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Research article

Optimization problems in rearrangement classes for fractional p-Laplacian equations

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Abstract: We discuss two optimization problems related to the fractional p-Laplacian. First, we prove the existence of at least one minimizer for the principal eigenvalue of the fractional p-Laplacian with Dirichlet conditions, with a bounded weight function varying in a rearrangement class. Then, we investigate the minimization of the energy functional for general nonlinear equations driven by the same operator, as the reaction varies in a rearrangement class. In both cases, we provide a pointwise relation between the optimizing datum and the corresponding solution.

Keywords: rearrangement class; fractional p-Laplacian; eigenvalues

1. Introduction and main results

The present paper deals with some optimization problems related to elliptic equations of nonlinear, nonlocal type, with data varying in rearrangement classes. For the reader's convenience, we recall here the basic definition, referring to Section 2 for details. Given a bounded smooth domain $\Omega \subset \mathbb{R}^N$ and a non-negative function $g_0 \in L^\infty(\Omega)$, we say that $g \in L^\infty(\Omega)$ lies in the rearrangement class of g_0, denoted $\mathcal{G}$, if for all $t \geqslant 0$

$$\Big|\{g > t\}\Big| = \Big|\{g_0 > t\}\Big|,$$

where we denote by $|\cdot|$ the N-dimensional Lebesgue measure of sets. We may define several functionals $\Phi : \mathcal{G} \to \mathbb{R}$ corresponding to variational problems, and study the optimization problems

$$\min_{g\in\mathcal{G}} \Phi(g), \quad \max_{g\in\mathcal{G}} \Phi(g).$$

We note that, since $\mathcal{G}$ is not a convex set, the problems above do not fall in the familiar case of convex optimization, whatever the nature of Φ. The following is a classical example. For all $g \in \mathcal{G}$ consider the Dirichlet problem

$$\begin{cases} -\Delta u = g(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

which, by classical results in the calculus of variations, admits a unique weak solution $u_g \in H_0^1(\Omega)$. So set

$$\Phi(g) = \int_\Omega g u_g \, dx.$$

The existence of a maximizer for Φ, i.e., of a datum $\hat{g} \in \mathcal{G}$ s.t. for all $g \in \mathcal{G}$

$$\Phi(\hat{g}) \geqslant \Phi(g),$$

was proved in [3,4], while the existence of a minimizer was investigated in [5]. One challenging feature of such problem is that, in general, the functional Φ turns out to be continuous (in a suitable sense) but the class $\mathcal{G}$ fails to be compact. Therefore, a possible strategy consists in optimizing Φ over the closure $\overline{\mathcal{G}}$ of $\mathcal{G}$ in the sequential weak* topology of $L^\infty(\Omega)$ (a much larger, and convex, set), and then proving that the maximizers and minimizers actually lie in $\mathcal{G}$ (which is far from being trivial).

In addition, due to the nature of the rearrangement equivalence and some functional inequalities, the maximizer/minimizer g may show some structural connection to the solution u_g of the corresponding variational problem. This has interesting consequences, for instance let g_0 be the characteristic function of some subdomain $D_0 \subset \Omega$, then the optimal g is as well the characteristic function of some $D \subset \Omega$ with $|D| = |D_0|$. Moreover, it is proved that $g = \eta \circ u_g$ in Ω for some nondecreasing η, while by the Dirichlet condition we have $u_g = 0$ on $\partial\Omega$. Therefore, any optimal domain D has a positive distance from $\partial\Omega$.

A similar approach applies to several variational problems and functionals. For instance, in [8] the authors consider the following p-Laplacian equation with $p > 1$, $q \in [0, p)$:

$$\begin{cases} -\Delta_p u = g(x) u^{q-1} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

which admits a unique non-negative solution $u_g \in W_0^{1,p}(\Omega)$, and study the maximum and minimum over $\mathcal{G}$ of the functional

$$\Phi(g) = \int_\Omega g u_g^q \, dx.$$

In [7], the following weighted eigenvalue problem is considered:

$$\begin{cases} -\Delta_p u = \lambda g(x) |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

It is well known that the problem above admits a principal eigenvalue $\lambda(g) > 0$ (see [20]), and the authors prove that the functional $\lambda(g)$ has a minimizer in $\mathcal{G}$.

In recent years, several researchers have studied optimization problems related to elliptic equations of fractional order (see [23] for a general introduction to such problems and the related variational methods). In the linear framework, the model operator is the s-fractional Laplacian with $s \in (0,1)$, defined by

$$(-\Delta)^s u(x) = C_{N,s} \lim_{\varepsilon \to 0^+} \int_{B^c_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy,$$

where $C_{N,s} > 0$ is a normalization constant. In [26], the following problem is examined:

$$\begin{cases} (-\Delta)^s u + h(x,u) = g(x) & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c, \end{cases}$$

where $h(x,\cdot)$ is nondecreasing and grows sublinearly in the second variable. The solution $u_g \in H^s_0(\Omega)$ is unique and is the unique minimizer of the energy functional

$$\Phi(g) = \frac{1}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}}\, dx\, dy + \int_\Omega [H(x,u) - gu]\, dx$$

(where $H(x,\cdot)$ denotes the primitive of $h(x,\cdot)$), so the authors investigate the minimization of $\Phi(g)$ over $\mathcal{G}$. Besides, in [1], the following nonlocal eigenvalue problem is considered:

$$\begin{cases} (-\Delta)^s u = \lambda g(x) u & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c. \end{cases}$$

The authors prove, among other results, the existence of a minimizer $g \in \mathcal{G}$ for the principal eigenvalue $\lambda(g)$. Optimization of the principal eigenvalue of fractional operators has significant applications in biomathematics, see [25]. In all the aforementioned problems, optimization in $\mathcal{G}$ also yields representation formulas and qualitative properties (e.g., Steiner symmetry over convenient domains) of the optimal data.

In the present paper, we focus on the following nonlinear, nonlocal operator:

$$\mathcal{L}_K u(x) = \lim_{\varepsilon \to 0^+} \int_{B^c_\varepsilon(x)} |u(x) - u(y)|^{p-2}(u(x) - u(y)) K(x,y)\, dy.$$

Here $N \geqslant 2$, $p > 1$, $s \in (0,1)$, and $K : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a measurable kernel s.t. for a.e. $x, y \in \mathbb{R}^N$,

(K_1) $K(x,y) = K(y,x)$;
(K_2) $C_1 \leqslant K(x,y)|x-y|^{N+ps} \leqslant C_2$ $(0 < C_1 \leqslant C_2)$.

If $C_1 = C_2 = C_{N,p,s} > 0$ (a normalization constant varying from one reference to the other), $\mathcal{L}_K$ reduces to the s-fractional p-Laplacian

$$(-\Delta)^s_p u(x) = C_{N,p,s} \lim_{\varepsilon \to 0^+} \int_{B^c_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{N+ps}}\, dy,$$

which in turn coincides with the s-fractional Laplacian seen above for $p = 2$. The nonlinear operator $\mathcal{L}_K$ arises from problems in game theory (see [2, 6]). Besides, the special case $(-\Delta)^s_p$ can be seen as either an approximation of the classical p-Laplace operator for fixed p and $s \to 1^-$ (see [19]), or an approximation of the fractional ∞-Laplacian for fixed s and $p \to \infty$, with applications to the problem of Hölder continuous extensions of functions (see [22]). Equations driven by the fractional p-Laplacian are the subject of a vast literature, dealing with existence, qualitative properties, and regularity of the solutions (see for instance [13–15, 24]).

Inspired by the cited references, we will examine two variational problems driven by $\mathcal{L}_K$, set on a bounded domain Ω with $C^{1,1}$-smooth boundary, with a datum g varying in a rearrangement class $\mathcal{G}$, and optimize the corresponding functionals. First, we consider the following nonlinear, nonlocal eigenvalue problem:

$$\begin{cases} \mathcal{L}_K u = \lambda g(x)|u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c. \end{cases} \tag{1.1}$$

Let $\lambda(g)$ be the principal eigenvalue of (1.1), defined by

$$\lambda(g) = \inf_{u\neq 0} \frac{\iint_{\mathbb{R}^N\times\mathbb{R}^N} |u(x)-u(y)|^p K(x,y)\,dx\,dy}{\int_\Omega g|u|^p\,dx},$$

and u_g be the (unique) associated eigenfunction s.t. $u_g > 0$ in Ω and

$$\int_\Omega g u_g^p\,dx = 1.$$

With such definitions, we will study the following optimization problem:

$$\min_{g\in\mathcal{G}} \lambda(g).$$

Precisely, we will prove that such problem admits at least one solution, that any solution actually minimizes $\lambda(g)$ over the larger set $\overline{\mathcal{G}}$, while all minimizers over $\overline{\mathcal{G}}$ lie in $\mathcal{G}$, and finally that any minimal weight can be represented as a nondecreasing function of the corresponding eigenfunction:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $C^{1,1}$-boundary, $p > 1$, $s \in (0,1)$, $K : \mathbb{R}^N\times\mathbb{R}^N \to \mathbb{R}$ be measurable satisfying* (K_1), (K_2), *$g_0 \in L^\infty(\Omega)_+ \setminus \{0\}$, $\mathcal{G}$ be the rearrangement class of g_0. For all $g \in \overline{\mathcal{G}}$, let $\lambda(g)$ be the principal eigenvalue of* (1.1). *Then,*

(*i*) *there exists $\hat{g} \in \mathcal{G}$ s.t. $\lambda(\hat{g}) \leqslant \lambda(g)$ for all $g \in \mathcal{G}$;*
(*ii*) *for all $\hat{g}$ as in* (*i*) *and $g \in \overline{\mathcal{G}} \setminus \mathcal{G}$, $\lambda(\hat{g}) < \lambda(g)$;*
(*iii*) *for all $\hat{g}$ as in* (*i*) *there exists a nondecreasing map $\eta : \mathbb{R} \to \mathbb{R}$ s.t. $\hat{g} = \eta \circ u_{\hat{g}}$ in Ω.*

Then, we will focus on the following general nonlinear Dirichlet problem:

$$\begin{cases} \mathcal{L}_K u + h(x,u) = g(x) & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c, \end{cases} \tag{1.2}$$

where, in addition to the previous hypotheses, we assume that $h : \Omega\times\mathbb{R} \to \mathbb{R}_+$ is a Carathéodory mapping satisfying the following conditions:

(h_1) $h(x,\cdot)$ is nondecreasing in $\mathbb{R}$ for a.e. $x \in \Omega$;
(h_2) $h(x,t) \leqslant C_0(1+|t|^{q-1})$ for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$, with $C_0 > 0$ and $q \in (1,p)$.

For all $g \in \overline{\mathcal{G}}$ problem (1.2) has a unique solution u_g, with associated energy

$$\Psi(g) = \frac{1}{p} \iint_{\mathbb{R}^N \times \mathbb{R}^N} |u_g(x) - u_g(y)|^p K(x, y)\, dx\, dy + \int_\Omega [H(x, u_g) - g u_g]\, dx$$

(where $H(x, \cdot)$ denotes the primitive of $h(x, \cdot)$). Our second result deals with following optimization problem:

$$\min_{g \in \mathcal{G}} \Psi(g),$$

and is stated as follows:

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $C^{1,1}$-boundary, $p > 1$, $s \in (0, 1)$, $K : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be measurable satisfying (K_1), (K_2), $h : \Omega \times \mathbb{R} \to \mathbb{R}_+$ be a Carathéodory mapping satisfying (h_1), (h_2), $g_0 \in L^\infty(\Omega)_+ \setminus \{0\}$, $\mathcal{G}$ be the rearrangement class of g_0. For all $g \in \overline{\mathcal{G}}$, let u_g be the solution of (1.2) and $\Psi(g)$ be the associated energy. Then,*

(*i*) *there exists $\hat{g} \in \mathcal{G}$ s.t. $\Psi(\hat{g}) \leqslant \Psi(g)$ for all $g \in \mathcal{G}$;*
(*ii*) *for all $\hat{g}$ as in (i) and $g \in \overline{\mathcal{G}} \setminus \mathcal{G}$, $\Psi(\hat{g}) < \Psi(g)$;*
(*iii*) *for all $\hat{g}$ as in (i) there exists a nondecreasing map $\eta : \mathbb{R} \to \mathbb{R}$ s.t. $\hat{g} = \eta \circ u_{\hat{g}}$ in Ω.*

Theorem 1.1 above extends [1, Theorem 1.1] to the nonlinear framework, which requires some delicate arguments due to the non-Hilbertian structure of the problem. Similarly, Theorem 1.2 extends [26, Theorem 3.2], also introducing the structure property of minimizers in (*iii*).

The dual problems, i.e., maximization of $\lambda(g)$ and $\Psi(g)$ respectively, remain open for now. The reason is easily understood, as soon as we recall that both $\lambda(g)$ and $\Psi(g)$ admit variational characterizations as minima of convenient functions on the Sobolev space $W^{s,p}_0(\Omega)$, so further minimizing with respect to g conjures a 'double minimization' problem. On the contrary, maximizing $\lambda(g)$, $\Psi(g)$, respectively, would result in a min-max problem, which requires a different approach.

The structure of the paper is the following: In Section 2 we recall some preliminaries on rearrangement classes and fractional order equations; in Section 3 we deal with the eigenvalue problem (1.1); and in Section 4 we deal with the general Dirichlet problem (1.2).

Notation. For all $\Omega \subset \mathbb{R}^N$, we denote by $|\Omega|$ the N-dimensional Lebesgue measure of Ω and $\Omega^c = \mathbb{R}^N \setminus \Omega$. For all $x \in \mathbb{R}^N$, $r > 0$ we denote by $B_r(x)$ the open ball centered at x with radius r. When we say that $g \geqslant 0$ in Ω, we mean $g(x) \geqslant 0$ for a.e. $x \in \Omega$, and similar expressions. Whenever X is a function space on the domain Ω, X_+ denotes the positive order cone of X. In any Banach space we denote by $\to$ strong (or norm) convergence, by $\rightharpoonup$ weak convergence, and by $\overset{*}{\rightharpoonup}$ weak* convergence. For all $q \in [1, \infty]$, we denote by $\|\cdot\|_q$ the norm of $L^q(\Omega)$. Finally, C denotes several positive constants, varying from line to line.

2. Preliminaries

In this section we collect some necessary preliminary results on rearrangement classes and fractional Sobolev spaces.

2.1. Rearrangement classes

Let $\Omega \subset \mathbb{R}^N$ ($N \geqslant 2$) be a bounded domain, $g_0 \in L^\infty(\Omega)$ be s.t. $0 \leqslant g_0 \leqslant M$ in Ω ($M > 0$), and $g_0 > 0$ on some subset of Ω with positive measure. We say that a function $g \in L^\infty(\Omega)$ is a rearrangement of g_0, denoted $g \sim g_0$, if for all $t \geqslant 0$

$$\left|\{x \in \Omega : \ g(x) > t\}\right| = \left|\{x \in \Omega : \ g_0(x) > t\}\right|.$$

Also, we define the rearrangement class

$$\mathcal{G} = \{g \in L^\infty(\Omega) : \ g \sim g_0\}.$$

Clearly, $0 \leqslant g \leqslant M$ in Ω for all $g \in \mathcal{G}$. Recalling that $L^\infty(\Omega)$ is the topological dual of $L^1(\Omega)$, we can endow such space with the weak* topology, characterized by the following type of convergence:

$$g_n \overset{*}{\rightharpoonup} g \iff \lim_n \int_\Omega g_n h \, dx = \int_\Omega g h \, dx \text{ for all } h \in L^1(\Omega).$$

We denote by $\overline{\mathcal{G}}$ the closure of $\mathcal{G}$ in $L^\infty(\Omega)$ with respect to such topology. It is proved in [3,4] that $\overline{\mathcal{G}}$ is a sequentially weakly* compact convex set, and that $0 \leqslant g \leqslant M$ in Ω for all $g \in \overline{\mathcal{G}}$. Therefore, given a sequentially weakly* continuous functional $\Phi : \overline{\mathcal{G}} \to \mathbb{R}$, there exist $\check{g}, \hat{g} \in \overline{\mathcal{G}}$ s.t. for all $g \in \overline{\mathcal{G}}$

$$\Phi(\check{g}) \leqslant \Phi(g) \leqslant \Phi(\hat{g}).$$

In general, the extrema are not attained at points of $\mathcal{G}$. As usual, we say that Φ is Gâteaux differentiable at $g \in \overline{\mathcal{G}}$, if there exists a linear functional $\Phi'(g) \in L^\infty(\Omega)^*$ s.t. for all $h \in \overline{\mathcal{G}}$

$$\lim_{\tau \to 0^+} \frac{\Phi(g + \tau(h - g)) - \Phi(g)}{\tau} = \langle \Phi'(g), h - g \rangle.$$

We remark that $g \in \overline{\mathcal{G}}$ being a minimizer (or maximizer) of Φ does not imply $\Phi'(g) = 0$ in general. Nevertheless, if Φ is convex, then for all $h \in \overline{\mathcal{G}}$

$$\Phi(h) \geqslant \Phi(g) + \langle \Phi'(g), h - g \rangle,$$

with strict inequality if Φ is strictly convex and $h \neq g$ (see [27] for an introduction to convex functionals and variational inequalities). Finally, let us recall a technical lemma on optimization of *linear* functionals over $\overline{\mathcal{G}}$, which also provides a representation formula:

Lemma 2.1. *Let $h \in L^1(\Omega)$. Then,*

(*i*) *there exists $\hat{g} \in \mathcal{G}$ s.t. for all $g \in \overline{\mathcal{G}}$*

$$\int_\Omega \hat{g} h \, dx \geqslant \int_\Omega g h \, dx;$$

(*ii*) *if $\hat{g}$ is unique, then there exists a nondecreasing map $\eta : \mathbb{R} \to \mathbb{R}$ s.t. $\hat{g} = \eta \circ h$ in Ω.*

Proof. By [3, Theorems 1, 4], there exists $\hat{g} \in \mathcal{G}$ which maximizes the linear functional

$$g \mapsto \int_\Omega gh\,dx$$

over $\mathcal{G}$. Given $g \in \overline{\mathcal{G}} \setminus \mathcal{G}$, we can find a sequence (g_n) in $\mathcal{G}$ s.t. $g_n \overset{*}{\rightharpoonup} g$. For all $n \in \mathbb{N}$ we have

$$\int_\Omega \hat{g}h\,dx \geqslant \int_\Omega g_n h\,dx,$$

so passing to the limit we get

$$\int_\Omega \hat{g}h\,dx \geqslant \int_\Omega gh\,dx,$$

thus proving (i). From [3, Theorem 5] we have (ii). □

2.2. *Fractional Sobolev spaces*

We recall some basic notions about the variational formulations of problems (1.1) and (1.2). For $p > 1$, $s \in (0, 1)$, all open $\Omega \subseteq \mathbb{R}^N$, and all measurable $u : \Omega \to \mathbb{R}$ we define the Gagliardo seminorm

$$[u]_{s,p,\Omega} = \Big[\iint_{\Omega\times\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}}\,dx\,dy \Big]^{\frac{1}{p}}.$$

The corresponding fractional Sobolev space is defined by

$$W^{s,p}(\Omega) = \{u \in L^p(\Omega) : [u]_{s,p,\Omega} < \infty\}.$$

If Ω is bounded and with a $C^{1,1}$-smooth boundary, we incorporate the Dirichlet conditions by defining the space

$$W^{s,p}_0(\Omega) = \{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ in } \Omega^c\},$$

endowed with the norm $\|u\|_{W^{s,p}_0(\Omega)} = [u]_{s,p,\mathbb{R}^N}$. This is a uniformly convex, separable Banach space with dual $W^{-s,p'}(\Omega)$, s.t. $C^\infty_c(\Omega)$ is a dense subset of $W^{s,p}_0(\Omega)$, and the embedding $W^{s,p}_0(\Omega) \hookrightarrow L^q(\Omega)$ is compact for all $q \in [1, p^*_s)$, where

$$p^*_s = \begin{cases} \dfrac{Np}{N - ps} & \text{if } ps < N \\ \infty & \text{if } ps \geqslant N. \end{cases}$$

For a detailed account on fractional Sobolev spaces, we refer the reader to [10, 21]. Now let $K : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}$ be a measurable kernel satisfying (K_1) and (K_2). We introduce an equivalent norm on $W^{s,p}_0(\Omega)$ by setting

$$[u]_K = \Big[\iint_{\mathbb{R}^N\times\mathbb{R}^N} |u(x) - u(y)|^p K(x, y)\,dx\,dy \Big]^{\frac{1}{p}}.$$

We can now rephrase more carefully the definitions given in Section 1, by defining the operator $\mathcal{L}_K : W^{s,p}_0(\Omega) \to W^{-s,p'}(\Omega)$ as the gradient of the C^1-functional

$$u \mapsto \frac{[u]^p_K}{p}.$$

Equivalently, for all $u, \varphi \in W_0^{s,p}(\Omega)$ we set

$$\langle \mathcal{L}_K u, \varphi \rangle = \iint_{\mathbb{R}^N \times \mathbb{R}^N} |u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))K(x, y)\, dx\, dy.$$

Both problems that we are going to study belong to the following class of nonlinear, nonlocal Dirichlet problems:

$$\begin{cases} \mathcal{L}_K u = f(x, u) & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c, \end{cases} \tag{2.1}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory mapping subject to the following subcritical growth conditions: there exist $C > 0$, $r \in (1, p_s^*)$ s.t. for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$

$$|f(x, t)| \leqslant C(1 + |t|^{r-1}). \tag{2.2}$$

We say that $u \in W_0^{s,p}(\Omega)$ is a weak solution of (2.1), if for all $\varphi \in W_0^{s,p}(\Omega)$

$$\langle \mathcal{L}_K u, \varphi \rangle = \int_\Omega f(x, u)\varphi\, dx.$$

There is a wide literature on problem (2.1), especially for the model case $\mathcal{L}_K = (-\Delta)_p^s$, see for instance [9, 14, 16, 17, 24]. We will only need to recall the following properties, which can be proved adapting [16, Proposition 2.3] and [9, Theorem 1.5], respectively:

Lemma 2.2. *Let f satisfy* (2.2)*, $u \in W_0^{s,p}(\Omega)$ be a weak solution of* (2.1)*. Then, $u \in L^\infty(\Omega)$.*

Lemma 2.3. *Let f satisfy* (2.2)*, and $\delta > 0$, $c \in C(\overline{\Omega})_+$ be s.t. for a.e. $x \in \Omega$ and all $t \in [0, \delta]$*

$$f(x, t) \geqslant -c(x)t^{p-1}.$$

Also, let $u \in W_0^{s,p}(\Omega)_+$ be a weak solution of (2.2)*. Then, either $u = 0$, or $u > 0$ in Ω.*

We will not cope with regularity of the weak solutions here. In the model case of the fractional p-Laplacian, under hypothesis (2.2), using Lemma 2.2 above and [15, Theorems 1.1, 2.7], it can be seen that whenever $u \in W_0^{s,p}(\Omega)$ solves (2.1), we have $u \in C^s(\mathbb{R}^N)$ and there exist $\alpha \in (0, s)$ depending only on the data of the problem, s.t. the function

$$\frac{u}{\operatorname{dist}(\cdot, \Omega^c)^s}$$

admits a α-Hölder continuous extension to $\overline{\Omega}$. The same result is not known for the general operator $\mathcal{L}_K$, except the linear case $p = 2$ with a special anisotropic kernel, see [28].

For future use, we prove here a technical lemma:

Lemma 2.4. *Let (g_n) be a sequence in $\overline{\mathcal{G}}$ s.t. $g_n \overset{*}{\rightharpoonup} g$, (u_n) be a bounded sequence in $W_0^{s,p}(\Omega)$, $r \in [1, p_s^*)$. Then, there exists $u \in W_0^{s,p}(\Omega)$ s.t. up to a subsequence*

$$\lim_n \int_\Omega g_n |u_n|^r\, dx = \int_\Omega g|u|^r.$$

Proof. By the compact embedding $W_0^{s,p}(\Omega) \hookrightarrow L^r(\Omega)$, passing if necessary to a subsequence we have $u_n \to u$ in $L^r(\Omega)$ and $u_n(x) \to u(x)$ for a.e. $x \in \Omega$, as $n \to \infty$. In particular $|u|^r \in L^1(\Omega)$, so

$$\lim_n \int_\Omega (g_n - g)|u|^r \, dx = 0.$$

Besides, recalling that $0 \leqslant g_n \leqslant M$ in Ω for all $n \in \mathbb{N}$, we have by Hölder's inequality

$$\begin{aligned}\int_\Omega [g_n|u_n|^r - g|u|^r] \, dx &\leqslant \int_\Omega g_n||u_n|^r - |u|^r| \, dx + \int_\Omega (g_n - g)|u|^r \, dx \\ &\leqslant C \int_\Omega [|u_n|^{r-1} + |u|^{r-1}]|u_n - u| \, dx + \int_\Omega (g_n - g)|u|^r \, dx \\ &\leqslant C[\|u_n\|_r^{r-1} + \|u\|_r^{r-1}]\|u_n - u\|_r + \int_\Omega (g_n - g)|u|^r \, dx,\end{aligned}$$

and the latter tends to 0 as $n \to \infty$. □

3. Optimization of the principal eigenvalue

In this section we consider the eigenvalue problem (1.1) and prove Theorem 1.1. Let Ω, p, s, K, g_0 be as in Section 1. For any $g \in \overline{\mathcal{G}}$, as in Subsection 2.2 we say that $u \in W_0^{s,p}(\Omega)$ is a (weak) solution of (1.1) if for all $\varphi \in W_0^{s,p}(\Omega)$

$$\langle \mathcal{L}_K u, \varphi \rangle = \lambda \int_\Omega g|u|^{p-2}u\varphi \, dx.$$

We say that $\lambda \in \mathbb{R}$ is an eigenvalue if (1.1) admits a solution $u \neq 0$, which is then called a λ-eigenfunction. Though a full description of the eigenvalues of (1.1) is missing, from [11, 12, 22] we know that for all $g \in L^\infty(\Omega)_+$ there exists a *principal eigenvalue* $\lambda(g) > 0$, namely the smallest positive eigenvalue, which admits the following variational characterization:

$$\lambda(g) = \inf_{u \neq 0} \frac{[u]_K^p}{\int_\Omega g|u|^p \, dx}. \tag{3.1}$$

In addition, from [11] we know that $\lambda(g)$ is an isolated eigenvalue, simple, with constant sign eigenfunctions, while for any eigenvalue $\lambda > \lambda(g)$ the associated λ-eigenfunctions change sign in Ω. So, recalling Lemma 2.3, there exists a unique normalized positive $\lambda(g)$-eigenfunction $u_g \in W_0^{s,p}(\Omega)$ s.t.

$$\int_\Omega g u_g^p \, dx = 1, \; [u_g]_K^p = \lambda(g).$$

In particular $g \mapsto \lambda(g)$ defines a real-valued functional defined in the rearrangement class of weights $\mathcal{G}$ (or in $\overline{\mathcal{G}}$), and we are interested in the minimizers of such functional. Equivalently, we may set for all $g \in \overline{\mathcal{G}}$

$$\Phi(g) = \frac{1}{\lambda(g)^2} = \sup_{u \neq 0} \frac{\left[\int_\Omega g|u|^p \, dx\right]^2}{[u]_K^{2p}},$$

and consider the maximization problem

$$\max_{g \in \mathcal{G}} \Phi(g).$$

First, we want to maximize $\Phi(g)$ over $\overline{\mathcal{G}}$, which is possible due to the following lemma:

Lemma 3.1. *The functional $\Phi(g)$ is sequentially weakly* continuous in $\overline{\mathcal{G}}$.*

Proof. Let (g_n) be a sequence in $\overline{\mathcal{G}}$ s.t. $g_n \overset{*}{\rightharpoonup} g$, and for simplicity denote $u_n = u_{g_n}$ for all $n \in \mathbb{N}$, and $u = u_g$. We need to prove that $\Phi(g_n) \to \Phi(g)$. Since $u^p \in L^1(\Omega)$, we have

$$\lim_n \int_\Omega g_n u^p \, dx = \int_\Omega g u^p \, dx = 1.$$

Also, by definition of Φ we have for all $n \in \mathbb{N}$

$$\Phi(g_n) \geqslant \frac{\left[\int_\Omega g_n u^p \, dx \right]^2}{[u]_K^{2p}},$$

and the latter tends to $\Phi(g)$ as $n \to \infty$. Therefore

$$\liminf_n \; \Phi(g_n) \geqslant \Phi(g). \tag{3.2}$$

In particular, for all $n \in \mathbb{N}$ we have

$$[u_n]_K = \Phi(g_n)^{-\frac{1}{2p}} \leqslant C,$$

so (u_n) is bounded in $W_0^{s,p}(\Omega)$. By reflexivity and the compact embedding $W_0^{s,p}(\Omega) \hookrightarrow L^p(\Omega)$, passing to a subsequence we have $u_n \rightharpoonup v$ in $W_0^{s,p}(\Omega)$, $u_n \to v$ in $L^p(\Omega)$, and $u_n(x) \to v(x)$ for a.e. $x \in \Omega$, as $n \to \infty$. In particular, $v \geqslant 0$ in Ω. By convexity we have

$$\liminf_n \, [u_n]_K^{2p} \geqslant [v]_K^{2p}.$$

By Lemma 2.4 (with $r = p$) we also have

$$\lim_n \int_\Omega g_n u_n^p \, dx = \int_\Omega g v^p \, dx.$$

So we get

$$\limsup_n \; \Phi(g_n) = \limsup_n \; \frac{\left[\int_\Omega g_n u_n^p \, dx \right]^2}{[u_n]_K^{2p}} \leqslant \frac{\left[\int_\Omega g v^p \, dx \right]^2}{[v]_K^{2p}} \leqslant \Phi(g).$$

This, besides (3.2), concludes the proof. □

Lemma 3.1, along with the compactness of $\overline{\mathcal{G}}$, proves that $\Phi(g)$ admits a minimizer and a maximizer in $\overline{\mathcal{G}}$. We next need to ensure that at least one maximizer lies in the smaller set $\mathcal{G}$. In the next lemmas we will investigate further properties of Φ.

Lemma 3.2. *The functional Φ is strictly convex in $\overline{\mathcal{G}}$.*

Proof. We introduce an alternative expression for Φ. For all $g \in \overline{\mathcal{G}}$, $u \in W_0^{s,p}(\Omega)_+$ set

$$F(g, u) = 2 \int_\Omega g u^p \, dx - [u]_K^{2p}.$$

We fix $g \in \overline{\mathcal{G}}$ and maximize $F(g, \cdot)$ over positive functions. For all $u \in W_0^{s,p}(\Omega)_+ \setminus \{0\}$ and $\tau > 0$, the function

$$F(g, \tau u) = 2\tau^p \int_\Omega g u^p \, dx - \tau^{2p} [u]_K^p$$

is differentiable in τ with derivative

$$\frac{\partial}{\partial \tau} F(g, \tau u) = 2p\tau^{p-1} \int_\Omega g u^p \, dx - 2p\tau^{2p-1} [u]_K^{2p}.$$

So the maximum of $\tau \mapsto F(g, \tau u)$ is attained at

$$\tau_0(u) = \frac{\left[\int_\Omega g u^p \, dx \right]^{\frac{1}{p}}}{[u]_K^2} > 0,$$

and amounts at

$$F(g, \tau_0(u) u) = \frac{\left[\int_\Omega g u^p \, dx \right]^2}{[u]_K^{2p}}.$$

Maximizing further over u, we obtain

$$\sup_{u>0} F(g, u) = \sup_{u \in W_0^{s,p}(\Omega)_+ \setminus \{0\}} \frac{\left[\int_\Omega g u^p \, dx \right]^2}{[u]_K^{2p}}.$$

Noting that $[|u|]_K \leqslant [u]_K$ for all $u \in W_0^{s,p}(\Omega)$, and recalling (3.1), we have for all $g \in \overline{\mathcal{G}}$

$$\Phi(g) = \sup_{u \in W_0^{s,p}(\Omega)_+ \setminus \{0\}} F(g, u) = \frac{1}{\lambda(g)^2}. \tag{3.3}$$

We claim that the supremum in (3.3) is attained at the unique function

$$\tilde{u}_g = \frac{u_g}{\lambda(g)^{\frac{2}{p}}} = \tau_0(u_g) u_g. \tag{3.4}$$

Indeed, by normalization of u_g we have

$$F(g, \tilde{u}_g) = \frac{2}{\lambda(g)^2} \int_\Omega g u_g^p \, dx - \frac{[u_g]_K^{2p}}{\lambda(g)^4} = \frac{1}{\lambda(g)^2}.$$

For uniqueness, first consider a function $u = \tau u_g$ with $\tau \neq \tau_0(u_g)$. By unique maximization in τ we have

$$F(g, \tau u_g) < F(g, \tau_0(u_g) u_g) = F(g, \tilde{u}_g) = \frac{1}{\lambda(g)^2}.$$

Besides, for all $v \in W_0^{s,p}(\Omega)_+ \setminus \{0\}$ which is not a $\lambda(g)$-eigenfunction, arguing as above with v replacing u, and recalling that the infimum in (3.1) is attained only at principal eigenfunctions, we have

$$F(g, v) \leqslant F(g, \tau_0(v) v) = \frac{\left[\int_\Omega g v^p \, dx \right]^2}{[v]_K^{2p}} < \frac{1}{\lambda(g)^2}.$$

So, $\tilde{u}_g$ is the unique maximizer of (3.3).

We now prove that Φ is convex. Let $g_1, g_2 \in \overline{\mathcal{G}}$, $\tau \in (0,1)$ and set

$$g_\tau = (1-\tau)g_1 + \tau g_2,$$

so $g_\tau \in \overline{\mathcal{G}}$ (a convex set, as seen in Subsection 2.1). For all $u \in W^{s,p}_0(\Omega)_+ \setminus \{0\}$, we have by (3.3)

$$\begin{aligned} F(g_\tau, u) &= 2(1-\tau)\int_\Omega g_1 u^p\,dx + 2\tau \int_\Omega g_2 u^p\,dx - [u]_K^{2p} \\ &= (1-\tau)F(g_1,u) + \tau F(g_2,u) \leqslant (1-\tau)\Phi(g_1) + \tau\Phi(g_2). \end{aligned}$$

Taking the supremum over u and using (3.3) again,

$$\Phi(g_\tau) \leqslant (1-\tau)\Phi(g_1) + \tau\Phi(g_2).$$

To prove that Φ is *strictly* convex, we argue by contradiction, assuming that for some $g_1 \neq g_2$ as above and $\tau \in (0,1)$

$$\Phi(g_\tau) = (1-\tau)\Phi(g_1) + \tau\Phi(g_2).$$

Set $\tilde{u}_i = \tilde{u}_{g_i}$ $(i = 1,2)$ and $\tilde{u}_\tau = \tilde{u}_{g_\tau}$ for brevity. Then, by (3.3) and the equality above

$$(1-\tau)F(g_1, \tilde{u}_\tau) + \tau F(g_2, \tilde{u}_\tau) = (1-\tau)F(g_1, \tilde{u}_1) + \tau F(g_2, \tilde{u}_2).$$

Recalling that $\tilde{u}_i$ is the only maximizer of $F(g_i, \cdot)$, the last inequality implies $\tilde{u}_1 = \tilde{u}_2 = \tilde{u}_\tau$, as well as

$$\Phi(g_1) = F(g_1, \tilde{u}_\tau) = F(g_2, \tilde{u}_\tau) = \Phi(g_2).$$

Therefore we have $\lambda(g_1) = \lambda(g_2) = \lambda$. Moreover, $\tilde{u}_\tau > 0$ is a λ-eigenfunction with both weights g_1, g_2, i.e., for all $\varphi \in W^{s,p}_0(\Omega)$

$$\lambda \int_\Omega g_1 \tilde{u}_\tau^{p-1}\varphi\,dx = \langle \mathcal{L}_K \tilde{u}_\tau, \varphi\rangle = \lambda \int_\Omega g_2 \tilde{u}_\tau^{p-1}\varphi\,dx.$$

So $g_1\tilde{u}_\tau^{p-1} = g_2\tilde{u}_\tau^{p-1}$ in Ω, which in turn, since $\tilde{u}_\tau > 0$, implies $g_1 = g_2$ a.e. in Ω, a contradiction. □

The next lemma establishes differentiability of Φ.

Lemma 3.3. *The functional Φ is Gâteaux differentiable in $\overline{\mathcal{G}}$, and for all $g, h \in \overline{\mathcal{G}}$*

$$\langle \Phi'(g), h-g\rangle = 2\int_\Omega (h-g)\tilde{u}_g^p\,dx,$$

where $\tilde{u}_g$ is the principal eigenfunction normalized as in (3.4).

Proof. First, let (g_n) be a sequence in $\overline{\mathcal{G}}$ s.t. $g_n \overset{*}{\rightharpoonup} g$, and set for brevity $\tilde{u}_n = \tilde{u}_{g_n}$, $\tilde{u} = \tilde{u}_g$. We claim that

$$\lim_n \int_\Omega |\tilde{u}_n - \tilde{u}|^p\,dx = 0. \tag{3.5}$$

Indeed, by normalization we have for all $n \in \mathbb{N}$

$$[\tilde{u}_n]_K^{2p} = \Phi(g_n),$$

and the latter is bounded from above, since $\Phi(g)$ has a maximizer in $\overline{\mathcal{G}}$. So, $(\tilde{u}_n)$ is bounded in $W_0^{s,p}(\Omega)$. By uniform convexity and the compact embedding $W_0^{s,p}(\Omega) \hookrightarrow L^p(\Omega)$, passing to a subsequence we have $\tilde{u}_n \rightharpoonup v$ in $W_0^{s,p}(\Omega)$, $\tilde{u}_n \to v$ in $L^p(\Omega)$, and $\tilde{u}_n(x) \to v(x)$ for a.e. $x \in \Omega$, as $n \to \infty$ (in particular $v \geqslant 0$ in Ω). By convexity, we see that

$$\liminf_n [\tilde{u}_n]_K^{2p} \geqslant [v]_K^{2p}.$$

By Lemma 2.4, we have

$$\lim_n \int_\Omega g_n \tilde{u}_n^p \, dx = \int_\Omega g v^p \, dx.$$

By Lemma 3.1, we have $\Phi(g_n) \to \Phi(g)$, so by (3.3) we get

$$\begin{aligned}
\Phi(g) &= \lim_n F(g_n, \tilde{u}_n) \\
&\leqslant 2 \lim_n \int_\Omega g_n \tilde{u}_n^p \, dx - \liminf_n [\tilde{u}_n]_K^{2p} \\
&\leqslant 2 \int_\Omega g v^p \, dx - [v]_K^{2p} \\
&= F(g, v) \leqslant \Phi(g).
\end{aligned}$$

Therefore v is a maximizer of $F(g, \cdot)$ over $W_0^{s,p}(\Omega)_+$, hence by uniqueness $v = \tilde{u}$. Then we have $\tilde{u}_n \to \tilde{u}$ in $L^p(\Omega)$, which is equivalent to (3.5).

We claim that for all $n \in \mathbb{N}$

$$\Phi(g) + 2 \int_\Omega (g_n - g)\tilde{u}^p \, dx \leqslant \Phi(g_n) \leqslant \Phi(g) + 2 \int_\Omega (g_n - g)\tilde{u}_n^p \, dx. \tag{3.6}$$

Indeed, by (3.3) we have

$$\begin{aligned}
\Phi(g) + 2 \int_\Omega (g_n - g)\tilde{u}^p \, dx &\leqslant \Phi(g_n) \\
&= F(g, \tilde{u}_n) + 2 \int_\Omega (g_n - g)\tilde{u}_n^p \, dx \\
&\leqslant \Phi(g) + 2 \int_\Omega (g_n - g)\tilde{u}_n^p \, dx.
\end{aligned}$$

Now fix $g, h \in \overline{\mathcal{G}}$, $g \neq h$, and a sequence (τ_n) in $(0, 1)$ s.t. $\tau_n \to 0$. By convexity of $\overline{\mathcal{G}}$, we have for all $n \in \mathbb{N}$

$$g_n = g + \tau_n(h - g) \in \overline{\mathcal{G}}.$$

Also, clearly $g_n \overset{*}{\rightharpoonup} g$. By (3.6), setting as usual $\tilde{u}_n = \tilde{u}_{g_n}$ and $\tilde{u} = \tilde{u}_g$, we have for all $n \in \mathbb{N}$

$$2\tau_n \int_\Omega (h - g)\tilde{u}^p \, dx \leqslant \Phi(g_n) - \Phi(g) \leqslant 2\tau_n \int_\Omega (h - g)\tilde{u}_n^p \, dx.$$

Dividing by $\tau_n > 0$ and recalling (3.5), we get

$$\lim_n \frac{\Phi(g + \tau_n(h-g)) - \Phi(g)}{\tau_n} = 2\int_\Omega (h-g)\tilde{u}^p\, dx.$$

Note that $2\tilde{u}^p \in L^1(\Omega) \subset L^\infty(\Omega)^*$, and by the arbitrariness of the sequence (τ_n) we deduce that Φ is Gâteaux differentiable at g with

$$\langle \Phi'(g), h-g\rangle = 2\int_\Omega (h-g)\tilde{u}^p\, dx,$$

which concludes the proof. □

We can now prove the main result of this section.

Proof of Theorem 1.1. We already know that Φ has a maximizer $\bar{g}$ over $\overline{\mathcal{G}}$. Set $\bar{w} = 2\tilde{u}_{\bar{g}}^p \in L^1(\Omega)$, then by Lemma 3.3 we have $\Phi'(\bar{g}) = \bar{w}$. Now we maximize on $\overline{\mathcal{G}}$ the linear functional

$$g \mapsto \int_\Omega g\bar{w}\, dx.$$

By Lemma 2.1 (*i*), there exists $\hat{g} \in \mathcal{G}$ s.t. for all $g \in \overline{\mathcal{G}}$

$$\int_\Omega \hat{g}\bar{w}\, dx \geqslant \int_\Omega g\bar{w}\, dx.$$

In particular we have

$$\int_\Omega \hat{g}\bar{w}\, dx \geqslant \int_\Omega \bar{g}\bar{w}\, dx. \tag{3.7}$$

By Lemma 3.2, the functional Φ is convex. Therefore, using also Lemma 3.3 and (3.7), we have

$$\Phi(\hat{g}) \geqslant \Phi(\bar{g}) + \int_\Omega (\hat{g} - \bar{g})\bar{w}\, dx \geqslant \Phi(\bar{g}).$$

Thus, $\hat{g} \in \mathcal{G}$ is as well a maximizer of Φ over $\overline{\mathcal{G}}$, which proves (*i*) since maximizers of Φ and minimizers of $\lambda(g)$ coincide. In addition, by the relation above we have

$$\int_\Omega (\hat{g} - \bar{g})\bar{w}\, dx = 0.$$

We will now prove that $\hat{g} = \bar{g}$, arguing by contradiction. Assume $\hat{g} \neq \bar{g}$, then by the strict convexity of Φ (Lemma 3.2 again) we have

$$\Phi(\hat{g}) > \Phi(\bar{g}) + \int_\Omega (\hat{g} - \bar{g})\bar{w}\, dx = \Phi(\bar{g}),$$

against the maximality of $\bar{g}$. So, any maximizer of Φ over $\overline{\mathcal{G}}$ actually lies in $\mathcal{G}$, which proves (*ii*). Finally, let $\hat{g} \in \mathcal{G}$ be any maximizer of Φ and set $\hat{w} = 2\tilde{u}_{\hat{g}}^p \in L^1(\Omega)$. By Lemmas 3.2 and 3.3, for all $g \in \overline{\mathcal{G}} \setminus \{\hat{g}\}$ we have

$$\Phi(\hat{g}) \geqslant \Phi(g) > \Phi(\hat{g}) + \int_\Omega (g - \hat{g})\hat{w}\, dx,$$

hence

$$\int_\Omega \hat{g}\hat{w}\,dx > \int_\Omega g\hat{w}\,dx.$$

Equivalently, $\hat{g}$ is the only maximizer over $\overline{\mathcal{G}}$ of the linear functional above, induced by the function $\hat{w}$. By Lemma 2.1 (*ii*), there exists a nondecreasing map $\tilde{\eta} : \mathbb{R} \to \mathbb{R}$ s.t. in Ω

$$\hat{g} = \tilde{\eta} \circ \hat{w}.$$

Now we recall (3.4) and the definition of $\hat{w}$, and by setting for all $t \geqslant 0$

$$\eta(t) = \tilde{\eta}\Big(\frac{2t^p}{\lambda(\hat{g})^2}\Big),$$

while $\eta(t) = \eta(0)$ for all $t < 0$, we immediately see that $\eta : \mathbb{R} \to \mathbb{R}$ is a nondecreasing map s.t. $\hat{g} = \eta \circ u_{\hat{g}}$ in Ω, thus proving (*iii*). □

4. Optimization of the energy functional

In this section we consider problem (1.2) and prove Theorem 1.2. Let Ω, p, s, K, g_0 be as in Section 1, and h satisfy (h_1), (h_2). For any $g \in \overline{\mathcal{G}}$, we say that $u \in W^{s,p}_0(\Omega)$ is a weak solution of (1.2) if for all $\varphi \in W^{s,p}_0(\Omega)$

$$\langle \mathcal{L}_K(u), \varphi \rangle + \int_\Omega h(x,u)\varphi\,dx = \int_\Omega g\varphi\,dx.$$

By classical results (see for instance [18] for the fractional p-Laplacian), for all $g \in \overline{\mathcal{G}}$ problem (1.2) has a unique solution $u_g \in W^{s,p}_0(\Omega)$. In addition, by Lemma 2.2 we have $u_g \in L^\infty(\Omega)$. Such solution is the unique minimizer in $W^{s,p}_0(\Omega)$ of the energy functional associated to (1.2). The corresponding energy, depending on $g \in \overline{\mathcal{G}}$, is given by

$$\Psi(g) = \frac{[u_g]_K^p}{p} + \int_\Omega [H(x,u_g) - gu_g]\,dx,$$

where for all $(x,t) \in \Omega \times \mathbb{R}$ we have set

$$H(x,t) = \int_0^t h(x,\tau)\,d\tau.$$

We are interested in the minimizers of $\Psi(g)$ over $\mathcal{G}$. Equivalently, we may set for all $g \in \overline{\mathcal{G}}$, $u \in W^{s,p}_0(\Omega)$

$$E(g,u) = \int_\Omega [gu - H(x,u)]\,dx - \frac{[u]_K^p}{p},$$

and maximize $E(g,\cdot)$ with respect to u, thus defining

$$\Phi(g) = \sup_{u \in W^{s,p}_0(\Omega)} E(g,u) = E(g,u_g). \tag{4.1}$$

So, as in Section 3, we are led to the maximization problem

$$\max_{g \in \mathcal{G}} \Phi(g).$$

First, we prove the continuity of Φ.

Lemma 4.1. *The functional Φ is sequentially weakly* continuous in $\overline{\mathcal{G}}$.*

Proof. Let (g_n) be a sequence in $\overline{\mathcal{G}}$ s.t. $g_n \overset{*}{\rightharpoonup} g$, and denote $u_n = u_{g_n}$, $u = u_g$. By (4.1), for all $n \in \mathbb{N}$ we have

$$\begin{aligned}\Phi(g_n) &= E(g_n, u_n) \\ &\geqslant E(g_n, u) \\ &= E(g, u) + \int_\Omega (g_n - g)u\,dx \\ &= \Phi(g) + \int_\Omega (g_n - g)u\,dx.\end{aligned}$$

Passing to the limit as $n \to \infty$ and using weak* convergence, we get

$$\liminf_n \Phi(g_n) \geqslant \Phi(g) + \lim_n \int_\Omega (g_n - g)u\,dx = \Phi(g). \tag{4.2}$$

From (1.2) with datum g_n and solution u_n, multiplying by u_n again, we get for all $n \in \mathbb{N}$

$$\int_\Omega [g_n - h(x, u_n)]u_n\,dx = [u_n]_K^p. \tag{4.3}$$

Since $\|g_n\|_\infty \leqslant M$ and by the continuous embedding $W_0^{s,p}(\Omega) \hookrightarrow L^1(\Omega)$, we have

$$\Big|\int_\Omega g_n u_n\,dx\Big| \leqslant C[u_n]_K,$$

with $C > 0$ independent of n. Also, by (h_2) and the continuous embedding $W_0^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$, we have

$$\Big|\int_\Omega h(x, u_n)u_n\,dx\Big| \leqslant C \int_\Omega [|u_n| + |u_n|^q]\,dx \leqslant C[u_n]_K + C[u_n]_K^q.$$

So (4.3) implies for all $n \in \mathbb{N}$

$$[u_n]_K^p \leqslant C[u_n]_K + C[u_n]_K^q.$$

Recalling that $q < p$, we deduce that (u_n) is bounded in $W_0^{s,p}(\Omega)$. Passing to a subsequence, we have $u_n \rightharpoonup v$ in $W_0^{s,p}(\Omega)$, $u_n \to v$ in $L^p(\Omega)$, and $u_n(x) \to v(x)$ for a.e. $x \in \Omega$, as $n \to \infty$. By convexity we have

$$\liminf_n [u_n]_K^p \geqslant [v]_K^p.$$

By Lemma 2.4 (with $r = 1$) we find

$$\lim_n \int_\Omega g_n u_n\,dx = \int_\Omega g v\,dx. \tag{4.4}$$

Finally, we have

$$\lim_n \int_\Omega H(x, u_n)\,dx = \int_\Omega H(x, v)\,dx. \tag{4.5}$$

Indeed, applying (h_2), Lagrange's rule, and Hölder's inequality, we get for all $n \in \mathbb{N}$

$$\int_\Omega \big|H(x,u_n) - H(x,v)\big|\,dx \leqslant C\int_\Omega [1 + |u_n|^{q-1} + |v|^{q-1}]|u_n - v|\,dx$$
$$\leqslant C\|u_n - v\|_1 + C[\|u_n\|_q^{q-1} + \|v\|_q^{q-1}]\|u_n - v\|_q,$$

and the latter tends to 0 as $n \to \infty$, by the continuous embeddings of $L^p(\Omega)$ into $L^1(\Omega)$, $L^q(\Omega)$, respectively, thus proving (4.5).

Next, we start from (4.1) and we apply (4.4) and (4.5):

$$\begin{aligned}\limsup_n \Phi(g_n) &= \limsup_n E(g_n,u_n)\\ &\leqslant \lim_n \int_\Omega [g_n u_n - H(x,u_n)]\,dx - \liminf_n \frac{[u_n]_K^p}{p}\\ &\leqslant \int_\Omega [gv - H(x,v)]\,dx - \frac{[v]_K^p}{p} = E(g,v),\end{aligned}$$

and the latter does not exceed $\Phi(g)$, so

$$\limsup_n \Phi(g_n) \leqslant \Phi(g). \tag{4.6}$$

Comparing (4.2) and (4.6), we have $\Phi(g_n) \to \Phi(g)$, which concludes the proof. □

By Lemma 4.1, Φ has both a minimizer and a maximizer over $\overline{\mathcal{G}}$. Next we prove strict convexity:

Lemma 4.2. *The functional* Φ *is strictly convex in* $\overline{\mathcal{G}}$.

Proof. The convexity of Φ follows as in Lemma 3.2, since $\Phi(g)$ is the supremum of linear functionals (in g). To prove strict convexity, we argue by contradiction. Let $g_1, g_2 \in \overline{\mathcal{G}}$ be s.t. $g_1 \neq g_2$, set for all $\tau \in (0,1)$

$$g_\tau = (1-\tau)g_1 + \tau g_2 \in \overline{\mathcal{G}},$$

and assume that for some $\tau \in (0,1)$

$$\Phi(g_\tau) = (1-\tau)\Phi(g_1) + \tau\Phi(g_2).$$

As usual, set $u_i = u_{g_i}$ $(i = 1,2)$ and $u_\tau = u_{g_\tau}$. By linearity of $E(g,u_\tau)$ in g and (4.1), the relation above rephrases as

$$(1-\tau)E(g_1,u_\tau) + \tau E(g_2,u_\tau) = (1-\tau)E(g_1,u_1) + \tau E(g_2,u_2).$$

Recalling that $E(g_i,u_\tau) \leqslant E(g_i,u_i)$ $(i = 1,2)$ and the uniqueness of the maximizer in (4.1), we deduce $u_1 = u_2 = u_\tau$. Now test (1.2) with an arbitrary $\varphi \in W_0^{s,p}(\Omega)$:

$$\int_\Omega g_1\varphi\,dx = \langle \mathcal{L}_K u_\tau, \varphi\rangle + \int_\Omega h(x,u_\tau)\varphi\,dx = \int_\Omega g_2\varphi\,dx.$$

So we have $g_1 = g_2$ a.e. in Ω, a contradiction. Thus, Φ is strictly convex. □

The last property we need is differentiability.

Lemma 4.3. *The functional* Φ *is Gâteaux differentiable in* $\overline{\mathcal{G}}$*, and for all* $g, k \in \overline{\mathcal{G}}$

$$\langle \Phi'(g), k - g\rangle = \int_\Omega (k-g)u_g \, dx.$$

Proof. First, let (g_n) be a sequence in $\overline{\mathcal{G}}$ s.t. $g_n \overset{*}{\rightharpoonup} g$, and let $u_n = u_{g_n}$, $u = u_g$. From Lemma 4.1 we know that $\Phi(g_n)$ tends to $\Phi(g)$, i.e.,

$$\lim_n E(g_n, u_n) = E(g, u). \tag{4.7}$$

We further claim that

$$\lim_n \int_\Omega |u_n - u|^p \, dx = 0. \tag{4.8}$$

Indeed, we recall that for all $n \in \mathbb{N}$

$$E(g_n, u_n) = \int_\Omega [g_n u_n - H(x, u_n)] \, dx - \frac{[u_n]_K^p}{p}.$$

Therefore, by (4.7), uniform boundedness of (g_n), (h_2), and the compact embeddings of $W_0^{s,p}(\Omega)$ into $L^1(\Omega)$, $L^q(\Omega)$, respectively, we have for all $n \in \mathbb{N}$

$$\frac{[u_n]_K^p}{p} \leqslant C + \int_\Omega [g_n u_n - H(x, u_n)] \, dx \leqslant C + C(\|u_n\|_1 + \|u_n\|_q^q) \leqslant C + C([u_n]_K + [u_n]_K^q).$$

Since $1 < q < p$, the sequence (u_n) is bounded in $W_0^{s,p}(\Omega)$. Passing to a subsequence, we have $u_n \rightharpoonup v$ in $W_0^{s,p}(\Omega)$, $u_n \to v$ in $L^p(\Omega)$, and $u_n(x) \to v(x)$ for a.e. $x \in \Omega$, as $n \to \infty$. Therefore, by convexity

$$\liminf_n [u_n]_K^p \geqslant [v]_K^p.$$

Also, by Lemma 2.4 and continuous embeddings we have

$$\lim_n \int_\Omega [g_n u_n - H(x, u_n)] \, dx = \int_\Omega [gv - H(x, v)] \, dx.$$

So, recalling (4.7), we get

$$E(g, u) = \lim_n E(g_n, u_n) \leqslant E(g, v),$$

which implies $u = v$ by uniqueness of the maximizer in (4.1). So $u_n \to u$ in $L^p(\Omega)$, which yields (4.8). In addition, for all $n \in \mathbb{N}$ we have

$$\Phi(g) + \int_\Omega (g_n - g)u \, dx \leqslant \Phi(g_n) \leqslant \Phi(g) + \int_\Omega (g_n - g)u_n \, dx. \tag{4.9}$$

Indeed, by definition of $\Phi(g)$ we have

$$\begin{aligned}\Phi(g) + \int_\Omega (g_n - g)u \, dx &= E(g_n, u) \leqslant \Phi(g_n) \\ &= E(g, u_n) + \int_\Omega (g_n - g)u_n \, dx \leqslant \Phi(g) + \int_\Omega (g_n - g)u_n \, dx.\end{aligned}$$

Now fix $k \in \overline{\mathcal{G}} \setminus \{g\}$ and a sequence (τ_n) in $(0, 1)$ s.t. $\tau_n \to 0$ as $n \to \infty$. Set

$$g_n = g + \tau_n(k - g) \in \overline{\mathcal{G}},$$

so that $g_n \overset{*}{\rightharpoonup} g$. By (4.9) with such choice of g_n, we have for all $n \in \mathbb{N}$

$$\int_\Omega (k - g)u\,dx \leqslant \frac{\Phi(g + \tau_n(k - g)) - \Phi(g)}{\tau_n} \leqslant \int_\Omega (k - g)u_n\,dx.$$

Passing to the limit for $n \to \infty$, and noting that by (4.8) we have in particular $u_n \to u$ in $L^1(\Omega)$, we get

$$\lim_n \frac{\Phi(g + \tau_n(k - g)) - \Phi(g)}{\tau_n} = \int_\Omega (k - g)u\,dx.$$

By arbitrariness of (τ_n), and noting that $u \in L^1(\Omega) \subset L^\infty(\Omega)^*$, we see that Φ is Gâteaux differentiable at g with

$$\langle \Phi'(g), k - g\rangle = \int_\Omega (k - g)u\,dx,$$

which concludes the proof. □

We can now prove our optimization result, with a similar argument as in Section 3.

Proof of Theorem 1.2. By Lemma 4.1 and sequential weak* compactness of $\overline{\mathcal{G}}$, there exists $\bar{g} \in \overline{\mathcal{G}}$ s.t. for all $g \in \overline{\mathcal{G}}$

$$\Phi(\bar{g}) \geqslant \Phi(g).$$

Set $\bar{u} = u_{\bar{g}} \in W^{s,p}_0(\Omega)$, then by Lemma 4.3 we have for all $k \in \overline{\mathcal{G}} \setminus \{\bar{g}\}$

$$\langle \Phi'(\bar{g}), k - \bar{g}\rangle = \int_\Omega (k - \bar{g})\bar{u}\,dx.$$

Since $\bar{u} \in L^1(\Omega)$, by Lemma 2.1 (i) there exists $\hat{g} \in \mathcal{G}$ s.t. for all $g \in \overline{\mathcal{G}}$

$$\int_\Omega \hat{g}\bar{u}\,dx \geqslant \int_\Omega g\bar{u}\,dx,$$

in particular

$$\int_\Omega \hat{g}\bar{u}\,dx \geqslant \int_\Omega \bar{g}\bar{u}\,dx. \tag{4.10}$$

By convexity of Φ (Lemma 4.2) and (4.10), we have

$$\Phi(\hat{g}) \geqslant \Phi(\bar{g}) + \int_\Omega (\hat{g} - \bar{g})\bar{u}\,dx \geqslant \Phi(\bar{g}).$$

Therefore, $\hat{g} \in \mathcal{G}$ is a maximizer of Φ over $\overline{\mathcal{G}}$, which proves (i). In fact we have $\hat{g} = \bar{g}$, otherwise by strict convexity (Lemma 4.2 again) and (4.10) we would have

$$\Phi(\hat{g}) > \Phi(\bar{g}) + \int_\Omega (\hat{g} - \bar{g})\bar{u}\,dx \geqslant \Phi(\bar{g}),$$

against maximality of $\bar{g}$. Thus, any maximizer of Φ over $\overline{\mathcal{G}}$ actually lies in $\mathcal{G}$, which proves (*ii*). Finally, let $\hat{g} \in \mathcal{G}$ be a maximizer of Φ and set $\hat{u} = u_{\hat{g}} \in W_0^{s,p}(\Omega)$. As we have seen before, $\hat{g}$ is the only maximizer in $\overline{\mathcal{G}}$ for the linear functional

$$g \mapsto \int_\Omega g\hat{u}\,dx,$$

hence by Lemma 2.1 (*ii*) there exists a nondecreasing map $\eta : \mathbb{R} \to \mathbb{R}$ s.t. $\hat{g} = \eta \circ \hat{u}$ in Ω, thus proving (*iii*). □

Remark 4.4. Theorem 1.2 is analogous to Theorem 1.1 above, while in fact the problem is easier since we do not need to consider normalization to ensure uniqueness, unlike in problem (1.1). On the other hand, in this case we have no information on the sign of the solution u_g.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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