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Research article

Weak solutions of generated Jacobian equations†

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Abstract: We prove two groups of relationships for weak solutions to generated Jacobian equations under proper assumptions on the generating functions and the domains, which are generalizations for the optimal transportation case and the standard Monge-Ampère case respectively. One group of weak solutions is Aleksandrov solution, Brenier solution and C-viscosity solution. The other group of weak solutions is Trudinger solution and L^p-viscosity solution.

Keywords: generated Jacobian equations; Aleksandrov solutions; Brenier solutions; C-viscosity solutions; Trudinger solutions; L^p-viscosity solutions

Dedicated to Professor Neil S. Trudinger on the occasion of his 80th birthday.

1. Introduction

In this paper, we study several weak solutions of the generated Jacobian equations (GJEs), (which were first introduced by Neil S. Trudinger in [34, 35]), subject to some boundary value conditions. These kinds of weak solutions of GJEs are proved to be equivalent under some necessary assumptions, which extends the known results in the optimal transportation case and the standard Monge-Ampère case.

We begin with the Jacobian determinant equations (JDEs) in [6],

$$\det DY = \psi(x), \quad \text{in } \Omega \subset \mathbb{R}^n, \tag{1.1}$$

which are the first order fully nonlinear underdetermined partial differential equations of the vector function $Y : \Omega \to \Omega$, where $\psi : \Omega \to \mathbb{R}^+$ is a given scalar function. The celebrated result established in [6] is one of the main tools for the correction of volume distortion in relation to the standard volume in Hölder spaces.

If Y and ψ in (1.1) depend also on u and Du for an unknown scalar function $u : \Omega \to \mathbb{R}$, we get the prescribed Jacobian equations (PJEs) as in [32, 33]. PJEs associated to the second boundary value conditions can be written as

$$\det DY(\cdot, u, Du) = \psi(\cdot, u, Du), \quad \text{in } \Omega, \tag{1.2}$$

$$Y(\cdot, u, Du)(\Omega) = \Omega^*, \tag{1.3}$$

where $\Omega, \Omega^* \subset \mathbb{R}^n$ are two given domains, $Y : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 mapping, Du denotes the gradient of the unknown function $u : \Omega \to \mathbb{R}$, and $\psi : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^+$ is a given scalar function. Here Ω and Ω^* are called the source domain and the target domain, respectively. Then Eq (1.2) can be regarded as the second order fully nonlinear partial differential equation of the unknown function u. The second boundary value condition (1.3) is usually called the natural boundary condition.

For $\Gamma \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, if the mapping Y, together with its dual function $Z : \Omega \times \mathbb{R} \times \mathbb{R}$, can be derived by generating functions $g \in C^4(\Gamma)$ through the equations

$$g_x(x, Y, Z) = Du, \quad g(x, Y, Z) = u, \tag{1.4}$$

then prescribed Jacobian Eq (1.2) is called *generated Jacobian equation* (GJE), and can be written as the following Monge-Ampère type form

$$\det[D^2u - A(\cdot, u, Du)] = B(\cdot, u, Du), \quad \text{in } \Omega, \tag{1.5}$$

where

$$A(\cdot, u, Du) = g_{xx}(\cdot, Y(\cdot, u, Du), Z(\cdot, u, Du)), \quad B(\cdot, u, Du) = \det E(\cdot, Y, Z)\psi(\cdot, u, Du). \tag{1.6}$$

The solvability of Y and Z from (1.4) and the form of Eq (1.5) are guaranteed by conditions A1 and A2 respectively, which will be introduced together with the matrix E in Section 2. In particular, we consider that the function ψ is separable in the sense that

$$\psi(\cdot, u, Du) = \frac{f(\cdot)}{f^* \circ Y(\cdot, u, Du)} \tag{1.7}$$

for positive functions $f \in L^1(\Omega)$ and $f^* \in L^1(\Omega^*)$ satisfying

$$\int_\Omega f = \int_{\Omega^*} f^*. \tag{1.8}$$

In applications of geometric optics and optimal transportation, condition (1.8) is called the conservation of energy and the mass balance condition, respectively.

Note that GJEs were introduced by Trudinger [35] to extend the Monge-Ampère theory in optimal transport problems to the near field geometric optics problems. In the near field optics, we refer the readers to the references [8, 13, 14, 35] for the explicit examples of generating functions. In the far field optics, the corresponding Monge-Ampère type has no u dependence, namely A and B in (1.5) are independent of u, see [39] for example.

On the other side, it is known in [8, 21, 22] that the structures and the underlying g-convexity theory of GJEs are also emerging in economics, in relation to both matching problems and principal/agent problems. In [22], the duality structure given by a generating function g yields a "Galois connection", which is already known in the economics literature and the computer science literature.

Recently, besides the applications of optics and economics, the theoretical and numerical aspects of GJEs themself have been extensively studied, see [9, 11–14, 19, 23–25, 36] for the theoretical aspect and [1, 3, 7, 28, 29] for the numerical aspect. So far, the study of GJEs has become an important research area.

The study of weak solutions to GJEs is both important in the theoretical study and the numerical analysis. For instance, the survey article [3] is a good introduction to GJEs, which proposes the theory for viscosity solutions of GJEs as a possible future direction. The aim of this paper is to show the relations and differences between several notions of weak solutions. One group of weak solutions is Aleksandrov solutions, Brenier solutions, C-viscosity solutions. The other group of weak solutions is Trudinger solutions and L^p-viscosity solutions.

We now formulate the main theorems of this paper. The terminologies in the main theorems will be introduced in Section 2.

Theorem 1.1. *Assume that positive functions $f \in L^1(\Omega)$ and $f^* \in L^1(\Omega^*)$ satisfy* (1.8)*, and conditions A1, A1*, A2, A3w and A4w are satisfied. Then the following relationships hold.*

(i). An Aleksandrov solution of (1.5) *is a Brenier solution of* (1.5)*. If Ω^* is g^*-convex with respect to $\Omega \times J$, then a Brenier solution of* (1.5) *is also an Aleksandrov solution of* (1.5)*.*

(ii). If f and f^* are continuous functions, then an Aleksandrov solution of (1.5) *is equivalent to a C-viscosity solution of* (1.5)*.*

Note that the relationship between Aleksandrov solution and Brenier solution in Theorem 1.1 extends the corresponding result for the optimal transportation case in [18] to the generated Jacoabian case. Also, the relationship between Aleksandrov solution and C-viscosity solution in Theorem 1.1 extends the corresponding result for the optimal transportation case in [16, 17] to the generated Jacoabian case.

In Theorem 1.1, when the conditions in (i) and (ii) are all satisfied, then Aleksandrov solution, Brenier solution and C-viscosity solution of (1.5) are all equivalent. Therefore, in this equivalent case, the Brenier solution and the C-viscosity solution of problem (1.5)–(1.3) can also have global C^3 regularity as the Aleksandrov solution under the additional assumptions A5, $f \in C^2(\bar{\Omega})$, $f^* \in C^2(\bar{\Omega}^*)$ and the uniform g-convexity of Ω and uniform g^*-convexity of Ω^* respectively, see Theorem 6.1 in [26]. Note that in [37], Trudinger is able to prove the C^3 regularity of Aleksandrov solution u without the monotonicity assumption A4w, see Corollary 4.1 in [37], where the strict g-convexity of u in [9] is applied. When A3w is strengthened to A3, the interior local C^2 estimate and interior local $C^{2,\alpha}$ estimate for Aleksandrov solution of (1.5) are proved in [27] when f is Dini continuous and Hölder continuous respectively. By Theorem 1.1, such interior local C^2 and $C^{2,\alpha}$ estimates also hold for Brenier solutions and C-viscosity solutions of (1.5).

In order to study Trudinger solutions and L^p-viscosity solutions, we consider equation (1.5) subject to the homogeneous Dirichlet boundary condition

$$u = 0, \quad \text{on } \partial\Omega. \tag{1.9}$$

Theorem 1.2. *Assume that $f \in L^p(\Omega)$ ($p \geq 1$) is a nonnegative function, f^* is a continuous positive function in $\bar{\Omega}^*$, then a weak solution v of problem* (1.5)–(1.9) *in the sense of Trudinger is an L^p-viscosity solution of problem* (1.5)–(1.9).

Theorem 1.2 extends the corresponding result for the standard Monge-Ampère case in [2] to the generated Jacobian case.

Note that in Theorem 1.1, B in (1.5) is only positive, while in Theorem 1.2, B in (1.5) is allowed to be nonnegative. This is the reason why we separate their statements into two theorems.

Although we have established equivalent results for various weak solutions under some conditions, it should be pointed out that these weak solutions are different in general. Aleksandrov solution and Brenier solution are defined in the sense of measure, which are fit for the measurable right hand side. If Ω^* is not g^*-convex with respect to $\Omega \times J$, then a Brenier solution of (1.5) will not be an Aleksandrov solution of (1.5). In this case, only partial regularity of Brenier solutions can be expected, see [12]. The C-viscosity and L^p-viscosity solutions are defined by C^2 and $W^{2,p}$ test functions using comparison principle, which can be applied to the cases when the right hand side terms are continuous and L^p functions, respectively. Trudinger solution is a notion in the sense of smooth approximations, which can also be applied to the case when the right hand side term is in L^p space.

This paper is organized as follows. In Section 2, we recall some conditions on the generating function g, and introduce the g-convexity of a function u and the g-convexity of the domain Ω, and finally give the definitions of the weak solutions of GJEs, which are presented in three subsections. In Sections 3 and 4, we give the proofs of Theorems 1.1 and 1.2 for the two groups of weak solutions, respectively. In particular, Trudinger solution and L^p-viscosity solution are linked by a kind of uniformly elliptic regularization in Section 4.

2. Preliminaries

In this section, we introduce the assumptions of the generating functions g, and give appropriate convexity notions of function u and domain Ω, and define various weak solutions. These preliminaries will be used in the next two sections.

2.1. Conditions on the generating function g

We first recall some standard conditions for the generating function g as in [14, 35, 36]. We assume $g \in C^4(\Gamma)$, where Γ has the property that the projections

$$I(x, y) = \{z \in \mathbb{R} |\ (x, y, z) \in \Gamma\}$$

are open intervals. Denoting

$$\mathcal{U} = \{(x, g(x, y, z), g_x(x, y, z)) |\ (x, y, z) \in \Gamma\}, \tag{2.1}$$

then we have the following conditions:

A1: For each $(x, u, p) \in \mathcal{U}$, there exists a unique point $(x, y, z) \in \Gamma$ satisfying

$$g(x, y, z) = u, \quad g_x(x, y, z) = p.$$

A2: $g_z < 0$, $\det E \neq 0$, in Γ, where E is the $n \times n$ matrix given by

$$E = [E_{i,j}] = g_{x,y} - (g_z)^{-1} g_{x,z} \otimes g_y. \tag{2.2}$$

Note that the sign of g_z in A2 can be changed to be positive as we wish. Here we fix the sign of g_z to be negative in accordance with [13, 35].

The strict monotonicity property of the generating function g with respect to z, enables us to define a dual generating function g^*,

$$g(x, y, g^*(x, y, u)) = u, \tag{2.3}$$

with $(x, y, u) \in \Gamma^* := \{(x, y, g(x, y, z)) | (x, y, z) \in \Gamma\}$, $g^*_x = -g_x/g_z$, $g^*_y = -g_y/g_z$ and $g^*_u = 1/g_z$, which leads to a dual condition to A1, namely

A1*: The mapping $Q := -g_y/g_z$ is one-to-one in x, for all $(x, y, z) \in \Gamma$.

Note that the Jacobian matrix of the mapping $x \to Q(x, y, z)$ is $-E^t/g_z$, where E^t is the transpose of E so its determinant will not vanish when condition A2 holds, that is A2 is self dual.

We also assume the following conditions on the generating function g which are expressed in terms of the matrix A. Extending the necessary assumption A3w for regularity in optimal transportation in [18, 32, 38], we assume the following regular condition for the matrix function A with respect to p, which we formulate together with its strict version [20].

A3w (**A3**): The matrix function A is regular (strictly regular) in $\mathcal{U}$, that is A is co-dimension one convex (strictly co-dimension one convex) with respect to p in the sense that,

$$A^{kl}_{ij}\xi_i\xi_j\eta_k\eta_l := (D_{p_k p_l} A_{ij})\xi_i\xi_j\eta_k\eta_l \geq 0, (> 0)$$

in $\mathcal{U}$, for all $\xi, \eta \in \mathbb{R}^n$ such that $\xi \cdot \eta = 0$.

We also need a monotonicity condition on the matrix A with respect to u, namely A4w or A4*w.

A4w (**A4*w**): The matrix A is monotone increasing (decreasing) with respect to u in $\mathcal{U}$, that is

$$D_u A_{ij}\xi_i\xi_j \geq 0, \ (\leq 0)$$

in $\mathcal{U}$, for all $\xi \in \mathbb{R}^n$.

We next have the following condition to guarantee the appropriate controls on $J_1[u]$, which is a refinement of condition G5 in [35], (see also [36]). Namely, writing $J(x, y) = g(x, y, \cdot)I(x, y)$, we assume:

A5: There exists an infinite open interval J_0 and a positive constant K_0, such that $J_0 \subset J(x, y)$ and

$$|g_x(x, y, z)| < K_0,$$

for all $x \in \bar{\Omega}, y \in \bar{\Omega}^*, g(x, y, z) \in J_0$.

Note that as in [14], we can assume that $J_0 = (m_0, \infty)$ for some constant $m_0 \geq -\infty$ or $J_0 = (-\infty, M_0)$ for a constant M_0. As in [37], we can also consider the situation when J_0 is a finite interval. We will assume some of the above conditions A1, A2, A1*, A3w and A4w in the discussions of weak solutions. In this paper, we will not use the conditions A3, A4*w and A5. As mentioned in the introduction, condition A5 was used in [26, 37] to guarantee the C^1 estimate and prove higher regularity of solutions.

2.2. g-convexity

In this subsection, we introduce the appropriate convexity notions of the function u and domain Ω with respect to the generating function g. Let Ω be a bounded domain, g be a generating function on Γ satisfying conditions A1 and A2, and I be an open interval in $\mathbb{R}$ as in the previous subsection. A function $u \in C^0(\Omega)$ is called g-convex in Ω, if for each $x_0 \in \Omega$, there exists $y_0 \in \Omega^*$ and $z_0 \in I(x_0, y_0)$ such that

$$\begin{aligned} u(x_0) &= g(x_0, y_0, z_0), \\ u(x) &\geq g(x, y_0, z_0) \end{aligned} \tag{2.4}$$

for all $x \in \Omega$. If u is differentiable at x_0, then $y_0 = T_u(x_0) = Y(x_0, u(x_0), Du(x_0))$, while if u is twice differentiable at x_0, then

$$D^2u(x_0) \geq D_x^2 g(x_0, y_0, z_0), \tag{2.5}$$

that is u is admissible for Eq (1.5) at x_0. If $u \in C^2(\Omega)$, we call u locally g-convex in Ω if this inequality holds for all $x_0 \in \Omega$. We refer to the function $g(\cdot, y_0, z_0)$ as g-affine function and as a g-support at x_0 if (2.4) is satisfied. Note that a locally g-convex function u satisfying (2.5) has a local g-support near x_0 and is g-convex in a neighbourhood of x_0.

Let $u \in C^0(\Omega)$ be g-convex in Ω. We define the g-normal mapping of u at $x_0 \in \Omega$ to be the set

$$T_u(x_0) = \{y_0 \in \mathcal{U}_\Omega |\, u(x) \geq g(x, y_0, g^*(x_0, y_0, u_0)) \text{ for all } x \in \Omega\}, \tag{2.6}$$

where $u_0 = u(x_0)$, and g^* is the dual generating function defined by $g(x, y, g^*(x, y, u)) = u$. Clearly T_u agrees with our previous terminology when u is differentiable and moreover in general

$$T_u(x_0) \subset Y(x_0, u(x_0), \partial u(x_0)), \tag{2.7}$$

where ∂u deonotes the subdifferential of u. Assume A1, A2, A1*, A3w, A4w hold in $\mathcal{U}$ and suppose $u \in C^0(\Omega)$ is g-convex in Ω, then by Lemma 2.2 in [35], we have

$$T_u(x_0) = Y(x_0, u(x_0), \partial u(x_0)) \tag{2.8}$$

for any $x_0 \in \Omega$.

We also recall the convexity notions of the domains in [14, 35, 37]. The domain Ω is g-convex with respect to $y_0 \in \mathcal{U}^*_\Omega$, $z_0 \in I(\Omega, y_0) = \cap_\Omega I(\cdot, y_0)$ if the image $Q_0(\Omega) := -\frac{g_y}{g_z}(\cdot, y_0, z_0)(\Omega)$ is convex in $\mathbb{R}^n$. The domain Ω^* is g^*-convex with respect to $(x_0, u_0) \in \Omega \times J$, where $J := J(x_0, \Omega^*) = \cap_{\Omega^*} J(x_0, \cdot)$, if the image $P_0(\Omega^*) := g_x[x_0, \cdot, g^*(x_0, \cdot, u_0)](\Omega^*)$ is convex in $\mathbb{R}^n$. Alternatively, we can also define the domain convexity with respect to the mapping Y, see [14, 37] for the detailed definitions. As in [14], g^*-convexity of Ω^* is equivalent to Y^*-convexity of Ω^*, while g-convexity of Ω can imply Y-convexity of Ω. Note that when we use condition A3w for convexity results and their consequences below, we assume at least that the convex hulls of the image $Q_0(\Omega)$ and $P_0(T_u(\Omega))$ lie in $Q(\Gamma) := -\frac{g_y}{g_z}(\Gamma)$ and $g_x(\Gamma)$, respectively.

Assume A1, A2, A1*, A3w and A4w hold in $\mathcal{U}$ and that $u \in C^2(\Omega)$ is a locally g-convex function in Ω, if Ω is g-convex with respect to each point in $(Y, Z)(\cdot, u, Du)(\Omega)$, then the function u is g-convex in Ω, see Lemma 2.1 in [35].

2.3. Definitions of weak solutions

In this subsection, we give the exact definitions of various weak solutions of (1.5), namely Aleksandrov solutions, Brenier solutions, C-viscosity solutions, L^p-viscosity solutions and Trudinger solutions.

In order to define the Aleksandrov solution, we introduce the generalized Monge-Ampère measure associated with the generating function g and the weight f^*.

Definition 2.1 (**Generalized Monge-Ampère measure).** *Let $f^* \in L^1_{loc}(\mathbb{R}^n)$, for a given g-convex function $u \in C(\Omega)$, the generalized Monge-Ampère measure of u associated with the generating function g and the weight f^* is the measure defined by*

$$\omega_g(f^*, u)(F) = \int_{T_u(F)} f^*(y)dy \tag{2.9}$$

for every Borel set $F \subset \Omega$. When $f^* \equiv 1$, we simply write the measure as $\omega_g(u)$.

From [35], the generalized Monge-Ampère measure $\omega_g(f^*, u)$ is a Borel measure. Moreover, since $f^* \in L^1_{loc}(\mathbb{R}^n)$, $\omega_g(f^*, u)$ is a Radon measure, which behaves well with respect to convergence, see also [35].

We are now in a position to define the Aleksandrov solution of (1.5).

Definition 2.2 (**Aleksandrov solution).** *A g-convex function $u \in C(\Omega)$ is said to be a generalized solution of* (1.5) *in the sense of Aleksandrov, or simply Aleksandrov solution of* (1.5)*, if*

$$\omega_g(f^*, u)(F) = \int_F f(x)dx \tag{2.10}$$

for any Borel set $F \subset \Omega$.

We can also define a generalized solution of the second boundary condition (1.4). If $\Omega^* \subset T_u(\bar{\Omega})$ and $|\{x|f(x) > 0 \text{ and } T_u(x)\backslash\bar{\Omega}^* \text{ is nonempty}\}| = 0$, u is said to be a generalized solution of the second boundary value condition (1.4).

Extending Brenier solution for the optimal transportation problem in [18], the Brenier solution of the second boundary value problem (1.5)–(1.4) can be defined as follows.

Definition 2.3 (**Brenier solution).** *A g-convex function $u \in C(\Omega)$ is said to be a weak solution of* (1.5) *in the sense of Brenier, or simply Brenier solution of* (1.5)*, if*

$$\int_{T_u^{-1}(F^*)} f(x)dx = \int_{F^*} f^*(y)dy, \tag{2.11}$$

for any Borel set $F^* \subset \Omega^*$.

Correspondingly, if $\Omega \subset T_u^{-1}(\bar{\Omega}^*)$ and $|\{y|f^*(y) > 0 \text{ and } T_u^{-1}(y)\backslash\bar{\Omega} \text{ is nonempty}\}| = 0$, u is said to be a Brenier solution of the second boundary value condition (1.4).

If we denote the source measure and target measure on Ω and Ω^* by μ and ν respectively, we can use the "pushback" $(T_u)^\#$ and the "pushforward" $(T_u)_\#$ as in [18] to simply denote the Aleksandrov solution and the Brenier solution of (1.5) respectively. In particular, when $d\mu = f dx$ and $d\nu = f^* dy$ respectively, Aleksandrov solution of (1.5) can be defined by

$$\mu = (T_u)^{\#}\nu, \tag{2.12}$$

and Brenier solution of (1.5) can be defined by

$$\nu = (T_u)_{\#}\mu. \tag{2.13}$$

Therefore, an Aleksandrov solution of (1.5) can be regarded as a weak solution u whose g-normal mapping T_u pushes back the target measure ν to the source measure μ, while a Brenier solution of (1.5) can be regarded as a weak solution u whose g-normal mapping T_u pushes forward the source measure μ to the target measure ν.

We next define the viscosity solutions of Eq (1.5), which are sometimes called "comparison solutions" or "Crandall–Lions solutions". Our C-viscosity solution definition of Eq (1.5) mainly follows that in [10], which will be used in the case when the function on right hand side of (1.5) is continuous. While L^p-viscosity solution definition of Eq (1.5) mainly follows from that in [5], which will be used in the case when the function on right hand side of (1.5) belongs to the L^p space.

Let

$$\mathcal{F}[u] := \mathbb{F}[u] - B(\cdot, u, Du), \tag{2.14}$$

where

$$\mathbb{F}[u] := \det[D^2 u - A(\cdot, u, Du)] \tag{2.15}$$

and A, B are matrix function and nonnegative scalar function satisfying (1.6). Now we can simply use $\mathcal{F}[u] = 0$ to denote Eq (1.5). Note that in the following definitions of weak solutions, we can allow B to be negative, but not merely positive.

We then give the definition of C-viscosity solution of Eq (1.5).

Definition 2.4 (**C-viscosity solution).** *Let u be an upper semi-continuous (respectively, lower semi-continuous) function, we say that u is a C-viscosity subsolution (resp., supersolution) of* (1.5)*, or equivalently, that $\mathcal{F}[u] \geq 0$ (resp., $\mathcal{F}[u] \leq 0$) in C-viscosity sense, if whenever $x_0 \in \Omega$ and g-convex function $\varphi \in C^2(\Omega)$ are such that $u - \varphi$ attains a local maximum (resp., minimum) at x_0, then*

$$\mathcal{F}[\varphi](x_0) \geq 0, \quad (resp., \leq 0). \tag{2.16}$$

A C-viscosity solution of (1.5) *is any continuous function u which is, at the same time, a C-viscosity supersolution of* (1.5) *and a C-viscosity subsolution of* (1.5)*. We shall also say that $\mathcal{F}[u] = 0$ in C-viscosity sense.*

A more restrictive notion of viscosity solution can be given by increasing the set of test functions from $C^2(\Omega)$ to $W^{2,p}(\Omega)$.

Definition 2.5 (**L^p-viscosity solution).** *Let u be an upper semi-continuous (respectively, lower semi-continuous) function, we say that u is an L^p-viscosity subsolution (resp., supersolution) of* (1.5)*, or equivalently, that $\mathcal{F}[u] \geq 0$ (resp., $\mathcal{F}[u] \leq 0$) in L^p-viscosity sense, if one of the following conditions holds:*

(i). If a g-convex function $\varphi \in W^{2,p}(\Omega)$ and $\delta > 0$ are such that

$$\mathcal{F}[\varphi] \leq -\delta < 0, \quad (resp., \geq \delta > 0) \tag{2.17}$$

almost everywhere in an open subset of Ω, then $u - \varphi$ cannot achieve a local maximum (resp., minimum) inside that set;

(ii). For every g-convex function $\varphi \in W^{2,p}(\Omega)$ and every $x_0 \in \Omega$ where $u-\varphi$ achieves a local maximum (resp., minimum), we have

$$\operatorname{ess}\liminf_{x\to x_0} \mathcal{F}[\varphi] \geq 0, \quad (resp., \operatorname{ess}\limsup_{x\to x_0} \mathcal{F}[\varphi] \leq 0) \tag{2.18}$$

where ess lim inf *(resp.,* ess lim sup*) means, as usual, the essential inferior (resp., superior) limit.*

An L^p-viscosity solution of (1.5) *is any continuous function u which is, at the same time, an L^p-viscosity supersolution of* (1.5) *and an L^p-viscosity subsolution of* (1.5)*. We shall also say that $\mathcal{F}[u] = 0$ in L^p-viscosity sense.*

From the above definitions, under the same assumptions of the known data, it is clear that an L^p-viscosity solution of (1.5) is a C-viscosity solution of (1.5). However, the definition of L^p-viscosity solution is particularly used in the "L^p-theory", namely that the right hand side function of Eq (1.5) is merely in the L^p space, see [5]. There is another well-known notion of weak solution in [31], which is also fit for the case of L^p right hand side. We shall call such a weak solution in [31] Trudinger solution. The relationship between L^p-viscosity solution and Trudinger solution is stated in Theorem 1.2, which will be proved in Section 4.

Letting

$$\tilde{\mathbb{F}}[u] := f^* \circ Y(x, u, Du) \det(E^{-1})\mathbb{F}[u], \tag{2.19}$$

we now define the weak solution of (1.5) in the sense of Trudinger [31].

Definition 2.6 (**Trudinger solution).** *Let u be a continuous function, we say that u is a weak subsolution of* (1.5) *in the sense of Trudinger, or simply Trudinger subsolution of* (1.5)*, if there exist sequences $\{u_m\} \subset C^2(\Omega)$ and $\{f_m\} \subset L^1_{loc}(\Omega)$ such that u_m is g-convex, $u_m \to u$ uniformly in Ω, $f_m \geq 0$, $f_m \to f$ in $L^1_{loc}(\Omega)$, and $\tilde{\mathbb{F}}[u_m] \geq f_m$.*

Let u be a continuous function, we say that u is a weak supersolution of (1.5) *in the sense of Trudinger, or simply Trudinger supersolution of* (1.5)*, if there exist sequences $\{u_m\} \subset C^2(\Omega)$ and $\{f_m\} \subset L^1_{loc}(\Omega)$ such that $u_m \to u$ uniformly in Ω, $f_m \geq 0$, $f_m \to f$ in $L^1_{loc}(\Omega)$, and $\tilde{\mathbb{F}}[u_m] \leq f_m$ whenever u_m is g-convex.*

A weak solution of (1.5) *in the sense of Trudinger, or simply Trudinger solution of* (1.5)*, is a continuous function u for which there exists a sequence $\{u_m\} \subset C^2(\Omega)$ of convex functions such that $u_m \to u$ uniformly in Ω and $\tilde{\mathbb{F}}[u_m] \to f$ in $L^1_{loc}(\Omega)$.*

3. Aleksandrov solutions, Brenier solutions and C-viscosity solutions

In this section, we discuss the relations of Aleksandrov solutions, Brenier solutions and C-viscosity solutions, and prove Theorem 1.1. We prove the assertions (i) and (ii) of Theorem 1.1 separately.

We first prove the relationship between Aleksandrov solutions and Brenier solutions.

Proof of Theorem 1.1 (i). **"Aleksandrov solutions ⇒ Brenier solutions"**. By Section 4 in [35], we have

$$|\{y \in \Omega^* | \, y \in T_u(x_1) \cap T_u(x_2) \text{ for some } x_1 \neq x_2, \; x_1, x_2 \in \Omega\}| = 0, \tag{3.1}$$

and from this $\omega_g(f^*, u)$ is countably additive, therefore $\omega_g(f^*, u)$ is a Randon measure. Since u is an Aleksandrov solution of (1.5), we have

$$\omega_g(f^*, u) = f dx, \tag{3.2}$$

namely, for any Borel set $F \subset \Omega$, we have

$$\int_{T_u(F)} f^*(y)dy = \omega_g(f^*, u)(F) = \int_F f(x)dx. \tag{3.3}$$

For any given Borel set $F^* \subset \Omega^*$, there exists a set $F \subset \Omega$ such that $T_u(F) = F^*$. Since the measures μ and ν have no singular parts and the property (3.1) holds, we have

$$|T_u^{-1}(F^*)| = |F|. \tag{3.4}$$

From (3.3) and (3.4), we then have

$$\begin{aligned} \int_{F^*} f^*(y)dy &= \int_{T_u(F)} f^*(y)dy \\ &= \int_F f(x)dx \\ &= \int_{T_u^{-1}(F^*)} f(x)dx, \end{aligned} \tag{3.5}$$

which implies that u is a Brenier solution of (1.5).

"Brenier solutions ⇒ Aleksandrov solutions". Let Ω, Ω^* be as above, suppose that u is a g-convex Brenier solution of (1.5), Ω^* is g^*-convex with respect to $\Omega \times J$, then we aim to show that u satisfies (1.5) in the Aleksandrov sense. Since u is a g-convex Brenier solution of (1.5), for any $h \in C(\mathbb{R}^n)$, we have

$$\int_\Omega h(T_u(x))f(x)dx = \int_{\Omega^*} h(y)f^*(y)dy. \tag{3.6}$$

Note that (3.6) is an analytical formulation of the measure equality (2.11) in Definition 2.3. For any compact set $F \subset \Omega$, the set $F^* = T_u(F)$ is compact. Taking the function $h \in C(\mathbb{R}^n)$ such that $h \geq \chi_{F^*}$, where χ_{F^*} denotes the characteristic function of F^*, we get from (3.6),

$$\begin{aligned} \int_{\Omega^*} h(y)f^*(y)dy &= \int_\Omega h(T_u(x))f(x)dx \\ &\geq \int_\Omega \chi_{F^*}(T_u(x))f(x)dx \\ &= \int_F f(x)dx. \end{aligned} \tag{3.7}$$

Letting h decrease to χ_{F^*}, from (3.7) we have

$$\omega_g(f^*, u)(F) \geq \int_F f(x)dx. \tag{3.8}$$

Since the target domain Ω^* is g^*-convexity with respect to $\Omega \times J$, by Lemma 4.3 in [35], we have

$$T_u(F) \subset \bar{\Omega}^*, \tag{3.9}$$

which leads to

$$|T_u(F)| = |T_u(F) \cap \Omega^*|. \tag{3.10}$$

Again, by taking the function $h \in C(\mathbb{R}^n)$ such that $h \geq \chi_{F^*}$, then we have

$$\begin{aligned}
\omega_g(f^*, u)(F) &= \int_{T_u(F)} f^*(y)dy \\
&= \int_{T_u(F)\cap\Omega^*} f^*(y)dy \\
&\leq \int_{\Omega^*} h(y)f^*(y)dy \\
&= \int_{\Omega} h(T_u(x))f(x)dx,
\end{aligned} \tag{3.11}$$

where (3.10) is used to obtain the second equality, and (3.6) is used to obtain the last equality. Letting h decrease to χ_{F^*}, from (3.11) we have

$$\omega_g(f^*, u)(F) \leq \int_F f(x)dx. \tag{3.12}$$

Combining (3.8) and (3.12), we get

$$\omega_g(f^*, u)(F) = \int_F f(x)dx \tag{3.13}$$

for any compact set $F \subset \Omega$.

Now, we have proved that (3.13) holds for any compact subset F of Ω. The regularity of the generalized Monge-Ampère measure $\omega_g(f^*, u)$ implies that (3.13) holds with compact F replaced by any Borel subset of Ω. Thus, u is an Aleksandrov solution of (1.5). □

Remark 1. In the above proof, we have proved that Aleksandrov solutions with no singular part are Brenier solutions, which do not need the g-convexity of the target domain Ω^*. While conversely, we do need the g-convexity of Ω^* as in the proof. For the particular case when $g(x, y, z) = x \cdot y - z$, one can refer to Lemma 2 in [4]. We give some heuristic explanations as follows. For Aleksandrov solutions of (1.5), whenever f and f^* are bounded away from zero and infinity on Ω and Ω^*, respectively, (3.2) implies that the multivalued map $x \to T_u x$ preserves the Lebesgue measure up to multiplicative constants, i.e., $|F| \simeq |T_u(F)|$ (the volumes of F and $T_u(F)$ are comparable) for any Borel set $F \subset X$. On the other hand, Brenier solutions of (1.5) can only see the regions where f and f^* live. Then for any Borel set $F \subset X$, we only have $|F| \simeq |T_u(F) \cap \Omega^*|$, but not $|F| \simeq |T_u(F)|$ as in the Aleksandrov situation. If one can ensure that the target always covers the image of $T_u(\Omega)$ so that $|T_u(F) \cap \Omega^*| = |T_u(F)|$ for all Borel set $F \subset \Omega$, then Brenier solutions of (1.5) will be the Aleksandrov solutions of (1.5). As in (3.11), it is the g-convexity of Ω^* with respect to Ω, that guarantees that the target Ω^* always covers the image $T_u(\Omega)$.

We now move to prove the relationship between Aleksandrov solutions and C-viscosity solutions.

Proof of Theorem 1.1 (ii). **"Aleksandrov solutions $\Rightarrow$ C-viscosity solutions"**. Let $\phi \in C^2(\Omega)$ be a g-convex function such that $u - \phi$ has a local maximum at $x_0 \in \Omega$. We can assume that $u(x_0) = \phi(x_0)$ and $u(x) < \phi(x)$ for all $0 < |x - x_0| < \delta$, where δ is some positive constant. This can be achieved by adding $r|x - x_0|^2$ to ϕ and letting $r \to 0$ at the end. Note that the perturbed function $\phi_r := \phi + r|x - x_0|^2$ is locally g-convex in $\Omega \cap B_\delta(x_0)$ for δ properly small, see Remark 2. For simplicity, we still denote ϕ_r by ϕ in the context.

Let $m = \min_{\frac{\delta}{2} \le |x - x_0| \le \delta} \{\phi(x) - u(x)\}$, by the above assumption, we have $m > 0$. Let $0 < \varepsilon < m$, we consider the set

$$S_\varepsilon := \{x \in B_\delta(x_0) : u(x) > \phi(x) - \varepsilon\}. \tag{3.14}$$

If $\frac{\delta}{2} \le |x - x_0| \le \delta$, then $\phi(x) - u(x) \ge m > \varepsilon$, so $x \notin S_\varepsilon$. Hence, we get $S_\varepsilon \subset B_{\frac{\delta}{2}}(x_0)$, $u = \phi - \varepsilon$ on ∂S_ε and $u > \phi - \varepsilon$ in S_ε. By condition A4w, (1.6) and the local g-convexity of ϕ, we have

$$\begin{aligned}
&D^2(\phi - \varepsilon) - g_{xx}(x, Y(x, \phi - \varepsilon, D(\phi - \varepsilon)), Z(x, \phi - \varepsilon, D(\phi - \varepsilon))) \\
=&D^2(\phi - \varepsilon) - A(x, \phi - \varepsilon, D(\phi - \varepsilon)) \\
=&D^2\phi - A(x, \phi - \varepsilon, D\phi) \\
\ge&D^2\phi - A(x, \phi, D\phi) \ge 0,
\end{aligned} \tag{3.15}$$

in $\Omega \cap B_\delta(x_0)$. From (2.5) and (3.15), the function $\phi - \varepsilon$ is locally g-convex in $\Omega \cap B_\delta(x_0)$. Hence, for some sufficiently small ε, $\phi - \varepsilon$ is g-convex in S_ε. Since both the functions u and $\phi - \varepsilon$ are g-convex in S_ε, by Lemma 4.4 in [35], we have

$$T_u(S_\varepsilon) \subset T_{\phi-\varepsilon}(S_\varepsilon). \tag{3.16}$$

Since u is an Aleksandrov solution of (1.5), we have

$$\begin{aligned}
&\int_{S_\varepsilon} f(x)dx = \omega_g(f^*, u)(S_\varepsilon) \le \omega_g(f^*, \phi - \varepsilon)(S_\varepsilon) \\
=&\int_{S_\varepsilon} f^* \circ Y(x, \phi - \varepsilon, D\phi) \det(E^{-1}) \det[D^2\phi - g_{xx}(x, Y(x, \phi - \varepsilon, D\phi), Z(x, \phi - \varepsilon, D\phi)]dx.
\end{aligned} \tag{3.17}$$

Letting $\varepsilon \to 0$ in (3.17), by the continuity of f, f^*, Y, Z and E, and the C^2 smoothness of ϕ, we get

$$\begin{aligned}
&\det[D^2\phi(x_0) - g_{xx}(x_0, Y(x_0, \phi(x_0), D\phi(x_0)), Z(x_0, \phi(x_0), D\phi(x_0))] \\
\ge&\det(E(x_0, Y(x_0, \phi(x_0), D\phi(x_0)), Z(x_0, \phi(x_0), D\phi(x_0))) \frac{f(x_0)}{f^* \circ Y(x_0, \phi(x_0), D\phi(x_0))},
\end{aligned} \tag{3.18}$$

which implies that u is a C-viscosity subsolution of (1.5). A similar argument shows that u is also a C-viscosity supersolution of (1.5), and thus a C-viscosity solution of (1.5).

“C-viscosity solutions $\Rightarrow$ Aleksandrov solutions”. Assuming $0 < \lambda \leq f(x) \leq \Lambda < +\infty$ in $\bar{\Omega}$, for given $x_0 \in \Omega$ and $0 < \eta < \frac{\lambda}{2}$, by the continuity of f, there exists $\varepsilon > 0$ such that

$$0 < f(x_0) - \eta < f(x) < f(x_0) + \eta \tag{3.19}$$

for all $x \in B_\varepsilon(x_0)$. Let u_τ, f^*_δ denote the the mollifications of u and f^* as $\tau \to 0$ and $\delta \to 0$, respectively. Let $u^\pm_{\delta,\tau}$ be smooth g-convex solutions to the Dirichlet problem

$$\begin{cases} f^*_\delta \circ Y(x, v, Dv) \det(E^{-1}) \det[D^2 v - g_{xx}(x, Y(x, v, Dv), Z(x, v, Dv))] \\ \qquad = f(x_0) \pm \eta, \qquad \text{in } B_\varepsilon(x_0), \\ v = u_\tau, \qquad \text{on } \partial B_\varepsilon(x_0). \end{cases} \tag{3.20}$$

Here note that f^*_δ, $\det(E^{-1})$ and u_τ are smooth function, the existence of smooth g-convex solutions for the Dirichlet problems in small balls is guaranteed by Lemma 4.6 in [35], where conditions A1, A2 and A3w are used, and the smallness of the radius ε is used. By Perron's method, let $u_{\delta,\tau}$ be a C-viscosity solution to the Dirichlet problem

$$\begin{cases} f^*_\delta \circ Y(x, v, Dv) \det(E^{-1}) \det[D^2 v - g_{xx}(x, Y(x, v, Dv), Z(x, v, Dv))] \\ \qquad = f(x), \qquad \text{in } B_\varepsilon(x_0), \\ v = u_\tau, \qquad \text{on } \partial B_\varepsilon(x_0). \end{cases} \tag{3.21}$$

Since (3.19) holds in $B_\varepsilon(x_0)$, by comparing the smooth solutions $u^\pm_{\delta,\tau}$ with the C-viscosity solution $u_{\delta,\tau}$, we get

$$u^+_{\delta,\tau} \leq u_{\delta,\tau} \leq u^-_{\delta,\tau}, \quad \text{in } B_\varepsilon(x_0), \tag{3.22}$$

where the comparison can be achieved by using Definition 2.4. Since $u^+_{\delta,\tau}$, $u_{\delta,\tau}$ and $u^-_{\delta,\tau}$ are equal on $\partial B_\varepsilon(x_0)$, using Lemma 4.4 in [35], we obtain from (3.22) that

$$T_{u^-_{\delta,\tau}}(B_\varepsilon(x_0)) \subset T_{u_{\delta,\tau}}(B_\varepsilon(x_0)) \subset T_{u^+_{\delta,\tau}}(B_\varepsilon(x_0)). \tag{3.23}$$

Consequently, from Definition 2.1, we have

$$\omega_g(f^*_\delta, u^-_{\delta,\tau})(B_\varepsilon(x_0)) \leq \omega_g(f^*_\delta, u_{\delta,\tau})(B_\varepsilon(x_0)) \leq \omega_g(f^*_\delta, u^+_{\delta,\tau})(B_\varepsilon(x_0)). \tag{3.24}$$

Since $u^+_{\delta,\tau}$, $u^-_{\delta,\tau}$ are smooth solutions in $B_\varepsilon(x_0)$, they are both Aleksandrov solutions. Therefore, we have

$$\omega_g(f^*_\delta, u^\pm_{\delta,\tau})(B_\varepsilon(x_0)) = \int_{B_\varepsilon(x_0)} [f(x_0) \pm \eta] dx = |B_\varepsilon(x_0)|(f(x_0) \pm \eta). \tag{3.25}$$

Combining (3.24) and (3.25), we get

$$|B_\varepsilon(x_0)|(f(x_0) - \eta) \leq \omega_g(f^*_\delta, u_{\delta,\tau})(B_\varepsilon(x_0)) \leq |B_\varepsilon(x_0)|(f(x_0) + \eta). \tag{3.26}$$

Here we observe that $|B_\varepsilon(x_0)|(f(x_0) - \eta)$ and $|B_\varepsilon(x_0)|(f(x_0) + \eta)$ in (3.26) are independent of δ and τ. From the stability property of viscosity solutions [10], we have $u_{\delta,\tau} \to u$, as $\delta, \tau \to 0$, where u is the assumed viscosity solution of Eq (1.5). Moreover, we have $f^*_\delta \to f^*$ as $\delta \to 0$. Then passing $\delta, \tau \to 0$ in (3.26), by the weak convergence of the generalized Monge-Ampère measure [26, 35], we have

$$|B_\varepsilon(x_0)|(f(x_0) - \eta) \leq \omega_g(f^*, u)(B_\varepsilon(x_0)) \leq |B_\varepsilon(x_0)|(f(x_0) + \eta). \tag{3.27}$$

From (3.27), we know that the measure $\omega_g(f^*, u)$ is absolutely continuous with respect to the Lebesgue measure. Therefore, there exists $\tilde{f} \in L^1_{loc}(\Omega)$ such that $\omega_g(f^*, u)(F) = \int_F \tilde{f}(x)dx$ for all Borel set $F \subset \Omega$. Dividing (3.27) by $|B_\varepsilon(x_0)|$ and letting $\varepsilon \to 0$, we get

$$f(x_0) - \eta \le \tilde{f}(x_0) \le f(x_0) + \eta, \tag{3.28}$$

for all $x_0 \in \Omega$ and for all sufficiently small η. Now we get that $\tilde{f} \equiv f$ in Ω. Hence the measure $\omega_g(f^*, u)$ has the density f, namely,

$$\omega_g(f^*, u)(F) = \int_F f(x)dx \tag{3.29}$$

for any Borel set $F \subset \Omega$. Thus, u is an Aleksandrov solution of (1.5). □

Remark 2. Here, we show that the perturbed function $\phi_r := \phi + r|x - x_0|^2$ $(r > 0)$ of a g-convex function ϕ is locally g-convex in $\Omega \cap B_\delta(x_0)$ for δ properly small and $x_0 \in \Omega$. Since ϕ and ϕ_r are C^2 functions, we can use (2.5) to check their local g-convexity. We denote the matrix $D^2u - A(x, u, Du) = D^2u - g_{xx}(\cdot, Y(\cdot, u, Du), Z(\cdot, u, Du))$ by $M[u]$. Since the function ϕ is C^2, from (2.5), the g-convexity of ϕ in Ω implies $M[\phi] \ge 0$ in Ω. Thus, we only need to prove $M[\phi_r] \ge 0$ in $\Omega \cap B_\delta(x_0)$ for some $\delta > 0$. Indeed, by calculations and mean value theorem, we have

$$\begin{aligned} M[\phi_r] &= D^2\phi + 2rI - A(x, \phi_r, D\phi_r) \\ &= M[\phi] + 2rI + A(x, \phi, D\phi) - A(x, \phi_r, D\phi_r) \\ &= M[\phi] + r[2I - D_uA(x, \hat{z}, D\phi)|x - x_0|^2 - 2\sum_{k=1}^n D_{p_k}A(x, \phi_r, \hat{p})(x - x_0)_k], \end{aligned} \tag{3.30}$$

where I is the identity matrix, $\hat{z} = \theta_1\phi + (1 - \theta_1)\phi_r$, $\hat{p} = \theta_2 D\phi + (1 - \theta_2)D\phi_r$ for some constants $\theta_1, \theta_2 \in (0, 1)$. Letting $\lambda = (\lambda_1, \cdots, \lambda_n)$ be the eigenvalues of the matrix $D_uA(x, \hat{z}, D\phi)$, $\lambda^{(k)} = (\lambda_1^{(k)}, \cdots, \lambda_n^{(k)})$ be the eigenvalues of the matrix $D_{p_k}A(x, \phi_r, \hat{p})$ for $k = 1, \cdots, n$, and setting $\tilde{\lambda} = (\tilde{\lambda}_1, \cdots, \tilde{\lambda}_n)$ with $\tilde{\lambda}_i = \sum_{k=1}^n |\lambda_i^{(k)}|$ for $i = 1, \cdots, n$, we use Λ_{D_uA} and Λ_{D_pA} to denote

$$\Lambda_{D_uA} = \max\{\sup_\Omega |\lambda_1|, \cdots, \sup_\Omega |\lambda_n|\}, \quad \text{and } \Lambda_{D_pA} = \max\{\sup_\Omega \tilde{\lambda}_1, \cdots, \sup_\Omega \tilde{\lambda}_n\},$$

respectively. Then, by taking $\delta \in (0, \min\{\frac{1}{\Lambda_{D_uA} + 2\Lambda_{D_pA}}, 1\})$, we have

$$M[\phi_r] \ge M[\phi] + rI[2 - \Lambda_{D_uA}\delta^2 - 2\Lambda_{D_pA}\delta] \ge rI > 0, \tag{3.31}$$

in $\Omega \cap B_\delta(x_0)$, for any $r > 0$. Therefore, the function ϕ_r $(r > 0)$ is locally g-convex in $\Omega \cap B_\delta(x_0)$.

Remark 3. When deriving (3.18) and (3.19), we have used the continuity of f and f^*. Similar continuity was also used in the proof of Theorem 1.1 in [16] by the author and X.-P. Yang. However, the continuity of the densities is missing in the statement of Theorem 1.1 in [16]. We take this opportunity to add the continuity of f and g to Theorem 1.1 in [16] so that a generalized solution for the optimal transportation equation is a C-viscosity solution of the optimal transportation equation.

4. Trudinger solutions and L^p-viscosity solutions

In this section, we discuss the relationship between Trudinger solutions and L^p-viscosity solutions, which gives the proof of Theorem 1.2.

If the right hand side term is continuous, Eq (1.5) can be studied in the framework of C-viscosity solutions [10]. If the right hand side term belongs to L^p space and is not continuous, Eq (1.5) should be treated in the framework of L^p-viscosity solutions [5]. However, the theory in [5] requires strong ellipticity of the equation, which is not satisfied for the Monge-Ampère type Eq (1.5) at the current stage. The notion of weak solution introduced by Trudinger in [31] works nicely for the Monge-Ampère case, (and furthermore the k-Hessian case). We will show that Trudinger solution of (1.5) is actually an L^p-viscosity solution.

Proof of Theorem 1.2. The proof is divided into four steps. In the first step, we treat a uniformly elliptic regularization of problem (1.5)–(1.9), which is called the vanishing viscosity approximation method in [2]. Here we shall use a feasible approximation scheme in [15]. Then in the second and third steps, we prove that the limit of the approximated solutions is an L^p-viscosity solution and a Trudinger solution, respectively. In the last step, we conclude that a Trudinger solution of (1.5)–(1.9) is actually an L^p-viscosity solution of (1.5)–(1.9).

Step 1. We treat a uniform approximated problem of (1.5)–(1.9). We consider the following approximated equation of (1.5) as in [15]:

$$\det[M[u] + \epsilon\,\mathrm{trace}(M[u])I] = B(x, u, Du), \tag{4.1}$$

which is a uniformly elliptic regularization of (1.5), where $M[u]$ denotes the augmented Hessian matrix $D^2u - A(x, u, Du)$ with A satisfying (1.6). In fact, for each $\epsilon > 0$, it is easy to check as in [15] that (4.1) is uniformly elliptic. Then by the theory in [5], for $\epsilon > 0$, there exists a unique L^p-viscosity solution $u_\epsilon \in W^{2,p}(\Omega)$ of (4.1)-(1.9), which satisfies (4.1) almost everywhere and $M[u] + \epsilon\,\mathrm{trace}(M[u])I \geq 0$ almost everywhere. Moreover, $u_\epsilon \in C^{0,1}(\bar{\Omega})$ and

$$\|u_\epsilon\|_{L^\infty} \leq C_1, \quad \|Du_\epsilon\|_{L^\infty} \leq C_2, \tag{4.2}$$

for some constants C_1 and C_2 independent of ϵ. Note that the uniform C^0 and C^1 estimates in (4.2) can be readily checked as in [15].

Step 2. The limit of approximated solutions is an L^p-viscosity solution. By (4.2) and Ascoli-Arzela theorem, u_ϵ has a subsequence that converges in $C(\bar{\Omega})$ to a Lipschitz continuous function u. Note that by the stability property of viscosity solutions [10], the limit function u does not depend on the choice of subsequence.

Next, we show that u is an L^p-viscosity solution of (1.5). We shall check that u is an L^p-viscosity supersolution of (1.5). Suppose that u is not an L^p-viscosity supersolution of (1.5), then there exist a point $x_0 \in \Omega$, a test function $\varphi \in W^{2,p}$ and two positive constants δ and r such that

$$\mathcal{F}[\varphi] \geq \delta \tag{4.3}$$

almost everywhere in $B_r(x_0) := \{x \in \Omega \,|\, |x - x_0| < r\}$, and $u - \varphi$ has a global strict minimum over $\overline{B_r(x_0)}$ at x_0. Assume that φ_ϵ is the test function for the solution u_ϵ of (4.1), and $\varphi_\epsilon \to \varphi$ uniformly in $B_r(x_0)$

as $\epsilon \to 0$, then by (4.3),

$$\mathcal{F}_\epsilon[\varphi_\epsilon] := \det[M[\varphi_\epsilon] + \epsilon\,\mathrm{trace}(M[\varphi_\epsilon])I] - B(x, \varphi_\epsilon, D\varphi_\epsilon) \geq \frac{\delta}{2} \tag{4.4}$$

holds almost everywhere in $B_r(x_0)$, for sufficiently small $\epsilon > 0$. By Weierstrass theorem, $u_\epsilon - \varphi_\epsilon$ has a maximum point x_ϵ in $\overline{B_r(x_0)}$. Since $u_\epsilon - \varphi_\epsilon \to u - \varphi$ uniformly in Ω, we have $x_\epsilon \to x_0$ up to a subsequence. Since x_0 is an interior point of $B_r(x_0)$, for sufficiently small ϵ, x_ϵ belongs to the open ball $B_r(x_0)$, which leads to a contradiction with Deifinition 2.5 (i). Therefore, u is an L^p-viscosity supersolution of (1.5). Similarly, we see that u is also an L^p-viscosity subsolution of (1.5). Thus, u is proved to be an L^p-viscosity solution of (1.5).

This step is completed by checking the availability of the function φ_ϵ satisfying (4.4). We set $\varphi_\epsilon = \varphi + Q_\epsilon$, where $Q_\epsilon := \frac{\epsilon}{2}|x - x_0|^2$. Then it is readily checked that

$$\begin{aligned}\mathcal{F}_\epsilon[\varphi_\epsilon] \geq&\det[M[\varphi_\epsilon]] + \det[\epsilon\,\mathrm{trace}(M[\varphi_\epsilon])I] - B(x, \varphi_\epsilon, D\varphi_\epsilon)\\ \geq&\mathcal{F}[\varphi] + \det[D^2 Q_\epsilon + A(x, \varphi, D\varphi) - A(x, \varphi + Q_\epsilon, D(\varphi + Q_\epsilon))]\\ &+ [\epsilon\mathrm{trace}(M[\varphi + Q_\epsilon])]^n + B(x, \varphi, D\varphi) - B(x, \varphi + Q_\epsilon, D(\varphi + Q_\epsilon))\\ \geq&\frac{\delta}{2},\end{aligned} \tag{4.5}$$

where the subadditivity of det and (4.3) are used, and

$$\begin{aligned}&\det[D^2 Q_\epsilon + A(x, \varphi, D\varphi) - A(x, \varphi + Q_\epsilon, D(\varphi + Q_\epsilon))]\\ &+[\epsilon\mathrm{trace}(M[\varphi + Q_\epsilon])]^n + B(x, \varphi, D\varphi) - B(x, \varphi + Q_\epsilon, D(\varphi + Q_\epsilon)) \geq -\frac{\delta}{2}\end{aligned} \tag{4.6}$$

is used in the last inequality by letting ϵ sufficiently small. Here when using the subadditivity of det, $M[\varphi] > 0$ and $D^2 Q_\epsilon + A(x, \varphi, D\varphi) - A(x, \varphi + Q_\epsilon, D(\varphi + Q_\epsilon)) > 0$ are used. The former one is guaranteed by the g-convexity of φ, and the latter one can be achieved by choosing r small.

Step 3. The limit of approximated solutions is a Trudinger solution. Let v be the Trudinger solution of (1.5)–(1.9), namely that there exists a sequence v_m of C^2 g-convex functions such that $v_m \to v$ uniformly in Ω and $\tilde{\mathbb{F}}[v_m] = f^* \circ Y(x, v_m, Dv_m) \det(E^{-1}) \det[M[v_m]] \to f$ in $L^1_{loc}(\Omega)$. Let u be the uniform limit of u_ϵ, we next prove $u = v$ in Ω.

To prove that $w = u - v \geq 0$, it suffices to check that $\mathcal{F}[w] \leq 0$ in the C-viscosity sense in Ω. Assume by contradiction that w is not a supersolution, then there exists a point $x_0 \in \Omega$, a test function $\varphi \in C^2$ and two positive constants δ and r such that

$$\mathcal{F}[\varphi] \geq \delta \tag{4.7}$$

in $B_r(x_0)$, and $w - \varphi$ has a global strict minimum over $\overline{B_r(x_0)}$ at x_0. We take a function of the form

$$\varphi_{\epsilon,m} := \varphi + v_m + Q_\epsilon, \tag{4.8}$$

where $Q_\epsilon := \frac{\epsilon}{2}|x - x_0|^2$. Similarly to (4.5), for small ϵ and large m, we have

$$\begin{aligned}\mathcal{F}_\epsilon[\varphi_{\epsilon,m}] \geq&\det[M[\varphi_{\epsilon,m}]] + \det[\epsilon\,\mathrm{trace}(M[\varphi_{\epsilon,m}])I] - B(x, \varphi_{\epsilon,m}, D\varphi_{\epsilon,m})\\ \geq&\mathcal{F}[\varphi] + \det[D^2(v_m + Q_\epsilon) + A(x, \varphi, D\varphi) - A(x, \varphi_{\epsilon,m}, D\varphi_{\epsilon,m})]\\ &+ [\epsilon\mathrm{trace}(M[\varphi_{\epsilon,m}])]^n + B(x, \varphi, D\varphi) - B(x, \varphi_{\epsilon,m}, D\varphi_{\epsilon,m})\\ \geq&\frac{\delta}{2}.\end{aligned} \tag{4.9}$$

Since the function $u_\epsilon - \varphi_{\epsilon,m}$ approaches $w - \varphi$ uniformly, it achieves a minimum inside the ball $B_r(x_0)$ at least for small ϵ and large m. By *Step 1*, we know that u_ϵ is an L^p-viscosity solution of (4.1). Then by Definition 2.5 (ii), we have

$$\operatorname{ess}\limsup_{x\to x_0} \mathcal{F}_\epsilon[\varphi_{\epsilon,m}] \le 0, \tag{4.10}$$

which contradicts with (4.9). Now, we have proved that $\mathcal{F}[w] \le 0$ in the C-viscosity sense in Ω, which leads to $u \ge v$ in Ω. Applying this argument again to $\tilde{w} := v - u$, we get $u \le v$ in Ω. Combining both inequalities, we have proved $u \equiv v$ in Ω.

Step 4. A Trudinger solution is an L^p-viscosity solution. From Definition 2.6, the Trudinger solution u is a uniform limit of a family of functions $\{u_m\}$, which is unique. Under the assumption that $f \in L^p(\Omega)$ ($p \ge 1$) is a nonnegative function, the L^p-viscosity solution of (1.5) may not be unique. Combining *Steps 2 and 3*, a Trudinger solution of (1.5)–(1.9) is equivalent to a vanishing viscosity solution of (1.5)–(1.9), and is an L^p-viscosity solution of (1.5)–(1.9). □

Remark 4. Note that $\operatorname{trace}(M[u]) = \Delta u - \sum_{i=1}^n A_{ii}$ in (4.1) involves the Laplacian of u, which can be regarded as a "viscosity term". For this reason, the method of using the approximation (4.1) and letting $\epsilon \to 0$ is also called the vanishing viscosity approximation method in some literature. The scheme of adding $\epsilon \operatorname{trace}(M[u])I$ to $M[u]$, (originates from [30] in the treatment of curvature equations), has been used in [15] for more general elliptic operators $\mathcal{F}$.

Remark 5. In [2], the authors considered the standard Monge-Ampère equation, which is just the case of $g(x, y, z) = x \cdot y - z$ in this paper. In this case, the relationship between Trudinger solution and L^p-viscosity solution is also studied in [2]. We refer the reader to [2] for detailed discussions about L^∞-viscosity solutions and the maximal L^p-viscosity solutions.

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Conflict of interest

The authors declare no conflict of interest.

References

1. F. Abedin, C. E. Gutiérrez, An iterative method for generated Jacobian equations, *Calc. Var.*, **56** (2017), 101. http://doi.org/10.1007/s00526-017-1200-2
2. A. L. Amadori, B. Brandolini, C. Trombetti, Viscosity solutions of the Monge-Ampère equation with the right hand side in L^p, *Rend. Lincei Mat. Appl.*, **18** (2007), 221–233. http://doi.org/10.4171/RLM/491
3. G. Awanou, Computational nonimaging geometric optics: Monge-Ampère, *Notices of the American Mathematical Society*, **68** (2021), 186–193. http://doi.org/10.1090/noti2220
4. L. Caffarelli, The regularity of mappings with a convex potential, *J. Amer. Math. Soc.*, **5** (1992), 99–104. http://doi.org/10.1090/S0894-0347-1992-1124980-8
5. L. Caffarelli, M. G. Crandall, M. Kocan, A. Święch, On viscosity solutions of fully nonlinear equations with measurable ingredients, *Commun. Pure Appl. Math.*, **49** (1996), 365–397. http://doi.org/10.1002/(SICI)1097-0312(199604)49:4<365::AID-CPA3>3.0.CO;2-A
6. B. Dacorogna, J. Moser, On a partial differential equation involving the Jacobian determinant, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **7** (1990), 1–26. http://doi.org/10.1016/S0294-1449(16)30307-9
7. A. Gallouët, Q. Mérigot, B. Thibert, A damped Newton algorithm for generated Jacobian equations, *Calc. Var.*, **61** (2022), 49. http://doi.org/10.1007/s00526-021-02147-7
8. N. Guillen, A primer on generated Jacobian equations: geometry, optics, economics, *Notices of the American Mathematical Society*, **66** (2019), 1401–1411. http://doi.org/10.1090/noti1956
9. N. Guillen, J. Kitagawa, Pointwise estimates and regularity in geometric optics and other generated Jacobian equations, *Commun. Pure Appl. Math.*, **70** (2017), 1146–1220. http://doi.org/10.1002/cpa.21691
10. H. Ishii, P. L. Lions, Viscosity solutions of fully nonlinear second-order elliptic partial differential equations, *J. Differ. Equations*, **83** (1990), 26–78. http://doi.org/10.1016/0022-0396(90)90068-Z
11. S. Jeong, Local Hölder regularity of solutions to generated Jacobian equations, *Pure and Applied Analysis*, **3** (2021), 163–188. http://doi.org/10.2140/paa.2021.3.163
12. Y. Jhaveri, Partial regularity of solutions to the second boundary value problem for generated Jacobian equations, *Methods Appl. Anal.*, **24** (2017), 445–475. http://doi.org/10.4310/MAA.2017.v24.n4.a2
13. F. Jiang, N. S. Trudinger, On Pogorelov estimates in optimal transportation and geometric optics, *Bull. Math. Sci.*, **4** (2014), 407–431. http://doi.org/10.1007/s13373-014-0055-5
14. F. Jiang, N. S. Trudinger, On the second boundary value problem for Monge-Ampère type equations and geometric optics, *Arch. Rational Mech. Anal.*, **229** (2018), 547–567. http://doi.org/10.1007/s00205-018-1222-8
15. F. Jiang, N. S. Trudinger, Oblique boundary value problems for augmented Hessian equations III, *Commun. Part. Diff. Eq.*, **44** (2019), 708–748. http://doi.org/10.1080/03605302.2019.1597113
16. F. Jiang, X. P. Yang, Weak solutions of Monge-Ampère type equations in optimal transportation, *Acta Math. Sci.*, **33** (2013), 950–962. http://doi.org/10.1016/S0252-9602(13)60054-5

17. J. Liu, Monge-Ampère type equations and optimal transportation, PhD Thesis, Australian National University, 2010.

18. G. Loeper, On the regularity of solutions of optimal transportation problems, *Acta Math.*, **202** (2009), 241–283. http://doi.org/10.1007/s11511-009-0037-8

19. G. Loeper, N. S. Trudinger, On the convexity theory of generating functions, arXiv:2109.04585.

20. X.-N. Ma, N. S. Trudinger, X.-J. Wang, Regularity of potential functions of the optimal transportation problem, *Arch. Rational Mech. Anal.*, **177** (2005), 151–183. http://doi.org/10.1007/s00205-005-0362-9

21. R. J. McCann, K. S. Zhang, On concavity of the monopolist's problem facing consumers with nonlinear price preferences, *Commun. Pure Appl. Math.*, **72** (2019), 1386–1423. http://doi.org/10.1002/cpa.21817

22. G. Nöldeke, L. Samuelson, The implementation duality, *Econometrica*, **86** (2018), 1283–1324. http://doi.org/10.3982/ECTA13307

23. C. Rankin, Distinct solutions to generated Jacobian equations cannot intersect, *Bull. Aust. Math. Soc.*, **102** (2020), 462–470. http://doi.org/10.1017/S0004972720000052

24. C. Rankin, Strict convexity and C^1 regularity of solutions to generated Jacobian equations in dimension two, *Calc. Var.*, **60** (2021), 221. http://doi.org/10.1007/s00526-021-02093-4

25. C. Rankin, Strict g-convexity for generated Jacobian equations with applications to global regularity, arXiv:2111.00448.

26. C. Rankin, Regularity and uniqueness results for generated Jacobian equations, PhD Thesis, Australian National University, 2021.

27. C. Rankin, First and second derivative Hölder estimates for generated Jacobian equations, arXiv:2204.07917.

28. L. B. Romijn, M. J. H. Anthonissen, J. H. M. ten Thije Boonkkamp, W. L. Ijzerman, Numerically solving generated Jacobian equations in freeform optical design, *EPJ Web Conf.*, **238** (2020), 02001. http://doi.org/10.1051/epjconf/202023802001

29. L. B. Romijn, J. H. M. ten Thije Boonkkamp, M. J. H. Anthonissen, W. L. Ijzerman, An iterative least-squares method for generated Jacobian equations in freeform optical design, *SIAM J. Sci. Comput.*, **43** (2021), B298–B322. http://doi.org/10.1137/20M1338940

30. N. S. Trudinger, The Dirichlet problem for the prescribed curvature equations, *Arch. Rational Mech. Anal.*, **111** (1990), 153–179. http://doi.org/10.1007/BF00375406

31. N. S. Trudinger, Weak solutions of Hessian equations, *Commun. Part. Diff. Eq.*, **22** (1997), 25–54. http://doi.org/10.1080/03605309708821299

32. N. S. Trudinger, Recent developments in elliptic partial differential equations of Monge-Ampère type, In: *International congress of mathematicians Madrid 2006 Volume III. Invited lectures*, 2006 291–302. http://doi.org/10.4171/022-3/15

33. N. S. Trudinger, On the prescribed Jacobian equation, In: *International conference for the 25th anniversary of viscosity solutions*, Tokyo, Japan, 2008, 243–255.

34. N. S. Trudinger, On generated prescribed Jacobian equations, *Oberwolfach Reports*, **8** (2011), 2194–2198.
35. N. S. Trudinger, On the local theory of prescribed Jacobian equations, *Discrete Contin. Dyn. Syst.*, **34** (2014), 1663–1681. http://doi.org/10.3934/dcds.2014.34.1663
36. N. S. Trudinger, On the local theory of prescribed Jacobian equations revisited, *Mathematics in Engineering*, **3** (2021), 1–17. http://doi.org/10.3934/mine.2021048
37. N. S. Trudinger, A note on second derivative estimates for Monge-Ampère type equations, arXiv:2204.01039.
38. N. S. Trudinger, X.-J. Wang, On the second boundary value problem for Monge-Ampère type equations and optimal transportation, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)*, **8** (2009), 143–174. https://doi.org/10.2422/2036-2145.2009.1.07
39. X.-J. Wang, On the design of a refector antenna, *Inverse Probl.*, **12** (1996), 351–375. http://doi.org/10.1088/0266-5611/12/3/013

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