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Research article

A "nonlinear duality" approach to $W_0^{1,1}$ solutions in elliptic systems related to the Keller-Segel model†

Lucio Boccardo*

Istituto Lombardo – Accademia di Scienze e Lettere - Palazzo di Brera, Milano, Italia

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* **Correspondence:** Email: boccardo@mat.uniroma1.it.

Abstract: In this paper, we prove existence of distributional solutions of a nonlinear elliptic system, related to the Keller-Segel model. Our starting point is the boundedness theorem (for solutions of elliptic equations) proved by Guido Stampacchia and Neil Trudinger.

Keywords: duality method; nonlinear elliptic system; $W_0^{1,1}$ solutions

1. Background: results by Guido Stampacchia and Neil Trudinger

The following results about the summability of the solutions of Dirichlet problems for equations with discontinuous coefficients are nowadays classical, since the papers [19] and [20] by Guido Stampacchia and Neil Trudinger [12].

If $f \in L^m(\Omega)$, $m \geq \frac{2N}{N+2}$, thanks to Lax-Milgram Theorem and Sobolev embedding, there exist a weak solution $u \in W_0^{1,2}(\Omega)$ of

$$u \in W_0^{1,2}(\Omega): \int_\Omega M(x)\nabla u \nabla v = \int_\Omega f(x)\, v(x), \quad \forall\, v \in W_0^{1,2}(\Omega), \tag{1.1}$$

where Ω is a bounded, open subset of $\mathbb{R}^N$, $N \geq 2$; the matrix $M(x)$ is symmetric, uniformly elliptic and bounded: there exist $\alpha > 0$ and $\beta > 0$ such that

$$M(x)\,\xi \cdot \xi \geq \alpha\, |\xi|^2, \qquad |M(x)| \leq \beta, \tag{1.2}$$

for every ξ in $\mathbb{R}^N$, and for almost every x in Ω.
Moreover:

1) If $f \in L^m(\Omega)$, $\frac{2N}{N+2} \le m \le \frac{N}{2}$, the summability of u (which belongs to $L^{m^{**}}(\Omega)$, $m^{**} = \frac{mN}{N-2m}$, if $\frac{2N}{N+2} \le m < \frac{N}{2}$ and it has exponential summability if $m = \frac{N}{2}$) was proved in [19] (see also [5, 13, 20, 21]).
2) If $f \in L^m(\Omega)$, $m > \frac{N}{2}$, the boundedness of u, was proved in [19] (see also [13, 20]).
3) If $f \in L^m(\Omega)$, $1 < m < \frac{2N}{N+2}$, in [8], is proved a (nonlinear) Calderon-Zygmund theory for operators with discontinuous coefficients, showing the existence of a distributional solution $u \in W_0^{1,m^*}(\Omega)$, $m^* = \frac{mN}{N-m}$. Note that, in this case $m^* \in (1^*, 2)$ and, thanks to the Sobolev embedding, $u \in L^{m^{**}}(\Omega)$ as in (1).
4) The existence in $W_0^{1,1^*}(\Omega)$ is proved, in [8], if $\int_\Omega |f| \log(1 + |f|) < \infty$.

Then a question arises: is it possible to prove (as in (3)) that

$$f \in L^m(\Omega),\ \frac{2N}{N+2} < m < N,\ \text{ implies } \nabla u \in (L^{m^*}(\Omega))^N \ ? \tag{1.3}$$

In [4], it is proved that the above statement is false if $\frac{N}{2} < m < N$.

The results recalled in (1), (2), (3) and (4) are crucial in the next proofs.

2. A stationary system weakly related to the Keller-Segel model

In this paper, we prove existence of distributional solutions of the following nonlinear elliptic boundary value problem:

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) + u = -\operatorname{div}\left(u\,\frac{M(x)\nabla\psi}{1+\psi}\right) + f(x) & \text{in } \Omega, \\ -\operatorname{div}(M(x)\nabla\psi) + \psi = u^{\sigma-1} & \text{in } \Omega, \\ u = 0 = \psi & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

with

$$0 \le f(x) \in L^\rho(\Omega),\ \rho > 1. \tag{2.2}$$

Of course, $f(x)$ needs not to be identically null. On the power σ we suppose $1 < \sigma < \frac{2N-2}{N-2}$, but it is preferable (in the following proofs) to split the assumption as

$$\frac{N}{N-2} < \sigma < \frac{2N-2}{N-2}, \tag{2.3}$$

or

$$\sigma = \frac{N}{N-2}, \tag{2.4}$$

or

$$1 < \sigma < \frac{N}{N-2}. \tag{2.5}$$

There are many theoretical models for chemotaxis; one of the most important is the Keller-Segel one (see [14, 16, 17] and also [3, 10]). Following [15] (see also [14]), one of the possible models is the "chemical signal driven logistic model" which leads, in the stationary case, to the system above. Note that the equation for u includes a chemotaxis term with nonlinear flux limitation having a kind of logarithmic dependence: $\frac{\nabla\psi}{1+\psi} = \nabla \log(1 + \psi)$.

The original model of chemotaxis presents a linear dependence of the gradient of the concentration of the chemical substance ψ in the equation of u in the form $-\operatorname{div}(\chi\, \mathrm{u}\, \nabla\psi)$, for a positive given constant χ.

2.1. Approximate problems

Note that, with our assumption, we have $\sigma - 1 > 0$. We define $f_n(x) = \frac{f(x)}{1+\frac{1}{n}f(x)}$ and we consider the following approximate Dirichlet problem

$$\begin{cases} u_n \in W_0^{1,2}(\Omega) : \ \forall\, v \in W_0^{1,2}(\Omega), \\ \displaystyle\int_\Omega M(x)\,\nabla u_n \nabla v + \int_\Omega u_n\, v = \int_\Omega T_n(u_n)\,\frac{M(x)\,\nabla\psi_n\nabla v}{(1+\frac{1}{n}\,|\nabla\psi_n|)(1+\psi_n)} + \int_\Omega f_n\, v, \\ \psi_n \in W_0^{1,2}(\Omega) : \ \forall\, \varphi \in W_0^{1,2}(\Omega), \\ \displaystyle\int_\Omega M(x)\,\nabla\psi_n\nabla\varphi + \int_\Omega \psi_n\varphi = \int_\Omega [T_{(n^3)}(u_n)]^{\sigma-1}\,\varphi, \end{cases}$$

where, $\forall\, k \in \mathbb{R}^+$,

$$T_k(s) = \begin{cases} s, \ \text{if } |s| \le k, \\ k\,\frac{s}{|s|}, \ \text{if } |s| > k. \end{cases}$$

Here we enumerate some properties of the solutions u_n, ψ_n.

A The existence of (u_n, ψ_n) is a consequence of Proposition 3.1 of [11] (with minimal changes).

B The positivity of $f(x)$ gives, in the first equation, $u_n \ge 0$ (see [3]), which, in the second equation, implies $\psi_n \ge 0$.

C In the first equation, $0 \le f_n(x) \le n$ and the modulus of the function in the divergence term is less than n^2, so that, with the boundedness theorem by G. Stampacchia ([19], see also [20]), we deduce $\|u_n\|_{L^\infty(\Omega)} \le C_0\, n^2$. Thus, in the second equation, we observe that $T_{(n^3)}(u_n) = u_n$ (for $n > n_0$) and we can rewrite the above system as

$$\begin{cases} 0 \le u_n \in W_0^{1,2}(\Omega) : \ \forall\, v \in W_0^{1,2}(\Omega), \\ \displaystyle\int_\Omega M(x)\,\nabla u_n \nabla v + \int_\Omega u_n\, v = \int_\Omega T_n(u_n)\,\frac{M(x)\,\nabla\psi_n\nabla v}{(1+\frac{1}{n}\,|\nabla\psi_n|)(1+\psi_n)} + \int_\Omega f_n\, v\,; \\ 0 \le \psi_n \in W_0^{1,2}(\Omega) : \ \forall\, \varphi \in W_0^{1,2}(\Omega), \\ \displaystyle\int_\Omega M(x)\,\nabla\psi_n\nabla\varphi + \int_\Omega \psi_n\varphi = \int_\Omega (u_n)^{\sigma-1}\,\varphi. \end{cases} \tag{2.6}$$

3. Nonlinear duality method

In the following lemma, in spite of the nonlinearity of the problem, we use a kind of duality, which will be advantageous to prove a priori estimates.

Lemma 3.1. *We assume* (1.2)*,* (2.2)*,* $1 < \sigma < \frac{2N-2}{N-2}$*. Let* $a \in (0,1)$*. Then the following "nonlinear dual" inequality holds*

$$\int_\Omega \frac{(u_n)^\sigma}{(1+\psi_n)} \le \frac{1}{a}\int_\Omega f_n(x)(\psi_n)^a. \tag{3.1}$$

Proof. In the above system, we use $\log(1+\psi_n)$ as test function in the first equation, $\dfrac{-\,u_n}{(1+\psi_n)}$ as test function in the second equation and we have

$$\begin{cases} \displaystyle\int_\Omega M(x)\,\nabla u_n \frac{\nabla\psi_n}{1+\psi_n} + \int_\Omega [u_n - f_n]\log(1+\psi_n) = \int_\Omega \frac{T_n(u_n)}{(1+\psi_n)^2}\frac{M(x)\,\nabla\psi_n\nabla\psi_n}{(1+\frac{1}{n}|\nabla\psi_n|)} \\ \displaystyle -\int_\Omega M(x)\,\nabla\psi_n\nabla u_n \frac{1}{(1+\psi_n)} + \int_\Omega M(x)\,\nabla\psi_n\nabla\psi_n \frac{u_n}{(1+\psi_n)^2} = \int_\Omega [\psi_n - (u_n)^{\sigma-1}]\frac{u_n}{(1+\psi_n)}. \end{cases}$$

Then, after simplifications $\Big($we use $0 \le \dfrac{T_n(u_n)}{1+\frac{1}{n}|\nabla\psi_n|} \le u_n\Big)$, we deduce that

$$\int_\Omega (u_n)^{\sigma-1}\frac{u_n}{(1+\psi_n)} + \int_\Omega u_n\Big[\log(1+\psi_n) - \frac{\psi_n}{(1+\psi_n)}\Big] \le \int_\Omega f_n(x)\log(1+\psi_n)$$

and, dropping a positive term, we prove the inequality

$$\int_\Omega \frac{(u_n)^\sigma}{(1+\psi_n)} \le \int_\Omega f_n(x)\log(1+\psi_n). \tag{3.2}$$

Now we use the inequality $0 \le \log(1+\psi_n) \le \frac{1}{a}(\psi_n)^a$, $a \in (0,1)$, and we have (3.1).

□

Lemma 3.2. *We assume* (1.2)*,* (2.2)*. Then the sequence* $\{u_n\}$ *is bounded in*

$$\begin{cases} L^{\frac{N}{N-2}}(\Omega),\ if\ \frac{N}{N-2} < \sigma < \frac{2N-2}{N-2}\ (assumption\,(2.3)); \\ L^r(\Omega),\ r < \frac{N}{N-2},\ if\ \sigma = \frac{N}{N-2}\ (assumption\,(2.4)); \\ L^\sigma(\Omega),\ if\ 1 < \sigma < \frac{N}{N-2}\ (assumption\,(2.5)). \end{cases}$$

Proof. First part: $\frac{N}{N-2} < \sigma < \frac{2N-2}{N-2}$ - Let $q < \sigma$. Then (we use Hölder inequality with exponents $\frac{\sigma}{q}$ and $\frac{\sigma}{\sigma-q}$) we have, using (3.1),

$$\int_\Omega (u_n)^q = \int_\Omega \frac{(u_n)^q}{(1+\psi_n)^{\frac{q}{\sigma}}}(1+\psi_n)^{\frac{q}{\sigma}} \le \Big[\int_\Omega \frac{(u_n)^\sigma}{(1+\psi_n)}\Big]^{\frac{q}{\sigma}}\Big[\int_\Omega (1+\psi_n)^{\frac{q}{\sigma-q}}\Big]^{\frac{\sigma-q}{\sigma}}$$

$$\le \Big[\frac{1}{a}\int_\Omega f_n(x)(\psi_n)^a\Big]^{\frac{q}{\sigma}} \|1+\psi_n\|^{\frac{q}{\sigma}}_{\frac{q}{\sigma-q}} \le \Big(\frac{1}{a}\Big)^{\frac{q}{\sigma}}\Big[\|\psi_n\|^a_{\frac{q}{\sigma-q}}\,\|f\|_{\frac{q}{q-(\sigma-q)a}}\Big]^{\frac{q}{\sigma}}\Big(C_1 + \|\psi_n\|_{\frac{q}{\sigma-q}}\Big)^{\frac{q}{\sigma}},$$

that is

$$\|u_n\|_q \le \Big(\frac{1}{a}\Big)^{\frac{1}{\sigma}}\Big[\|\psi_n\|^a_{\frac{q}{\sigma-q}}\,\|f\|_{\frac{q}{q-(\sigma-q)a}}\Big]^{\frac{1}{\sigma}}\Big(C_1 + \|\psi_n\|_{\frac{q}{\sigma-q}}\Big)^{\frac{1}{\sigma}}$$

Define $q = \frac{N}{N-2}$ and $p = \frac{N}{(\sigma-1)(N-2)}$; we note that $p > 1$ since $\sigma < \frac{2N-2}{N-2}$.

Then we use Calderon-Zygmund type estimates for Dirichlet problems with infinite energy solutions, proved in [8, 19] (see (1) and (3)) and we have

$$\|u_n\|_q \le \Big(\frac{1}{a}\Big)^{\frac{1}{\sigma}}\Big[\|\psi_n\|^a_{p^{**}}\,\|f\|_{\frac{q}{q-(\sigma-q)a}}\Big]^{\frac{1}{\sigma}}\Big(C_1 + \|\psi_n\|_{p^{**}}\Big)^{\frac{1}{\sigma}}$$

$$\|u_n\|^\sigma_q \le \Big(\frac{1}{a}\Big)\Big[C_p\|u_n^{\sigma-1}\|^a_p\,\|f\|_{\frac{q}{q-(\sigma-q)a}}\Big]\Big(C_1 + C_p\|u_n^{\sigma-1}\|_p\Big)$$

We note that $p(\sigma - 1) = q$ and we rewrite the last inequality as

$$\|u_n\|_q^{\sigma} \leq \Big(\frac{1}{a}\Big)\Big[C_q \|u_n\|_q^{(\sigma-1)a}\ \|f\|_{\frac{q}{q-(\sigma-q)a}}\Big]\Big(C_1 + C_q\|u_n\|_q^{\sigma-1}\Big)$$

Thus for $a > 0$ close to zero, we have proved the following estimate, where $\rho > 1$ is close to one,

$$\|u_n\|_{\frac{N}{N-2}} \leq C_0(\|f\|_{\rho}) \tag{3.3}$$

Second part: $\sigma = \frac{N}{N-2}$ - There is only a slight change with respect to the previous case: $p(\sigma - 1) < q$.
Third part: $1 \leq \sigma < \frac{N}{N-2}$ - Recall the following L^{∞} estimate (proved in [19], see also [20]), concerning the second equation,

$$\|\psi_n\|_{\infty} \leq C_0\|(u_n)^{\sigma-1}\|_p, \quad p > \frac{N}{2}.$$

Then we deduce directly from (3.2)

$$\frac{1}{(1 + \|\psi_n\|_{\infty})}\int_{\Omega}(u_n)^{\sigma} \leq \log(1 + \|\psi_n\|_{\infty})\int_{\Omega} f(x)$$

and

$$\int_{\Omega}(u_n)^{\sigma} \leq (1 + \|\psi_n\|_{\infty})\frac{1}{a}\|\psi_n\|_{\infty}^{a}\|f\|_1 \leq (1 + C_0\|(u_n)^{\sigma-1}\|_p)\frac{1}{a}[C_0\|(u_n)^{\sigma-1}\|_p]^a\|f\|_1.$$

Let $p = \sigma'$ (which implies $\sigma < \frac{N}{N-2}$). Then

$$\int_{\Omega}(u_n)^{\sigma} \leq C(\|f\|_1).$$

□

Corollary 3.3. *We assume* (1.2), (2.2). *As a consequence of the previous lemma, the sequence* $\{(u_n)^{\sigma-1}\}$ *is bounded in*

$$\begin{cases} L^{\frac{N}{(N-2)(\sigma-1)}}(\Omega), \ \textit{if}\ \frac{N}{N-2} < \sigma < \frac{2N-2}{N-2}; \\ L^{s}(\Omega), \ s < \frac{N}{2}, \ \textit{if}\ \sigma = \frac{N}{N-2}; \\ L^{\sigma'}(\Omega), \ \textit{if}\ 1 < \sigma < \frac{N}{N-2}. \end{cases}$$

Thus the right hand side of the second equation is bounded in $L^1(\Omega)$ *if* $\frac{N}{(N-2)(\sigma-1)} \geq 1$*; that is, if* $\sigma \leq \frac{2N-2}{N-2}$.

Corollary 3.4. *If, in the second equation of* (2.6), *we take as test function* $\dfrac{\psi_n}{1+\psi_n}$, *(following [2, 3]), we have*

$$\alpha\int_{\Omega}\frac{|\nabla\psi_n|^2}{(1+\psi_n)^2} \leq \int_{\Omega}(u_n)^{\sigma-1} \leq C_1. \tag{3.4}$$

Corollary 3.5. *The sequence* $\{\psi_n\}$ *is bounded in* $W_0^{1,2}(\Omega)$ *if the right hand side of the second equation is bounded in* $L^{\frac{2N}{N+2}}(\Omega)$ *that is if*

$$\begin{cases} \sigma \leq \frac{3N-2}{2(N-2)}, \ \textit{if}\ \frac{N}{N-2} < \sigma < \frac{2N-2}{N-2}; \\ \textit{always}, \ \textit{if}\ \sigma = \frac{N}{N-2}; \\ \textit{always}, \ \textit{if}\ 1 < \sigma < \frac{N}{N-2}. \end{cases}$$

Corollary 3.6. *If in the first equation of* (2.6) *we take as test function* $\dfrac{u_n}{1+u_n}$*, following [2, 3], and we use Young inequality, we have*

$$\frac{\alpha}{2}\int_\Omega \frac{|\nabla u_n|^2}{(1+u_n)^2} \le \frac{\beta^2}{2\alpha}\int_\Omega \frac{|\nabla\psi_n|^2}{(1+\psi_n)^2} + \int_\Omega f. \tag{3.5}$$

That is, the sequence $\{\frac{|\nabla u_n|}{(1+u_n)}\}$ *is bounded in* $L^2(\Omega)$*; with this boundedness, in [3], is proved that there exists a measurable function* $u(x)$ *such that*

$$u_n(x) \text{ converges a.e. to } u(x). \tag{3.6}$$

Corollary 3.7. *If in the first equation of* (2.6) *we take as test function* $T_k(u_n)$*, following [2, 3], we deduce*

$$\frac{\alpha}{2}\int_\Omega |\nabla T_k(u_n)|^2 \le k^2\frac{\beta^2}{2\alpha}\int_\Omega \frac{|\nabla\psi_n|^2}{(1+\psi_n)^2} + k\int_\Omega f,$$

so that we can add to (3.6) *the following weak convergence*

$$T_k(u_n) \text{ converges weakly in } W_0^{1,2}(\Omega) \text{ to } T_k(u),\ \forall\, k\in\mathbb{R}^+. \tag{3.7}$$

Corollary 3.8. *If* $1<\sigma<\frac{2N-2}{N-2}$*, the sequence* $\{(u_n)^{\sigma-1}\}$ *is bounded in* $L^\nu(\Omega)$*,* $\nu>1$ *(and more: in* $L^{\sigma'}(\Omega)$ *if* $1<\sigma<\frac{N}{N-2}$*). Then the above a.e. convergence* (3.6) *and the Vitali theorem say that the sequence* $\{(u_n)^{\sigma-1}\}$ *converges in* $L^1(\Omega)$ *to* $\{u^{\sigma-1}\}$*.*

Then (see [7, 8]) the sequence $\{\psi_n\}$ *is compact in* $W_0^{1,q}(\Omega)$*,* $q<\frac{N}{N-1}$*, at least; in Corollary 3.5 is proved a stronger result for a smaller subset of exponents* σ*. Define* ψ *a cluster point of* $\{\psi_n\}$ *in* $W_0^{1,q}(\Omega)$*.*

Corollary 3.9. *A result by Leone-Porretta ([18]) states that the sequence* $\{\nabla T_k(\psi_n)\}$ *is* L^2 *compact, because the right hand side of the second equation in* (2.6) *is* L^1 *compact (Corollary 3.8).*

Lemma 3.10. *The sequence*

$$\Big\{\frac{|\nabla\psi_n|}{1+\psi_n}\Big\} \text{ is } L^2 \text{ compact.} \tag{3.8}$$

Proof. If in the second equation of (2.6) we take $\Big[\dfrac{\psi_n}{1+\psi_n}-\dfrac{k}{1+k}\Big]^+$ as test function and we use Hölder inequality, we have (recall (3.3))

$$\alpha\int_{\{k<\psi_n\}}\frac{|\nabla\psi_n|^2}{(1+\psi_n)^2} \le \int_{\{k<\psi_n\}}(u_n)^{\sigma-1} \le (C_0\|f\|_\rho)^{\sigma-1}\Big|\{k<\psi_n\}\Big|^{1-\frac{(\sigma-1)(N-2)}{N}}. \tag{3.9}$$

Now we use this inequality to prove the L^1 equi-integrability of the sequence $\Big\{\dfrac{|\nabla\psi_n|^2}{(1+\psi_n)^2}\Big\}$. Indeed, for every measurable subset $E\subset\Omega$, we have

$$\begin{aligned}\int_E\frac{|\nabla\psi_n|^2}{(1+\psi_n)^2} &\le \int_{\{k<\psi_n\}}\frac{|\nabla\psi_n|^2}{(1+\psi_n)^2} + \int_{E\cap\{\psi_n\le k\}}\frac{|\nabla\psi_n|^2}{(1+\psi_n)^2}\\ &\le \frac{1}{\alpha}(C_0\|f\|_\rho)^{\sigma-1}\Big|\{k<\psi_n\}\Big|^{1-\frac{(\sigma-1)(N-2)}{N}} + \int_E|\nabla T_k(\psi_n)|^2.\end{aligned}$$

Now Corollary 3.9 says that, for every $k \in \mathbb{R}^+$, the last integral is small (uniformly with respect to n) if $|E|$ is small. Here $|E|$ denotes the measure of the subset E.

Moreover $\big|\{k < \psi_n\}\big|$ is small (uniformly with respect to n) for k large enough. Thus the last two sentences prove that

$$\text{the sequence } \Big\{\frac{|\nabla\psi_n|}{1+\psi_n}\Big\} \text{ is } L^2 \text{ equi-integrable.} \tag{3.10}$$

Furthermore a result proved in [8] implies that the sequences $\{\nabla\psi_n(x)\}$ and $\{\psi_n(x)\}$ converge almost everywhere, so that these a.e. convergences, (3.10) and Vitali theorem yield (3.8). □

Corollary 3.11. *In the first equation of* (2.6) *we take as test function* $\Big[\dfrac{u_n}{1+u_n} - \dfrac{k}{1+k}\Big]^+$, $k \in \mathbb{R}^+$, *(following [2, 3]) we use the Young inequality and we have*

$$\frac{\alpha}{2}\int_{\{k<u_n\}} \frac{|\nabla u_n|^2}{[1+u_n]^2} \le \frac{\beta^2}{2\alpha}\int_{\{k<u_n\}} \frac{|\nabla\psi_n|^2}{(1+\psi_n)^2} + \int_{\{k<u_n\}} f. \tag{3.11}$$

Moreover, there is a second important consequence of (3.10)*: the a priori estimates on the sequence* $\{u_n\}$ *imply that* $\big|\{k < u_n\}\big|$ *is small for k large (uniformly with respect to n), so that, in* (3.11)*, the term* $\int_{\{k<u_n\}} \frac{|\nabla\psi_n|^2}{(1+\psi_n)^2}$ *is small (uniformly with respect to n) if k is large enough and then the term*

$$\int_{\{k<u_n\}} \frac{|\nabla u_n|^2}{[1+u_n]^2} \text{ is also small (uniformly with respect to n) if k is large enough.} \tag{3.12}$$

3.1. Entropy solutions

Following [3] and [1] we recall the definition of entropy solution, useful in cases (as here) of very singular framework, where the definition of distributional solution is meaningless.

Note that, if $N > 4$, $u \notin L^2(\Omega)$, so that the term $u\frac{\nabla\psi}{1+\psi}$ does not belong to L^1.

Definition 3.12. *A measurable function u is an entropy solution of the first equation of our system if*

$$\begin{cases} T_k(u) \in W_0^{1,2}(\Omega), \ \forall\, k \in \mathbb{R}^+; \\ \displaystyle\int_\Omega M(x)\nabla u \nabla T_k[u-\varphi] + \int_\Omega u\, T_k[u-\varphi] \\ \displaystyle\qquad \le \int_\Omega u\,\frac{M(x)\,\nabla\psi\cdot\nabla T_k[u-\varphi]}{1+\psi} + \int_\Omega f(x) T_k[u-\varphi], \\ \forall\, k \in \mathbb{R}^+, \forall\, \varphi \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega). \end{cases} \tag{3.13}$$

Thanks to (3.6), Corollary 3.8, (3.8), we can use the above definition for our problem, we can repeat the proof of Theorem 3.9 of [3] and we prove the following result.

Theorem 3.13. *Assume* (1.2), (2.2), $1 < \sigma < \frac{2N-2}{N-2}$. *Then there exists an entropy solution* $u \ge 0$ *of the first equation in the sense of Definition 3.12. Moreover there exists a weak solution* $0 \le \psi \in W_0^{1,2}(\Omega)$ *of the second equation, if* $\sigma \le \frac{3(N+2)}{2(N-2)}$ *and* $\frac{N}{N-2} < \sigma < \frac{2N-2}{N-2}$, *or a distributional solution* $0 \le \psi \in W_0^{1,q}(\Omega)$, *in the other range of value of* σ.

Remark 3.14. *Note that we have not proved that u, entropy solution of the first equation, belongs to some Sobolev space; we only have, from* (3.5), *that* $\log(1+u)$ *belongs to* $W_0^{1,2}(\Omega)$.

3.2. Distributional solutions

In this subsection we study a case of distributional solutions u, that is a case of $\nabla u \in L^1$.
Observe that Lemma 3.2 says that the sequence $\{u_n\}$ is bounded in $L^2(\Omega)$ if

$$\begin{cases} N \leq 4, \text{ under the assumption (2.3);} \\ N < 4, \text{ under the assumptions (2.4) and (2.5).} \end{cases} \tag{3.14}$$

Lemma 3.15. *Assume* (3.14). *Then the sequence* $\{\nabla u_n\}$ *is equi-integrable and the sequence* $\{u_n\}$ *is* L^1 *compact.*

Proof. Here we follow an approach of [9] (see also [6]). Since we observed that the sequence $\{u_n\}$ is bounded in $L^2(\Omega)$, we use the Hölder inequality, (3.11) and we have

$$\int_{\{k<u_n\}} |\nabla u_n| = \int_{\{k<u_n\}} \frac{|\nabla u_n|}{[1+u_n]}[1+u_n] \leq \Big[\int_{\{k<u_n\}} \frac{|\nabla u_n|^2}{[1+u_n]^2}\Big]^{\frac{1}{2}}\Big[\int_\Omega [1+u_n]^2\Big]^{\frac{1}{2}}$$

$$\leq \Big[\frac{\beta^2}{2\alpha}\int_{\{k<u_n\}} \frac{|\nabla \psi_n|^2}{(1+\psi_n)^2} + \int_{\{k<u_n\}} f\Big]^{\frac{1}{2}} C_1[1+\|u_n\|_2] = \omega_k.$$

In (3.12) is proved that ω_k is small (uniformly with respect to n) if k is large enough. Then, for every measurable subset $E \subset \Omega$, we deduce that

$$\int_E |\nabla u_n| \leq \int_{\{k<u_n\}} |\nabla u_n| + \int_E |\nabla T_k(u_n)| \leq \omega_k + |E|^{\frac{1}{2}}\Big[\int_\Omega \nabla T_k(u_n)|^2\Big]^{\frac{1}{2}}$$

which implies (recall Corollary 3.7)

$$\lim_{|E|\to 0} \int_E |\nabla u_n| \leq \omega_k,$$

that is the equi-integrability.

The above inequalities, with $k = 0$, give the L^1 boundedness of of the sequence $\{\nabla u_n\}$. Then the L^1 compactness of the sequence $\{u_n\}$ is a consequence of the Rellich theorem.

Thus we improved (3.6):

$$\nabla u_n \text{ converges weakly in } L^1 \text{ to } \nabla u. \tag{3.15}$$

□

Now we can state the existence of distributional solutions.

Theorem 3.16. *Under the assumptions of Theorem 3.13, let assume* (3.14). *Then there exist distributional solutions* $0 \leq u \in W_0^{1,1}(\Omega)$ *and* $0 \leq \psi \in W_0^{1,q}(\Omega)$, $q < \frac{N}{N-1}$, *of system* (2.1)*; that is, we have that*

$$\int_\Omega M(x)\,\nabla u \cdot \nabla v + \int_\Omega u\,v = \int_\Omega u\,\frac{M(x)\,\nabla\psi \cdot \nabla v}{(1+\psi)} + \int_\Omega f\,v\,,$$

for every v *in* $C_0^1(\Omega)$*, and*

$$\int_\Omega M(x)\,\nabla\psi \cdot \nabla\varphi + \int_\Omega \psi\varphi = \int_\Omega u^{\sigma-1}\,\varphi\,,$$

for every φ *in* $C_0^1(\Omega)$.

3.3. A direct approach to the boundedness of the sequence $\{\psi_n\}$

In this subsection, we assume (1.2), $f \in L^1(\Omega)$, $1 < \sigma < \frac{3}{2} + \frac{1}{N}$.

Following [3], we prove the following a priori estimate

$$\int_\Omega |u_n| \le \int_\Omega |f|. \tag{3.16}$$

Indeed, if we take $\dfrac{u_n}{h + |u_n|}$ as test function in the first equation, we have (thanks to the Young inequality)

$$\frac{\alpha h}{2} \int_\Omega \frac{|\nabla u_n|^2}{(h + |u_n|)^2} + \int_\Omega \frac{|u_n|^2}{h + |u_n|} \le \frac{h}{2\alpha} \int_\Omega \beta^2 \frac{|\nabla \psi_n|^2}{(1 + \psi_n)^2} + \int_\Omega |f|,$$

which implies, dropping a positive term and letting $h \to 0$, the estimate (3.16).

Thus, for the right hand side of the second equation we have the estimate

$$\int_\Omega (u_n^{\sigma-1})^{\frac{1}{\sigma-1}} \le \int_\Omega |f|$$

and, if $\frac{1}{\sigma-1} > \frac{N}{2}$ (that is $\sigma - 1 < \frac{2}{N}$), the right hand side of the second equation is bounded in $L^s(\Omega)$, $s > \frac{N}{2}$, which implies that the sequence of the solutions $\{\psi_n\}$ is bounded in $L^\infty(\Omega)$; if $\frac{1}{\sigma-1} \ge \frac{2N}{N+2}$ (that is $\sigma - 1 \le \frac{1}{2} + \frac{1}{N}$), the right hand side of the second equation is bounded in $L^{\frac{2N}{N+2}}(\Omega)$, which implies that the sequence of the solutions $\{\psi_n\}$ is bounded in $W_0^{1,2}(\Omega)$.

Summarizing, with this approach,

$$\sigma - 1 \le \tfrac{1}{2} + \tfrac{1}{N} \text{ yields the boundedness of the sequence } \{\psi_n\} \text{ in } W_0^{1,2}(\Omega), \tag{3.17}$$

with the use of the estimate (3.16).

3.4. General nonlinearities

It is possible to adapt our approach (nonlinear duality) to the case of the system

$$\int_\Omega M(x)\,\nabla u \cdot \nabla v + \int_\Omega u\,v = \int_\Omega u\,\frac{M(x)\,\nabla\psi \cdot \nabla v}{(1+\psi)^\gamma} + \int_\Omega f\,v, \quad \forall\, v \in C_0^1(\Omega),$$

$$\int_\Omega M(x)\,\nabla\psi \cdot \nabla\varphi + \int_\Omega \psi\varphi = \int_\Omega u^{\sigma-1}\,\varphi, \quad \forall\, \varphi \in C_0^1(\Omega),$$

with $\gamma \in \mathbb{R}^+$. A possible approach (which we only sketch here) is

- define an approximate system (as in (2.6));
- use as test functions $\Big(g(\psi_n), \dfrac{u_n}{h(\psi_n)}\Big)$

with

$$\begin{cases} g(t) = \displaystyle\int_0^t e^{\frac{(1+s)^{1-\gamma}}{\gamma-1}}\,ds \\ h(t) = e^{-\frac{(1+t)^{1-\gamma}}{\gamma-1}}. \end{cases}$$

Conflict of interest

The author declares no conflict of interest.

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