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Editorial

Calculus of variations and nonlinear analysis: advances and applications†

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Calculus of variations and nonlinear analysis are research areas of longstanding tradition, nevertheless, they frequently appear as disjoint fields of investigation.

Our main motivation in editing this special issue is to pursue an exchange of questions and ideas in these fields, by presenting some of the most recently investigated topics.

Clearly, we do not cover all, and not even most of the trends, but we believe that creating occasions of gathering topics in these research lines might be an inspiration for our community.

The main issues included here, with intertwining aspects, concern shape optimization problems, regularity theory, geometric diffusion flows, nonlocal diffusions, spectral theory, fully non-linear equations and Schrödinger systems.

A classic shape optimization problem is treated in [9], where the anisotropic version of the Kirchhoff-Plateau problem is studied, showing the existence of solutions and performing a dimensional reduction via $\Gamma-$convergence.

Shape optimization problems are of interest also thanks to their natural interpretation in the applied sciences. For instance, looking for the optimal way to arrange the features of a material in order to optimize its performance leads to the study of a two-phase problem. This model is treated in [14], where it is shown a bang-bang property for minimizers, by exploiting rearrangement techniques.

In the study of regularity properties of solutions to PDEs, a classical tool is the so-called Calderon-Zygmund theory. Results in this context are proved in [8], where (local) minima of multi-phase variational integrals are studied. By means of a reverse Hölder inequality, regularity results of Calderon-Zygmund type are obtained.

Harnack type inequalities for classical solutions of semilinear parabolic PDEs on complete Riemannian manifolds without boundary are proved in [1], yielding local bounds on the solutions in terms of the geometric quantities involved.

In the last few years, a renewed interest on geometric flows has arisen. This special issue contains some contributions in this field, with a focus on nonlocal flows. This is also motivated from applications, as nonlocal features of differential problems naturally appear in several contexts of the applied sciences.

In the paper [10], the authors review and develop the theory of the modified Mullins–Sekerka flow and the surface diffusion flow of smooth sets, starting from a detailed analysis of the nonlocal area functionals under a volume constraint. The deep study performed in this contribution, makes [10] a precious reading to approach this topic.

The evolution of the geometric features of a set is strongly affected by the driving diffusion. For instance, in [2] it is analyzed the dependence of the normal velocity of smooth sets evolving by a nonlocal diffusion guided by the s-fractional Laplacian. In the highly nonlocal regime ($s < 1/2$), the normal velocity is shown to be nearly proportional to the fractional mean curvature of the initial set. The contribution [6] deals with the fractional capacity: a lower bound for the isocapacitary deficit in terms of the Fraenkel asymmetry is proved with the help of the so called Caffarelli-Silvestre extension to get rid of the nonlocality of the operator.

In [12] nonlocal convolution-type functionals defined on suitable Sobolev spaces are used as an approximation of a nonisotropic Griffith type energy, pursuing the study, via Γ-convergence, of brittle fractures in linearly elastic materials.

Nonlocal functionals are used also in [15], where a systematic study of the gradient flow for harmonic and half-harmonic mappings is carried on, with a focus on the possibility of bubbling in finite time.

When dealing with double nonlocalities, both in the diffusion and in the internal interaction, important applications arise in the study of exotic stars; in this direction in [5] it is proved the existence, together with regularity and decay properties, of a least action solution to a fractional Schrödinger equation with Hartree type nonlinearities. The existence result is obtained by variational methods, as the equation under study is the Euler equation of an associated action functional.

On the other hand, when passing to the study of coupled systems of Schrödinger equations, this approach may fail, depending on the coupling terms, so that in this context bifurcation theory is a powerful tool to find solutions. In this spirit, a largely exploited approach is based on the Lyapunov-Schmidt reduction which enables to find solutions with a prescribed asymptotic profile. The result in [11] follows this philosophy to tackle a Schrödinger system with mixed aggregating and segregating internal forces. The authors find a solution whose components are subdivided into groups so that each component within a given group concentrates around the same point, while components in different groups concentrate in different points.

Bifurcation type results naturally rely on the deep knowledge of the linearized equations associated to the nonlinear problem, and in particular on its spectrum. This is only an example of the crucial role played by the linear theory underneath the nonlinear one.

Fundamental information in the study of the spectrum of Schrödinger operator are the so-called Lieb-Thirring inequalities: these kind of inequalities are investigated for the perturbed Lamé operators of elasticity and for the damped elastic wave equation in [4, 7]. The key point in the argument is to put into relation the eigenvalue problem with the one corresponding to suitable Lamé operators with non self-adjoint perturbations; this allows to show quantitative bounds on the location of the point spectrum of the elastic wave operator, in terms of suitable norms of the damping coefficient. In [7] various estimates of Lieb-Thirring type on discrete eigenvalues of the Lamé operator are provided again by reformulating the eigenvalue problem for a suitable integral operator.

Spectral estimates are also obtained in [3], where the study of sharp Sobolev-Poincaré constants is translated into a convex minimization problem with a divergence type constraint. This allows to show lower bounds for associated principal frequencies.

Another largely studied topic in which variational methods cannot be exploited is the case of fully nonlinear equations. A prototype of diffusion operators in this context are the extremal Pucci operators. In [13] the study of a class of nonlinear equations driven by Pucci extremal operators and subjected to a Hénon type weight is addressed. Shooting methods, together with a deep analysis of the associated ODE system, are exploited to find positive radial solutions in annuli or exterior domains.

Conflict of interest

The authors declare no conflict of interest.

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