condition for $\varepsilon > 0$.

Remark 1.2. *The discontinuous points of ρ are of the first kind. If the function $f \in Lip_s(\mathbb{R})$, then f can be decomposed into*

$$f = f_J + f_C, \tag{1.7}$$

where f_J represents the jump part of function f; f_C represents the absolutely continuous part of the function f and is a Lipschitz continuous function.

Remark 1.3. *If the curve $x(t; x_0, 0)$ is from the initial discontinuity point $(x_0, 0)$, the discontinuity will propagate along this curve. On the other hand, according to the Rankine-Hugoniot condition, $[u] = 0$ leads to*

$$\frac{dx}{dt} = \frac{[\rho u]}{[\rho]} = u. \tag{1.8}$$

Then, the discontinuous curve $x(t; x_0, 0)$ satisfies

$$\begin{cases} \frac{dx(t;x_0,0)}{dt} = u(x(t; x_0, 0), t) \\ x(0; x_0, 0) = x_0. \end{cases} \tag{1.9}$$

For fixed $\varepsilon > 0$, the two eigenvalues of Aw–Rascle model consist: the genuinely nonlinear one, and the linearly degenerate one. For the well-posedness of Aw–Rascle model, we start with local existence and gradient blow-up.

For the local existence, if the solution is C^1 function without vacuum, (1.1) is equivalent to strict hyperbolic system. In this case, the local existence of various types of function spaces, including C^1 function class, H^s function class and BV function class, has been studied. Under the condition of $\rho > 0$ in the whole space, Li and Yu [13] could provide the local well-posedness of one-dimensional C^1 solutions on each compact characteristic triangle.

For gradient blow-up of conservation laws, in 1964, Lax [12] proved this conclusion for one-dimensional 2×2 genuinely nonlinear hyperbolic system. The results show that for a strictly hyperbolic system, if the initial value is a small smooth perturbation near a constant state, then the initial compression in any genuinely nonlinear characteristic field will produce a gradient blow-up in finite time. John [11], Li et al. [14, 15], and Liu [16] have proven the generation of shock waves for $n \times n$ conservation law equations under different conditions.

The Riemann problem of (1.1) with $p(\rho) = \rho^\gamma$ was solved in [1]. Pan and Han [18] introduced the Chaplygin pressure function into the Aw–Rascle traffic model and solved the respective Riemann problem. It is noticeable that under the generalized Rankine–Hugoniot condition and entropy condition, they establish the existence and uniqueness of δ-wave. Cheng and Yang [6] solved the Riemann problem for the Aw–Rascle model with the modified Chaplygin gas pressure. Godvik and Hanche-Olsen [8] proved the existence of the weak entropy solution for the Cauchy problem with vacuum. By the compensated compactness method, Lu [17] proved the global existence of bounded entropy weak solutions for the Cauchy problem of general nonsymmetric systems of Keyfitz–Kranzer type. When the parameter $n = 1$, the system is the Aw–Rascle model.

When the pressure $P(\rho) \equiv 0$ in the Aw–Rascle model, the equations are simplified to pressureless Euler equations:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + \left(\rho u^2\right)_x = 0. \end{cases} \tag{1.10}$$

The system (1.3) was used to describe both the process of the motion of free particles sticking under collision [3] and the formation of large scale in the universe [21, 25]. The research on the pressureless Euler equations mainly focuses on the well-posedness of weak solutions. Brenier and Grenier [3] and Weinan et al. [25] independently obtained the existence of global weak solutions, and E-Sinai obtained the explicit expression of weak solutions by using the generalized variational principle. Wang et al. [24] prove the global existence of generalized solution to the L^∞ initial data. Boudin [2] proved that the weak solution is the limit of the solution of the viscous pressureless Euler equations. Furthermore, Huang [9] proved the existence of entropy solutions for general pressureless Euler equations. Wang and Ding [23] proved the uniqueness of the weak solution of the Cauchy problem satisfying the Oleinik entropy condition when the initial value ρ_0 is a bounded measurable function. Bouchut and James [7] also obtained similar results. Huang and Wang [10] proved the uniqueness of the weak solution when the initial value is the Radon measure.

To consider the relationship between the Aw–Rascle model and the pressureless Euler equations, a natural idea is the vanishing pressure limit, which considers limits of $\varepsilon \to 0$ as the pressure in the form of (1.2). The first study of vanishing pressure limit is on the Riemann solution of the isentropic Euler equations by Chen and Liu [4], the limiting solution in which includes δ-wave by concentration and vacuum by cavitation. Later, they extended the above result to the full Euler case [5]. Recently, Peng and Wang [19] studied the case of C^1 solutions by a new level set argument. They showed that: For compressive initial data, the continuous solutions converge to a mass-concentrated solution of the pressureless Euler system; For rarefaction initial data, the solutions instead converge globally to a continuous solution. In [20], the authors studied the hypersonic limitation for C^1 solution, and showed the convergence of blow-up time. For the Aw–Rascle model, Shen and Sun [22] proved that as $\varepsilon \to 0$, the Riemann solutions of the perturbed Aw–Rascle system converge to the ones of the pressureless Euler equations (1.10). Pan and Han [18] proved that for the Riemann problem, as the Chaplygin pressure $P(\rho) = -\frac{\varepsilon}{\rho}$ vanishes, the Riemann solutions of the Aw–Rascle traffic model converge to the respective solutions of the pressureless gas dynamics model (1.10).

This paper addresses two fundamental problems for the Aw–Rascle traffic flow model: (i) the well-posedness of contact discontinuity solutions with initial velocity u_0 and piecewise Lipschitz initial density ρ_0 satisfying the ε-condition, and (ii) the vanishing pressure limit as $\varepsilon \to 0$. For problem (i), the analysis begins by establishing a strictly positive lower bound for the density through a partition of the domain, and Lagrangian coordinate transformations. This lower bound leads to a dichotomy: Compressive initial data induce finite time velocity gradient blow-up, whereas rarefactive initial data guarantee global existence of solutions. For problem (ii), uniform density estimates independent of ε are derived via level set argument. It is proven that Aw–Rascle solutions converge to pressureless Euler solutions as $\varepsilon \to 0$: compressive data lead to mass-concentrated solutions, while rarefactive data yield globally regular solutions, with matching $O(\varepsilon^2)$ convergence rates for velocity fields and characteristic triangles. Furthermore, under the conditions of pressure, convergence of the blow-up time is established through enhanced regularity analysis in non-discontinuous regions.

We have the following two theorems.

Theorem 1.4. *For fixed* $\varepsilon > 0$, *and* (1.1)–(1.4), *if* $\lim_{\rho\to 0} p(\rho)$ *satisfies one of the following two cases:*
Case 1: $\lim_{\rho\to 0} p(\rho) = 0$ *and the initial data satisfies the* ε*-condition;*
Case 2: $\lim_{\rho\to 0} p(\rho) = -\infty$*;*
then there exists a time T_b^ε *such that on* $\mathbb{R}\times\left[0, T_b^\varepsilon\right)$, *there exists contacted discontinuity solution* $(\rho^\varepsilon, u^\varepsilon)$ *satisfying:*

(1) If $\inf_{x\in\mathbb{R}} u_0'(x) \geq 0$, $T_b^\varepsilon = +\infty$, *the solution exists globally,*

$$\rho^\varepsilon \in Lip_s(\mathbb{R}\times[0,+\infty)), \quad u^\varepsilon \in Lip(\mathbb{R}\times[0,+\infty)).$$

(2) If $\inf_{x\in\mathbb{R}} u_0'(x) < 0$, T_b^ε *is finite and there exists at least one* X_b^ε *such that as* $(x,t) \to \left(X_b^\varepsilon, T_b^\varepsilon\right)$, $u_x^\varepsilon(x,t) \to -\infty$ *while* $\rho^\varepsilon(x,t)$ *is upper and lower bounded. And the solution stands*

$$\rho^\varepsilon \in Lip_s(\mathbb{R}\times[0,T_b^\varepsilon)), \quad u^\varepsilon \in Lip(\mathbb{R}\times[0,T_b^\varepsilon)).$$

And, $\Gamma^\varepsilon = \left\{(x,t)\in\mathbb{R}\times\left[0,T_b^\varepsilon\right) : x = x_2^\varepsilon(t)\right\}$ *is the discontinuous curve and* $\left(x_2^\varepsilon\right)'(t)$ *is a Lipschitz function with respect to t, satisfying*

$$\frac{dx_2^\varepsilon(t)}{dt} = u^\varepsilon\left(x_2^\varepsilon(t), t\right).$$

Theorem 1.5. *For* $\varepsilon > 0$, $(\rho^\varepsilon, u^\varepsilon)$ *are the unique solution of* (1.1)–(1.4) *with* $0-$*condition on* $\mathbb{R}\times\left[0, T_b^\varepsilon\right)$. *And,* $(\bar\rho, \bar u)$ *are the unique solution of* (1.10) *with initial data* (1.4) *on* $\mathbb{R}\times[0,T_b)$.

(1) For any $0 < T_* < T_b$, *there exists a* $\varepsilon_* > 0$, *such that, for* $0 < \varepsilon < \varepsilon_*$, $T_* < T_b^\varepsilon$. *As* $\varepsilon \to 0$,

$$\rho^\varepsilon \to \bar\rho \text{ in } \mathcal{M}(\mathbb{R}\times[0,T_*]), \quad u^\varepsilon \to \bar u \text{ in } Lip(\mathbb{R}\times[0,T_*]).$$

And, $T_b \leq \underline{\lim}_{\varepsilon\to 0} T_b^\varepsilon$. *Furthermore, there are the following convergence rates: For* $i = 1, 2$,

$$|u^\varepsilon - \bar u| \sim O\left(\varepsilon^2\right), \quad \left|\lambda_i^\varepsilon - \bar u\right| \sim O\left(\varepsilon^2\right), \quad \left|x_2^\varepsilon - \bar x\right| \sim O\left(\varepsilon^2\right),$$

where λ_i *are the eigenvalues of* (1.1)*;* x_2^ε *and* $\bar x$ *are the discontinuity lines of* (1.1) *and* (1.10) *respectively.*

(2) If $\rho_0 \in C_s^1(\mathbb{R})$, $u_0 \in C^1(\mathbb{R})$, $I(\rho) := \frac{2\rho p'(\rho) + (\rho)^2 p''(\rho)}{(\rho p'(\rho))^2}$ *satisfies following conditions:*

(a) $I(\rho)$ *is an increasing function with respect to* ρ*;*

(b) There exists a small δ *such that* $\int_0^\delta \frac{I(s)}{s^2} ds = +\infty$.

Then, the blow-up time convergences: $\lim_{\varepsilon\to 0} T_b^\varepsilon = T_b$.

Comparing with classical solutions of the compressible Euler equations, contact discontinuity solutions of the Aw–Rascle model and their vanishing pressure limits pose unique analytical challenges. First, the characteristic structure of the Aw–Rascle system, comprising both a linearly degenerate field and a genuinely nonlinear field, induces distinct regularity properties in Riemann invariants, with initial discontinuities propagating along linearly degenerate characteristic curves.

Second, the coupling between evolving discontinuity curves and the solution itself characterizes the problem as a free boundary problem. Third, density-dependent degeneracy in the Riccati-type equations governing the system necessitates new density lower-bound estimation, distinct from those for the compressible Euler equations. Fourth, discontinuities inherently reduce solution regularity, mandating uniform estimates in tailored function spaces to rigorously establish convergence. Finally, precise analysis of blow-up times requires delicate estimates in non-discontinuity regions, where enhanced regularity can be exploited.

To overcome these challenges, three key ideas are introduced: (i) Lagrangian coordinate transformations: that map evolving discontinuity curves to the fixed boundaries; (ii) Density Lower bound analysis: the derivatives of Riemann invariants in smooth regions and the jump condition analysis at discontinuities; (iii) Time-directional derivative techniques: avoiding spatial discontinuities in Lagrangian. Level set argument are further developed to track invariant derivatives and establish uniform estimates for the vanishing pressure limit.

The paper is organized as follows: Section 2 proves the existence of classical solutions to the Aw–Rascle system through Lagrangian coordinate transformations, on avoiding vacuum formation. Section 3 establishes the well-posedness of contact discontinuity solutions using uniform L^∞ estimates for time-directional derivatives in Lagrangian coordinates. Section 4 studies the vanishing pressure limit, while Section 5 quantifies blow-up time convergence via modulus of continuity estimates in non-discontinuity domains. Finally, Section 6 determines sharp convergence rates ($O(\varepsilon^2)$) for characteristic curves and Riemann invariants.

2. The well-posedness of Aw–Rascle model with C^1 initial data

2.1. Lagrangian coordinates

In this section, we consider the Lagrangian transformations for fixed $\varepsilon > 0$ that does not involve estimations, so we drop ε of $(\rho^\varepsilon, u^\varepsilon)$.

The eigenvalues of (1.1) are

$$\begin{cases} \lambda_1 = u - \varepsilon^2 \rho p'(\rho), \\ \lambda_2 = u, \end{cases} \tag{2.1}$$

and Riemann invariants are

$$\begin{cases} w = u, \\ z = u + \varepsilon^2 p(\rho), \end{cases} \tag{2.2}$$

while $p(\rho)$ can be represented by Riemann invariants as $p(\rho) = \frac{z-u}{\varepsilon^2}$. Then, for $\rho > 0$, (1.1) is equivalent to

$$\begin{cases} u_t + \left(u - \varepsilon^2 \rho p'(\rho)\right) u_x = 0, \\ \left(u + \varepsilon^2 p(\rho)\right)_t + u\left(u + \varepsilon^2 p(\rho)\right)_x = 0. \end{cases} \tag{2.3}$$

The derivatives along the charismatics are: For $i = 1, 2$,

$$D_i = \partial_t + \lambda_i \partial_x, \tag{2.4}$$

and the respective characteristic lines passing through $(\tilde{x}, \tilde{t})$ are defined as:

$$\begin{cases} \frac{dx_i(t;\tilde{x},\tilde{t})}{dt} = \lambda_i\left(x_i\left(t;\tilde{x},\tilde{t}\right),t\right), \\ x_i\left(\tilde{t};\tilde{x},\tilde{t}\right) = \tilde{x}. \end{cases} \tag{2.5}$$

Combining (2.1) and (2.3), we have

$$\begin{cases} D_1 u = 0, \\ D_2 z = 0. \end{cases} \tag{2.6}$$

Since the eigenvalue λ_1 is genuinely nonlinear and λ_2 is linearly degenerate, there will be contact discontinuity in density along the second family of eigenvalues. So we consider the following Lagrangian transformation: Let $\tau = t$ and

$$\begin{cases} \frac{d}{d\tau}x(\tau;y) = u\left(x(\tau;y),\tau\right), \\ x(0;y) = y. \end{cases} \tag{2.7}$$

Let $u\left(x(\tau;y),\tau\right) = v(y,\tau)$, $J = \frac{\partial x}{\partial y}$ is the Jacobian determinant of the coordinate transformation satisfying:

$$\begin{cases} \frac{\partial}{\partial\tau}J = v_y(y,\tau), \\ J(y,0) = 1. \end{cases} \tag{2.8}$$

In Lagrangian coordinates, we denote

$$g(y,\tau) := \rho\left(x(\tau;y),\tau\right), \quad Z(y,\tau) := z\left(x(\tau;y),\tau\right), \tag{2.9}$$

and $Z(y,\tau) = v(y,\tau) + \varepsilon^2 p\left(g(y,\tau)\right)$.

Next, for the relationship between J and g, in Lagrangian coordinates, $(1.1)_1$ is equivalent to

$$g_\tau + gJ^{-1}J_\tau = 0, \tag{2.10}$$

which equals to $(\ln(gJ))_\tau = 0$. For $g > 0$, integrating on τ leads to

$$J(y,\tau) = \frac{g_0(y)}{g(y,\tau)}, \tag{2.11}$$

where $g_0(y) = g(y,0)$ is the initial density. Combining with (2.8), in Lagrangian coordinates, (2.3) is equivalent to

$$\begin{cases} J_\tau = v_y, \\ v_\tau + \mu v_y = 0, \\ Z_\tau = 0, \end{cases} \tag{2.12}$$

where $\mu = -\varepsilon^2 g p'(g) J^{-1} = -\frac{\varepsilon^2}{g_0} g^2 p'(g)$. And, the initial data are

$$(J,v,Z)|_{\tau=0} = \left(1, v_0(y), Z_0(y)\right). \tag{2.13}$$

In the Lagrangian coordinates, one could introduce following direction derivative:

$$D = \partial_\tau + \mu\partial_y, \tag{2.14}$$

and the characteristic line passing through $(\tilde{y}, \tilde{\tau})$

$$\begin{cases} \frac{dy_1(\tau;\tilde{y},\tilde{\tau})}{d\tau} = \mu\left(y_1\left(\tau;\tilde{y},\tilde{\tau}\right),\tau\right), \\ y_1\left(\tilde{\tau};\tilde{y},\tilde{\tau}\right) = \tilde{y}. \end{cases} \tag{2.15}$$

Along the characteristic line, $(2.12)_2$ is equivalent to

$$Dv = 0, \tag{2.16}$$

which leads to

$$v\left(y,\tau\right) = v_0\left(y_1\left(0;y,\tau\right)\right). \tag{2.17}$$

And, by $(2.12)_3$, one could have

$$Z(y,\tau) = Z_0(y). \tag{2.18}$$

By the expression of p we have

$$p\left(g(y,\tau)\right) = \frac{Z(y,\tau) - v(y,\tau)}{\varepsilon^2} = \frac{Z_0(y) - v_0\left(y_1\left(0;y,\tau\right)\right)}{\varepsilon^2}. \tag{2.19}$$

If $g(y,\tau) > 0$,

$$g = p^{-1}\left(p\left(g_0(y)\right) + \frac{v_0(y) - v\left(y,\tau\right)}{\varepsilon^2}\right). \tag{2.20}$$

Remark 2.1. *If absence of vacuum ($g(y,\tau) > 0$), the density g are uniquely determined by the velocity field $v(y,\tau)$ and Lagrangian coordinate y. Consequently, the velocity field v is governed by the transport equation:*

$$v_\tau + \mu\left(v,y\right)v_y = 0. \tag{2.21}$$

For initial data containing jump, the coefficient $\mu(v,y)$ becomes discontinuous along y-direction. This necessitates estimating v through time τ-derivatives via Lagrangian evolution instead of spatial y-derivatives, thereby avoiding the jumps. Since the ε-condition in this paper ensures that the density has a lower bound at jump. The structure of (2.21) *is the basis of the regularity of the solution in this paper. Even in the presence of contact discontinuities, its structure enables deriving the appropriate gradient estimates.*

And in Lagrangian coordinates, ε-condition and 0-condition equivalence to:

- ε-condition: If $g_0(y_i^-) < g_0(y_i^+)$ at jump point y_i, for $y > y_i$,

$$v_0(y_i) + \varepsilon^2 p\left(g_0\right)\left(y_i^-\right) > v_0\left(y\right). \tag{2.22}$$

- 0-condition: If $g_0(y_i^-) < g_0(y_i^+)$ at jump point y_i, for $y > y_i$

$$v_0\left(y_i\right) > y_0\left(y\right). \tag{2.23}$$

In the paper, we do not distinguish the ε-condition and 0-condition in the Eulerian coordinates or Lagrangian coordinates.

2.2. The well-posedness and blow-up C^1 solutions

In this section, we consider v_0, $Z_0 \in C^1$, by [13], we could have the local existence of C^1 solution with $g(y,\tau) > 0$ for (2.12)-(2.13). Next, we further consider the sharp life-span of C^1 solution of (1.1)–(1.3). First, we have the following lemma.

Lemma 2.2. *For the C^1 solution $(J^\varepsilon, v^\varepsilon, Z^\varepsilon)$ of* (2.12)-(2.13)*, g^ε is upper bounded with respect to ε; v^ε is uniformly bounded with respect to ε.*

Proof. By (2.17),

$$\min_{y\in\mathbb{R}} v_0(y) \le v^\varepsilon(y,\tau) \le \max_{y\in\mathbb{R}} v_0(y). \tag{2.24}$$

On the other hand, for $p(g^\varepsilon)$

$$\begin{aligned} p\left(g^\varepsilon(y,\tau)\right) &= \frac{Z^\varepsilon(y,\tau) - v^\varepsilon(y,\tau)}{\varepsilon^2} \\ &= \frac{\varepsilon^2 p\left(g_0(y)\right) + v_0(y) - v_0\left(y_1^\varepsilon(0;y,\tau)\right)}{\varepsilon^2} \\ &\le \max_{y\in\mathbb{R}} p(g_0) + \frac{2\max_{y\in\mathbb{R}} v_0}{\varepsilon^2} \le C(\varepsilon). \end{aligned} \tag{2.25}$$

Combining (1.3), g^ε has an upper bound with respect to ε, denoted by $\bar{g}^\varepsilon$. □

We have the following lower bound estimate for density g^ε.

Proposition 2.3. *For the C^1 solutions $(J^\varepsilon, v^\varepsilon, Z^\varepsilon)$ of* (2.12)-(2.13)*, g^ε has a uniform lower bound*

$$g^\varepsilon(y,\tau) \ge \frac{A_1}{1 + A_2 B\tau}, \tag{2.26}$$

where $A_1 = \min_{y\in\mathbb{R}} g_0$, $A_2 = \max_{y\in\mathbb{R}} g_0$, $B = \max_{y\in\mathbb{R}} \frac{((Z_0^\varepsilon)')_+}{g_0}$ with $(f)_+ = \max(f,0)$.

Proof. To estimate g^ε

$$p'(g^\varepsilon)Dg^\varepsilon = Dp(g^\varepsilon) = \frac{1}{\varepsilon^2}DZ^\varepsilon = \frac{1}{\varepsilon^2}\left(Z_\tau^\varepsilon + \mu^\varepsilon Z_y^\varepsilon\right) = \frac{1}{\varepsilon^2}\mu^\varepsilon Z_y^\varepsilon = -g_0^{-1}(g^\varepsilon)^2 p'(g^\varepsilon)\left(Z_0^\varepsilon\right)_y. \tag{2.27}$$

Then, we have

$$Dg^\varepsilon = -g_0^{-1}(g^\varepsilon)^2\left(Z_0^\varepsilon\right)_y. \tag{2.28}$$

Dividing both sides of the above equation by $(g^\varepsilon)^2$, by the definition of B, we have

$$D\left(\frac{1}{g^\varepsilon}\right) = \frac{\left(Z_0^\varepsilon\right)_y}{g_0} \le B. \tag{2.29}$$

Integrating s from 0 to τ along the characteristic line $y_1^\varepsilon(\tau;y,\tau)$

$$g^\varepsilon(y,\tau) \ge \frac{g_0\left(y_1^\varepsilon(0;y,\tau)\right)}{1 + g_0\left(y_1^\varepsilon(0;y,\tau)\right)B\tau} \ge \frac{A_1}{1 + A_2 B\tau}. \tag{2.30}$$

□

So for the C^1 initial data, we have the following proposition.

Proposition 2.4. *For fixed $\varepsilon > 0$ and (1.1)–(1.3), $(\rho_0, u_0) \in \left(C^1(\mathbb{R})\right)^2$, there exists a time T_b^ε such that on $\mathbb{R} \times \left[0, T_b^\varepsilon\right)$:*

(1) If $\inf_{x\in\mathbb{R}} u_0'(x) \geq 0$, $T_b^\varepsilon = +\infty$, the solution exists globally, $(\rho^\varepsilon, u^\varepsilon) \in (C^1(\mathbb{R} \times [0, +\infty)))^2$.

(2) If $\inf_{x\in\mathbb{R}} u_0'(x) < 0$, T_b^ε is finite and there exists at least one X_b^ε such that as $(x,t) \to \left(X_b^\varepsilon, T_b^\varepsilon\right)$, $u_x^\varepsilon(x,t) \to -\infty$ while $\rho^\varepsilon(x,t)$ is upper and lower bounded. And the solution stands $(\rho^\varepsilon, u^\varepsilon) \in (C^1\left(\mathbb{R} \times \left[0, T_b^\varepsilon\right)\right))^2$.

Proof. Since the lower bound of density, by Remark 2.1, we just need to consider the gradient of v^ε. Taking ∂_τ on $(2.12)_2$ leads to

$$D(v_\tau^\varepsilon) = (v_\tau^\varepsilon)_\tau + \mu^\varepsilon (v_\tau^\varepsilon)_y = -\mu_\tau^\varepsilon v_y^\varepsilon. \tag{2.31}$$

To compute $\mu_\tau^\varepsilon v_y^\varepsilon$, we have

$$g_\tau^\varepsilon = -g^\varepsilon (J^\varepsilon)^{-1} v_y^\varepsilon = -\frac{(g^\varepsilon)^2 v_y^\varepsilon}{g_0} = \frac{(g^\varepsilon)^2 v_\tau^\varepsilon}{g_0 \mu^\varepsilon} = -\frac{v_\tau^\varepsilon}{\varepsilon^2 p'(g^\varepsilon)}, \tag{2.32}$$

which indicates

$$\begin{aligned} \mu_\tau^\varepsilon v_y^\varepsilon &= -\frac{\varepsilon^2}{g_0}\left((g^\varepsilon)^2 p'(g^\varepsilon)\right)_\tau v_y^\varepsilon \\ &= -\frac{\varepsilon^2}{g_0}\left(2g^\varepsilon p'(g^\varepsilon) + (g^\varepsilon)^2 p''(g^\varepsilon)\right) g_\tau^\varepsilon v_y^\varepsilon \\ &= \frac{2g^\varepsilon p'(g^\varepsilon) + (g^\varepsilon)^2 p''(g^\varepsilon)}{\varepsilon^2 (g^\varepsilon p'(g^\varepsilon))^2} (v_\tau^\varepsilon)^2. \end{aligned} \tag{2.33}$$

By

$$I(g) := \frac{2gp'(g) + (g)^2 p''(g)}{(gp'(g))^2},$$

one could get

$$D(v_\tau^\varepsilon) = -\frac{I(g^\varepsilon)}{\varepsilon^2} (v_\tau^\varepsilon)^2. \tag{2.34}$$

Since $D(v_\tau^\varepsilon) \leq 0$, v_τ^ε is upper bounded:

$$v_\tau^\varepsilon(y,\tau) \leq \max_{y\in\mathbb{R}} v_\tau^\varepsilon(y,0).$$

Next, we focus on the lower bound of v_τ^ε. Dividing both sides of the above formula by $(v_\tau^\varepsilon)^2$, we have

$$D\left(-\frac{1}{v_\tau^\varepsilon}\right) = D\left(-\frac{1}{\varepsilon^2 g^\varepsilon p'(g^\varepsilon)(J^\varepsilon)^{-1} v_y^\varepsilon}\right) = -\frac{I(g^\varepsilon)}{\varepsilon^2}. \tag{2.35}$$

Integrating s from 0 to τ along the characteristic line $y_1^\varepsilon(\tau;\xi,0)$

$$\frac{1}{\left(g^\varepsilon p'(g^\varepsilon)(J^\varepsilon)^{-1} v_y^\varepsilon\right)\left(y_1^\varepsilon(\tau;\xi,0),\tau\right)} - \frac{1}{\left(g_0 p'(g_0) v_0'\right)(\xi)} = \frac{1}{\varepsilon^2}\int_0^\tau I\left(g^\varepsilon\left(y_1^\varepsilon(s;\xi,0), s\right)\right) ds. \tag{2.36}$$

By (2.36), we have

$$\left(g^{\varepsilon}p'\left(g^{\varepsilon}\right)\left(J^{\varepsilon}\right)^{-1}v_{y}^{\varepsilon}\right)\left(y_{1}^{\varepsilon}\left(\tau;\xi,0\right),\tau\right)=\frac{\left(g_{0}p'\left(g_{0}\right)v_{0}'\right)\left(\xi\right)}{1+\left(g_{0}p'\left(g_{0}\right)v_{0}'\right)\left(\xi\right)\int_{0}^{\tau}I\left(g^{\varepsilon}\left(y_{1}^{\varepsilon}\left(s;\xi,0\right),s\right)\right)ds}. \tag{2.37}$$

If $v_0'(\xi)\geq 0$, by (2.37), we have

$$\left(J^{\varepsilon}\right)^{-1}v_{y}^{\varepsilon}\left(y_{1}^{\varepsilon}\left(\tau;\xi,0\right),\tau\right)\geq 0$$

for all $\tau\geq 0$. If $v_0'(\xi)<0$, we need to estimate $\int_0^{\tau}I\left(g^{\varepsilon}\left(y_1^{\varepsilon}(s;\xi,0),s\right)\right)ds$. By (2.25) and (2.26), for g^{ε}, we have

$$\frac{A_1}{1+A_2B\tau}\leq g^{\varepsilon}(y,\tau)\leq\bar{g}^{\varepsilon}. \tag{2.38}$$

Then, since $p(g^{\varepsilon})\in C^2(\mathbb{R}^+)$, for $g^{\varepsilon}\in\left[\frac{A_1}{1+A_2B\tau},\bar{g}^{\varepsilon}\right]$, by (1.3), a positive lower bound $\underline{I}^{\varepsilon}$ such that

$$I(g^{\varepsilon})\geq\underline{I}^{\varepsilon}>0. \tag{2.39}$$

Then we have the following inequality:

$$\int_0^{\tau}I\left(g^{\varepsilon}\left(y_1^{\varepsilon}(s;\xi,0),s\right)\right)ds\geq\int_0^{\tau}\underline{I}^{\varepsilon}ds. \tag{2.40}$$

Thus, if $v_0'(\xi)<0$, when τ increases from 0 to some $T_b^{\varepsilon}(\xi)$, we have

$$1+\left(g_0p'(g_0)v_0'\right)(\xi)\int_0^{T_b^{\varepsilon}(\xi)}I\left(g^{\varepsilon}\left(y_1^{\varepsilon}(s;\xi,0),s\right)\right)ds=0, \tag{2.41}$$

which indicate $\left(J^{\varepsilon}\right)^{-1}v_y^{\varepsilon}\to-\infty$ as $\tau\to T_b^{\varepsilon}(\xi)$.

For ξ run over all the points satisfying $v_0'(\xi)<0$, we could have the minimum life-span

$$T_b^{\varepsilon}:=\inf\left\{T_b^{\varepsilon}(\xi)\right\}>0. \tag{2.42}$$

In Lagrangian coordinates, for fixed $\varepsilon>0$, g^{ε} is upper and lower bounded in $\left[0,T_b^{\varepsilon}\right)$, so the global solution exists if and only if, for $y\in\mathbb{R}$

$$v_0'(y)\geq 0. \tag{2.43}$$

On the other side, if $v_0'(y)<0$, $\left(J^{\varepsilon}\right)^{-1}v_y^{\varepsilon}$ will goes to $-\infty$ in the finite time. Then, there exists at least one point $\left(Y_b^{\varepsilon},T_b^{\varepsilon}\right)$ such that on $\mathbb{R}\times\left[0,T_b^{\varepsilon}\right)$, as $(y,\tau)\to\left(Y_b^{\varepsilon},T_b^{\varepsilon}\right)$,

$$\left(J^{\varepsilon}\right)^{-1}v_y^{\varepsilon}\to-\infty. \tag{2.44}$$

Next, we want to discuss the transformation of the solution between the Lagrangian coordinates and the Eulerian coordinates. For $y\in[-L,L]$, there is a characteristic triangle

$$\left\{(y,\tau)\mid -L\leq y\leq y_1^{\varepsilon}(s;L,0),0\leq s\leq\tau\right\}, \tag{2.45}$$

where $y_1^{\varepsilon}(\tau;L,0)=-L$. And, in the characteristic triangle, each (y,τ) has a unique (x,t) satisfying

$$\begin{cases}\frac{d}{dt}x(t;y)=v^{\varepsilon}(y,t),\\ x(0;y)=y,\end{cases} \tag{2.46}$$

which could be expressed as

$$x(y,\tau) = y + \int_0^\tau v^\varepsilon(y,s)ds. \tag{2.47}$$

We let $L \to +\infty$, for any $y \in \mathbb{R}$, each (y,τ) has a unique (x,t) by (2.46). According to the transformation of Eulerian coordinates and Lagrangian coordinates, we have

$$\begin{cases} u_x^\varepsilon = (J^\varepsilon)^{-1} v_y^\varepsilon, \\ u_t^\varepsilon = -\left(v^\varepsilon - \varepsilon^2 g^\varepsilon p'(g^\varepsilon)\right)(J^\varepsilon)^{-1} v_y^\varepsilon. \end{cases} \tag{2.48}$$

And, for ρ^ε we have

$$\begin{cases} \rho_x^\varepsilon = (J^\varepsilon)^{-1} g_y^\varepsilon = (J^\varepsilon)^{-1} \frac{v_0' + \varepsilon^2 p'(g_0) g_0' - v_y^\varepsilon}{\varepsilon^2 p'(g^\varepsilon)}, \\ \rho_t^\varepsilon = -v^\varepsilon (J^\varepsilon)^{-1} g_y^\varepsilon - g^\varepsilon (J^\varepsilon)^{-1} v_y^\varepsilon. \end{cases} \tag{2.49}$$

In Eulerian coordinates, for fixed $\varepsilon > 0$, u^ε and ρ^ε is upper and lower bounded in $\left[0, T_b^\varepsilon\right)$, so the global solution exists if and only if, for $x \in \mathbb{R}$

$$u_0'(x) \geq 0. \tag{2.50}$$

On the contrary, if $u_0'(x) < 0$, u_x^ε will goes to $-\infty$ in the finite time. Then, respecting to $\left(Y_b^\varepsilon, T_b^\varepsilon\right)$, there exists $\left(X_b^\varepsilon, T_b^\varepsilon\right)$ such that on $\mathbb{R} \times \left[0, T_b^\varepsilon\right)$, as $(x,t) \to \left(X_b^\varepsilon, T_b^\varepsilon\right)$, $u_x^\varepsilon(x,t) \to -\infty$.

Further, modulus of continuity estimates of the solution see [13]. □

3. The well-posedness contact discontinuous solution

In this section we consider the Eq (1.1) with piecewise Lipshcitzs initial data (2.13), where $v_0 \in Lip(\mathbb{R})$ and $Z_0 \in Lip_s(\mathbb{R})$. Without loss of generality, we can assume that Z_0 only has a jump discontinuity at $y = 0$.

3.1. The lower bound of density

First, we would to clarify the influence of the jump. Here, we denote the characteristic line starting from $y = 0$ as $y_1^\varepsilon(\tau)$, which is defined in (2.15). And the characteristic line leading back from $(\tilde{y}, \tilde{\tau})$ to the initial data is denoted as $y_1^\varepsilon(s; \tilde{y}, \tilde{\tau})$. For whether the characteristic line $y_1^\varepsilon(s; \tilde{y}, \tilde{\tau})$ crosses the discontinuity line or not, we can divide $\Omega := \mathbb{R} \times [0, +\infty)$ into the following regions:

$$\Omega = \Omega_+ \cup \Omega_{\mathrm{I}}^\varepsilon \cup \Omega_{\mathrm{II}}^\varepsilon \cup \{y = 0\}, \tag{3.1}$$

where

$$\Omega_+ = \{(y,\tau) \mid y > 0, \tau \geq 0\}, \ \Omega_{\mathrm{I}}^\varepsilon = \{(y,\tau) \mid y_1^\varepsilon(\tau) \leq y < 0, \tau \geq 0\},$$

and

$$\Omega_{\mathrm{II}}^\varepsilon = \{(y,\tau) \mid y < y_1^\varepsilon(\tau), \tau \geq 0\}.$$

For $(\tilde{y}, \tilde{\tau}) \in \Omega_+ \cup \Omega_{\mathrm{II}}^\varepsilon$, $y_1^\varepsilon(s; \tilde{y}, \tilde{\tau})$ does not cross the discontinuity line. From the proof of Proposition 2.4, we can obtain the local existence of Lipschitz solutions in the characteristic triangle of domain $\Omega_+ \cup \Omega_{\mathrm{II}}^\varepsilon$.

For $(\tilde{y}, \tilde{\tau}) \in \Omega_{\mathrm{I}}^{\varepsilon}$, its backward characteristic line $y_1^{\varepsilon}(s; \tilde{y}, \tilde{\tau})$ must reach $(0, \tau_0)$. So for $v(\tilde{y}, \tilde{\tau})$ and $Z^{\varepsilon}(\tilde{y}, \tilde{\tau})$ we have

$$v(\tilde{y}, \tilde{\tau}) = v(0, \tau_0), \quad Z^{\varepsilon}(\tilde{y}, \tilde{\tau}) = Z_0^{\varepsilon}(\tilde{y}). \tag{3.2}$$

Therefore, if $g > 0$, then g in $\Omega_{\mathrm{I}}^{\varepsilon}$ can be expressed as

$$g^{\varepsilon}(\tilde{y}, \tilde{\tau}) = p^{-1}\left(\frac{Z_0^{\varepsilon}(\tilde{y}) - v(0, \tau_0)}{\varepsilon^2}\right). \tag{3.3}$$

Without loss of generality, for different jump cases where Z_0^{ε} has only one discontinuity at $y = 0$, we discuss the lower bound estimate of density for different regions. The key point is based on the results of Proposition 2.3, we have the following proposition.

Proposition 3.1. *For fixed* $\varepsilon > 0$ *and the solutions* $(J^{\varepsilon}, v^{\varepsilon}, Z^{\varepsilon})$ *of* (2.12)-(2.13)*:*
Case 1: $\lim_{g \to 0} p(g) = 0$ *and the initial data satisfies the* ε*-condition;*
Case 2: $\lim_{g \to 0} p(g) = -\infty$ *(* ε*-condition is not required);*
then g^{ε} *has a lower bound with respect to* ε*.*

Proof. (1) Region $\Omega_+ \cup \Omega_{\mathrm{II}}^{\varepsilon}$: For any point $(\tilde{y}, \tilde{\tau})$ in $\Omega_+ \cup \Omega_{\mathrm{II}}^{\varepsilon}$, by the method in Proposition 2.3, we can get the uniform lower bound estimate of density:

$$g^{\varepsilon}(y, \tau) \geq \frac{A_1}{1 + A_2 B \tau}. \tag{3.4}$$

where $A_1 = \min_{y \in \mathbb{R}} g_0$, $A_2 = \max_{y \in \mathbb{R}} g_0$, and B can be defined as

$$B = \frac{\left(Lip\left(Z_0^{\varepsilon}\right)\right)_+}{g_0}, \tag{3.5}$$

here $\left(Z_0^{\varepsilon}\right)_C = Z_0^{\varepsilon} - \left(Z_0^{\varepsilon}\right)_J$ is the absolute continuous part of Z_0, and $\left(Lip(Z_0^{\varepsilon})\right)_+$ is the Lipschitzs constant of $\left(Z_0^{\varepsilon}\right)_C$ without decreasing.

(2) The discontinuous curve $y = 0$: If $g_0(0^-) > g_0(0^+)$ at $y = 0$, since v_0 is continuous, this means $Z_0(0^-) > Z_0(0^+)$. By (2.29), similar to the case in $\Omega_+ \cup \Omega_{\mathrm{II}}^{\varepsilon}$, there exists a constant B such that the density has a lower bound (3.4).

Otherwise, if $g_0(0^-) < g_0(0^+)$, which means $Z_0(0^-) < Z_0(0^+)$, for $(0, \tilde{\tau})$ and $p(g^{\varepsilon}(0^-, \tilde{\tau}))$, we have

$$p(g^{\varepsilon}(0^-, \tilde{\tau})) = \frac{Z_0^{\varepsilon}(0^-) - v_0\left(y_1^{\varepsilon}(0; 0, \tilde{\tau})\right)}{\varepsilon^2}. \tag{3.6}$$

For (3.6), we need to discuss the following two cases:

Case 1: $\lim_{g \to 0} p(g) = 0$ and the initial data satisfies the ε-condition. By (1.5), there exists a constant A^{ε} which may depend on ε such that

$$p(g^{\varepsilon}(0^-, \tilde{\tau})) \geq A^{\varepsilon} > 0. \tag{3.7}$$

So we have

$$g^{\varepsilon}(0^+, \tilde{\tau}) > g^{\varepsilon}(0^-, \tilde{\tau}) \geq p^{-1}(A^{\varepsilon}) > 0. \tag{3.8}$$

Case 2: $\lim_{g\to 0} p(g) = -\infty$. By (3.6), for $p(g)$ we have

$$p\left(g^{\varepsilon}\left(0^{-},\tilde{\tau}\right)\right) = p\left(g_0(0^{-})\right) + \frac{v_0(0) - v_0\left(y_1^{\varepsilon}\left(0;0,\tilde{\tau}\right)\right)}{\varepsilon^2} \geq p\left(\underline{g_0}\right) - \frac{2\max_{y\in\mathbb{R}}|v_0(y)|}{\varepsilon^2} > -\infty. \tag{3.9}$$

By (1.3), $g^{\varepsilon}(0^{-},\tilde{\tau})$ has a lower bound with respect to ε.

(3) Region $\Omega_{\mathrm{I}}^{\varepsilon}$: For any point $(\tilde{y},\tilde{\tau}) \in \Omega_{\mathrm{I}}^{\varepsilon}$, its backward characteristic line must reach $(0,\tau_0)$,

$$y_1^{\varepsilon}\left(\tau_0;\tilde{y},\tilde{\tau}\right) = 0, \tag{3.10}$$

by (2.29) we have

$$D\left(\frac{1}{g^{\varepsilon}}\right) = \frac{\left(Z_0^{\varepsilon}\right)_y}{g_0}. \tag{3.11}$$

Integrating s from τ_0 to $\tilde{\tau}$ along the characteristic line $y_1^{\varepsilon}(\tau;\tilde{y},\tilde{\tau})$, there exists a constant B such that

$$B = \frac{\left(Lip\left(Z_0^{\varepsilon}\right)\right)_+}{g_0}, \tag{3.12}$$

such that

$$\frac{1}{g^{\varepsilon}\left(\tilde{y},\tilde{\tau}\right)} - \frac{1}{\min_{0\leq\tau_0\leq\tilde{\tau}} g^{\varepsilon}\left(0^{-},\tau_0\right)} \leq \frac{1}{g^{\varepsilon}\left(\tilde{y},\tilde{\tau}\right)} - \frac{1}{g^{\varepsilon}\left(0^{-},\tau_0\right)} = \int_{\tau_0}^{\tilde{\tau}} \frac{\left(Lip\left(Z_0^{\varepsilon}\right)\right)_+}{g_0} ds \leq B\left(\tilde{\tau} - \tau_0\right). \tag{3.13}$$

So for fixed ε, we have

$$g^{\varepsilon}\left(\tilde{y},\tau_0\right) \geq \frac{A_1^{\varepsilon}}{1 + A_1^{\varepsilon}B\tau}, \tag{3.14}$$

where

$$A_1^{\varepsilon} = \min_{0\leq\tau_0\leq\tilde{\tau}} g^{\varepsilon}\left(0^{-},\tau_0\right), \quad B = \frac{\left(Lip\left(Z_0^{\varepsilon}\right)\right)_+}{g_0}.$$

Combining the three cases, we obtain that the density has a uniform lower bound in Ω under the Lagrangian coordinate, denoted by $\underline{g}^{\varepsilon}$.

$$\underline{g}^{\varepsilon} := \min\left\{\frac{A_1}{1 + A_2B\tau}, p^{-1}\left(A^{\varepsilon}\right), \frac{A_1^{\varepsilon}}{1 + A_1^{\varepsilon}B\tau}\right\}. \tag{3.15}$$

□

Remark 3.2. *For* $\lim_{g\to 0} p(g) = 0$*, if the assumption in* (1.5) *becomes an equality, there exists at least one point* $y_0 > 0$ *such that*

$$Z_0^{\varepsilon}\left(0^{-}\right) = v_0\left(y_0\right). \tag{3.16}$$

Then there exists a time $T_{y_0}^{\varepsilon}$ *such that*

$$y_1^{\varepsilon}\left(T_{y_0}^{\varepsilon};y_0,0\right) = 0. \tag{3.17}$$

So we have

$$p\left(g^{\varepsilon}\left(0^{-},T_{y_0}^{\varepsilon}\right)\right)=\frac{Z^{\varepsilon}\left(0^{-},T_{y_0}^{\varepsilon}\right)-v\left(y_1^{\varepsilon}\left(T_{y_0}^{\varepsilon};y_0,0\right),T_{y_0}^{\varepsilon}\right)}{\varepsilon^2}=\frac{Z_0^{\varepsilon}\left(0^{-}\right)-v_0\left(y_0\right)}{\varepsilon^2}=0. \tag{3.18}$$

This means there exists a finite time $T_{y_0}^{\varepsilon}$ such that

$$g^{\varepsilon}\left(0^{-},T_{y_0}^{\varepsilon}\right)=0. \tag{3.19}$$

Based on the above discussion, (1.5) *is a necessary and sufficient condition for the absence of vacuum.*

Remark 3.3. *For the case of* $\lim_{g\to 0}p\left(g\right)=-\infty$*, we do not need to use* ε*-condition to get the lower bound of density.*

3.2. The well-posedness of Aw–Rascle model

Based on the above discussion, we prove Theorem 1.4.

Proof. By (2.37), in Eulerian coordinates, we have

$$u_x^{\varepsilon}\left(x_1^{\varepsilon}\left(t;\eta,0\right),t\right)=\frac{1}{\rho^{\varepsilon}p'\left(\rho^{\varepsilon}\right)\left(x_1^{\varepsilon}\left(t;\eta,0\right),t\right)}\cdot\frac{\left(\rho_0p'\left(\rho_0\right)u_0'\right)\left(\eta\right)}{1+\left(\rho_0p'\left(\rho_0\right)u_0'\right)\left(\eta\right)\int_0^t I\left(\rho^{\varepsilon}\left(x_1^{\varepsilon}\left(s;\eta,0\right),s\right)\right)ds}. \tag{3.20}$$

If $u_0'\left(\eta\right)\geq 0$, by (3.20), we have $u_x^{\varepsilon}\left(x_1^{\varepsilon}\left(t;\eta,0\right),t\right)\geq 0$ for all $t\geq 0$. If $u_0'\left(\eta\right)<0$, we need to estimate $\int_0^{\tau}I\left(\rho^{\varepsilon}\left(x_1^{\varepsilon}\left(s;\eta,0\right),s\right)\right)ds$. By Proposition 3.1 and (2.25), for ρ^{ε}, we have

$$\underline{\rho}^{\varepsilon}\leq\rho^{\varepsilon}\left(x_1^{\varepsilon}\left(t;\eta,0\right),t\right)\leq\bar{\rho}^{\varepsilon}. \tag{3.21}$$

Then, since $p\left(\rho^{\varepsilon}\right)\in C^2\left(\mathbb{R}^+\right)$, for $\rho^{\varepsilon}\in\left[\underline{\rho}^{\varepsilon},\bar{\rho}^{\varepsilon}\right]$, $I\left(\rho^{\varepsilon}\right)$ has a lower bound $\underline{I}^{\varepsilon}$ such that

$$I\left(\rho^{\varepsilon}\right)\geq\underline{I}^{\varepsilon}. \tag{3.22}$$

Then we have the following inequality:

$$\int_0^t I\left(\rho^{\varepsilon}\left(x_1^{\varepsilon}\left(s;\eta,0\right),s\right)\right)ds\geq\int_0^t\underline{I}^{\varepsilon}ds. \tag{3.23}$$

Thus, if $u_0'\left(\eta\right)<0$, when t increases from 0 to some $T_b^{\varepsilon}\left(\eta\right)$, we have

$$1+\left(\rho_0p'\left(\rho_0\right)u_0'\right)\left(\eta\right)\int_0^{T_b^{\varepsilon}\left(\eta\right)}I\left(\rho^{\varepsilon}\left(x_1^{\varepsilon}\left(s;\eta,0\right),s\right)\right)ds=0, \tag{3.24}$$

which indicate $\left(J^{\varepsilon}\right)^{-1}v_y^{\varepsilon}\to-\infty$ as $\tau\to T_b^{\varepsilon}(\eta)$.

To consider $\eta\in\mathbb{R}$, we could introduce the minimum life-span

$$T_b^{\varepsilon}:=\inf\left\{T_b^{\varepsilon}\left(\eta\right)\right\}>0, \tag{3.25}$$

where η run over all the points satisfying $u_0'(\eta) < 0$. In Eulerian coordinates, for fixed $\varepsilon > 0$, ρ^ε is upper and lower bounded in $\left[0, T_b^\varepsilon\right)$, so the global solution exists if and only if, for $x \in \mathbb{R}$

$$u_0'(x) \geq 0. \tag{3.26}$$

On the other side, if $u_0'(x) < 0$, u_x^ε will goes to $-\infty$ in the finite time. Then, there exists $\left(X_b^\varepsilon, T_b^\varepsilon\right)$ such that on $\mathbb{R} \times \left[0, T_b^\varepsilon\right)$, as $(x,t) \to \left(X_b^\varepsilon, T_b^\varepsilon\right)$, $u_x^\varepsilon(x,t) \to -\infty$. Therefore, by (2.48) and (2.49), ρ^ε and u^ε satisfy

$$\rho^\varepsilon \in Lip_s\left(\mathbb{R} \times [0, T_b^\varepsilon)\right), \quad u^\varepsilon \in Lip\left(\mathbb{R} \times [0, T_b^\varepsilon)\right). \tag{3.27}$$

Next, we consider the regularity of discontinuous line. The discontinuous line in Eulerian coordinates is

$$\frac{dx_2^\varepsilon(t)}{dt} = u^\varepsilon\left(x_2^\varepsilon(t), t\right), \tag{3.28}$$

which has the following implicit expression

$$x_2^\varepsilon(t) = x_0 + \int_0^t u^\varepsilon\left(x_2^\varepsilon(s), s\right) ds. \tag{3.29}$$

Therefore, by (3.29), for $h > 0$, $t \geq 0$, we have

$$\begin{aligned} \left|(x_2^\varepsilon)'(t+h) - (x_2^\varepsilon)'(t)\right| &= \left|u^\varepsilon\left(x_2^\varepsilon(t+h), t+h\right) - u^\varepsilon\left(x_2^\varepsilon(t), t\right)\right| \\ &\leq Lip(u^\varepsilon)\left(\left|x_2^\varepsilon(t+h) - x_2^\varepsilon(t)\right| + h\right) \\ &\leq Lip(u^\varepsilon)\left(\|u\|_{L^\infty} + 1\right) h. \end{aligned} \tag{3.30}$$

So $(x^\varepsilon)'$ is a Lipschitz function with respect to t.

□

Remark 3.4. *We have proved the well-posedness of the contacted discontinuity solution where ρ_0 has one contact discontinuity. For a general piecewise Lipschitz functions function ρ_0, we can follow the above idea. Since the discontinuity points where $[\rho_0] > 0$ are separable, we can repeat the above procedure for each characteristic triangle containing only one such discontinuity. Then we can glue the fragments together to establish local existence. By repeating this procedure, we can extend the solution's existence time forward until a nonlinear singularity appears. Therefore we proved the well-posedness for general piecewise Lipschitz functions ρ_0.*

For fixed $\varepsilon > 0$, if the initial data are $\rho_0, u_0 \in C^1$, by the method in [13], for a point (x,t) in the domain $\mathbb{R} \times \left[0, T_b^\varepsilon\right)$, we can obtain the uniform modulus of continuity estimation of u_x and u_t, and then obtain the existence of C^1 solution. However, for the case where there is a discontinuity, since the backward characteristic line of the point in the region $\Omega_+ \cup \Omega_{\mathrm{II}}^\varepsilon$ does not pass through the discontinuity, the regularity of the solution in the region can be improved to C^1 by the uniform modulus of continuity estimation. However, the backward characteristic line of the points in the region $\Omega_{\mathrm{I}}^\varepsilon$ will pass through the discontinuity, we cannot obtain the uniform modulus of continuity estimation of u_x and u_t along the spatial direction, so the Lipschitz regularity is optimal.

3.3. Pressureless fluid case

The blow-up of pressureless fluid in Eulerian coordinates can refer to [19], and we have similar argument in Lagrangian coordinates. For a smooth solution $(\bar{\rho}, \bar{u})$ of (1.10), $(1.10)_2$ is equivalent to Burgers' equation

$$\bar{u}_t + \bar{u}\bar{u}_x = 0. \tag{3.31}$$

For pressureless fluid, we introduce the following derivative

$$D_0 = \partial_t + \bar{u}\partial_x. \tag{3.32}$$

And the characteristic line passing through $(\tilde{x}, \tilde{t})$ is defined as

$$\begin{cases} \frac{d\bar{x}(t;\tilde{x},\tilde{t})}{dt} = \bar{u}\left(\bar{x}\left(t; \tilde{x}, \tilde{t}\right), t\right), \\ \bar{x}\left(\tilde{t}; \tilde{x}, \tilde{t}\right) = \tilde{x}. \end{cases} \tag{3.33}$$

Under the Lagrangian transformation, the Eq (1.10) is equivalent to

$$\begin{cases} \bar{J}_\tau = \bar{v}_y, \\ \bar{v}_\tau = 0. \end{cases} \tag{3.34}$$

and the respective initial data are

$$\left(\bar{J}, \bar{v}\right)|_{\tau=0} = \left(1, v_0\left(y\right)\right). \tag{3.35}$$

Next, for the smooth solution $\left(\bar{J}, \bar{v}\right)$ of (3.34). By $(3.34)_2$ we have

$$\bar{v}\left(y, \tau\right) = v_0\left(y\right) \quad \text{and} \quad \bar{v}_y(y, \tau) = v_0'(y). \tag{3.36}$$

Integrating the first equation in (3.34), we have

$$\bar{J}\left(y, \tau\right) = 1 + v_0'\left(y\right)\tau. \tag{3.37}$$

Combining (2.11),

$$\bar{g}\left(y, \tau\right) = \frac{g_0(y)}{J(y, \tau)} = \frac{g_0\left(y\right)}{1 + v_0'\left(y\right)\tau}. \tag{3.38}$$

Then, by (3.36) and (3.37), one could get

$$\bar{J}^{-1}\bar{v}_y\left(y, \tau\right) = \frac{v_0'\left(y\right)}{1 + v_0'\left(y\right)\tau}. \tag{3.39}$$

According to (3.39), if $v_0'\left(y\right) < 0$, when τ increases from 0 to some $T_b\left(y\right)$, such that

$$1 + v_0'\left(y\right)T_b\left(y\right) = 0, \tag{3.40}$$

which indicate $\bar{J}^{-1}\bar{v}_y \to -\infty$ and $\bar{g} \to +\infty$ as $\tau \to T_b(y)$.

For y run over all the points such that $v_0'\left(y\right) < 0$, we could have the minimum life-span

$$T_b := \inf\left\{T_b\left(y\right)\right\} = \inf\left\{-\frac{1}{v_0'\left(y\right)}\right\} > 0. \tag{3.41}$$

Thus, from (3.39), we see that singularity of $\bar{v}_y$ first happens at $\tau = T_b$. When $\tau \to T_b$, there is a point such that $\bar{g}$ goes to $+\infty$, which corresponds to the mass concentration.

According to the transformation of Eulerian coordinates and Lagrangian coordinates, we have

$$\begin{cases} \bar{u}_x = \bar{J}^{-1}\bar{v}_y, \\ \bar{u}_t = -\bar{v}\bar{J}^{-1}\bar{v}_y. \end{cases} \tag{3.42}$$

Therefore, in Eulerian coordinates, we have

$$\bar{\rho}\left(\bar{x}(t; x, 0), t\right) = \frac{\rho_0(x)}{1 + u_0'(x)\, t}, \tag{3.43}$$

and

$$\bar{u}_x\left(\bar{x}(t; x, 0), t\right) = \frac{u_0'(x)}{1 + u_0'(x)\, t}. \tag{3.44}$$

According to the above discussion, for $t \in [0, T_b)$, if

$$\rho_0 \in Lip_s(\mathbb{R}), \quad u_0 \in Lip(\mathbb{R}), \tag{3.45}$$

then $\bar{\rho} \in Lip_s(\mathbb{R} \times [0, T_b))$, $\bar{u} \in Lip(\mathbb{R} \times [0, T_b))$.

Based on the above discussion, for the case of pressureless fluid, similar to Theorem 1.4, we have the following proposition.

Proposition 3.5. *For* (1.10) *with initial data* (3.45)*, there exists a time T_b such that on $\mathbb{R} \times [0, T_b)$, there exists a solution $(\bar{\rho}, \bar{u})$ satisfying*

(1) If $\inf_{x\in\mathbb{R}} u_0'(x) \geq 0$, the solution exists globally. For $\bar{\rho}$ and $\bar{u}$, there are

$$\bar{\rho} \in Lip_s(\mathbb{R} \times [0, +\infty)), \quad \bar{u} \in Lip(\mathbb{R} \times [0, +\infty)). \tag{3.46}$$

(2) If $\inf_{x\in\mathbb{R}} u_0'(x) < 0$, there exists a finite T_b and at least one X_b such that as $(x, t) \to (X_b, T_b)$, $\bar{u}_x(x, t) \to -\infty$ and $\bar{\rho}(x, t) \to +\infty$. And the solution stands

$$\bar{\rho} \in Lip_s(\mathbb{R} \times [0, T_b)), \quad \bar{u} \in Lip(\mathbb{R} \times [0, T_b)). \tag{3.47}$$

And, $\bar{\Gamma} = \{(x, t) \in \mathbb{R} \times [0, T_b) : x = \bar{x}(t)\}$ is the discontinuous curve and $\bar{x}'(t)$ is a Lipschitz function with respect to t, satisfying

$$\frac{d\bar{x}(t)}{dt} = \bar{u}\left(\bar{x}(t), t\right). \tag{3.48}$$

4. On vanishing pressure limit

From the above discussion, we know that for Aw–Rascle model, if $u_0' < 0$ initially, u_x will goes to $-\infty$ in finite time. In this section, for any fixed T, as $\varepsilon \to 0$, we consider the convergence of v^ε and g^ε on $\mathbb{R} \times [0, T]$ in Lagrangian coordinates. Without loss of generality, we assume that $\varepsilon < 1$. First, we introduce the level set on the lower bound of $(J^\varepsilon)^{-1} v_y^\varepsilon$:

$$m^\varepsilon(\tau) := \inf_{y\in\mathbb{R}} \left\{ (J^\varepsilon)^{-1} v_y^\varepsilon(y, \tau) \right\}. \tag{4.1}$$

For any fixed $\varepsilon > 0$ and the compressive initial data, there exists a finite life-span T_b^ε defined in Proposition 2.4, which is $+\infty$ for the rarefaction initial data. And we have

$$\lim_{\tau \uparrow T_b^\varepsilon} m^\varepsilon(\tau) = -\infty. \tag{4.2}$$

Further, for $M > 0$, we define τ_M^ε

$$\tau_M^\varepsilon = \sup\left\{ s : -M \leq \inf_{\tau\in[0,s]} m^\varepsilon(\tau), s \leq T \right\}. \tag{4.3}$$

According to the definition, $\left\{\tau_M^\varepsilon\right\}$ is a monotone sequence with respect to M, and

$$\lim_{M\to+\infty} \tau_M^\varepsilon = \min\{T_b^\varepsilon, T\} =: \tau_b^\varepsilon. \tag{4.4}$$

First, we need to proof that the density has a uniform lower bound with respect to ε, and we have the following proposition.

Proposition 4.1. *For any $\varepsilon > 0$ and the solution $(J^\varepsilon, v^\varepsilon, Z^\varepsilon)$ of* (2.12)-(2.13) *with* $0-$*condition, g^ε has a uniform lower bound with respect to ε.*

Proof. (1) Region $\Omega_+ \cup \Omega_{\rm II}^\varepsilon$: For any point $(\tilde{y}, \tilde{\tau})$ in $\Omega_+ \cup \Omega_{\rm II}^\varepsilon$, by Proposition 3.1, the density has a uniform lower bound:

$$g^\varepsilon(y,\tau) \geq \frac{A_1}{1 + A_2 B\tau}. \tag{4.5}$$

where $A_1 = \min_{y\in\mathbb{R}} g_0$, $A_2 = \max_{y\in\mathbb{R}} g_0$, and B can be defined as

$$B = \frac{\left(Lip\left(Z_0^\varepsilon\right)\right)_+}{g_0}, \tag{4.6}$$

here $\left(Z_0^\varepsilon\right)_C = Z_0^\varepsilon - \left(Z_0^\varepsilon\right)_J$ is the absolute continuous part of Z_0, and $\left(Lip\left(Z_0^\varepsilon\right)\right)_+$ is the Lipschitzs constant of the continuous part of Z_0^ε without decreasing.

(2) The discontinuous curve $y = 0$: If $g_0(0^-) > g_0(0^+)$ at $y = 0$. Since v_0 is continuous, $Z_0(0^-) > Z_0(0^+)$. By (2.29), similar to the case in $\Omega_+ \cup \Omega_{\rm II}^\varepsilon$, there exists a constant B such that the density has a lower bound (3.4).

If $g_0(0^-) < g_0(0^+)$ at $y = 0$, for $(0, \tilde{\tau})$ and $p\left(g^\varepsilon(0^-, \tilde{\tau})\right)$, we have

$$p\left(g^\varepsilon(0^-,\tilde{\tau})\right) = \frac{Z_0^\varepsilon(0^-) - v_0\left(y_1^\varepsilon(0;0,\tilde{\tau})\right)}{\varepsilon^2} = p\left(g_0(0^-)\right) + \frac{v_0(0^-) - v_0\left(y_1^\varepsilon(0;0,\tilde{\tau})\right)}{\varepsilon^2}. \tag{4.7}$$

By (1.6), we have

$$p\left(g^\varepsilon(0^-,\tilde{\tau})\right) \geq p\left(g_0(0^-)\right) \geq p\left(\underline{g_0}\right). \tag{4.8}$$

So we have

$$g^\varepsilon(0^+,\tilde{\tau}) > g^\varepsilon(0^-,\tilde{\tau}) \geq \underline{g_0}. \tag{4.9}$$

(3) Region $\Omega_{\rm I}^\varepsilon$: For any point $(\tilde{y}, \tilde{\tau}) \in \Omega_{\rm I}^\varepsilon$, by Proposition 3.1, we have

$$g^\varepsilon(\tilde{y},\tilde{\tau}) \geq \frac{\min g^\varepsilon(0^-,\tau_0)}{1 + \min g^\varepsilon(0^-,\tau_0)\, B(\tilde{\tau} - \tau_0)} \geq \frac{A_1'}{1 + A_1' B\tau}, \tag{4.10}$$

where

$$A_1' = \min g^\varepsilon(0^-, \tau_0), \quad B = \frac{\left(Lip\left(Z_0^\varepsilon\right)\right)_+}{g_0}.$$

Combining the above cases, we obtain that the density has a lower bound in Ω under the Lagrangian coordinate, denoted by $\underline{g}$:

$$\underline{g} := \min\left\{\frac{A_1}{1 + A_2 B\tau}, \underline{g_0}, \frac{A_1'}{1 + A_1' B\tau}\right\}. \tag{4.11}$$

□

Remark 4.2. *In order to get a uniform lower bound of g^ε, for constant M, we need*

$$p\left(g^\varepsilon(y,\tau)\right) = p\left(g_0(y)\right) + \frac{v_0(y) - v_0\left(y_1^\varepsilon(0; y, \tau)\right)}{\varepsilon^2} \geq M > -\infty. \tag{4.12}$$

The above equation is equivalent to

$$v_0(y) \geq v_0\left(y_1^\varepsilon(0; y, \tau)\right) + \varepsilon^2\left(M - p\left(g_0(y)\right)\right). \tag{4.13}$$

As $\varepsilon \to 0$, the above equation is equivalent to the 0*-condition, which is necessary and sufficient for both* $\lim_{g\to 0} p(g) = 0$ *case and* $\lim_{g\to 0} p(g) = -\infty$ *case.*

For fixed sufficiently large M, let $\underline{\tau}_M := \underline{\lim}_{\varepsilon\to 0} \tau_M^\varepsilon$, see Lemma 4.5 for specific proof. Next, we have the following L^∞ estimates lemma.

Lemma 4.3 (L^∞ uniform estimations)**.** *On $\mathbb{R} \times \left[0, \underline{\tau}_M\right]$, for any $\varepsilon > 0$, g^ε has uniform bound with respect to ε in $L^\infty\left(\mathbb{R} \times \left[0, \underline{\tau}_M\right]\right)$; and v^ε has uniform bound with respect to ε in $Lip\left(\mathbb{R} \times \left[0, \underline{\tau}_M\right]\right)$.*

Proof. On $\mathbb{R} \times \left[0, \underline{\tau}_M\right]$, we have

$$\ln\frac{g^\varepsilon(y,\tau)}{g_0(y)} = \int_0^\tau \left(\ln g^\varepsilon(y,s)\right)_\tau ds = -\int_0^\tau \left(J^\varepsilon\right)^{-1} v_y^\varepsilon(y,s)\, ds \leq MT. \tag{4.14}$$

Therefore, on $\mathbb{R} \times \left[0, \underline{\tau}_M\right]$, let $\bar{g}_0 := \max_{y\in\mathbb{R}} g_0$ we obtain the uniform upper bound of g^ε.

$$g^\varepsilon \leq \bar{g}_0 e^{MT} =: \bar{g}_M. \tag{4.15}$$

Combining (4.11), then g^ε has uniform bound with respect to ε on $\mathbb{R} \times \left[0, \underline{\tau}_M\right]$:

$$\underline{g} \leq g^\varepsilon \leq \bar{g}_M. \tag{4.16}$$

By (2.17), v^ε has uniform bound with respect to ε

$$\min_{y\in\mathbb{R}} v_0(y) \leq v^\varepsilon(y,\tau) \leq \max_{y\in\mathbb{R}} v_0(y). \tag{4.17}$$

Next, we estimate the bound of $\left(J^\varepsilon\right)^{-1} v_y^\varepsilon$, due to

$$D\left(v_\tau^\varepsilon\right) = -\frac{I\left(g^\varepsilon\right)}{\varepsilon^2}\left(v_\tau^\varepsilon\right)^2 \leq 0. \tag{4.18}$$

So we have

$$\varepsilon^2 \left(g^\varepsilon p'\left(g^\varepsilon\right)\left(J^\varepsilon\right)^{-1} v_y^\varepsilon\right)(y,\tau) \leq \varepsilon^2 \left(g_0 p'\left(g_0\right) v_0'\right)\left(y_1^\varepsilon\left(0;y,\tau\right)\right). \tag{4.19}$$

g^ε is uniformly bounded for any ε and $p \in C^2\left(\mathbb{R}^+\right)$, so $p'\left(g^\varepsilon\right)$ is uniformly bounded. So we obtain the uniform upper bound of $\left(J^\varepsilon\right)^{-1} v_y^\varepsilon$ on $\mathbb{R}\times\left[0,\underline{\tau}_M\right]$

$$\left(\left(J^\varepsilon\right)^{-1} v_y^\varepsilon\right)(y,\tau) \leq \frac{\left(g_0 p'\left(g_0\right) v_0'\right)\left(y_1^\varepsilon\left(0;y,\tau\right)\right)}{\left(g^\varepsilon p'\left(g^\varepsilon\right)\right)(y,\tau)} \leq C_1(M), \tag{4.20}$$

On the other hand, due to

$$\left(J^\varepsilon\right)^{-1} v_y^\varepsilon \geq -M, \tag{4.21}$$

on $\mathbb{R}\times\left[0,\underline{\tau}_M\right]$, we have

$$\left\|\left(J^\varepsilon\right)^{-1} v_y^\varepsilon\right\|_\infty \leq \max\left\{C_1(M), M\right\} \leq C(M). \tag{4.22}$$

□

According to the transformation of Eulerian coordinates and Lagrangian coordinates, we have

$$\begin{cases} u_x^\varepsilon = \left(J^\varepsilon\right)^{-1} v_y^\varepsilon, \\ u_t^\varepsilon = -\left(v^\varepsilon - \varepsilon^2 g^\varepsilon p'\left(g^\varepsilon\right)\right)\left(J^\varepsilon\right)^{-1} v_y^\varepsilon. \end{cases} \tag{4.23}$$

Correspondingly we have the following lemma.

Lemma 4.4. *On $\mathbb{R}\times\left[0,\underline{t}_M\right]$, for any $\varepsilon>0$, ρ^ε has uniform bound with respect to ε in $L^\infty\left(\mathbb{R}\times\left[0,\underline{t}_M\right]\right)$; and u^ε has uniform bound with respect to ε in $Lip\left(\mathbb{R}\times\left[0,\underline{t}_M\right]\right)$.*

Before we discuss the convergence of ρ^ε and u^ε, we need to estimate the uniform lower bound of t_M^ε with respect to ε, which means that for M large enough, t_M^ε does not go to 0 as ε goes to 0.

Lemma 4.5. *For M is large enough, t_M^ε has a uniform positive lower bound.*

Proof. This claim is obviously true for $u_0' \geq 0$, we then prove it by contradiction for $u_0' < 0$. If the claim is false, we can find a subsequence $\{\varepsilon_n\}$ such that $t_M^{\varepsilon_n} \to 0$ as $n\to\infty$. By (2.37), in Eulerian coordinates, we can find a point $\left(x_M^{\varepsilon_n}, t_M^{\varepsilon_n}\right)$ such that

$$\begin{aligned} -M &= u_x^{\varepsilon_n}\left(x_M^{\varepsilon_n}, t_M^{\varepsilon_n}\right) \\ &= \frac{1}{\left(\rho^{\varepsilon_n} p'\left(\rho^{\varepsilon_n}\right)\right)\left(x_M^{\varepsilon_n}, t_M^{\varepsilon_n}\right)} \cdot \frac{\left(\rho_0 p'\left(\rho_0\right) u_0'\right)\left(x_1^{\varepsilon_n}\left(0; x_M^{\varepsilon_n}, t_M^{\varepsilon_n}\right)\right)}{1+\left(\rho_0 p'\left(\rho_0\right) u_0'\right)\left(x_1^{\varepsilon_n}\left(0; x_M^{\varepsilon_n}, t_M^{\varepsilon_n}\right)\right)\int_0^{t_M^{\varepsilon_n}} I\left(\rho^{\varepsilon_n}\left(x_M^{\varepsilon_n}, s\right)\right) ds}. \end{aligned} \tag{4.24}$$

For $\varepsilon_n > 0$, ρ^{ε_n} has uniform upper and lower bounds

$$\underline{\rho} \leq \rho^{\varepsilon_n} \leq \bar{\rho}_0 e^{M t_M^{\varepsilon_n}}. \tag{4.25}$$

Therefore, for $\rho^{\varepsilon_n} \in \left[\underline{\rho}, \bar{\rho}_0 e^{M t_M^{\varepsilon_n}}\right]$, $I(\rho)$ is a continuous function respect to ρ, $I\left(\rho^{\varepsilon_n}\right)$ has a uniform upper and lower bounds. By condition (1.3) satisfied by $p(\rho)$, we have

$$\left(\rho^2 p'(\rho)\right)' = 2\rho p'(\rho) + \rho^2 p''(\rho) > 0. \tag{4.26}$$

Combining the (4.25) and monotonicity of $\rho^2 p'(\rho)$, we have

$$\rho^{\varepsilon_n} p'(\rho^{\varepsilon_n}) = \frac{(\rho^{\varepsilon_n})^2 p'(\rho^{\varepsilon_n})}{\rho^{\varepsilon_n}} \geq \frac{\left(\underline{\rho}\right)^2 p'\left(\underline{\rho}\right)}{\bar{\rho}_0 e^{Mt_M^{\varepsilon_n}}}. \tag{4.27}$$

So, as $n \to \infty$,

$$\begin{aligned} -M \quad &\geq \quad \frac{\bar{\rho}_0 e^{Mt_M^{\varepsilon_n}}}{\left(\underline{\rho}\right)^2 p'\left(\underline{\rho}\right)} \cdot \frac{\min_{x\in\mathbb{R}}\left(\rho_0 p'(\rho_0) u_0'\right)}{1 + \left(\rho_0 p'(\rho_0) u_0'\right)\left(x_1^{\varepsilon_n}\left(0; x_M^{\varepsilon_n}, t_M^{\varepsilon_n}\right)\right)\int_0^{t_M^{\varepsilon_n}} I\left(\rho^{\varepsilon_n}\left(x_M^{\varepsilon_n}, s\right)\right) ds} \\ &\to \quad \frac{\bar{\rho}_0 \cdot \min_{x\in\mathbb{R}}\left(\rho_0 p'(\rho_0) u_0'\right)}{\left(\underline{\rho}\right)^2 p'\left(\underline{\rho}\right)}. \end{aligned} \tag{4.28}$$

On the other hand, when M is sufficiently large

$$\frac{\bar{\rho}_0 \cdot \min_{x\in\mathbb{R}}\left(\rho_0 p'(\rho_0) u_0'\right)}{\left(\underline{\rho}\right)^2 p'\left(\underline{\rho}\right)} \geq -M. \tag{4.29}$$

This contradicts with (4.28). Therefore, for each fix M large enough, t_M^{ε} has a uniformly positive lower bound, denoted by $\underline{T}$. □

Based on the uniform estimates discussed above, we then prove Theorem 1.5(1). There is a uniform domain defined as $\underline{t}_M := \underline{\lim}_{\varepsilon\to 0} t_M^{\varepsilon}$. For the compact set $H \subset \mathbb{R} \times \left[0, \underline{t}_M\right]$, the estimates we did above are uniform in H, we have

$$u^{\varepsilon} \to \bar{u} \text{ in } Lip(H), \quad \rho^{\varepsilon} \to \bar{\rho} \text{ in } \mathcal{M}(H). \tag{4.30}$$

As $\varepsilon \to 0$, combine the (2.5) and the convergence of $(u^{\varepsilon}, \rho^{\varepsilon})$, we have

$$x^{\varepsilon}(t) \to \bar{x}(t), \tag{4.31}$$

where $\bar{x}(t)$ is a Lipschitz function with respect to t.

Remark 4.6. *In Lagrangian coordinates, for $\tau \leq \underline{\tau}_M$ and any point (y, τ) in $\Omega_{\mathrm{I}}^{\varepsilon}$, when ε is sufficiently small, the backward characteristic line of this point is included in Ω_-. That is, when $\varepsilon \to 0$, $\Omega_{\mathrm{II}}^{\varepsilon} \to \Omega_-$ and $\Omega = \bar{\Omega}_+ \cup \Omega_-$.*

Next, we consider the consistency. For $\varepsilon > 0$ and $\varphi \in C_c^{\infty}(H)$, we have

$$\int_0^{\infty}\int_{\mathbb{R}} \rho^{\varepsilon}\varphi_t + \rho^{\varepsilon}u^{\varepsilon}\varphi_x dxdt + \int_{\mathbb{R}} \rho_0 \varphi(x, 0)\, dx = 0, \tag{4.32}$$

$$\begin{aligned} &\int_0^{\infty}\int_{\mathbb{R}} \rho^{\varepsilon}\left(u^{\varepsilon} + \varepsilon^2 p(\rho^{\varepsilon})\right)\varphi_t + \rho^{\varepsilon}u^{\varepsilon}\left(u^{\varepsilon} + \varepsilon^2 p(\rho^{\varepsilon})\right)\varphi_x dxdt \\ &+ \int_{\mathbb{R}} \rho_0\left(u_0 + \varepsilon^2 p(\rho_0)\right)\varphi(x, 0)\, dx = 0. \end{aligned} \tag{4.33}$$

As $\varepsilon \to 0$, by the convergence of $(\rho^\varepsilon, u^\varepsilon)$, the above equalities turn to

$$\int_0^\infty \int_{\mathbb{R}} \bar{\rho}\varphi_t + \bar{\rho}\bar{u}\varphi_x dxdt + \int_{\mathbb{R}} \rho_0 \varphi(x,0)\,dx = 0, \tag{4.34}$$

$$\int_0^\infty \int_{\mathbb{R}} \bar{\rho}\bar{u}\varphi_t + \bar{\rho}\bar{u}^2\varphi_x dxdt + \int_{\mathbb{R}} \rho_0 u_0 \varphi(x,0)\,dx = 0, \tag{4.35}$$

which show $(\bar{\rho}, \bar{u})$ satisfies (1.10) with initial value (1.4) in H. Finally, we consider the convergence of u_x^ε. For $\varphi \in C_c^\infty(H)$, as $\varepsilon \to 0$,

$$\int_0^\infty \int_{\mathbb{R}} (u_x^\varepsilon - \bar{u}_x)\,\varphi dxdt = -\int_0^\infty \int_{\mathbb{R}} (u^\varepsilon - \bar{u})\,\varphi_x dxdt \to 0. \tag{4.36}$$

So we have $u_x^\varepsilon \rightharpoonup \bar{u}_x$. Since for each H, u_x^ε are uniformly bounded respect to ε, then $u_x^\varepsilon \to \bar{u}_x$ in $L^\infty(H)$.

Then, in Lagrangian coordinates, we could have as $\varepsilon \to 0$

$$m^\varepsilon(\tau) \to m(\tau) := \inf_{y\in\mathbb{R}} \left\{\bar{J}^{-1}\bar{v}_y(y,\tau)\right\} \textit{ and } \lim_{\tau\uparrow T_b} m(\tau) = -\infty, \tag{4.37}$$

where T_b is defined in (3.41). Similar to τ_M^ε, for $M > 0$, we could introduce

$$\tau_M = \sup\left\{s : -M \le \inf_{\tau\in[0,s]} m(\tau)\,, s \le T\right\}. \tag{4.38}$$

According to the definition, $\{\tau_M\}$ is a monotone sequence with respect to M, and as $M \to \infty$

$$\lim_{M\to+\infty} \tau_M = \min\{T_b, T\} =: \tau_b. \tag{4.39}$$

By the convergence of the level set, we have

$$\lim_{\varepsilon\to 0} \tau_M^\varepsilon = \tau_M. \tag{4.40}$$

Next, we shows that

$$T_b \le \varliminf_{\varepsilon\to 0} T_b^\varepsilon. \tag{4.41}$$

Since $\tau_M^\varepsilon \le T_b^\varepsilon$. So we have

$$\lim_{M\to+\infty}\lim_{\varepsilon\to 0} \tau_M^\varepsilon \le \lim_{M\to+\infty}\varliminf_{\varepsilon\to 0} T_b^\varepsilon. \tag{4.42}$$

The left hand side holds

$$\lim_{M\to+\infty}\lim_{\varepsilon\to 0} \tau_M^\varepsilon = \lim_{M\to+\infty} \tau_M = T_b. \tag{4.43}$$

So we get (4.41).

5. Convergence of blow-up time

In this section, if $\rho_0 \in C_s^1(\mathbb{R})$, $u_0 \in C^1(\mathbb{R})$, we can further obtain the convergence of the blow-up time T_b^ε. First, we need to introduce the non-discontinuous regions as:

$$\Omega_M^\varepsilon = \Omega_+ \cup \Omega_{\mathrm{II}}^\varepsilon.$$

Similar to the previous definition, we define

$$\bar{m}^\varepsilon(\tau) := \inf_{(y,\tau)\in\Omega_M^\varepsilon} \left\{ (J^\varepsilon)^{-1} v_y^\varepsilon(y,\tau) \right\}. \tag{5.1}$$

For any fixed $\varepsilon > 0$ and the compressive initial data, there exists a finite life-span $\bar{T}_b^\varepsilon$, which is $+\infty$ for the rarefaction initial data. And we have

$$\lim_{\tau\uparrow\bar{T}_b^\varepsilon} \bar{m}^\varepsilon(\tau) = -\infty. \tag{5.2}$$

Further, for $M > 0$, we define $\bar{\tau}_M^\varepsilon$

$$\bar{\tau}_M^\varepsilon = \sup\left\{ s : -M \le \inf_{\tau\in[0,s]} \bar{m}^\varepsilon(\tau), s \le T \right\}. \tag{5.3}$$

According to the definition, $\left\{\bar{\tau}_M^\varepsilon\right\}$ is a monotone sequence with respect to M, and

$$\lim_{M\to+\infty} \bar{\tau}_M^\varepsilon = \min\left\{\bar{T}_b^\varepsilon, T\right\} =: \bar{\tau}_b^\varepsilon. \tag{5.4}$$

Before discussing the convergence of the blow-up time, we need to discuss the modulus of continuity estimation in Ω_M^ε.

The modulus of continuity of a function $f(x,t)$ is defined by the following non-negative function:

$$\eta(f \mid \varepsilon)(s) = \sup\left\{ |f(x_1,s) - f(x_2,s)| : s \in [0,t], |x_1 - x_2| < \varepsilon \right\}. \tag{5.5}$$

For modulus of continuity, there are the following properties, for more details please refer to [13].

Lemma 5.1. *For modulus of continuity*

(1) If $\epsilon_1 < \epsilon_2$, $\eta(f \mid \epsilon_1) < \eta(f \mid \epsilon_2)$.

(2) For any positive number C, $\eta(f \mid C\epsilon) \le (C+1)\eta(f \mid \epsilon)$.

(3) For f and g that are two continuous functions, $\eta(f \pm g \mid \epsilon) \le \eta(f \mid \epsilon) + \eta(g \mid \epsilon)$, and $\eta(fg \mid \epsilon) \le \eta(f \mid \epsilon)\|g\|_{L^\infty} + \eta(g \mid \epsilon)\|f\|_{L^\infty}$.

(4) If $f \in C^\alpha$, $0 < \alpha \le 1$, if and only if there exists a constant L such that $\eta(f \mid \epsilon) \le L\epsilon^\alpha$.

Define

$$\eta(f_1, f_2, \ldots, f_n \mid \epsilon)(t) := \max\left\{ \eta(f_i \mid \epsilon)(t) : i = 1, \ldots, n \right\}. \tag{5.6}$$

By definition, $\eta(f_1, f_2, \ldots, f_n \mid \epsilon)(t)$ has similar properties in Lemma 5.1. So we have the following lemma:

Lemma 5.2. *If $u_0 \in C^1(\mathbb{R})$ and $g_0 \in C^1(\mathbb{R}_- \cup \mathbb{R}_+)$. Then on the Ω_M^ε, the modulus of continuity of $(J^\varepsilon)^{-1} v_y^\varepsilon$ and v_τ^ε is uniform, which means $(J^\varepsilon)^{-1} v_y^\varepsilon$ and v_τ^ε is equi-continuous.*

Proof. By (1.1), we have the following equation

$$D\left(\frac{v_y^\varepsilon}{J^\varepsilon}\right) = -\left(2 + \frac{g^\varepsilon p''(g^\varepsilon)}{p'(g^\varepsilon)}\right)\left(\frac{v_y^\varepsilon}{J^\varepsilon}\right)^2 + \left(1 + \frac{g^\varepsilon p''(g^\varepsilon)}{p'(g^\varepsilon)}\right)\frac{g^\varepsilon}{g_0} Z_y^\varepsilon \frac{v_y^\varepsilon}{J^\varepsilon}. \tag{5.7}$$

And v_τ can be expressed by $\frac{v_y^\varepsilon}{J^\varepsilon}$ and the lower order terms. Thus we only need to prove the modulus of continuity of $\frac{v_y^\varepsilon}{J^\varepsilon}$. To prove this, we first prove the modulus of continuity of the characteristics line. Then combine with the regularity of the lower order terms, we obtain the modulus of continuity of $\frac{v_y^\varepsilon}{J^\varepsilon}$.

By the Lemma 4.4, we have that g^ε and J^ε are uniformly Lipschitzs functions on the Ω_M^ε. Thus for the characteristic line, for $\epsilon > 0$ and $|\tilde{y}_1 - \tilde{y}_2| < \epsilon$, without loss of generality, we assume $s < \tau$

$$\begin{aligned}
& \left|y_1^\varepsilon(\tau; \tilde{y}_1, s) - y_1^\varepsilon(\tau; \tilde{y}_2, s)\right| \\
\leq\ & |\tilde{y}_1 - \tilde{y}_2| + \int_s^\tau \left|\mu^\varepsilon\left(y_1^\varepsilon(s'; \tilde{y}_1, s), s'\right) - \mu^\varepsilon\left(y_1^\varepsilon(s'; \tilde{y}_2, s), s'\right)\right| ds' \\
\leq\ & |\tilde{y}_1 - \tilde{y}_2| + C(M) \int_s^\tau \left|y_1^\varepsilon(s'; \tilde{y}_1, s) - y_1^\varepsilon(s'; \tilde{y}_2, s)\right| ds'.
\end{aligned} \tag{5.8}$$

By Grönwall's inequality, we conclude that

$$\left|y_1^\varepsilon(\tau; \tilde{y}_1, s) - y_1^\varepsilon(\tau; \tilde{y}_2, s)\right| \leq |\tilde{y}_1 - \tilde{y}_2|\, e^{C(M)(\tau - s)} \leq |\tilde{y}_1 - \tilde{y}_2|\, e^{C(M)\tau}. \tag{5.9}$$

Then we consider the modulus of continuity of $\frac{v_y^\varepsilon}{J^\varepsilon}$. We consider the estimation on two directions: characteristic direction and y-direction. For characteristic direction, because g^ε and v^ε are uniform bounded in $C^1(\Omega_M^\varepsilon)$,

$$\left|\frac{v_y^\varepsilon}{J^\varepsilon}\left(y_1^\varepsilon(\tau_1; \tilde{y}, \tilde{\tau}), \tau_1\right) - \frac{v_y^\varepsilon}{J^\varepsilon}\left(y_1^\varepsilon(\tau_2; \tilde{y}, \tilde{\tau}), \tau_2\right)\right| \leq \left|\int_{\tau_1}^{\tau_2} D\left(\frac{v_y^\varepsilon}{J^\varepsilon}\right) ds\right| \leq C(M)\,|\tau_1 - \tau_2|. \tag{5.10}$$

Now we consider the $y-$direction in which we need the (5.9). For convenient, we use the denotation $Y^i(s) = \left(y^i(s), s\right) = \left(y_1^\varepsilon(s; \tilde{y}_i, 0), s\right)$. Thus for any $(\tilde{y}_i, 0) \in \Omega_M^\varepsilon$, $i = 1, 2$. Let

$$G(g^\varepsilon) = \frac{g^\varepsilon p''(g^\varepsilon)}{p'(g^\varepsilon)}. \tag{5.11}$$

By (5.7) we have

$$\begin{aligned}
& \left|\frac{v_y^\varepsilon}{J^\varepsilon}\left(Y^1(\tau)\right) - \frac{v_y^\varepsilon}{J^\varepsilon}\left(Y^2(\tau)\right)\right| \\
\leq\ & \left|v_0'(\tilde{y}_1) - v_0'(\tilde{y}_2)\right| + C(M)\int_0^\tau \left|(Z_0^\varepsilon)_y\left(y^1(s)\right) - (Z_0^\varepsilon)_y\left(y^2(s)\right)\right| ds \\
& + C(M)\int_0^\tau \left(\left|g^\varepsilon\left(Y^1(s)\right) - g^\varepsilon\left(Y^2(s)\right)\right| + \left|G\left(g^\varepsilon\left(Y^1(s)\right)\right) - G\left(g^\varepsilon\left(Y^2(s)\right)\right)\right|\right) ds \\
& + C(M)\int_0^\tau \left|\frac{v_y^\varepsilon}{J^\varepsilon}\left(Y^1(s)\right) - \frac{v_y^\varepsilon}{J^\varepsilon}\left(Y^2(s)\right)\right| ds
\end{aligned}$$

$$\leq \eta\left(v_0'\middle| |\tilde{y}_1-\tilde{y}_2|\right)(0)+C(M)\int_0^\tau \eta\left((Z_0^\varepsilon)_y\middle|\left|y^1(s)-y^2(s)\right|\right)(0)\,ds$$
$$+C(M)\int_0^\tau\left(\eta\left(g^\varepsilon\middle|\left|Y^1(s)-Y^2(s)\right|\right)(s)+\eta\left(G(g^\varepsilon)\middle|\left|Y^1(s)-Y^2(s)\right|\right)(s)\right)ds$$
$$+C(M)\int_0^\tau \eta\left(\frac{v_y^\varepsilon}{J^\varepsilon}\middle|\left|Y^1(s)-Y^2(s)\right|\right)(s)\,ds. \tag{5.12}$$

For first term we have

$$\eta\left(v_0'\middle| |\tilde{y}_1-\tilde{y}_2|\right)(0)\leq \eta\left(v_0'|\epsilon\right)(0). \tag{5.13}$$

The second term we have

$$\eta\left((Z_0^\varepsilon)_y\middle|\left|y^1(s)-y^2(s)\right|\right)(0)\leq \eta\left((Z_0^\varepsilon)_y\middle|e^{C(M)s}\epsilon\right)(0)\leq\left(e^{C(M)s}+1\right)\eta\left((Z_0^\varepsilon)_y\middle|\epsilon\right)(0). \tag{5.14}$$

For the third term we have

$$\eta\left(g^\varepsilon\middle|\left|Y^1(s)-Y^2(s)\right|\right)(s)\leq C(M)\left|y^1(s)-y^2(s)\right|\leq C(M)e^{C(M)s}\epsilon, \tag{5.15}$$

and

$$\eta\left(G(g^\varepsilon)\middle|\left|Y^1(s)-Y^2(s)\right|\right)(s)\leq\eta\left(G(g^\varepsilon)\middle|C(M)e^{C(M)s}\epsilon\right)(s)\leq\left(C(M)e^{C(M)s}+1\right)\eta\left(G\middle|\epsilon\right)(s). \tag{5.16}$$

For the last term, we have

$$\eta\left(\frac{v_y^\varepsilon}{J^\varepsilon}\middle|\left|Y^1(s)-Y^2(s)\right|\right)(s)\leq\eta\left(\frac{v_y^\varepsilon}{J^\varepsilon}\middle|e^{C(M)s}\epsilon\right)(s)\leq\left(e^{C(M)s}+1\right)\eta\left(\frac{v_y^\varepsilon}{J^\varepsilon}\middle|\epsilon\right)(s). \tag{5.17}$$

There exists a constant $C'(M)$ such that

$$\eta\left(\frac{v_y^\varepsilon}{J^\varepsilon}\middle|\epsilon\right)(s)\leq C'(M)\left(\left(\epsilon+\eta\left(v_0',(Z_0^\varepsilon)_y\middle|\epsilon\right)(0)\right)+\int_0^\tau\eta\left(G\middle|\epsilon\right)(s)\,ds\right)$$
$$+C'(M)\int_0^\tau\eta\left(\frac{v_y^\varepsilon}{J^\varepsilon}\middle|\epsilon\right)(s)\,ds. \tag{5.18}$$

For any $\varepsilon>0$, because $\tilde{y}_i$ are chosen arbitrary on the Ω_M^ε. Through the Grönwall's inequality, the above inequality equals to

$$\eta\left(\frac{v_y^\varepsilon}{J^\varepsilon}\middle|\epsilon\right)\leq C'(M)e^{C'(M)T}\left(\epsilon+\eta\left(v_0',(Z_0^\varepsilon)_y\middle|\epsilon\right)(0)+\int_0^\tau\eta\left(G\middle|\epsilon\right)(s)\,ds\right). \tag{5.19}$$

Since G is a continuous function, the modulus of continuity of $\frac{v_y^\varepsilon}{J^\varepsilon}$ on Ω_M^ε is uniform. □

For a compact set $H\subset\Omega_M^\varepsilon$, we could lift the convergence as $(J^\varepsilon)^{-1}v_y^\varepsilon\to\bar{J}^{-1}\bar{v}_y$ in $C(H)$ as $\varepsilon\to0$. Then, for $(y,\tau)\in H$, we could have as $\varepsilon\to0$,

$$\bar{m}^\varepsilon(\tau)\to m(\tau). \tag{5.20}$$

By the convergence of the level set, we have

$$\lim_{\varepsilon\to0}\bar{\tau}_M^\varepsilon=\tau_M. \tag{5.21}$$

Furthermore, we consider the convergence of the blow-up time for $I(g)$ satisfying the following conditions:

(a) $I(g)$ is a increasing function with respect to g;

(b) There exists a small δ such that $\int_0^\delta \frac{I(s)}{s^2} ds = +\infty$.

For example, $p(g) = \ln g$ meets the above conditions. Using the idea in [20], we have the following lemma.

Lemma 5.3. *For $M > 0$, if $I(g)$ meet the above conditions (1) and (2), then we have*

$$\lim_{\varepsilon \to 0} T_b^\varepsilon = T_b. \tag{5.22}$$

Proof. **Case 1:** $\inf \left\{ v_0'(y) \right\} \geq 0$.

In this case, $T_b = +\infty$. Let $\alpha^\varepsilon := g^\varepsilon p'(g^\varepsilon)(J^\varepsilon)^{-1} v_y^\varepsilon$, by (2.37)

$$\alpha^\varepsilon(y,0) = \alpha_0(y) = g_0 p'(g_0) v_0'(y) \geq 0. \tag{5.23}$$

In this case we want to show $\lim_{\varepsilon \to 0} T_b^\varepsilon = +\infty$. If the claim is false, we can find a subsequence $\{\varepsilon_n\}$ such that $\left\{ T_b^{\varepsilon_n} \right\}$ is bounded. For any ε_n, there exist $y_1^{\varepsilon_n}$ defined by (2.15) such that when $\tau \uparrow T_b^{\varepsilon_n}$,

$$\lim_{\tau \uparrow T_b^{\varepsilon_n}} \alpha^{\varepsilon_n} = -\infty, \tag{5.24}$$

which

$$1 + \alpha^{\varepsilon_n}\left(y_1^{\varepsilon_n}(0;y,\tau),0\right) \int_0^{T_b^{\varepsilon_n}} I\left(g^{\varepsilon_n}\left(y_1^{\varepsilon_n}(s;y,\tau),s\right)\right) ds = 0. \tag{5.25}$$

The above equation is equivalent to

$$-\frac{1}{\alpha^{\varepsilon_n}\left(y_1^{\varepsilon_n}(0;y,\tau)\right)} = \int_0^{T_b^{\varepsilon_n}} I\left(g^{\varepsilon_n}\left(y_1^{\varepsilon_n}(s;y,\tau),s\right)\right) ds. \tag{5.26}$$

The right hand side is uniformly bounded by a finite positive constant

$$\int_0^{T_b^{\varepsilon_n}} I\left(g^{\varepsilon_n}\left(y_1^{\varepsilon_n}(s;y,\tau),s\right)\right) ds \leq C. \tag{5.27}$$

For α^{ε_n}

$$\alpha^{\varepsilon_n}\left(y_1^{\varepsilon_n}(0;y,\tau)\right) \leq -\frac{1}{C} < 0. \tag{5.28}$$

This contradicts with (5.23), so we have

$$\lim_{\varepsilon \to 0} T_b^\varepsilon = +\infty = T_b. \tag{5.29}$$

Case 2: $\inf \left\{ v_0'(y) \right\} < 0$.

For large M, we first proof that $\bar{T}_b^\varepsilon$ is uniformly bounded. If the claim is false, we can find a subsequence $\{\varepsilon_n\}$ such that $\bar{T}_b^{\varepsilon_n} \to +\infty$ as $n \to +\infty$. And there exist $y_1^{\varepsilon_n}$ defined by (2.15) such that when $\tau \uparrow \bar{T}_b^{\varepsilon_n}$,

$$1 + \alpha^{\varepsilon_n}\left(y_1^{\varepsilon_n}(0;y,\tau),0\right) \int_0^{\bar{T}_b^{\varepsilon_n}} I\left(g^{\varepsilon_n}\left(y_1^{\varepsilon_n}(s;y,\tau),s\right)\right) ds = 0. \tag{5.30}$$

By (4.11) and the monotonicity of $I(\rho)$, there exists a constant C such that

$$C \geq -\frac{1}{\alpha^{\varepsilon_n}\left(y_1^{\varepsilon_n}(0;y,\tau)\right)} = \int_0^{\bar{T}_b^{\varepsilon_n}} I\left(g^{\varepsilon_n}\left(y_1^{\varepsilon_n}(s;y,\tau),s\right)\right)ds \geq \int_0^{\bar{T}_b^{\varepsilon_n}} I\left(\frac{A_1}{1+A_2Bs}\right)ds. \tag{5.31}$$

Let $n \to \infty$ we have

$$C \geq \int_0^{+\infty} I\left(\frac{A_1}{1+A_2Bs}\right)ds = A_2B\int_0^{+\infty} I\left(\frac{A_1}{1+s}\right)ds. \tag{5.32}$$

The definitions of A_1, A_2 and B, see Proposition 4.1. Let $y = \frac{A_1}{1+s}$, we have

$$C \geq A_1A_2B\int_0^{A_1} \frac{I(y)}{y^2}ds = A_2B\int_0^{+\infty} I\left(\frac{A_1}{1+s}\right)ds. \tag{5.33}$$

On the other hand, there exists δ such that

$$\int_0^{\delta} \frac{I(y)}{y^2}ds = +\infty. \tag{5.34}$$

This contradicts with (5.33), so $\bar{T}_b^{\varepsilon}$ is bounded with respect to ε.

We set

$$-2\delta_0 := \inf\{v_0'\} < 0, \tag{5.35}$$

where δ_0 is a constant independent from ε. When ε is small enough such that

$$\varepsilon^2\left\|(p(g_0))_y\right\|_{L^\infty} < \frac{\delta_0}{2}. \tag{5.36}$$

Then on Ω_M^{ε} there exists $y_1^{\varepsilon}(s;\tilde{y},0)$

$$v_0'(\tilde{y}) = -2\delta_0, \tag{5.37}$$

when $\varepsilon < \varepsilon_0$ is small enough, $\tilde{y}$ and $y_1^{\varepsilon}\left(\bar{T}_b^{\varepsilon};\tilde{y},0\right)$ are close enough such that for $y \in \left[y_1^{\varepsilon}\left(\bar{T}_b^{\varepsilon};\tilde{y},0\right),\tilde{y}\right]$,

$$v_0'(y) \leq -\frac{3}{2}\delta_0. \tag{5.38}$$

For $y \in \left[y_1^{\varepsilon}\left(\bar{T}_b^{\varepsilon};\tilde{y},0\right),\tilde{y}\right]$, we have

$$(Z_0^{\varepsilon})_y(y) = v_0'(y) + \varepsilon^2(p(g_0(y)))_y \leq -\delta_0. \tag{5.39}$$

Next, we want to show, for ε small enough, there exists M_0 independent of M such that

$$\bar{T}_b^{\varepsilon} - \bar{\tau}_M^{\varepsilon} \leq \frac{M_0}{M}. \tag{5.40}$$

For g^{ε}, we have

$$\int_{\bar{\tau}_M^{\varepsilon}}^{\bar{T}_b^{\varepsilon}} D\left(\frac{1}{g^{\varepsilon}}\right)(y_1^{\varepsilon}(s;\tilde{y},0),s)\,ds = \int_{\bar{\tau}_M^{\varepsilon}}^{\bar{T}_b^{\varepsilon}} g_0^{-1}(Z_0^{\varepsilon})_y(y_1^{\varepsilon}(s;\tilde{y},0),s)\,ds. \tag{5.41}$$

The above equation is equivalent to

$$\begin{aligned}\frac{1}{g^{\varepsilon}\left(y_1^{\varepsilon}\left(\bar{\tau}_M^{\varepsilon};\tilde{y},0\right),\bar{\tau}_M^{\varepsilon}\right)} &\geq \frac{1}{g^{\varepsilon}\left(y_1^{\varepsilon}\left(\bar{\tau}_M^{\varepsilon};\tilde{y},0\right),\bar{\tau}_M^{\varepsilon}\right)}-\frac{1}{g^{\varepsilon}\left(y_1^{\varepsilon}\left(\bar{T}_b^{\varepsilon};\tilde{y},0\right),\bar{T}_b^{\varepsilon}\right)}\\ &= \int_{\bar{\tau}_M^{\varepsilon}}^{\bar{T}_b^{\varepsilon}} -g_0^{-1}\left(Z_0^{\varepsilon}\right)_y\left(y_1^{\varepsilon}\left(s;\tilde{y},0\right)\right)\\ &\geq \frac{\delta_0}{\bar{g}_0}\left(\bar{T}_b^{\varepsilon}-\bar{\tau}_M^{\varepsilon}\right). \end{aligned} \tag{5.42}$$

When M is large enough, by the convergence of $(g^{\varepsilon}, v^{\varepsilon})$ and the explicit expression of $(\bar{g}, \bar{v})$, we have for $\tau \in [0, \tau_M]$

$$\left|\frac{\bar{J}^{-1}\bar{v}_y\left(\tilde{y},\tau_M\right)}{g^{\varepsilon}\left(y_1^{\varepsilon}\left(\bar{\tau}_M^{\varepsilon};\tilde{y},0\right),\bar{\tau}_M^{\varepsilon}\right)}\right| \to \left|\frac{v_0'(y)}{g_0(y)}\right|. \tag{5.43}$$

There exists M_0 independent of M such that

$$\left|\frac{v_0'(y)}{g_0(y)}\right| < \frac{M_0}{2}. \tag{5.44}$$

Then, for ε small enough

$$\left|\frac{\bar{J}^{-1}\bar{v}_y\left(\tilde{y},\tau_M\right)}{g^{\varepsilon}\left(y_1^{\varepsilon}\left(\bar{\tau}_M^{\varepsilon},\tilde{y},0\right),\bar{\tau}_M^{\varepsilon}\right)}\right| < M_0. \tag{5.45}$$

Thus

$$\frac{1}{g^{\varepsilon}\left(y_1^{\varepsilon}\left(\bar{\tau}_M^{\varepsilon},\tilde{y},0\right),\bar{\tau}_M^{\varepsilon}\right)} < \frac{M_0}{\left|\bar{J}^{-1}\bar{v}_y\left(\tilde{y},\tau_M\right)\right|} = \frac{M_0}{M}. \tag{5.46}$$

By (5.42) and (5.46) we obtain the following inequality

$$\bar{T}_b^{\varepsilon} - \bar{\tau}_M^{\varepsilon} \leq \frac{\bar{g}_0}{\delta_0}\cdot\frac{M_0}{M}. \tag{5.47}$$

Next, we prove that $\lim_{\varepsilon\to 0}\bar{T}_b^{\varepsilon} = T_b$. There exists y such that

$$\tau_M = -\frac{1}{v_0'(y)} - \frac{1}{M}, \quad T_b = -\frac{1}{v_0'(y)}. \tag{5.48}$$

Thus we have

$$|\tau_M - T_b| \leq \frac{1}{M}. \tag{5.49}$$

Then we can choose M large enough such that

$$\left|\bar{T}_b^{\varepsilon} - \bar{\tau}_M^{\varepsilon}\right| + |\tau_M - T_b| < \frac{\delta}{2}. \tag{5.50}$$

Since $(g^{\varepsilon}, v^{\varepsilon})$ uniformly converges to $(\bar{g}, \bar{v})$ on any level set, there exists ε_1 such that for $\varepsilon \in (0, \varepsilon_1]$, we have

$$\left|\bar{\tau}_M^{\varepsilon} - \tau_M\right| < \frac{\delta}{2}. \tag{5.51}$$

Combined with the above discussion, when M is sufficiently large, for any $\delta > 0$, there exists ε_1 such that for $\varepsilon \in (0, \varepsilon_1]$

$$\left|\bar{T}_b^\varepsilon - T_b\right| \leq \left|\bar{T}_b^\varepsilon - \bar{\tau}_M^\varepsilon\right| + \left|\bar{\tau}_M^\varepsilon - \tau_M\right| + |\tau_M - T_b| < \delta. \tag{5.52}$$

Therefore, we have

$$\lim_{\varepsilon\to 0} \bar{T}_b^\varepsilon = T_b. \tag{5.53}$$

Finally, we need to prove $\lim_{\varepsilon\to 0} T_b^\varepsilon = T_b$. By the definition of $\bar{T}_b^\varepsilon$, we have

$$T_b = \lim_{\varepsilon\to 0} \bar{T}_b^\varepsilon \geq \overline{\lim_{\varepsilon\to 0}}\, T_b^\varepsilon \geq \underline{\lim_{\varepsilon\to 0}}\, T_b^\varepsilon. \tag{5.54}$$

On the other hand, by (4.41) we have

$$T_b \leq \underline{\lim_{\varepsilon\to 0}}\, T_b^\varepsilon. \tag{5.55}$$

Therefore, we have

$$\overline{\lim_{\varepsilon\to 0}}\, T_b^\varepsilon = \underline{\lim_{\varepsilon\to 0}}\, T_b^\varepsilon = \lim_{\varepsilon\to 0} T_b^\varepsilon = T_b. \tag{5.56}$$

□

6. Convergence rates

In this section, we will consider the convergence rates of solution on each $[0, t_M]$ in Eulerian coordinates. For u^ε and z^ε, we have the following lemma.

Lemma 6.1. *For $M > 0$, on $\mathbb{R} \times [0, t_M]$, $u^\varepsilon \sim z^\varepsilon \sim \lambda_1^\varepsilon \sim \lambda_2^\varepsilon \sim \bar{x} \sim x_2^\varepsilon \sim \bar{u}\left(O\left(\varepsilon^2\right)\right)$.*

Remark 6.2. *$\lambda_1^\varepsilon \sim \lambda_2^\varepsilon \sim \bar{u}\left(O\left(\varepsilon^2\right)\right)$, the characteristics triangle vanishing as the ε^2 order.*

Proof. In order to simplify the calculation, we set

$$\begin{cases} x_1^\varepsilon(s) = x_1^\varepsilon(s; \tilde{x}, \tilde{t}), \\ x_2^\varepsilon(s) = x_2^\varepsilon(s; \tilde{x}, \tilde{t}), \\ \bar{x}^\varepsilon(s) = \bar{x}^\varepsilon(s; \tilde{x}, \tilde{t}), \end{cases} \tag{6.1}$$

which are defined at (2.5) and (3.33). On $\mathbb{R} \times [0, t_M]$

$$\begin{aligned} \left|x_1^\varepsilon(s) - x_2^\varepsilon(s)\right| &= \left|\int_{\tilde{t}}^s \left(\lambda_1^\varepsilon\left(x_1^\varepsilon(t), t\right) - \lambda_2^\varepsilon\left(x_2^\varepsilon(t), t\right)\right) dt\right| \\ &\leq \int_{\tilde{t}}^s \left|\lambda_1^\varepsilon\left(x_1^\varepsilon(t), t\right) - \lambda_2^\varepsilon\left(x_2^\varepsilon(t), t\right)\right| dt \\ &\leq \int_{\tilde{t}}^s \left|u^\varepsilon\left(x_1^\varepsilon(t), t\right) - u^\varepsilon\left(x_2^\varepsilon(t), t\right)\right| dt + C'(M)\, s\varepsilon^2 \\ &\leq C''(M) \int_{\tilde{t}}^s \left|x_1^\varepsilon(t) - x_2^\varepsilon(t)\right| dt + C'(M)\, s\varepsilon^2. \end{aligned} \tag{6.2}$$

By Grönwall's inequality

$$\left|x_1^\varepsilon(s) - x_2^\varepsilon(s)\right| \leq C'(M)\, s\varepsilon^2 \left(e^{C''(M)\left(s-\tilde{t}\right)} - 1\right) \leq C(M)\,\varepsilon^2. \tag{6.3}$$

Here, $C(M)$ is a uniform constant with respect to ε. Thus, we can see the different characteristic lines converge with the rate $O\left(\varepsilon^2\right)$ on $\mathbb{R}\times[0,t_M]$. Then we can have the following estimate of z^ε and u^ε

$$\begin{aligned}\left|z^\varepsilon\left(x_2^\varepsilon(t),t\right)-u^\varepsilon\left(x_1^\varepsilon(t),t\right)\right| &\le \left|u^\varepsilon\left(x_2^\varepsilon(t),t\right)-u^\varepsilon\left(x_1^\varepsilon(t),t\right)\right|+C'(M)\varepsilon^2\\ &\le C''(M)\left|x_1^\varepsilon(s)-x_2^\varepsilon(s)\right|+C'(M)\varepsilon^2\\ &\le \left(C''(M)C(M)+C'(M)\right)\varepsilon^2\\ &\le C'''(M)\varepsilon^2. \end{aligned}\tag{6.4}$$

It means that z^ε converges to u^ε with the rate of ε^2. By the same way, we can prove that $\lambda_1^\varepsilon\sim\lambda_2^\varepsilon\sim z^\varepsilon\sim u^\varepsilon\left(O\left(\varepsilon^2\right)\right)$ on $\mathbb{R}\times[0,t_M]$. Next, we consider the convergence rate of u^ε to $\bar{u}$,

$$\begin{aligned}\left|\bar{x}(s)-x_1^\varepsilon(s;\bar{x}(0),0)\right| &\le \int_0^s\left|\bar{u}(\bar{x}(t),t)-\lambda_1^\varepsilon\left(x_1^\varepsilon(t;\bar{x}(0),0),t\right)\right|dt\\ &\le \int_0^s\left|\bar{u}(\bar{x}(t),t)-u^\varepsilon\left(x_1^\varepsilon(t;\bar{x}(0),0),t\right)\right|dt+C'(M)s\varepsilon^2\\ &\le \int_0^s\left|u_0(\bar{x}(0))-u_0(\bar{x}(0))\right|ds+C'(M)s\varepsilon^2\\ &\le C(M)\varepsilon^2. \end{aligned}\tag{6.5}$$

Then, we can have the following estimate of u^ε and $\bar{u}$

$$\begin{aligned}\left|u^\varepsilon(\tilde{x},\tilde{t})-\bar{u}(\tilde{x},\tilde{t})\right| &\le \left|u^\varepsilon(\tilde{x},\tilde{t})-u^\varepsilon\left(x_1^\varepsilon(\tilde{t};\bar{x}(0),0),\tilde{t}\right)\right|+\left|u^\varepsilon\left(x_1^\varepsilon(\tilde{t};\bar{x}(0),0),\tilde{t}\right)-\bar{u}(\bar{x}(\tilde{t}),\tilde{t})\right|\\ &\le \left|u^\varepsilon(\tilde{x},\tilde{t})-u^\varepsilon\left(x_1^\varepsilon(\tilde{t};\bar{x}(0),0),\tilde{t}\right)\right|+\left|u_0(\bar{x}(0))-u_0(\bar{x}(0))\right|\\ &\le C''(M)\left|\tilde{x}-x_1^\varepsilon(\tilde{t};\bar{x}(0),0)\right|\\ &\le C''(M)\left|\bar{x}(\tilde{t})-x_1^\varepsilon(\tilde{t};\bar{x}(0),0)\right|\\ &\le C''(M)C(M)\varepsilon^2. \end{aligned}\tag{6.6}$$

We conclude that $z^\varepsilon\sim u^\varepsilon\sim\bar{u}\left(O\left(\varepsilon^2\right)\right)$.

Finally, we prove the convergence of discontinuous lines $\bar{x}(t)$ and $x_1^\varepsilon(t)$. Without loss of generality, we consider that there is only one discontinuity at $x=0$, and discuss the convergence of the discontinuity lines passing through 0. For the case of separable discontinuous points, we have similar results. By (6.6) and the uniform boundedness of u_x^ε, we have

$$\begin{aligned}&\left|\bar{x}(s;0,0)-x_2^\varepsilon(s;0,0)\right|\\ \le\ &\int_0^s\left|\bar{u}(\bar{x}(t;0,0),t)-u^\varepsilon\left(x_2^\varepsilon(t;0,0),t\right)\right|dt\\ \le\ &\int_0^s\left|\bar{u}(\bar{x}(t;0,0),t)-u^\varepsilon(\bar{x}(t;0,0),t)\right|+\left|u^\varepsilon(\bar{x}(t;0,0),t)-u^\varepsilon\left(x_2^\varepsilon(t;0,0),t\right)\right|dt\\ \le\ &C(M)s\varepsilon^2+\int_0^s\left|u^\varepsilon(\bar{x}(t;0,0),t)-u^\varepsilon\left(x_2^\varepsilon(t;0,0),t\right)\right|dt\\ \le\ &C(M)s\varepsilon^2+C'(M)\int_0^s\left|\bar{x}(t;0,0)-x_2^\varepsilon(t;0,0)\right|dt. \end{aligned}\tag{6.7}$$

By Grönwall's inequality

$$\left|\bar{x}(s;0,0) - x_2^{\varepsilon}(s;0,0)\right| \le C(M)s\varepsilon^2 e^{C'(M)s} \le C''(M)\varepsilon^2. \tag{6.8}$$

We conclude that the discontinuous lines converges at the rate of ε^2. □

7. Conclusions

This paper investigates the well-posedness of contact discontinuity solutions and the vanishing pressure limit for the Aw-Rascle traffic flow model.

For the well-posedness of contact discontinuity solutions, the Lagrangian coordinate transformation is employed to fix the discontinuity boundary. A novel method establishes the positive lower bound of density at the discontinuity, governed by a necessary and sufficient ε-condition based on the initial density jump. The result show: rarefactive initial data yields global solutions; compressive data causes finite-time singularity. The discontinuity curve is C^1 with a Lipschitz tangent. Well-posedness for the pressureless fluid model is also proven.

For the vanishing pressure limit to the pressureless fluid model, a 0-condition ensures the uniform lower bound of density during the limit. By employing level sets in the Lagrangian coordinates, uniform boundedness estimates for density and velocity derivatives are derived, proving convergence of the solutions. For a class of pressure functions, blow-up times away from the discontinuity converge as the pressure vanishes. Finally, we establish the convergence rate of the solution.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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