http://www.aimspress.com/journal/mine

Mathematics in Engineering, 5(3): 1–7.
DOI:10.3934/mine.2023061
Received: 26 October 2022
Revised: 16 November 2022
Accepted: 16 November 2022
Published: 23 November 2022

Research article

A symmetry theorem in two-phase heat conductors[†]

Hyeonbae Kang[1] **and Shigeru Sakaguchi**[2,*]

[1] Department of Mathematics and Institute of Applied Mathematics, Inha University, Incheon 22212, S. Korea

[2] Graduate School of Information Sciences, Tohoku University, Sendai, 980-8579, Japan

[†] **This contribution is part of the Special Issue:** When analysis meets geometry – on the 50th birthday of Serrin's problem
Guest Editors: Giorgio Poggesi; Lorenzo Cavallina
Link: `www.aimspress.com/mine/article/5924/special-articles`

* **Correspondence:** Email: sigersak@tohoku.ac.jp.

Abstract: We consider the Cauchy problem for the heat diffusion equation in the whole Euclidean space consisting of two media with different constant conductivities, where initially one medium has temperature 0 and the other has temperature 1. Under the assumptions that one medium is bounded and the interface is of class $C^{2,\alpha}$, we show that if the interface is stationary isothermic, then it must be a sphere. The method of moving planes due to Serrin is directly utilized to prove the result.

Keywords: heat diffusion equation; two-phase heat conductors; Cauchy problem; stationary isothermic surface; method of moving planes; transmission conditions

1. Introduction

In the previous paper [7], we considered the Cauchy problem for the heat diffusion equation in the whole Euclidean space consisting of two media with different constant conductivities, where initially one medium has temperature 0 and the other has temperature 1. There, the large time behavior, either stabilization to a constant or oscillation, of temperature was studied. The present paper deals with the case where one medium is bounded and the interface is of class $C^{2,\alpha}$, and introduces an overdetermined problem with the condition that the interface is stationary isothermic.

To be precise, let Ω consist of a finite number, say m, of bounded domains $\{\Omega_j\}$ in $\mathbb{R}^N$ with $N \geq 2$, where each $\partial\Omega_j$ is of class $C^{2,\alpha}$ for some $0 < \alpha < 1$ and $\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset$ if $i \neq j$. Denote by $\sigma = \sigma(x)$ ($x \in$

$\mathbb{R}^N$) the conductivity distribution of the whole medium given by

$$\sigma = \begin{cases} \sigma_+ & \text{in } \Omega = \bigcup_{j=1}^{m} \Omega_j, \\ \sigma_- & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where σ_-, σ_+ are positive constants with $\sigma_- \neq \sigma_+$. The diffusion over such multiphase heat conductors has been dealt with also in [3, 4, 9–11].

We consider the unique bounded solution $u = u(x, t)$ of the Cauchy problem for the heat diffusion equation:

$$u_t = \operatorname{div}(\sigma \nabla u) \quad \text{in } \mathbb{R}^N \times (0, +\infty) \text{ and } u = \mathcal{X}_\Omega \text{ on } \mathbb{R}^N \times \{0\}, \tag{1.2}$$

where $\mathcal{X}_\Omega$ denotes the characteristic function of the set Ω. The maximum principle gives

$$0 < u(x, t) < 1 \text{ for every } (x, t) \in \mathbb{R}^N \times (0, +\infty). \tag{1.3}$$

Our symmetry theorem is stated as follows.

Theorem 1.1. *If there exists a function* $a : (0, +\infty) \to (0, +\infty)$ *satisfying*

$$u(x, t) = a(t) \text{ for every } (x, t) \in \partial\Omega \times (0, +\infty), \tag{1.4}$$

then Ω *must be a ball.*

If $\partial\Omega$ is of class C^6, then Theorem 1.1 can be proved by the method employed in [3, Theorem 1.5 with the proof, pp. 335–341], where concentric balls are characterized. The proof there consists of four steps summarized as follows: (i) reduction of (1.2) to elliptic problems by the Laplace-Stieltjes transform $\lambda \int_0^\infty e^{-\lambda t} u(x, t)dt$ for all sufficiently large $\lambda > 0$, (ii) construction of precise barriers based on the formal WKB approximation where the fourth derivatives of the distance function to $\partial\Omega$ together with the assumption (1.4) are used, (iii) showing that the mean curvature of $\partial\Omega$ is constant with the aid of the precise asymptotics as $\lambda \to \infty$ and the transmission conditions on the interface $\partial\Omega$, (iv) Alexandrov's soap bubble theorem [1] from which we conclude that $\partial\Omega$ must be a sphere.

The approach of the present paper is different from that in [3] and only requires $\partial\Omega$ to be of class $C^{2,\alpha}$ for some $\alpha > 0$. Here the proof consists of two ingredients: (i) reduction to elliptic problems by the Laplace-Stieltjes transform $\lambda \int_0^\infty e^{-\lambda t} u(x, t)dt$ for some λ, for instance $\lambda = 1$, (ii) the method of moving planes due to Serrin [6, 8, 12, 13] with the aid of the transmission conditions on $\partial\Omega$. To apply the method of moving planes, the solutions need to be of class C^2 up to the interface $\partial\Omega$ from each side, which is guaranteed if $\partial\Omega$ is of class $C^{2,\alpha}$.

2. Introducing a Laplace-Stieltjes transform

Let $u = u(x, t)$ be the unique bounded solution of (1.2) satisfying (1.4). We use the Gaussian bounds for the fundamental solutions of diffusion equations due to Aronson [2, Theorem 1, p. 891] (see also [5, p. 328]). Let $g = g(x, \xi, t)$ be the fundamental solution of $u_t = \operatorname{div}(\sigma \nabla u)$. Then there exist two positive constants $\lambda < \Lambda$ such that

$$\lambda t^{-\frac{N}{2}} e^{-\frac{|x-\xi|^2}{\lambda t}} \leq g(x, \xi, t) \leq \Lambda t^{-\frac{N}{2}} e^{-\frac{|x-\xi|^2}{\Lambda t}} \tag{2.1}$$

for all $(x, t), (\xi, t) \in \mathbb{R}^N \times (0, +\infty)$. Note that u is represented as

$$u(x, t) = \int_\Omega g(x, \xi, t) d\xi \quad \text{for } (x, t) \in \mathbb{R}^N \times (0, +\infty). \tag{2.2}$$

Define the function $v = v(x)$ by

$$v(x) = \int_0^\infty e^{-t} u(x, t) dt \quad \text{for } x \in \mathbb{R}^N. \tag{2.3}$$

With the function a in (1.4), we set $a^* = \int_0^\infty e^{-t} a(t) dt$. Then, (1.3) yields that $0 < a^* < 1$. Set

$$v^+ = v \quad \text{for } x \in \overline{\Omega} \quad \text{and} \quad v^- = v \quad \text{for } x \in \mathbb{R}^N \setminus \Omega. \tag{2.4}$$

Then we observe that

$$a^* < v^+ < 1 \quad \text{and} \quad -\sigma_+ \Delta v^+ + v^+ = 1 \quad \text{in } \Omega, \tag{2.5}$$

$$0 < v^- < a^* \quad \text{and} \quad -\sigma_- \Delta v^- + v^- = 0 \quad \text{in } \mathbb{R}^N \setminus \overline{\Omega}, \tag{2.6}$$

$$v^+ = v^- = a^* \quad \text{and} \quad \sigma_+ \frac{\partial v^+}{\partial n} = \sigma_- \frac{\partial v^-}{\partial n} \quad \text{on } \partial\Omega, \tag{2.7}$$

$$\lim_{|x| \to \infty} v^-(x) = 0. \tag{2.8}$$

Here, n denotes the outward unit normal vector to $\partial\Omega$, the inequalities in (2.5) and (2.6) follow from the maximum principle, (2.7) expresses the transmission conditions on the interface $\partial\Omega$, and (2.8) follows from (2.1) and (2.2).

3. Proof of Theorem 1.1

Let us apply directly the method of moving planes due to Serrin [6, 8, 12, 13] to our problem in order to show that Ω must be a ball. The point is to apply the method to both the interior Ω and the exterior $\mathbb{R}^N \setminus \overline{\Omega}$ at the same time. For the method of moving planes for $\mathbb{R}^N \setminus \overline{\Omega}$, we refer to [8, 12]. In this procedure, the supposition that Ω is not symmetric will lead us to the contradiction that the transmission conditions (2.7) do not hold.

Let γ be a unit vector in $\mathbb{R}^N$, $\lambda \in \mathbb{R}$, and let π_λ be the hyperplane $x \cdot \gamma = \lambda$. For large λ, π_λ is disjoint from $\overline{\Omega}$; as λ decreases, π_λ intersects $\overline{\Omega}$ and cuts off from Ω an open cap $\Omega_\lambda = \Omega \cap \{x \in \mathbb{R}^N : x \cdot \gamma > \lambda\}$.

Denote by Ω^λ the reflection of Ω_λ with respect to the plane π_λ. Then, Ω^λ is contained in Ω at the beginning, and remains in Ω until one of the following events occurs:

(i) Ω^λ becomes internally tangent to $\partial\Omega$ at some point $p \in \partial\Omega \setminus \pi_\lambda$;

(ii) π_λ reaches a position where it is orthogonal to $\partial\Omega$ at some point $q \in \partial\Omega \cap \pi_\lambda$ and the direction γ is not tangential to $\partial\Omega$ at every point on $\partial\Omega \cap \{x \in \mathbb{R}^N : x \cdot \gamma > \lambda\}$.

Let λ_* denote the value of λ at which either (i) or (ii) occurs. We claim that Ω is symmetric with respect to π_{λ_*}.

Suppose that Ω is not symmetric with respect to π_{λ_*}. Denote by D the reflection of $\left(\mathbb{R}^N \setminus \overline{\Omega}\right) \cap \{x \in \mathbb{R}^N : x \cdot \gamma > \lambda_*\}$ with respect to π_{λ_*}. Let Σ be the connected component of $\left(\mathbb{R}^N \setminus \overline{\Omega}\right) \cap \{x \in \mathbb{R}^N : x \cdot \gamma < \lambda_*\}$

whose boundary contains the points p and q in the respective cases (i) and (ii). Since $\Omega^{\lambda_*} \subset \Omega$, we notice that

$$\Sigma \subset \left(\mathbb{R}^N \setminus \overline{\Omega}\right) \cap \{x \in \mathbb{R}^N : x \cdot \gamma < \lambda_*\} \subset D.$$

Let x^{λ_*} denote the reflection of a point $x \in \mathbb{R}^N$ with respect to π_{λ_*}, namely,

$$x^{\lambda_*} = x + 2[\lambda_* - (x \cdot \gamma)]\gamma. \tag{3.1}$$

Using the functions $v^{\pm}$ defined in (2.4), we introduce the functions $w^{\pm} = w^{\pm}(x)$ by

$$\begin{aligned} w^+(x) &:= v^+(x) - v^+(x^{\lambda_*}) \quad \text{for } x \in \overline{\Omega^{\lambda_*}}, \\ w^-(x) &:= v^-(x) - v^-(x^{\lambda_*}) \quad \text{for } x \in \overline{\Sigma}. \end{aligned} \tag{3.2}$$

It then follows from (2.5)–(2.8) that

$$-\sigma_+ \Delta w^+ + w^+ = 0 \ \text{ in } \Omega^{\lambda_*} \quad \text{and} \ \ w^+ \geq 0 \ \text{ on } \partial\Omega^{\lambda_*}, \tag{3.3}$$

$$-\sigma_- \Delta w^- + w^- = 0 \ \text{ in } \Sigma \quad \text{and} \ \ w^- \geq 0 \ \text{ on } \partial\Sigma, \tag{3.4}$$

and hence by the maximum principle

$$w^+ \geq 0 \ \text{ in } \Omega^{\lambda_*} \quad \text{and} \quad w^- > 0 \ \text{ in } \Sigma. \tag{3.5}$$

Note that w^+ can be zero in Ω^{λ_*} since some connected component Ω_j of Ω can be symmetric with respect to π_{λ_*} and, in such a case, $w^+ \equiv 0$ in Ω_j. But w^- is strictly positive in Σ since Ω is not symmetric with respect to π_{λ_*}.

Let us first consider the case (i). The first equality in (2.7) yields that $w^+(p) = w^-(p) = 0$. Then, it follows from (3.5) and Hopf's boundary point lemma that

$$\frac{\partial w^+}{\partial n}(p) \leq 0 < \frac{\partial w^-}{\partial n}(p), \tag{3.6}$$

where we used the fact that n is the outward unit normal vector to $\partial\Omega$ as well as the inward unit normal vector to $\partial\Sigma$. It thus follows from the definition (3.2) of $w^{\pm}$ that

$$\left.\frac{\partial v^+(x)}{\partial n}\right|_{x=p} \leq \left.\frac{\partial (v^+(x^{\lambda_*}))}{\partial n}\right|_{x=p} \quad \text{and} \quad \left.\frac{\partial v^-(x)}{\partial n}\right|_{x=p} > \left.\frac{\partial (v^-(x^{\lambda_*}))}{\partial n}\right|_{x=p}.$$

Reflection symmetry with respect to the plane π_{λ_*} yields that

$$\left.\frac{\partial (v^{\pm}(x^{\lambda_*}))}{\partial n}\right|_{x=p} = \frac{\partial v^{\pm}}{\partial n}(p^{\lambda_*}). \tag{3.7}$$

Indeed, we observe that

$$n(p) \cdot \gamma = -n(p^{\lambda_*}) \cdot \gamma \ \text{ and } \ n(p) - (n(p) \cdot \gamma)\gamma = n(p^{\lambda_*}) - (n(p^{\lambda_*}) \cdot \gamma)\gamma,$$

and by using (3.1), we see that

$$\nabla(v^{\pm}(x^{\lambda_*})) = (\nabla v^{\pm})(x^{\lambda_*}) - 2\left((\nabla v^{\pm})(x^{\lambda_*}) \cdot \gamma\right)\gamma.$$

Then, combing these equalities yields (3.7). It thus follows that

$$\frac{\partial v^+}{\partial n}(p) \le \frac{\partial v^+}{\partial n}(p^{\lambda_*}) \quad \text{and} \quad \frac{\partial v^-}{\partial n}(p) > \frac{\partial v^-}{\partial n}(p^{\lambda_*}). \tag{3.8}$$

On the other hand, the second equality in (2.7) shows that

$$\sigma_+ \frac{\partial v^+}{\partial n}(p) = \sigma_- \frac{\partial v^-}{\partial n}(p) \quad \text{and} \quad \sigma_+ \frac{\partial v^+}{\partial n}(p^{\lambda_*}) = \sigma_- \frac{\partial v^-}{\partial n}(p^{\lambda_*}),$$

which contradict (3.8).

Let us proceed to the case (ii). As in [13], by a translation and a rotation of coordinates, we may assume:

$$\gamma = (1, 0, \dots, 0), \ \ q = 0, \ \ \lambda_* = 0 \ \text{ and } \ n(q) = (0, \dots, 0, 1).$$

Since $\partial\Omega$ is of class C^2, there exists a C^2 function $\varphi : \mathbb{R}^{N-1} \to \mathbb{R}$ such that in a neighborhood of $q = 0$, $\partial\Omega$ is represented as a graph $x_N = \varphi(\hat{x})$ where $\hat{x} = (x_1, \dots, x_{N-1}) \in \mathbb{R}^{N-1}$, where

$$\varphi(0) = 0, \quad \nabla\varphi(0) = 0, \quad \text{and} \quad n = \frac{1}{\sqrt{1 + |\nabla\varphi|^2}}(-\nabla\varphi, 1).$$

Since the event (ii) occurs at $\lambda = 0$, we observe that the function $\frac{\partial\varphi}{\partial x_1}(0, x_2, \dots, x_{N-1})$ achieves its local maximum 0 at $(x_2, \dots, x_{N-1}) = 0 \in \mathbb{R}^{N-2}$, and hence

$$\frac{\partial^2\varphi}{\partial x_1 \partial x_j}(0) = 0 \ \text{ for } j = 2, \dots, N-1. \tag{3.9}$$

Notice that

$$w^\pm(x) = v^\pm(x_1, x_2, \dots, x_N) - v^\pm(-x_1, x_2, \dots, x_N), \tag{3.10}$$

since $x^{\lambda_*} = (-x_1, x_2, \dots, x_N)$.

The equalities (2.7) at $(\hat{x}, \varphi(\hat{x}))$ in a neighborhood of $q = 0$ are read as

$$v^\pm = a^*, \tag{3.11}$$

$$\sigma_+ \left(-\sum_{k=1}^{N-1} \frac{\partial\varphi}{\partial x_k} \frac{\partial v^+}{\partial x_k} + \frac{\partial v^+}{\partial x_N} \right) = \sigma_- \left(-\sum_{k=1}^{N-1} \frac{\partial\varphi}{\partial x_k} \frac{\partial v^-}{\partial x_k} + \frac{\partial v^-}{\partial x_N} \right). \tag{3.12}$$

Differentiating (3.11) in x_i for $i = 1, \dots, N-1$ yields that at $(\hat{x}, \varphi(\hat{x}))$

$$\frac{\partial v^\pm}{\partial x_i} + \frac{\partial v^\pm}{\partial x_N} \frac{\partial\varphi}{\partial x_i} = 0. \tag{3.13}$$

Then, differentiating (3.13) in x_j for $j = 1, \dots, N-1$ yields that at $(\hat{x}, \varphi(\hat{x}))$

$$\frac{\partial^2 v^\pm}{\partial x_j \partial x_i} + \frac{\partial^2 v^\pm}{\partial x_N \partial x_i} \frac{\partial\varphi}{\partial x_j} + \frac{\partial^2 v^\pm}{\partial x_j \partial x_N} \frac{\partial\varphi}{\partial x_i} + \frac{\partial^2 v^\pm}{\partial x_N^2} \frac{\partial\varphi}{\partial x_i} \frac{\partial\varphi}{\partial x_j} + \frac{\partial v^\pm}{\partial x_N} \frac{\partial^2\varphi}{\partial x_j \partial x_i} = 0. \tag{3.14}$$

By letting $\hat{x} = 0$ in these equalities, we obtain from (3.9) that

$$\frac{\partial v^\pm}{\partial x_i}(0) = \frac{\partial^2 v^\pm}{\partial x_1 \partial x_j}(0) = 0 \ \text{ for } i = 1, \dots, N-1 \text{ and } j = 2, \dots, N-1. \tag{3.15}$$

Next, differentiating (3.12) in x_i for $i = 1, \ldots, N-1$ and letting $\hat{x} = 0$ give

$$\sigma_+ \frac{\partial^2 v^+}{\partial x_i \partial x_N}(0) = \sigma_- \frac{\partial^2 v^-}{\partial x_i \partial x_N}(0) \ \text{ for } i = 1, \ldots, N-1. \tag{3.16}$$

Since the functions $w^\pm$ are expressed as (3.10), with the aid of (3.15) we have that

$$w^\pm(0) = \frac{\partial w^\pm}{\partial x_j}(0) = \frac{\partial^2 w^\pm}{\partial x_1 \partial x_j}(0) = 0 \ \text{ for } j = 1, \ldots, N-1. \tag{3.17}$$

The relations (3.3)–(3.5) enable us to apply Serrin's corner point lemma (see [6, Lemma S, p. 214] or [8, Serrin's Corner Lemma, p. 393]) to show that

$$\frac{\partial^2 w^+}{\partial s_+^2}(0) \geq 0 \ \text{ and } \ \frac{\partial^2 w^-}{\partial s_-^2}(0) > 0 \ \text{ with } s_\pm = -\gamma \mp n = (-1, 0, \ldots, 0, \mp 1), \tag{3.18}$$

where $\frac{\partial^2 w^\pm}{\partial s_\pm^2}$ denotes the second derivative of $w^\pm$ in the direction of $s_\pm$. Note that each of the directions $s_\pm$ respectively enters $\Omega^{\lambda_*}, \Sigma$, transversally to both of the hypersurfaces $\partial\Omega$ and π_{λ_*}. Thus, we have from (3.10) and (3.17) that

$$\frac{\partial^2 w^\pm}{\partial s_\pm^2}(0) = \pm 2 \frac{\partial^2 w^\pm}{\partial x_1 \partial x_N}(0) = \pm 4 \frac{\partial^2 v^\pm}{\partial x_1 \partial x_N}(0). \tag{3.19}$$

It then follows from (3.18) that

$$\frac{\partial^2 v^-}{\partial x_1 \partial x_N}(0) < 0 \leq \frac{\partial^2 v^+}{\partial x_1 \partial x_N}(0), \tag{3.20}$$

which contradicts (3.16) with $i = 1$. Thus Ω is symmetric with respect to π_{λ_*}. Since the unit vector γ is arbitrary, Ω must be a ball and Theorem 1.1 is proved.

Acknowledgments

This research was partially supported by the Grants-in-Aid for Scientific Research (B) and (C) (♯ 18H01126 and ♯ 22K03381) of Japan Society for the Promotion of Science and National Research Foundation of S. Korea grant 2022R1A2B5B01001445.

Conflict of interest

The authors declare no conflict of interest.

References

1. A. D. Alexandrov, Uniqueness theorems for surfaces in the large V, *Vestnik Leningrad Univ.*, **13** (1958), 5–8.
2. D. G. Aronson, Bounds for the fundamental solutions of a parabolic equation, *Bull. Amer. Math. Soc.*, **73** (1967), 890–896. https://doi.org/10.1090/S0002-9904-1967-11830-5

3. L. Cavallina, R. Magnanini, S. Sakaguchi, Two-phase heat conductors with a surface of the constant flow property, *J. Geom. Anal.*, **31** (2021), 312–345. https://doi.org/10.1007/s12220-019-00262-8
4. L. Cavallina, S. Sakaguchi, S. Udagawa, A characterization of a hyperplane in two-phase heat conductors, *Commun. Anal. Geom.*, in press.
5. E. B. Fabes, D. W. Stroock, A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash, *Arch. Rational Mech. Anal.*, **96** (1986), 327–338. https://doi.org/10.1007/BF00251802
6. B. Gidas, W.-M. Ni, L. Nirenberg, Symmetry and related properties via maximum principle, *Commun. Math. Phys.*, **68** (1979), 209–243. https://doi.org/10.1007/BF01221125
7. H. Kang, S. Sakaguchi, Large time behavior of temperature in two-phase heat conductors, *J. Differ. Equations*, **303** (2021), 268–276. https://doi.org/10.1016/j.jde.2021.09.027
8. W. Reichel, Radial symmetry for elliptic boundary-value problems on exterior domains, *Arch. Rational Mech. Anal.*, **137** (1997), 381–394. https://doi.org/10.1007/s002050050034
9. S. Sakaguchi, Two-phase heat conductors with a stationary isothermic surface, *Rend. Ist. Mat. Univ. Trieste*, **48** (2016), 167–187. https://doi.org/10.13137/2464-8728/13155
10. S. Sakaguchi, Two-phase heat conductors with a stationary isothermic surface and their related elliptic overdetermined problems, *RIMS Kôkyûroku Bessatsu*, **B80** (2020), 113–132.
11. S. Sakaguchi, Some characterizations of parallel hyperplanes in multi-layered heat conductors, *J. Math. Pure. Appl.*, **140** (2020), 185–210. https://doi.org/10.1016/j.matpur.2020.06.007
12. B. Sirakov, Symmetry for exterior elliptic problems and two conjectures in potential theory, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **18** (2001), 135–156. https://doi.org/10.1016/S0294-1449(00)00052-4
13. J. Serrin, A symmetry problem in potential theory, *Arch. Rational Mech. Anal.*, **43** (1971), 304–318. https://doi.org/10.1007/BF00250468