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Research article

Local Calderón-Zygmund estimates for parabolic equations in weighted Lebesgue spaces†

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Abstract: We prove local Calderón-Zygmund type estimates for the gradient of weak solutions to degenerate or singular parabolic equations of p-Laplacian type with $p > \frac{2n}{n+2}$ in weighted Lebesgue spaces L^q_w. We introduce a new condition on the weight w which depends on the intrinsic geometry concerned with the parabolic p-Laplace problems. Our condition is weaker than the one in [13], where similar estimates were obtained. In particular, in the case $p = 2$, it is the same as the condition of the usual parabolic A_q weight.

Keywords: parabolic equation; p-Laplacian; Calderón-Zygmund estimate; weighted Lebesgue space

Dedicated to Giuseppe Rosario Mingione, on the occasion of his 50th birthday.

1. Introduction

We study local regularity theory for weak solutions to the following parabolic equations of p-Laplacian type:

$$u_t - \operatorname{div} \mathbf{a}(Du) = -\operatorname{div}\left(|F|^{p-2}F\right) \quad \text{in } \Omega_T := \Omega \times (0, T], \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ ($n \geqslant 2$) is an open set, T is a positive constant, $u = u(x,t)$ is a real valued function with $(x,t) \in \Omega \times (0,T] = \Omega_T$, u_t is the partial derivative of u with respect to the time variable t, and $Du \in \mathbb{R}^n$ is the gradient of u with respect to the space variable x (i.e., $Du = D_x u$). For given $p \in (1, \infty)$, we assume that $\mathbf{a} : \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following p-growth and p-ellipticity conditions:

$$|\mathbf{a}(\xi)| + |D_\xi \mathbf{a}(\xi)||\xi| \leqslant L|\xi|^{p-1} \tag{1.2}$$

and

$$D_\xi \mathbf{a}(\xi)\, \eta \cdot \eta \geqslant \nu |\xi|^{p-2}|\eta|^2, \tag{1.3}$$

for every $\xi, \eta \in \mathbb{R}^n \setminus \{0\}$, and for some constants ν and L with $0 < \nu \leqslant 1 \leqslant L$. The prototype of $\mathbf{a}$ is

$$\mathbf{a}(\xi) = |\xi|^{p-2}\xi.$$

L^q-regularity theory with Calderón-Zygmund estimates for partial differential equations is originated from the classical result of Calderón and Zygmund [16] about the boundedness of linear operators including the Laplace operator. For the following p-Laplacian type equation

$$\operatorname{div}\left(|Du|^{p-2}Du\right) = \operatorname{div}\left(|F|^{p-2}F\right),$$

a fundamental L^q-regularity theory, which is also called (nonlinear) Calderón-Zygmund theory, is to show the following implication:

$$|F|^p \in L^q \implies |Du|^p \in L^q, \qquad q > 1, \tag{1.4}$$

and obtain corresponding estimates, so-called *Calderón-Zygmund estimates*. In this regard, Iwaniec [26] first obtained Calderón-Zygmund estimates in the whole space $\mathbb{R}^n$ when $p \geqslant 2$, and DiBenedetto and Manfredi [21] extended this result to the corresponding system with $1 < p < \infty$. Thereafter, Caffarelli and Peral [15] considered general elliptic equations with p-growth, for instance, the stationary case of (1.1), applying a new approach by means of maximal functions and a covering argument obtained by Krylov and Safonov based on the Calderón-Zygmund decomposition. We further refer to e.g., [1, 11, 14, 17, 28] for L^q-regularity theory with corresponding Calderón-Zygmund estimates for elliptic problems.

Difficulty of the study on regularity theory for parabolic problems of p-Laplacian type is originated from the absence of the scaling invariant property: a constant multiple of a solution of the parabolic p-Laplace equation $u_t - \operatorname{div}(|Du|^{p-2}Du) = 0$ does not become a solution. It could be overcome by considering intrinsic parabolic cylinders depending on solutions, instead of the usual parabolic cylinders. This idea was introduced by DiBenedetto and Friedman in [19, 20], see also the monograph [18], where Hölder regularity for parabolic p-Laplace systems had been established.

For the L^q-regularity theory, on the other hand, the approaches used in [26] and [15] are not directly applicable to the parabolic p-Laplacian type problems when $p \neq 2$, since the intrinsic geometry prevents the use of maximal functions. Finally, Acerbi and Mingione [2] established local Calderón-Zygmund estimates when $p > \frac{2n}{n+2}$ with a new approach, hence proved the implication (1.4) in the local sense. We also refer to the higher integrability result of Kinnunen and Lewis in [27], where the implication (1.4) is obtained when q is sufficiently close to 1. Note that the condition $p > \frac{2n}{n+2}$ is essential since there exists an unbounded weak solution to the parabolic p-Laplace system in this case, see [18]. It is worth pointing out that the approach in [2] does not employ maximal functions and the covering argument by Krylov and Safonov but a new covering argument used in [31] which is based on the Vitali covering lemma. It has led to the development of the Calderón-

Zygmund theory for parabolic problems. For instance, we refer to [5, 7, 10] for global Calderón Zygmund theory in bounded domains, [6, 33] for parabolic obstacle problems, [4, 9] for parabolic problems with variable exponent, [25, 32] for parabolic problems with growth and [35] for parabolic variational problems.

Research on Calderón-Zygmund estimates in general function spaces such as weighted Lebesgue spaces, Orlicz spaces, variable exponent Lebesgue spaces, Lorentz spaces have been actively conducted for the last decade, e.g., [3, 8, 12, 29, 36]. In particular, estimates in weighted Lebesgue spaces are crucial since these imply estimates in various function spaces by extrapolation argument, see [24, Section 5]. For parabolic problems with p-growth as in (1.1), Byun and Ryu [13] obtained global Calderón-Zygmund estimates in the weight Lebesgue spaces L^q_w hence proved the following implication:

$$|F|^p \in L^q_w \implies |Du|^p \in L^q_w, \tag{1.5}$$

with $q > 1$ and the weight w satisfying the following Muckenhoupt type condition:

$$\sup_{Q\in C} \left(\fint_Q w \, dz \right) \left(\fint_Q w^{-\frac{1}{q-1}} \, dz \right)^{q-1} < \infty, \tag{1.6}$$

where C is the set of all cylinders of the form $B_r(x_0) \times (t_1, t_2) \subset \mathbb{R}^n \times \mathbb{R}$. On the other hand, if $p = 2$, the same implication can be obtained for every usual parabolic A_q weight w, that is, w satisfies (2.1) with C the set of all parabolic cylinders of the form $B_r(x_0) \times (t_0 - r^2, t_0 + r^2)$. Therefore, there is a drastic change between the conditions of weights when $p = 2$ and $p \neq 2$.

In this paper, we introduce a new parabolic Muckenhoupt type condition depending on the intrinsic geometry concerned with the parabolic p-Laplacian setting, see Definition 2.1. We emphasize that our condition on weights depends on p, and is weaker than the one in [13] and exactly the same as the parabolic A_q condition when $p = 2$. With this condition we prove the implication (1.5) in the local sense by obtaining corresponding Calderón-Zymund estimates.

Now, we state our main result. Notation and the definition of the p-intrinsic A_q weight are introduced in next section. We say that $u \in C^0(0, T; L^2(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega))$ is a weak solution to (1.1) if

$$-\int_{\Omega_T} u\, \zeta_t \, dz + \int_{\Omega_T} \mathbf{a}(Du) \cdot D\zeta \, dz = \int_{\Omega_T} |F|^{p-2} F \cdot D\zeta \, dz$$

holds for every $\zeta \in C_0^\infty(\Omega_T)$.

Theorem 1.1. *Let $p > \frac{2n}{n+2}$ and $u \in C^0(0, T; L^2(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega))$ be a weak solution to* (1.1) *with $F \in L^p(\Omega_T, \mathbb{R}^n)$. If w is a p-intrinsic A_q weight with $q > 1$ and $|F|^p \in L^q_{w,\mathrm{loc}}(\Omega_T)$, then $|Du|^p \in L^q_{w,\mathrm{loc}}(\Omega_T)$.*

Furthermore, there exists $R_0 = R_0(n, \nu, L, p, q, [w]_q, Du, F) > 0$ such that for every $Q_{2r} \Subset \Omega_T$ with $2r < R_0$,

$$\begin{aligned} &\left(\frac{1}{w(Q_r)} \int_{Q_r} |Du|^{pq} w \, dz \right)^{\frac{1}{q}} \\ &\leqslant c \left(\fint_{Q_{2r}} [|Du|^p + |F|^p + 1] \, dz \right)^d + c \left(\frac{1}{w(Q_{2r})} \int_{Q_{2r}} |F|^{pq} w \, dz \right)^{\frac{1}{q}} \end{aligned} \tag{1.7}$$

for some $c = c(n, \nu, L, p, q, [w]_q) > 0$*, where*

$$d := \begin{cases} \frac{2p}{p(n+2)-2n}, & \text{if } \frac{2n}{n+2} < p < 2, \\ \frac{p}{2}, & \text{if } p \geqslant 2. \end{cases} \tag{1.8}$$

Remark 1.1. In the above theorem, R_0 will be chosen as in (3.1). Furthermore, when $p = 2$ we may put $R_0 = \infty$.

Remark 1.2. (Possible extensions) In this paper, we deal with only scalar problems without coefficients for simplicity. We can consider more general problems such as general non-autonomous parabolic equations with p-growth

$$u_t - \operatorname{div} \mathbf{a}(x, t, Du) = -\operatorname{div}\left(|F|^{p-2}F\right),$$

where $\mathbf{a}(x, t, \xi)$ satisfies (1.2), (1.3) and a VMO condition (see [10]), and parabolic p-Laplace systems with coefficients

$$u_t - \operatorname{div}\left((A(x,t)Du : Du)^{\frac{p-2}{2}} A(x,t)Du\right) = -\operatorname{div}\left(|F|^{p-2}F\right),$$

where $u : \Omega_T \to \mathbb{R}^N$ and $A(x,t) : \mathbb{R}^{n+1} \to \mathbb{R}^{n^2N^2}$ satisfies a VMO condition (see [2]). Moreover, as in [13], we can also consider global Calderón-Zygmund estimates in Reifenberg flat domains.

The remaining part of the paper is organized as follows. In Section 2, we introduce notation, weights with their main assumption, and comparison and regularity estimates for corresponding homogeneous problems. In Section 3, we prove our main theorem, Theorem 1.1.

2. Preliminaries

2.1. Notation

Let $z_0 = (x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$ with $x_0 = (x_0^1, \dots, x_0^n)$, $r, \alpha, \lambda > 0$ and $1 < p < \infty$. We define an α-parabolic cylinder by $Q_{r,\alpha}(z_0) = B_r(x_0) \times (t_0 - \alpha r^2, t_0 + \alpha r^2)$ and an α-parabolic cube by $\tilde{Q}_{r,\alpha}(z_0) = C_r(x_0) \times (t_0 - \alpha r^2, t_0 + \alpha r^2)$, where $B_r(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < r\}$ and $C_r(x_0) := \{x = (x^1, \dots, x^n) \in \mathbb{R}^n : \max\{|x^1 - x_0^1|, \dots, |x^n - x_0^n|\} < r\}$. Note that $Q_r(z_0) := Q_{r,1}(z_0)$ and $\tilde{Q}_r(z_0) := \tilde{Q}_{r,1}(z_0)$ is the usual parabolic cylinder and cube, respectively, and we denote $\partial_{\mathrm{p}} Q_r(z_0) := (B_r(x_0) \times \{t = t_0 - r^2\}) \cup (\partial B_r(x_0) \times [t_0 - r^2, t_0 + r^2))$ Furthermore, when $\alpha = \lambda^{2-p}$ we write $Q_r^\lambda(z_0) = Q_{r,\lambda^{2-p}}(z_0) = B_r(x_0) \times (t_0 - \lambda^{2-p} r^2, t_0 + \lambda^{2-p} r^2)$ which is usually called a p-intrinsic parabolic cylinder since we will consider λ related to the weak solution to (1.1).

For an integrable function $f : U \to \mathbb{R}^m$ with $U \subset \mathbb{R}^{n+1}$ and $0 < |U| < \infty$, we write $(f)_U = \fint_U f\,dz := \frac{1}{|U|}\int_U f\,dz$, where $|U|$ is the Lebesgue measure of U in $\mathbb{R}^{n+1}$.

2.2. Weights

We say that $w : \mathbb{R}^{n+1} \to \mathbb{R}$ is a *weight* if it is nonnegative and locally integrable. For a weight w and a bounded open set $U \subset \mathbb{R}^{n+1}$, we write

$$w(U) := \int_U w\,dz,$$

and define by weighted Lebesgue space $L^q_w(U)$, $1 \leqslant q < \infty$, the set of all measurable function f on U such that

$$\|f\|_{L^q_w(U)} := \left(\int_U |f|^q w\, dz\right)^{\frac{1}{q}} < \infty.$$

We introduce the assumption of the weight w in Theorem 1.1.

Definition 2.1. Let $p, q \in (1, \infty)$. We say that weight $w : \mathbb{R}^{n+1} \to \mathbb{R}$ is a *p-intrinsic parabolic A_q weight* if it satisfies that

$$[w]_q := \sup_{Q \in C_p} \left(\fint_Q w\, dz\right)\left(\fint_Q w^{-\frac{1}{q-1}}\, dz\right)^{q-1} < \infty, \tag{2.1}$$

where

$$C_p := \left\{Q_{r,\alpha}(z_0) \,:\, z_0 \in \mathbb{R}^{n+1},\ \ r > 0,\ \ \alpha = \lambda^{2-p},\ \ 1 \leqslant \lambda \leqslant \max\{1, r^{-\frac{n+2}{2}}\}\right\}.$$

Here, the α-parabolic cylinders $Q_{r,\alpha}(z_0)$ can be replaced by the α-parabolic cubes $\tilde{Q}_{r,\alpha}(z_0)$.

Note that in the definition of the class C_p, the range of α with respect to p is following:

$$\begin{cases} 1 \leqslant \alpha \leqslant \max\{1, r^{\frac{(p-2)(n+2)}{2}}\} & \text{if } \ p < 2, \\ \alpha = 1 & \text{if } \ p = 2, \\ \min\{1, r^{\frac{(p-2)(n+2)}{2}}\} \leqslant \alpha \leqslant 1 & \text{if } \ p > 2. \end{cases}$$

Hence, C_p contains all the parabolic cylinders $Q_r(z_0)$ and, in particular, C_2 (i.e., $p = 2$) consists of only the parabolic cylinders. Moreover, since $r^{-\frac{n+2}{2}} \leqslant \rho^{-\frac{n+2}{2}}$ for $\rho \in (0, r]$, we have

$$Q_{r,\alpha} \in C_p \quad \Longrightarrow \quad Q_{\rho,\alpha} \in C_p \ \text{ for every } \ \rho \in (0, r].$$

From this fact, we can obtain the following properties for p-intrinsic parabolic A_q weights, which are well known properties of the usual A_q weights, see e.g., [23, Section 7.2]. Proofs are exactly the same as the ones in there with just replacing cubes and the dimension n by the α-parabolic cubes and $n + 2$, respectively. Therefore, we omit their proofs.

Proposition 2.1. *Let $p, q \in (1, \infty)$ and $w : \mathbb{R}^{n+1} \to \mathbb{R}$ be a p-intrinsic parabolic A_q weight.*

(1) For every $f \in L^q_w(Q)$ with $Q \in C_p$,

$$\left(\fint_Q |f|\, dz\right)^q \leqslant \frac{[w]_q}{w(Q)} \int_Q |f|^q\, w\, dz. \tag{2.2}$$

(2) There exist $\gamma, c > 0$ depending on n, q and $[w]_q$ such that

$$\left(\fint_Q w^{1+\gamma}\, dz\right)^{\frac{1}{1+\gamma}} \leqslant c \fint_Q w\, dz.$$

(3) There exist $\gamma_1 > 0$ and $c_1, c_2 \geqslant 1$ depending on n, q and $[w]_q$ such that for every $Q \in C_p$ and $E \subset Q$

$$\frac{1}{c_1}\left(\frac{|E|}{|Q|}\right)^q \leqslant \frac{w(E)}{w(Q)} \leqslant c_2 \left(\frac{|E|}{|Q|}\right)^{\gamma_1}. \tag{2.3}$$

(4) w is a p-intrinsic parabolic A_{q_1} weight for every $q_1 > q$. Moreover, w is a p-intrinsic parabolic $A_{q'}$ weight for some $q' \in (1, q)$, where q' and $[w]_{q'}$ depend on n, q and $[w]_q$.

Example. On $\mathbb{R}^{n+1}$, the function $w(x,t) = \max\{|x|, \sqrt{|t|}\}^\gamma$ is a p-intrinsic A_q weight for $p \geqslant 2$, $q \in (1, \infty)$ when $-n < \gamma < n(q-1)$. Indeed, we first write

$$I[Q] := \left(\fint_Q \max\{|x|, \sqrt{|t|}\}^\gamma \, dz\right)\left(\fint_Q \max\{|x|, \sqrt{|t|}\}^{-\frac{\gamma}{q-1}} \, dz\right)^{q-1}, \qquad Q \in C_p.$$

We divide the α-parabolic cylinders $Q_{r,\lambda^{2-p}}(z_0)$ in C_p for $z_0 = (x_0, t_0) \in \mathbb{R}^{n+1}$ into three cases:

(i) $\min\{|x_0|, \sqrt{|t_0|}\} \geqslant 3r$,
(ii) $|x_0| < 3r$ and $3\lambda^{\frac{2-p}{2}} r \leqslant \sqrt{|t_0|} < 3r$,
(iii) $|x_0| < 3r$ and $\sqrt{|t_0|} < 3\lambda^{\frac{2-p}{2}} r$.

Note that if $x \in B_r(x_0)$ and $|x_0| \geqslant 3r$,

$$\frac{2}{3}|x_0| \leqslant |x_0| - |x - x_0| \leqslant |x| \leqslant |x - x_0| + |x_0| \leqslant \frac{4}{3}|x_0|,$$

and if $t_0 - \lambda^{2-p} r^2 \leqslant t \leqslant t_0 + \lambda^{2-p} r^2$ and $\sqrt{|t_0|} > 3\lambda^{\frac{2-p}{2}} r$,

$$\frac{\sqrt{8}}{3}\sqrt{|t_0|} \leqslant \sqrt{|t_0| - |t - t_0|} \leqslant \sqrt{|t|} \leqslant \sqrt{|t - t_0| + |t_0|} \leqslant \frac{\sqrt{10}}{3}\sqrt{|t_0|}.$$

In Case (i), if $(x,t) \in Q_{r,\lambda^{2-p}}(z_0)$, we have $|x| \approx |x_0|$ and $|t| \approx |t_0|$, hence

$$\begin{aligned}
&I[Q_{r,\lambda^{2-p}}(z_0)] \\
&\leqslant c \underbrace{\left(\fint_{Q_{r,\lambda^{2-p}}(z_0)} \max\{|x_0|, \sqrt{|t_0|}\}^\gamma \, dz\right)\left(\fint_{Q_{r,\lambda^{2-p}}(z_0)} \max\{|x_0|, \sqrt{|t_0|}\}^{-\frac{\gamma}{q-1}} \, dz\right)^{q-1}}_{=1} = c.
\end{aligned}$$

In Case (ii), if $(x,t) \in Q_{r,\lambda^{2-p}}(z_0)$, then $|x| < 4r$ and $|t| \approx |t_0| < 9r^2$, and hence

$$\begin{aligned}
I[Q_{r,\lambda^{2-p}}(z_0)] &\leqslant c\left(\fint_{t_0-\lambda^{2-p}r^2}^{t_0+\lambda^{2-p}r^2} \fint_{B_{4r}(0)} \max\{|x|, \sqrt{|t_0|}\}^\gamma \, dxdt\right) \\
&\quad \times \left(\fint_{t_0-\lambda^{2-p}r^2}^{t_0+\lambda^{2-p}r^2} \fint_{B_{4r}(0)} \max\{|x|, \sqrt{|t_0|}\}^{-\frac{\gamma}{q-1}} \, dx\right)^{q-1} \\
&\leqslant c\left[\frac{1}{r^n}\left(\int_0^{\sqrt{|t_0|}} |t_0|^{\frac{\gamma}{2}} \rho^{n-1} \, d\rho + \int_{\sqrt{|t_0|}}^{4r} \rho^{\gamma+n-1} \, d\rho\right)\right] \\
&\quad \times \left[\frac{1}{r^n}\left(\int_0^{\sqrt{|t_0|}} |t_0|^{-\frac{\gamma}{2(q-1)}} \rho^{n-1} \, d\rho + \int_{\sqrt{|t_0|}}^{4r} \rho^{-\frac{\gamma}{q-1}+n-1} \, d\rho\right)\right]^{q-1} \\
&\leqslant \frac{c}{r^{nq}}\left[\frac{\gamma|t_0|^{\frac{\gamma+n}{2}}}{n(n+\gamma)} + \frac{(4r)^{\gamma+n}}{\gamma+n}\right]\left[\frac{-\gamma|t_0|^{\frac{nq-n-\gamma}{2(q-1)}}}{n(nq-n-\gamma)} + \frac{(q-1)(4r)^{\frac{nq-n-\gamma}{q-1}}}{nq-n-\gamma}\right]^{q-1} \\
&\leqslant \frac{c}{r^{nq}} r^{\gamma+n} r^{nq-n-\gamma} = c,
\end{aligned}$$

where we have used $-n < \gamma < n(q-1)$.

In Case (iii), if $(x,t) \in Q_{r,\lambda^{2-p}}(z_0)$, then $|x| \leqslant 4r$ and $|t| \leqslant 10\lambda^{2-p}r^2$, and hence, by the similar computation as in Case (ii), we have

$$I[Q_{r,\lambda^{2-p}}(z_0)] \leqslant \frac{c}{r^{nq}}\Big[\frac{\gamma}{n(n+\gamma)}\fint_{-10\lambda^{2-p}r^2}^{10\lambda^{2-p}r^2}|t|^{\frac{\gamma+n}{2}}\,dt + \frac{(4r)^{\gamma+n}}{\gamma+n}\Big]$$
$$\times\Big[\frac{-\gamma}{n(nq-n-\gamma)}\fint_{-10\lambda^{2-p}r^2}^{10\lambda^{2-p}r^2}|t|^{\frac{n(q-1)-\gamma}{2(q-1)}}\,dt + \frac{(q-1)(4r)^{\frac{nq-n-\gamma}{q-1}}}{nq-n-\gamma}\Big]^{q-1}.$$

Finally, using the facts that $-n < \gamma < n(q-1)$ and $\lambda \geqslant 1$,

$$\begin{aligned}
I[Q_{r,\lambda^{2-p}}(z_0)] &\leqslant \frac{c}{r^{nq}}\Big[\gamma(\lambda^{2-p}r^2)^{\frac{\gamma+n}{2}} + r^{\gamma+n}\Big]\Big[-\gamma(\lambda^{2-p}r^2)^{\frac{n(q-1)-\gamma}{2(q-1)}} + r^{\frac{nq-n-\gamma}{q-1}}\Big]^{q-1}\\
&\leqslant \frac{c}{r^{nq}}\Big(\gamma\lambda^{\frac{(2-p)(\gamma+n)}{2}}r^{\gamma+n} + r^{\gamma+n}\Big)\Big(-\gamma\lambda^{\frac{(2-p)(nq-n-\gamma)}{2(q-1)}}r^{\frac{nq-n-\gamma}{q-1}} + r^{\frac{nq-n-\gamma}{q-1}}\Big)^{q-1}\\
&\leqslant \frac{c}{r^{nq}}r^{\gamma+n}r^{nq-n-\gamma} = c.
\end{aligned}$$

2.3. *Comparison and Lipschitz regularity of homogeneous problems*

We consider the following homogeneous problem in simple parabolic cylinder $Q_2 = Q_2(0)$:

$$\begin{cases} h_t - \operatorname{div}\mathbf{a}(Dh) = 0 & \text{in } Q_2,\\ h = u & \text{on } \partial_{\mathrm{p}}Q_2, \end{cases} \tag{2.4}$$

where $u \in C^0(-2^2,2^2;L^2(B_2)) \cap L^p(-2^2,2^2;W^{1,p}(B_2))$ is a weak solution to (1.1) with replacing Ω_T by Q_2. For the existence and the uniqueness of the weak solution $h \in C^0(-2^2,2^2;L^2(B_2)) \cap L^p(-2^2,2^2;W^{1,p}(B_2))$ to the above equation, we refer to e.g., [34, Section III.4]. Then, we obtain the following regularity estimates for h and comparison estimate between u and h.

Lemma 2.1. *Let u be a weak solution to* (1.1) *in Q_2 with*

$$\fint_{Q_2}|Du|^p dz \leqslant 1 \qquad \text{and} \qquad \fint_{Q_2}|F|^p dz \leqslant \delta^p \tag{2.5}$$

for some $\delta \in (0,1)$, and let h be the weak solution to (2.4)*. Then*

$$\|Dh\|_{L^\infty(Q_1,\mathbb{R}^n)} \leqslant c\left(\fint_{Q_2}|Dh|^p\,dz + 1\right)^{\frac{d}{p}} \leqslant c_{\mathrm{Lip}} \tag{2.6}$$

for some $c, c_{\mathrm{Lip}} \geqslant 1$ depending on n,ν,L and p, where $d \geqslant 1$ is from (1.8)*.*

Moreover, for any $\varepsilon \in (0,1)$, there exists small $\delta = \delta(n,\nu,L,p,\varepsilon) \in (0,1)$ such that

$$\fint_{Q_2}|Du - Dh|^p dz \leqslant \varepsilon. \tag{2.7}$$

Proof. In view of [18, Section VIII.5], we have the first inequality in (2.6). We note that the Lipschitz regularity estimates in [18] are obtained for the parabolic p-Laplace systems. However, the same argument can apply to equations of p-Laplacian type such as (1.1) with the nonlinearity $\mathbf{a}$ satisfying (1.2) and (1.3).

Regarding (2.7) and the second inequality in (2.6), similar comparison estimates can be found in numerous papers, see e.g., [2, 5, 10]. But, we shall prove them in details for completeness.

We take $\zeta = u - h$ as a test function in (1.1) and (2.4) to obtain

$$\int_{Q_2} u_t(u-h)\,dz + \int_{Q_2} \mathbf{a}(Du)\cdot(Du-Dh)\,dz = \int_{Q_2} |F|^{p-2}F\cdot(Du-Dh)\,dz$$

and

$$\int_{Q_2} h_t(u-h)\,dz + \int_{Q_2} \mathbf{a}(Dh)\cdot(Du-Dh)\,dz = 0.$$

We notice that u and h are not differentiable for t. However, by considering their Steklov averages (see e.g., [18, Section I.3] and [6]), we may assume that they are differentiable for t. Then we have

$$\int_{Q_2} (u-h)_t(u-h)\,dz + \int_{Q_2} (\mathbf{a}(Du)-\mathbf{a}(Dh))\cdot(Du-Dh)\,dz = \int_{Q_2} |F|^{p-2}F\cdot(Du-Dh)\,dz.$$

Note that

$$\begin{aligned}\int_{Q_2} (u-h)_t(u-h)\,dz &= \int_{Q_2} \frac{1}{2}\frac{\partial}{\partial t}(u-h)^2\,dz\\ &= \frac{1}{2}\int_{B_r} (u-h)^2\Big|_{t=4}\,dx - \frac{1}{2}\int_{B_r} \underbrace{(u-h)^2\Big|_{t=-4}}_{\equiv 0}\,dx \geqslant 0.\end{aligned}$$

We remark that the condition (1.3) implies the monotonicity condition:

$$(\mathbf{a}(\xi)-\mathbf{a}(\eta))\cdot(\xi-\eta) \geqslant c(p,\nu)\left(|\xi|^2+|\eta|^2\right)^{\frac{p-2}{2}}|\xi-\eta|^2$$

for every $\xi,\eta \in \mathbb{R}^n \setminus \{0\}$. Then we see

$$(|Du|^2+|Dh|^2)^{\frac{p-2}{2}}|Du-Dh|^2 \leqslant c\,(\mathbf{a}(Du)-\mathbf{a}(Dh))\cdot(Du-Dh).$$

Therefore, by the above estimates, Young's inequality and the second inequality in (2.5), we have that for any $\kappa_1 \in (0,1)$,

$$\begin{aligned}\fint_{Q_2} (|Du|^2+|Dh|^2)^{\frac{p-2}{2}}|Du-Dh|^2\,dz &\leqslant c\kappa_1 \fint_{Q_2} |Du-Dh|^p\,dz + c\kappa_1^{-\frac{1}{p-1}} \fint_{Q_2} |F|^p\,dz\\ &\leqslant c\kappa_1 \fint_{Q_2} |Du-Dh|^p\,dz + c\kappa_1^{-\frac{1}{p-1}}\delta^p.\end{aligned}$$

If $p \geqslant 2$, since $|Du-Dh|^p \leqslant (|Du|^2+|Dh|^2)^{\frac{p-2}{2}}|Du-Dh|^2$, by taking sufficiently small $\kappa_1 = \kappa_1(n,\nu,L,p) > 0$ we have

$$\fint_{Q_2} |Du-Dh|^p\,dz \leqslant c\delta^p.$$

The second inequality in (2.6) follows from the first inequality in (2.5) together with $\delta \leqslant 1$. Moreover, by choosing small δ depending on ε, we get (2.7).

If $\frac{2n}{n+2} < p < 2$, on the other hand, applying Young's inequality, we have that for any $\kappa_2 \in (0,1)$,

$$\begin{aligned}
\fint_{Q_2} |Du - Dh|^p \, dz &= \fint_{Q_2} (|Du|^2 + |Dh|^2)^{\frac{p(2-p)}{4}} (|Du|^2 + |Dh|^2)^{\frac{p(p-2)}{4}} |Du - Dh|^p \, dz \\
&\leqslant c\kappa_2 \fint_{Q_2} (|Du|^2 + |Dh|^2)^{\frac{p}{2}} \, dz \\
&\quad + c\kappa_2^{-\frac{2-p}{p}} \fint_{Q_2} (|Du|^2 + |Dh|^2)^{\frac{p-2}{2}} |Du - Dh|^2 \, dz \\
&\leqslant c\kappa_2 \fint_{Q_2} \left[|Du|^p + |Dh|^p \right] dz \\
&\quad + c\kappa_2^{-\frac{2-p}{p}} \kappa_1 \fint_{Q_2} |Du - Dh|^p \, dz + c\kappa_2^{-\frac{2-p}{p}} \kappa_1^{-\frac{1}{p-1}} \delta^p.
\end{aligned}$$

Hence by choosing κ_1 sufficiently small depending on κ_2 we have

$$\fint_{Q_2} |Du - Dh|^p \, dz \leqslant c\kappa_2 \left(\fint_{Q_2} |Dh|^p \, dz + \fint_{Q_2} |Du|^p \, dz \right) + c(\kappa_2)\delta^p. \tag{2.8}$$

We first note that

$$\fint_{Q_2} |Dh|^p \, dz \leqslant c\kappa_2 \fint_{Q_2} |Dh|^p \, dz + c \fint_{Q_2} |Du|^p \, dz + +c(\kappa_2)\delta^p.$$

Then by choosing κ_2 sufficiently small and using the first inequality in (2.5) and $\delta \leqslant 1$ we have the second inequality in (2.6). Finally, applying the second inequalities of (2.5) and (2.6) to (2.8), we have

$$\fint_{Q_2} |Du - Dh|^p \, dz \leqslant c\kappa_2 + c(\kappa_2)\delta^p.$$

Finally, choosing κ_2 and δ sufficiently small depending on ε we get (2.7). □

3. Proof of Theorem 1.1

Now we start with the proof of the main theorem, Theorem 1.1. As we mentioned in the introduction, we follow the approach introduced in [2], see also [13] for the case of the weighted Lebesgue space. We divide the proof into five steps.

Step 1. (Setting and stopping time argument)

Let $\delta \in (0,1)$, which will be determined as a small constant depending only on n, ν, L, p, q and $[w]_q$ in below (3.17). Then there exists $R_0 > 0$ satisfying that

$$\int_{Q_{R_0}(z_0) \cap \Omega_T} \left[|Du|^p + \left| \frac{F}{\delta} \right|^p \right] dz \leqslant \frac{2|B_1|}{5^{n+2}} \quad \text{for all } z_0 \in \Omega_T. \tag{3.1}$$

We fix any $Q_{2r} = Q_{2r}(z_0) \Subset \Omega_T$ with $2r < R_0$. For simplicity, we write $Q_\rho = Q_\rho(z_0)$, $\rho \in (0, 2r]$. In addition, for $\rho > 0$ and $\lambda > 0$, we define the super level set

$$E(\rho, \lambda) := \{z \in Q_\rho : |Du(z)| > \lambda\},$$

and

$$\lambda_0^{\frac{p}{d}} := \fint_{Q_{2r}} \left[|Du|^p + \left| \frac{F}{\delta} \right|^p + 1 \right] dz \geqslant 1, \tag{3.2}$$

where $d \geqslant 1$ is from (1.8).

Let $r \leqslant r_1 < r_2 \leqslant 2r$ and consider any λ satisfying the following:

$$\lambda \geqslant B\lambda_0 \quad \text{with } B := \left(\frac{20r}{r_2 - r_1} \right)^{\frac{d(n+2)}{p}}. \tag{3.3}$$

We notice that $Q_\rho^\lambda(\tilde{z}) \subset Q_{r_2} \subset Q_{2r}$ for any $\tilde{z} = (\tilde{x}, \tilde{t}) \in E(r_1, \lambda)$ and all $\rho < \rho_0$ where

$$\rho_0 := \begin{cases} \lambda^{\frac{p-2}{2}}(r_2 - r_1) & \text{if } \frac{2n}{n+2} < p < 2, \\ r_2 - r_1 & \text{if } p \geqslant 2. \end{cases}$$

Then we obtain the following Vitali type covering result for the super-level set $E(r_1, \lambda)$.

Lemma 3.1. *For each $r \leqslant r_1 < r_2 \leqslant 2r$ and $\lambda \geqslant B\lambda_0$, there exist $z_i \in E(r_1, \lambda)$ and $\rho_i \in \left(0, \frac{\rho_0}{10}\right)$, $i = 1, 2, 3, \cdots$, such that the intrinsic parabolic cylinders $Q_{\rho_i}^\lambda(z_i)$ are mutually disjoint,*

$$E(r_1, \lambda) \setminus \mathcal{N} \subset \bigcup_{i=1}^{\infty} Q_{5\rho_i}^\lambda(z_i)$$

for some Lebesgue measure zero set $\mathcal{N}$,

$$\fint_{Q_{\rho_i}^\lambda(z_i)} \left[|Du|^p + \left| \frac{F}{\delta} \right|^p \right] dz = \lambda^p \tag{3.4}$$

and

$$\fint_{Q_\rho^\lambda(z_i)} \left[|Du|^p + \left| \frac{F}{\delta} \right|^p \right] dz < \lambda^p \quad \text{for all } \rho \in (\rho_i, r_2 - r_1]. \tag{3.5}$$

Proof. For $\tilde{z} \in E(r_1, \lambda)$ and $\rho \in \left[\frac{\rho_0}{10}, \rho_0\right)$, by (3.2) and (3.3), we derive

$$\begin{aligned} \fint_{Q_\rho^\lambda(\tilde{z})} \left[|Du|^p + \left| \frac{F}{\delta} \right|^p \right] dz &\leqslant \frac{|Q_{2r}|}{|Q_\rho^\lambda(\tilde{z})|} \fint_{Q_{2r}} \left[|Du|^p + \left| \frac{F}{\delta} \right|^p + 1 \right] dz \\ &= \frac{|Q_{2r}| \lambda_0^{\frac{p}{d}}}{|Q_\rho^\lambda(\tilde{z})|} \\ &\leqslant \lambda^p. \end{aligned}$$

To attain the last bound, we consider two cases $p < 2$ and $p \geqslant 2$. When $p \geqslant 2$, we see $\frac{p}{d} = 2$ and so

$$\frac{|Q_{2r}|\lambda_0^{\frac{p}{d}}}{|Q_\rho^\lambda(\tilde z)|} = \frac{(2r)^{n+2}\lambda_0^2}{\lambda^{2-p}\rho^{n+2}} \leqslant \left(\frac{20r}{r_2-r_1}\right)^{n+2}\lambda^{p-2}\lambda_0^2 \leqslant \left(\frac{20r}{r_2-r_1}\right)^{n+2}\lambda^p(B\lambda_0)^{-2}\lambda_0^2 = \lambda^p.$$

When $p < 2$, we see $\frac{p}{d} = \frac{(p-2)(n+2)}{2} + 2$ and $\rho \geqslant \frac{\lambda^{\frac{p-2}{2}}(r_2-r_1)}{10}$ and so

$$\begin{aligned}\frac{|Q_{2r}|\lambda_0^{\frac{p}{d}}}{|Q_\rho^\lambda(\tilde z)|} &= \frac{(2r)^{n+2}\lambda_0^{\frac{p}{d}}}{\lambda^{2-p}\rho^{n+2}}\\ &\leqslant \left(\frac{20r}{\lambda^{\frac{p-2}{2}}(r_2-r_1)}\right)^{n+2}\lambda^{p-2}\lambda_0^{\frac{p}{d}}\\ &= \left(\frac{20r}{r_2-r_1}\right)^{n+2}\left(\frac{\lambda_0}{\lambda}\right)^{\frac{p}{d}}\lambda^p\\ &\leqslant \left(\frac{20r}{r_2-r_1}\right)^{n+2}\left(\frac{\lambda_0}{B\lambda_0}\right)^{\frac{p}{d}}\lambda^p = \lambda^p.\end{aligned}$$

Moreover, from the parabolic Lebesgue differentiation theorem, we deduce that, for almost every $\tilde z \in E(r_1, \lambda)$,

$$\lim_{\rho\to 0^+} \fint_{Q_\rho^\lambda(\tilde z)} \left[|Du|^p + \left|\frac{F}{\delta}\right|^p\right] dz \geqslant |Dw(\tilde z)|^p > \lambda^p.$$

Since the map $\rho \mapsto \fint_{Q_\rho^\lambda(\tilde z)} \left[|Du|^p + \left|\frac{F}{\delta}\right|^p\right] dz$ is continuous, there exists $\rho_{\tilde z} \in \left(0, \frac{r_2-r_1}{10}\right)$ such that

$$\fint_{Q_{\rho_{\tilde z}}^\lambda(\tilde z)} \left[|Du|^p + \left|\frac{F}{\delta}\right|^p\right] dz = \lambda^p$$

and

$$\fint_{Q_\rho^\lambda(\tilde z)} \left[|Du|^p + \left|\frac{F}{\delta}\right|^p\right] dz < \lambda^p \quad \text{for all } \rho \in (\rho_{\tilde z}, r_2 - r_1].$$

Hence we apply Vitali's covering lemma for $\{Q_{\rho_{\tilde z}}^\lambda(\tilde z) : \tilde z \in E(r_1, \lambda)\}$ to complete the proof. $\square$

From now on, let us set for $i = 1, 2, 3, \ldots,$

$$Q_i^{(0)} := Q_{\rho_i}^\lambda(z_i) \quad \text{and} \quad Q_i^{(j)} := Q_{5^j\rho_i}^\lambda(z_i), \quad j = 1, 2.$$

Step 2. (Estimates of super-level sets)

With the result in Lemma 3.1, we first estimates the Lebesgue measure of super-level set

$$|\{z \in Q_i^{(1)} : |Du(z)| > A\lambda\}| \quad \text{with } \lambda \geqslant B\lambda_0,$$

where $A \geqslant 1$ will be determined below in (3.9), by using estimates in Lemma 2.1. Note from (3.5) that

$$\fint_{Q_i^{(2)}} \left[|Du|^p + \left|\frac{F}{\delta}\right|^p\right] dz < \lambda^p. \tag{3.6}$$

We consider the following rescaled functions:

$$\mathbf{a}_\lambda(\xi) := \frac{\mathbf{a}(\lambda\xi)}{\lambda^{p-1}} \quad \text{for } \xi \in \mathbb{R}^n,$$

$$u_{\lambda,i}(z) := \frac{u(Z_i)}{5\rho_i\lambda} \quad \text{and} \quad F_{\lambda,i}(z) := \frac{F(Z_i)}{\lambda} \quad \text{for } Z_i = z_i + \left(5\rho_i x, \lambda^{2-p}(5\rho_i)^2 t\right)$$

with $z = (x,t) \in Q_2$. Then it is obvious that $\mathbf{a}_\lambda(\xi)$ satisfies (1.2) and (1.3) with $\Omega_T = Q_2(0) = Q_2$. Then we see that $u_{\lambda,i}$ is a weak solution to

$$(u_{\lambda,i})_t - \operatorname{div} \mathbf{a}_\lambda(Du_{\lambda,i}) = -\operatorname{div}(|F_{\lambda,i}|^{p-2}F_{\lambda,i}) \text{ in } Q_2.$$

Moreover we have from (3.6) that

$$\fint_{Q_2} \left[|Du_{\lambda,i}|^p + \left|\frac{F_{\lambda,i}}{\delta}\right|^p\right] dz = \frac{1}{\lambda^p} \fint_{Q_i^{(2)}} \left[|Du|^p + \left|\frac{F}{\delta}\right|^p\right] dz < 1,$$

which implies

$$\fint_{Q_2} |Du_{\lambda,i}|^p \, dz \leqslant 1 \qquad \text{and} \qquad \fint_{Q_2} |F_{\lambda,i}|^p \, dz \leqslant \delta^p. \tag{3.7}$$

In addition, let $\widetilde{h}_{\lambda,i}$ be a weak solution to

$$(\widetilde{h}_{\lambda,i})_t - \operatorname{div} \mathbf{a}_\lambda(D\widetilde{h}_{\lambda,i}) = 0 \text{ in } Q_2, \qquad \text{and} \qquad \widetilde{h}_{\lambda,i} = u_{\lambda,i} \text{ on } \partial_p Q_2.$$

Now, we consider sufficiently small constant $\varepsilon > 0$ which will be determined below in (3.17). Then by applying Lemma 2.1, one can find $\delta = \delta(n, \nu, L, p, \varepsilon) > 0$ satisfying (3.7) such that

$$\fint_{Q_1} |Du_{\lambda,i} - D\widetilde{h}_{\lambda,i}|^p dz \leqslant \varepsilon \qquad \text{and} \qquad \|D\widetilde{h}_{\lambda,i}\|_{L^\infty(Q_1)} \leqslant c_{\mathrm{Lip}}.$$

Remark that both δ and c_{Lip} are independent of λ and i. Therefore setting

$$h_{\lambda,i}(z) = h_{\lambda,i}(x,t) := 5\rho_i\lambda\,\widetilde{h}_{\lambda,i}\left(\frac{x-y_i}{5\rho_i}, \frac{t-\tau_i}{\lambda^{2-p}(5\rho_i)^2}\right)$$

where $z_i = (y_i, \tau_i)$, we obtain

$$\fint_{Q_i^{(1)}} |Du - Dh_{\lambda,i}|^p dz \leqslant \varepsilon\lambda^p, \quad \text{and} \quad \|Dh_{\lambda,i}\|_{L^\infty(Q_i^{(1)})} \leqslant c_{\mathrm{Lip}}\lambda. \tag{3.8}$$

We set

$$A := 2c_{\mathrm{Lip}} > 1. \tag{3.9}$$

Then since

$$\begin{aligned} &\{z \in Q_i^{(1)} : |Du(z)| > A\lambda\} \\ &\quad \subset \left\{z \in Q_i^{(1)} : |Du(z) - Dh_{\lambda,i}(z)| > \frac{A\lambda}{2}\right\} \cup \left\{z \in Q_i^{(1)} : |Dh_{\lambda,i}(z)| > \frac{A\lambda}{2}\right\}, \end{aligned}$$

we have from the estimates in (3.8) that

$$\begin{aligned}
|\{z \in Q_i^{(1)} : |Du(z)| > A\lambda\}| &\leqslant |\{z \in Q_i^{(1)} : |Du(z) - Dh_{\lambda,i}(z)| > c_{\mathrm{Lip}}\lambda\}| \\
&\quad + \underbrace{|\{z \in Q_i^{(1)} : |Dh_{\lambda,i}(z)| > c_{\mathrm{Lip}}\lambda\}|}_{=0} \\
&\leqslant \frac{1}{\lambda^p} \int_{Q_i^{(1)}} |Du - Dh_{\lambda,i}|^p \, dz \leqslant \varepsilon |Q_i^{(1)}|
\end{aligned}$$

which implies

$$\frac{|\{z \in Q_i^{(1)} : |Du(z)| > A\lambda\}|}{|Q_i^{(1)}|} \leqslant \varepsilon. \tag{3.10}$$

Step 3. (weighted estimates of supper-level sets)

In this step, we estimate the weighted measure of super-level set

$$w(E(r_1, A\lambda)) \quad \text{with} \ \ \lambda \geqslant B\lambda_0.$$

We first observe from (3.1) and (3.4) that

$$\lambda^p \leqslant \frac{1}{|Q_i^{(0)}|} \int_{Q_{2r}} \left[|Du|^p + \left| \frac{F}{\delta} \right|^p \right] dz \leqslant \frac{2|B_1| 5^{-n-2}}{2|B_1| \rho_i^{n+2} \lambda^{2-p}},$$

hence

$$\lambda \leqslant (5\rho_i)^{-\frac{n+2}{2}}.$$

This and the fact $\lambda \geqslant 1$ from (3.3) imply $Q_i^{(1)} \in C_p$ for every $i = 1, 2, 3, \ldots$. Then we obtain from (2.3) and (3.10) that

$$\frac{w(\{z \in Q_i^{(1)} : |Du(z)| > A\lambda\})}{w(|Q_i^{(1)}|)} \leqslant c_2 \varepsilon^{\gamma_1}. \tag{3.11}$$

By Proposition 2.1, w is a p-intrinsic $A_{q'}$ for some $q' \in (1, q)$. Now we suppose that

$$\int_{Q_{2r}} |Du|^{pq'} w \, dz < \infty. \tag{3.12}$$

Then by (3.4) and (2.2) with q replaced by q' we have

$$\begin{aligned}
\lambda^{pq'} &\leqslant 2^{q'-1} \left(\fint_{Q_i^{(0)}} |Du|^p \, dz \right)^{q'} + 2^{q'-1} \left(\fint_{Q_i^{(0)}} \left| \frac{F}{\delta} \right|^p \, dz \right)^{q'} \\
&\leqslant \frac{2^{q'-1} [w]_{q'}}{w(Q_i^{(0)})} \left(\int_{Q_i^{(0)}} |Du|^{pq'} w \, dz + \int_{Q_i^{(0)}} \left| \frac{F}{\delta} \right|^{pq'} w \, dz \right) \\
&\leqslant \frac{2^{q'-1} [w]_{q'}}{w(Q_i^{(0)})} \Bigg(\int_{Q_i^{(0)} \cap \{|Du| > \frac{\lambda}{c_0}\}} |Du|^{pq'} w \, dz \\
&\qquad + \int_{Q_i^{(0)} \cap \{\frac{|F|}{\delta} > \frac{\lambda}{c_0}\}} \left| \frac{F}{\delta} \right|^{pq'} w \, dz + 2 c_0^{-pq'} \lambda^{pq'} w(Q_i^{(0)}) \Bigg),
\end{aligned}$$

where $c_0 := (2^{q'+1}[w]_{q'})^{\frac{1}{pq'}}$. Note that the right hand side is finite by the assumptions $F \in L^{pq}_{w,\mathrm{loc}}(\Omega_T)$ and (3.12). The above estimate means

$$w(Q_i^{(0)}) \leqslant \frac{2^{q'}[w]_{q'}}{\lambda^{pq'}} \left(\int_{Q_i^{(0)} \cap \{|Du| > \frac{\lambda}{c_0}\}} |Du|^{pq'} w \, dz + \int_{Q_i^{(0)} \cap \{\frac{|F|}{\delta} > \frac{\lambda}{c_0}\}} \left| \frac{F}{\delta} \right|^{pq'} w \, dz \right). \tag{3.13}$$

Therefore, using Lemma 3.1, (3.11), (2.3) and (3.13), we obtain

$$\begin{aligned}
w(E(r_1, A\lambda)) &= w(\{z \in Q_{r_1} : |Du(z)| > A\lambda\}) \\
&\leqslant \sum_{i=1}^{\infty} w(\{z \in Q_i^{(1)} : |Du(z)| > A\lambda\}) \\
&\leqslant c\varepsilon^{\gamma_1} \sum_{i=1}^{\infty} w(Q_i^{(1)}) \\
&\leqslant c\varepsilon^{\gamma_1} \sum_{i=1}^{\infty} \left(\frac{|Q_i^{(1)}|}{|Q_i^{(0)}|} \right)^q w(Q_i^{(0)}) \\
&\leqslant c\frac{\varepsilon^{\gamma_1}}{\lambda^{pq'}} \sum_{i=1}^{\infty} \left(\int_{Q_i^{(0)} \cap \{|Du| > \frac{\lambda}{c_0}\}} |Du|^{pq'} w \, dz + \int_{Q_i^{(0)} \cap \{\frac{|F|}{\delta} > \frac{\lambda}{c_0}\}} \left| \frac{F}{\delta} \right|^{pq'} w \, dz \right) \\
&\leqslant c\frac{\varepsilon^{\gamma_1}}{\lambda^{pq'}} \left(\int_{Q_{r_2} \cap \{|Du| > \frac{\lambda}{c_0}\}} |Du|^{pq'} w \, dz + \int_{Q_{r_2} \cap \{\frac{|F|}{\delta} > \frac{\lambda}{c_0}\}} \left| \frac{F}{\delta} \right|^{pq'} w \, dz \right).
\end{aligned} \tag{3.14}$$

Step 4. (A priori estimates)

We prove the estimate (1.7) under the additional assumption

$$\int_{Q_{2r}} |Du|^{pq} w \, dz < \infty, \tag{3.15}$$

where $2r < R_0$ and R_0 satisfies (3.1). Note that (3.15) implies (3.12).

Fix any $r \leqslant r_1 < r_2 \leqslant 2r$. Observe that

$$\begin{aligned}
\int_{Q_{r_1}} |Du|^{pq} w \, dz &= pqA^{pq} \int_0^{\infty} w\,(E(r_1, A\lambda))\, \lambda^{pq-1} d\lambda \\
&= pqA^{pq} \int_0^{B\lambda_0} w\,(E(r_1, A\lambda))\, \lambda^{pq-1} d\lambda + pqA^{pq} \underbrace{\int_{B\lambda_0}^{\infty} w\,(E(r_1, A\lambda))\, \lambda^{pq-1} d\lambda}_{=:I} \\
&= (AB\lambda_0)^{pq} w(Q_{2r}) + pqA^{pq} I,
\end{aligned} \tag{3.16}$$

where A and B are from (3.9) and (3.3). We estimate the second term I. Applying (3.14), we derive

$$\begin{aligned} I &\leqslant c\varepsilon^{\gamma_1}\int_{B\lambda_0}^{\infty}\left(\int_{Q_{r_2}\cap\{|Du|>\frac{\lambda}{c_0}\}}|Du|^{pq'}w\,dz+\int_{Q_{r_2}\cap\{\frac{|F|}{\delta}>\frac{\lambda}{c_0}\}}\left|\frac{F}{\delta}\right|^{pq'}w\,dz\right)\lambda^{pq-pq'-1}d\lambda \\ &\leqslant c\varepsilon^{\gamma_1}\int_{0}^{\infty}\left(\int_{Q_{r_2}\cap\{|Du|>\frac{\lambda}{c_0}\}}(c_0|Du|)^{pq'}w\,dz+\int_{Q_{r_2}\cap\{\frac{|F|}{\delta}>\frac{\lambda}{c_0}\}}\left|\frac{c_0F}{\delta}\right|^{pq'}w\,dz\right)\lambda^{pq-pq'-1}d\lambda \\ &\leqslant c\varepsilon^{\gamma_1}\left(\int_{Q_{r_2}}|Du|^{pq}w\,dz+\int_{Q_{r_2}}\left|\frac{F}{\delta}\right|^{pq}w\,dz\right). \end{aligned}$$

In the last inequality we apply the following elementary identity with $g=c_0|Du|$ or $\frac{c_0|F|}{\delta}$, $\beta_2=pq$, $\beta_1=pq'$ and $U=Q_{r_2}$:

$$\int_U g^{\beta_2}w\,dz=(\beta_2-\beta_1)\int_0^{\infty}\lambda^{\beta_2-\beta_1-1}\int_{\{z\in U:g(z)>\lambda\}}g^{\beta_1}w\,dz\,d\lambda,\quad \beta_2>\beta_1>1.$$

Inserting the estimate for I into (3.16) and recalling the definitions of A and B and the fact that $\varepsilon\in(0,1)$, we have

$$\int_{Q_{r_1}}|Du|^{pq}w\,dz\leqslant c_*\varepsilon^{\gamma_1}\int_{Q_{r_2}}|Du|^{pq}w\,dz+\frac{cw(Q_{2r})\lambda_0^{pq}r^{d(n+2)q}}{(r_2-r_1)^{d(n+2)q}}+c\int_{Q_{2r}}\left|\frac{F}{\delta}\right|^{pq}w\,dz,$$

where the constants c_*, γ_1 and c depend on n,p,ν,L,q and $[w]_q$. At this stage, we choose $\varepsilon=\varepsilon(n,p,\nu,L,q,[w]_q)$ such that

$$c_*\varepsilon^{\gamma_1}\leqslant\frac{1}{2},\tag{3.17}$$

hence δ is also determined as a small constant depending on n,p,ν,L,q and $[w]_q$. Therefore we obtain

$$\int_{Q_{r_1}}|Du|^{pq}w\,dz\leqslant\frac{1}{2}\int_{Q_{r_2}}|Du|^{pq}w\,dz+\frac{c\lambda_0^{pq}w(Q_{2r})r^{d(n+2)q}}{(r_2-r_1)^{d(n+2)q}}+c\int_{Q_{2r}}|F|^{pq}w\,dz$$

for every $r\leqslant r_1<r_2\leqslant 2r$. Finally, applying Lemma 3.2 below with $\Psi(\rho)=\int_{Q_\rho}|Du|^{pq}w\,dz$ with $R_1=r$ and $R_2=2r$ and recalling (3.2), we have that

$$\begin{aligned} \int_{Q_r}|Du|^{pq}w\,dz &\leqslant cw(Q_{2r})\lambda_0^{pq}+c\int_{Q_{2r}}|F|^{pq}w\,dz \\ &\leqslant cw(Q_{2r})\left(\fint_{Q_{2r}}\left[|Du|^p+|F|^p+1\right]\,dz\right)^{dq}+c\int_{Q_{2r}}|F|^{pq}w\,dz. \end{aligned}$$

This together with (2.3) implies (1.7).

Lemma 3.2 (Lemma 6.1 in [22])**.** *Let $\Psi:[R_1,R_2]\to[0,\infty)$ be a bounded function. Suppose that for any r_1 and r_2 with $0<R_1\leqslant r_1<r_2\leqslant R_2$,*

$$\Psi(r_1)\leqslant\vartheta\Psi(r_2)+\frac{C}{(r_2-r_1)^{\kappa}}+D$$

where $C>0$ and $D\geqslant 0$, $\kappa>0$ and $\vartheta\in[0,1)$. Then there exists $c=(\vartheta,\kappa)>0$ such that

$$\Psi(R_1)\leqslant c(\vartheta,\kappa)\left[\frac{A}{(R_2-R_1)^{\kappa}}+B\right].$$

Step 5. (Approximation)

Finally, we remove the a priori assumption (3.15) by a standard approximation argument. Suppose w is a p-intrinsic A_q weight and $F \in L^{pq}_{w,\mathrm{loc}}(\Omega_T, \mathbb{R}^n)$. Fix any $Q_{2r} = Q_{2r}(z_0) \Subset \Omega_T$ with $2r < R_0$. Then there exists $Q_R = Q_R(z_0)$ such that $Q_{2r} \Subset Q_R \Subset \Omega_T$. Note that by Proposition 2.1 (4), w is a p-intrinsic A_{pq} weight hence a usual parabolic A_{pq} weight. Therefore, $C^\infty_{\mathrm{c}}(\mathbb{R}^{n+1})$ is dense in $L^{pq}_w(\mathbb{R}^{n+1})$, see e.g., [30, Lemma 2.1]. Therefore there exist $F_k \in C^\infty_{\mathrm{c}}(\mathbb{R}^{n+1}, \mathbb{R}^n)$, $k = 1, 2, 3, \ldots$, such that

$$F_k \longrightarrow F \quad \text{in } L^{pq}_w(Q_R, \mathbb{R}^n) \quad \text{as } k \to \infty,$$

hence by (2.2),

$$F_k \longrightarrow F \quad \text{in } L^p(Q_R, \mathbb{R}^n) \quad \text{as } k \to \infty. \tag{3.18}$$

We further assume that

$$\int_{Q_R} |F_k|^p \, dz \leqslant 2 \int_{Q_R} |F|^p \, dz. \tag{3.19}$$

Let $u_k \in C^0(t_0 - R^2, t_0 + R^2; L^2(B_R(x_0)) \cap L^p(t_0 - R^2, t_0 + R^2; W^{1,p}(B_R(x_0))$ be the unique weak solution to

$$\begin{cases} (u_k)_t - \operatorname{div} \mathbf{a}(Du_k) &= \operatorname{div}(|F_k|^{p-2} F_k) \quad \text{in } Q_R, \\ u_k &= u \quad \text{on } \partial_{\mathrm{p}} Q_R, \end{cases} \tag{3.20}$$

see e.g., [34, Section III.4] for the existence of such u_k. In view of [2], we have at least $|Du_k|^p \in L^\gamma_{\mathrm{loc}}(Q_R)$ for every $\gamma > 1$ since $|F_k|^p \in L^\gamma(Q_R)$. In particular, by Hölder's inequality with Proposition 2.1 (2),

$$\int_{Q_{2r}} |Du_k|^{pq} w \, dz \leqslant \left(\int_{Q_{2r}} |Du_k|^{\frac{pq(1+\gamma)}{\gamma}} \, dz \right)^{\frac{\gamma}{1+\gamma}} \left(\int_{Q_{2r}} w^{1+\gamma} \, dz \right)^{\frac{1}{1+\gamma}} < \infty,$$

which implies the a priori assumption in (3.15) for u_k hence it follows from the previous results in *Step 4* that

$$\begin{aligned} &\left(\frac{1}{w(Q_r)} \int_{Q_r} |Du_k|^{pq} w \, dz \right)^{\frac{1}{q}} \\ &\qquad \leqslant c \left(\fint_{Q_{2r}} [|Du_k|^p + |F_k|^p + 1] \, dz \right)^d + c \left(\frac{1}{w(Q_{2r})} \int_{Q_{2r}} |F_k|^{pq} w \, dz \right)^{\frac{1}{q}}. \end{aligned} \tag{3.21}$$

Now we take $u - u_k$ as a test function in the weak forms of (1.1) and (3.20) to get

$$\begin{aligned} \int_{Q_R} (u - u_k)_t (u - u_k) \, dz + &\int_{Q_R} (\mathbf{a}(Du) - \mathbf{a}(Du_k)) \cdot (Du - Du_k) \, dz \\ &= \int_{Q_R} (|F|^{p-2} F - |F_k|^{p-2} F_k) \cdot (Du - Du_k) \, dz. \end{aligned}$$

Then, in a similar way as in the proof of Lemma 2.1, we derive

$$\begin{aligned} &\int_{Q_R} (|Du_k|^2 + |Du|^2)^{\frac{p-2}{2}} |Du_k - Du|^2 \, dz \\ &\qquad \leqslant c \int_{Q_R} ||F_k|^{p-2} F_k - |F|^{p-2} F||Du_k - Du| \, dz \\ &\qquad \leqslant c \tau_1^{-\frac{1}{p-1}} \int_{Q_R} ||F_k|^{p-2} F_k - |F|^{p-2} F|^{\frac{p}{p-1}} \, dz + \tau_1 \int_{Q_R} |Du_k|^p + |Du|^p \, dz \end{aligned} \tag{3.22}$$

for any $\tau_1 \in (0,1)$, by applying Young's inequality.

If $p \geqslant 2$, since $|Du_k - Du|^p \leqslant (|Du_k|^2 + |Du|^2)^{\frac{p-2}{2}}|Du_k - Du|^2$, we infer

$$\int_{Q_R} |Du_k|^p \, dz \leqslant c \int_{Q_R} |F|^p + |Du|^p \, dz$$

by taking sufficiently small $\tau_1 > 0$ and (3.19). If $\frac{2n}{n+2} < p < 2$, applying Young's inequality, we have that for any $\tau_2 \in (0,1)$,

$$\begin{aligned}\int_{Q_R} |Du_k - Du|^p \, dz \leqslant \tau_2 \int_{Q_R} & (|Du_k|^2 + |Du|^2)^{\frac{p}{2}} \, dz \\ & + c\tau_2^{-\frac{2-p}{p}} \int_{Q_R} (|Du_k|^2 + |Du|^2)^{\frac{p-2}{2}} |Du_k - Du|^2 \, dz \\ & \leqslant c\big(\tau_2 + \tau_1 \tau_2^{-\frac{2-p}{p}}\big) \int_{Q_R} \big[|Du_k|^p + |Du|^p\big] \, dz \\ & + c\tau_1^{-\frac{1}{p-1}} \tau_2^{-\frac{2-p}{p}} \int_{Q_R} \big||F_k|^{p-2}F_k - |F|^{p-2}F\big|^{\frac{p}{p-1}} \, dz\end{aligned}$$

and then by taking sufficiently small $\tau_1, \tau_2 > 0$ and (3.19),

$$\int_{Q_R} |Du_k|^p \, dz \leqslant c \int_{Q_R} |F|^p + |Du|^p \, dz.$$

Eventually, for any $p > \frac{2n}{n+2}$, we obtain that

$$\int_{Q_R} |Du_k|^p \, dz \leqslant c \int_{Q_R} |F|^p + |Du|^p \, dz < \infty \quad \text{for all } \ k = 1, 2, 3, \ldots. \tag{3.23}$$

Moreover, from (3.18) we see

$$\int_{Q_R} \big||F_k|^{p-2}F_k - |F|^{p-2}F\big|^{\frac{p}{p-1}} \, dz \to 0 \quad \text{ as } k \to \infty.$$

Then taking into account (3.22) with (3.23),

$$\limsup_{k\to\infty} \int_{Q_R} (|Du_k|^2 + |Du|^2)^{\frac{p-2}{2}} |Du_k - Du|^2 \, dz \leqslant c\tau_1 \int_{Q_R} |F|^p + |Du|^p \, dz.$$

Since $\tau_1 \in (0,1)$ is arbitrary, we have that

$$\int_{Q_R} (|Du_k|^2 + |Du|^2)^{\frac{p-2}{2}} |Du_k - Du|^2 \, dz \to 0 \quad \text{ as } \ k \to \infty.$$

Now, if $\frac{2n}{n+2} < p < 2$, the Hölder inequality yields

$$\begin{aligned}\int_{Q_R} |Du_k - Du|^p \, dz &= \int_{Q_R} (|Du_k|^2 + |Du|^2)^{\frac{p(2-p)}{4}} (|Du_k|^2 + |Du|^2)^{\frac{p(p-2)}{4}} |Du_k - Du|^p \, dz \\ &\leqslant \Big(\int_{Q_R} (|Du_k|^2 + |Du|^2)^{\frac{p}{2}} \, dz\Big)^{\frac{2-p}{2}} \Big(\int_{Q_R} (|Du_k|^2 + |Du|^2)^{\frac{p-2}{2}} |Du_k - Du|^2 \, dz\Big)^{\frac{p}{2}}\end{aligned}$$

and therefore by virtue of (3.23), we obtain

$$\int_{Q_R} |Du_k - Du|^p \, dz \to 0 \quad \text{as } k \to \infty.$$

This also holds in case $p \geqslant 2$ from (3.22), because

$$\int_{Q_R} |Du_k - Du|^p \, dz \leqslant \int_{Q_R} (|Du_k|^2 + |Du|^2)^{\frac{p-2}{2}} |Du_k - Du|^2 \, dz.$$

In turn, we obtain that for every $p > \frac{2n}{n+2}$

$$Du_k \longrightarrow Du \quad \text{in } L^p(Q_R, \mathbb{R}^n) \subset L^p(Q_{2r}, \mathbb{R}^n) \quad \text{as } k \to \infty.$$

In particular, we also have that $Du_k \longrightarrow Du$ a.e. in $L^p(Q_{2r}, \mathbb{R}^n)$ as $k \to \infty$, up to subsequence.

Finally by passing $k \to \infty$ from (3.21) and applying the above convergence results for F_k and Du_k with Fatou's lemma, we obtain

$$\begin{aligned}
\left(\frac{1}{w(Q_r)} \int_{Q_r} |Du|^{pq} w \, dz \right)^{\frac{1}{q}} &\leqslant \liminf_{k\to\infty} \left(\frac{1}{w(Q_r)} \int_{Q_r} |Du_k|^{pq} w \, dz \right)^{\frac{1}{q}} \\
&\leqslant \liminf_{k\to\infty} \left[c \left(\fint_{Q_{2r}} [|Du_k|^p + |F_k|^p + 1] \, dz \right)^d + c \left(\frac{1}{w(Q_{2r})} \int_{Q_{2r}} |F_k|^{pq} w \, dz \right)^{\frac{1}{q}} \right] \\
&= c \left(\fint_{Q_{2r}} [|Du|^p + |F|^p + 1] \, dz \right)^d + c \left(\frac{1}{w(Q_{2r})} \int_{Q_{2r}} |F|^{pq} w \, dz \right)^{\frac{1}{q}}.
\end{aligned}$$

Therefore, we complete the proof. □

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Conflict of interest

The authors declare no conflict of interest.

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