http://www.aimspress.com/journal/mine

Mathematics in Engineering, 5(3): 1–38.
DOI:10.3934/mine.2023058
Received: 16 December 2021
Revised: 13 October 2022
Accepted: 17 October 2022
Published: 07 November 2022

Research article

Half-harmonic gradient flow: aspects of a non-local geometric PDE†

Jerome D. Wettstein[1,2,*]

[1] Department of Mathematical Sciences, Florida Institute of Technology, Melbourne, FL, USA

[2] 150 West University Blvd, Melbourne, FL 32901, USA

† **This contribution is part of the Special Issue:** Calculus of Variations and Nonlinear Analysis: Advances and Applications
Guest Editors: Dario Mazzoleni; Benedetta Pellacci
Link: www.aimspress.com/mine/article/5983/special-articles

* **Correspondence:** Email: jwettstein@fit.edu; Tel: +1(321)6743028.

Abstract: The goal of this paper is to discuss some of the results in the author's previous papers and expand upon the work there by proving two new results: a global weak existence result as well as a first bubbling analysis for the half-harmonic gradient flow in finite time. In addition, an alternative local existence proof to the one provided in [47] is presented based on a fixed-point argument. This preliminary bubbling analysis leads to two potential outcomes for the possibility of finite-time bubbling until a conjecture by Sire, Wei and Zheng, see [40], is settled: Either there always exists a global smooth solution to the half-harmonic gradient flow without concentration of energy in finite-time, which still allows for the formation of half-harmonic bubbles as $t \to +\infty$, or finite-time bubbling may occur in a similar way as for the harmonic gradient flow due to energy concentration in finitely many points. In the first part of the introduction to this paper, we provide a survey of the theory of harmonic and fractional harmonic maps and the associated gradient flows. For clarity's sake, we restrict our attention to the case of spherical target manifolds S^{n-1}, but our discussion extends to the general case after taking care of technicalities associated with arbitrary closed target manifolds N (cf. [48]).

Keywords: fractional Laplacian; half-harmonic map; gradient flow; finite-time bubbling; non-local PDE

---

1. Introduction

Among the most fundamental and prominent partial differential equations in geometric analysis is the harmonic map equation. Its relevance derives from the way it emerges naturally* as well as the fact that the associated PDE, especially in the critical realm where the domain is two-dimensional, has inspired the creation of powerful techniques such as Hélein's moving frames method (Hélein [20]), a suitable Gauge-construction à la Uhlenbeck (Uhlenbeck [45]) by Rivière [32] and bubbling techniques (Sacks-Uhlenbeck [34]). The harmonic map equation has later on also sparked the introduction of fractional harmonic maps in Da Lio-Rivière [10, 11]. The corresponding regularity theory for fractional harmonic maps, based on the contributions of a variety of authors including Da Lio-Rivière [11], Schikorra [35], Da Lio [6], Da Lio-Schikorra [12, 13], Mazowiecka-Schikorra [25], and bubbling analysis (Da Lio [5], Da Lio-Laurain-Rivière [7]) have led to generalisations of various ideas from the realm of local PDEs, such as Wente/Coifman-Lions-Meyer-Semmes-type estimates, gauge techniques, Pohozaev identities and many others to fractional geometric PDEs.

Let us take a step back and recall the main definitions that we shall be using. For the moment, let us take (M, g), (N, γ) to be arbitrary closed Riemannian manifolds. We mention here that we shall usually assume N to be isometrically embedded in $\mathbb{R}^n$, a property guaranteed by Nash's embedding theorem for sufficiently large dimensions n of the prospective ambient space. Using M and N, one is naturally led to define the *Dirichlet energy* $E(u)$ for maps $u : M \to N$

$$E(u) := \frac{1}{2}\int_M g^{\alpha\beta}(x)\gamma_{ij}(u(x))\frac{\partial u^i}{\partial x_\alpha}(x)\frac{\partial u^j}{\partial x_\beta}(x)dx, \tag{1.1}$$

where we assume for convenience that M, N are embedded submanifolds of the Euclidean space and use Einstein's summation convention. Naturality of this definition becomes apparent if one realises that given $M = \mathbb{T}^m$ the m-dimensional torus

$$\mathbb{T}^m = \underbrace{S^1 \times \ldots \times S^1}_{m \text{ times}},$$

as a domain as well as the target space $N = \mathbb{R}^n$ with the corresponding natural (flat) Riemannian metrics, the energy simplifies to the usual Dirichlet energy

$$E(u) = \frac{1}{2}\int_{\mathbb{T}^m} |\nabla u|^2 dx. \tag{1.2}$$

By standard considerations, critical points of (1.2) satisfy the equation

$$-\Delta u = 0,$$

which immediately, by means of elliptic regularity, implies $u \in C^\infty(\mathbb{T}^m)$. A natural question thus pertains to the regularity of critical points of (1.1) with the target space restriction. One may interpret the condition that u takes values in N as a Lagrange-multiplier, already hinting at the more involved nature of the Euler-Lagrange equation associated with critical points. Thus, we are led to the first key definition:

*The action induced by the Dirichlet energy in Quantum Field Theory leads to the so-called sigma model, connecting harmonic maps with instatons. See also Chapter 2.4 in Jost [23].

Definition 1.1. *A map $u \in H^1(M;N)$ is called* ***harmonic****, if and only if it is a critical point of the energy function E as defined in* (1.1) *among competitors in $H^1(M;N)$. This means that for all $\varphi \in H^1 \cap L^\infty(M;\mathbb{R}^n)$, we have*

$$\frac{d}{dt}E\left(\pi\left(u+t\cdot\varphi\right)\right)\Big|_{t=0}=0,$$

where we used the closest point projection onto N defined in a tubular neighbourhood of N.

Here, we define $H^1(M;N)$ to be the collection of all Sobolev functions $u \in H^1(M;\mathbb{R}^n)$ which also satisfy $u(x) \in N$ for almost every $x \in M$. Consequently, the choice of Sobolev space is appropriate for both the energy functional $E(u)$ as well as the condition of u assuming values in N. It should be emphasised at this point that critical points do not only include (global) extrema, but also saddle points.

As per usual with variational problems, the first step consists of computing the Euler-Lagrange equation in order to obtain insight into the regularity and nature of critical points. In the case of harmonic maps, we arrive at the following characterisation

$$\begin{aligned} u \text{ is harmonic } &\Leftrightarrow (-\Delta)_M u \perp T_u N \text{ in } \mathcal{D}'(M) &(1.3)\\ &\Leftrightarrow (-\Delta)_M u = A(u)(\nabla u, \nabla u), &(1.4)\end{aligned}$$

where A denotes the second fundamental form of N and Δ_M being the Laplace-Beltrami operator[†] on the domain M. This already hints at the intimate connection between harmonic maps and curvature as well as the geometry of the manifolds involved. Indeed, existence of solutions may be tied to conditions on the curvature, see for example Schoen-Yau [37]. A particularly striking feature of the PDE (1.4) is its quadratic structure quantified by

$$|A(u)(\nabla u, \nabla u)| \leq C|\nabla u|^2,$$

for some constant $C > 0$. This property immediately singles out the case of M being two-dimensional as being critical for bootstrap considerations. To be precise, assuming M to be 2-dimensional and $u \in H^1(M;N)$, the RHS of (1.4) is in $L^1(M)$. As a result, Caldéron-Zygmund theory is not applicable and one may merely deduce $\nabla u \in L^{2,\infty}$. To summarise, we have only obtained regularity of the same homogeneity as initially given and for this reason, standard bootstrapping techniques are not immediately applicable to prove regularity. What is even worse, there is no general regularity theory for solutions of PDEs with smooth quadratic non-linearity. This may be seen by example with the scalar PDE for $u \in H^1(B_1(0);\mathbb{R})$, $B_1(0)$ denoting here the unit ball in $\mathbb{R}^2$, given below

$$-\Delta u = |\nabla u|^2. \tag{1.5}$$

[†]In local coordinates, the Laplace-Beltrami operator takes the form

$$(-\Delta)_M u = -\sum_{i,j}\frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j u\right),$$

where $|g|$ denotes the determinante of the Riemannian metric $(g_{ij})_{i,j}$ in local coordinates and $(g^{ij})_{i,j}$ the inverse of the Riemannian metric considered as a matrix. The second fundamental form A is defined at every point $x \in N$ and all $V, W \in T_x N$ by

$$A(x)(V,W) := -\sum_j \langle dn_i(x)V, W\rangle \cdot n_i,$$

where $n_1, \ldots, n_{n-\dim N}$ is an orthonormal basis of the normal bundle close to the fiber at $x \in N$.

In this case, by using $v := e^u$, one may immediately construct a counterexample to regularity by using the observation that the above PDE is equivalent to $-\Delta v = 0$

$$0 = -\Delta v = -\Delta\left(e^u\right) = -e^u\Delta u - e^u|\nabla u|^2 = e^u\left(-\Delta u - |\nabla u|^2\right).$$

Employing the fundamental solution in place of v, we find that

$$u = \log\left(\log\left(\frac{2}{|x|}\right)\right),$$

provides a solution to (1.5) which is discontinuous at $x = 0$. We refer to Rivière [33, p.32–34] for further details.

Fortunately, in the case of the harmonic map Eq (1.4), the non-linearity behaves more benign due to its geometric nature. For this reason, by the combined efforts over decades of various authors including (but not limited to) Grüter [18] (regularity of stationary weakly harmonic maps on domains of dimension 2), Hélein [19] (regularity of arbitrary weakly harmonic maps on domains of dimension 2), Morrey [27] (regularity of minimizing harmonic maps on domains of dimension 2), Rivière [32] (regularity of conformally invariant variational problems on domains of dimension 2, in particular harmonic maps), Shatah [39] (regularity of harmonic maps on domains of dimension 2 taking values in the spheres), just to name a few, we know that harmonic maps possess better regularity properties than solutions to arbitrary PDEs with quadratic non-linearities. A common feature among proofs of regularity properties is the use of compensation results based on properties of 2D-jacobians as summarised, for example, in Wente's estimate, see Wente [46], presented below:

Proposition 1.2 ([46]). *Let $r > 0$ be arbitrary, $1 \leq p < 2$ and $B_r(0) \subset \mathbb{R}^2$ be the ball of radius r around 0. If $u \in W_0^{1,p}(B_r(0))$*[‡] *and $a, b \in W^{1,2}(B_r(0))$ are such that*

$$\begin{aligned} -\Delta u &= \partial_x a \partial_y b - \partial_x b \partial_y a =: \nabla a \cdot \nabla^{\perp} b, && \text{in } B_r(0), \\ u &= 0 && \text{on } \partial B_r(0) \end{aligned} \tag{1.6}$$

Then u is continuous and the following estimate holds true

$$\|u\|_{L^\infty} + \|\nabla u\|_{L^{2,1}} + \|\nabla^2 u\|_{L^1} \lesssim \|\nabla a\|_{L^2}\|\nabla b\|_{L^2}. \tag{1.7}$$

A similar result continues to hold for the RHS of (1.6), if it is a product of a divergence-free and a rotation-free vector field for higher dimensional domains (see Coifman-Lions-Meyer-Semmes [4]). The estimate in Proposition 1.2 is then a special case of this more general result. In fact, the underlying reason for the improved regularity is a Hardy-regularity estimate for the RHS[§]. One may wonder how this estimate helps us establish regularity for equations such as (1.4). The central idea is that, by either

‡ $W_0^{1,p}(B_r(0))$ denotes the linear subspace of $W^{1,p}(B_r(0))$ of Sobolev maps with vanishing trace.

§ In contrast to $L^1(\mathbb{R}^m)$, the Hardy space $\mathcal{H}^1(\mathbb{R}^m)$ is well-behaved with respect to Caldéron-Zygmund operators and thus elliptic regularity results apply in this case, see Coifman-Lions-Meyer-Semmes [4]. The Hardy space $\mathcal{H}^1(\mathbb{R}^m)$ is the subspace of $L^1(\mathbb{R}^m)$-functions such that

$$M_\Phi(f)(x) := \sup_{t>0} |\Phi_t * f|(x) \in L^1(\mathbb{R}),$$

where Φ is a Schwartz function on $\mathbb{R}$ with $\int \Phi dx = 1$ and $\Phi_t(x) = 1/t\Phi(x/t)$. Various other, sometimes simpler characterisations (for example $\mathcal{H}^1(\mathbb{R}^m) \simeq F^0_{1,2}(\mathbb{R}^m)$) exist. We refer to Grafakos [17], Chapter 6.4, for details on these spaces.

employing Hélein's moving frames [20] or Rivière's change of gauge approach [32], one exposes an underlying structure of $2D$-jacobians inherent to the harmonic map equation. This is best illustrated in the case $M = \mathbb{T}^2, N = S^{n-1}$, where the harmonic map equation reads

$$-\Delta u = u|\nabla u|^2. \tag{1.8}$$

Shatah [39] observed that the Eq (1.8) is actually equivalent to the following set of conservation laws

$$\forall i, j \in \{1, \dots n\} : \operatorname{div}\left(u_i \nabla u_j - u_j \nabla u_i\right) = 0. \tag{1.9}$$

Thus, we have (see Hélein [20, Section 2.6])

$$\begin{aligned} -\Delta u_i &= \sum_j u_i \nabla u_j \cdot \nabla u_j \\ &= \sum_j \left(u_i \nabla u_j - u_j \nabla u_i\right) \cdot \nabla u_j + u_j \nabla u_i \cdot \nabla u_j \\ &= \sum_j \left(u_i \nabla u_j - u_j \nabla u_i\right) \cdot \nabla u_j, \end{aligned} \tag{1.10}$$

where in the last line we used the observation that $\sum_j u_j \nabla u_j = 0$, as u takes values in S^{n-1} and thus this may be interpreted as a vector of scalar products between u, which belongs to the normal space of S^{n-1}, and partial derivatives of u, which assume values in the tangent space of S^{n-1}. Consequently, it becomes apparent that $\sum_j u_j \nabla u_j = 0$ holds. Keeping Shatah's conservation laws (1.9) in mind, the structure required for the application of Wente's estimate is now revealed in the equation above. More precisely, we have a product of a divergence-free and a rotation-free part, and applying Hodge decompositions leads to 2D-jacobians in the equation. By suitable localisation arguments, one is able to conclude that $u \in W^{1,q}_{loc}$, for any $q > 1$. The details of the argument may be found in Hélein [20, Section 4.1].

A non-local generalisation of the notion of harmonic maps was developed later by Da Lio and Rivière in [10] following the spirit above. Let us denote by $H^s(S^1; \mathbb{R}^n)$ the space of functions, such that

$$\forall s \in \mathbb{R} : H^s(S^1; \mathbb{R}^n) := \left\{ u : S^1 \to \mathbb{R}^n \text{ measurable} \;\Big|\; \sum_{n \in \mathbb{Z}} (1 + |n|)^{2s} |\hat{u}(n)\,|^2 < +\infty \right\}, \tag{1.11}$$

where $\hat{u}(n)$ denotes the n-th Fourier coefficient of u. The space $H^1(S^1; N)$ is then defined as the subspace of $H^1(S^1; \mathbb{R}^n)$ such that $u(x) \in N$ for almost every $x \in S^1$. With the induced Riemannian structure on N and by using the *s-Dirichlet energy* which, for any $s > 0$, is defined by

$$E_s(u) := \int_{S^1} |(-\Delta)^{s/2} u|^2 dx, \quad \forall u \in H^s(S^1, N),$$

one may introduce the following notion:

Definition 1.3. *A map $u \in H^s(S^1; N)$ is called **s-harmonic**, if and only if it is a critical point of the energy function E_s among variations in the space $H^s(S^1; N)$. This means that for all $\varphi \in H^s \cap L^\infty(M; \mathbb{R}^n)$, we have*

$$\frac{d}{dt} E_s\left(\pi\left(u + t \cdot \varphi\right)\right)\Big|_{t=0} = 0,$$

where we used the closest point projection onto N defined in a tubular neighbourhood of N.

Fractional Laplacians $(-\Delta)^{s/2}$ may be defined by Fourier multipliers or using principal value integrals, we refer to Nezza-Palatucci-Valdinocci [28] for some exposition as well as the next section where both kinds of definitions are introduced. Half-harmonic maps are related to free-boundary minimal discs (Da Lio-Rivière [10], Da Lio-Pigati [9], Millot-Sire [26]) and singular limits of Ginzburg-Landau approximations (Millot-Sire [26]), so there are again interesting connections to geometry.

For the case of greatest interest to our current considerations, namely when $s = 1/2$, there is a-priori no reason to prefer S^1 over $\mathbb{R}$ (or vice versa) as the domain. Indeed, if $s = 1/2$, $E_{1/2}(u)$ is conformally invariant (under the trace of Möbius transformations) and the stereographic projection (see Da Lio [8, Prop. 1.1]), a conformal mapping between $S^1 \setminus \{(0,1)\}$ and $\mathbb{R}$, enables us to switch between $\mathbb{R}$ and S^1 seemlessly and thus transfer results from one domain to the other. For the remainder of this paper, however, we shall restrict our attention to $s = 1/2$ and the domain being the unit circle S^1:

As a very simple first case, one may consider the simple target $N = \mathbb{R}$. In this case, the Euler-Lagrange equation turns out to be

$$(-\Delta)^{1/2} u = 0,$$

which, similar to the harmonic case, immediately proves regularity by elliptic regularity. For an arbitrary closed target manifold N, the Euler-Lagrange equation has a very similar structure to (1.4), as it becomes

$$u \text{ is } \frac{1}{2}\text{-harmonic } \Leftrightarrow (-\Delta)^{1/2} u \perp T_u N \text{ in } \mathcal{D}'(S^1). \tag{1.12}$$

In Da Lio-Rivière [10], the equation above has been rewritten using three-term commutators to reveal compensation structures due to the inherent geometric nature of the non-linearity. In addition, in the case $N = S^{n-1}$, we may push the similarity with (1.8) even further due to the equivalence of (1.12) and

$$(-\Delta)^{1/2} u = u|d_{1/2} u|^2, \tag{1.13}$$

where we use the framework of fractional gradients as introduced in Mazowiecka-Schikorra [25]

$$d_{1/2} u(x,y) = \frac{u(x) - u(y)}{|x-y|^{1/2}}, \quad |d_{1/2} u|^2(x) := \int_{S^1} |d_{1/2} u(x,y)|^2 \frac{dy}{|x-y|}.$$

At this point, let us mention that we use the natural metric $|x - y| = 2|\sin(\frac{x-y}{2})|$ on S^1. It should be noted that the fractional gradients are natural non-local objects, as they relate to the Gagliardo-Sobolev spaces and Bessel potential spaces (see Prats [29], Prats-Saksman [30]).

It is interesting to observe that the key features of the non-linearity in (1.4) are also present in (1.12). Both possess a quadratic RHS (notice the emerging term $u|d_{1/2}u|^2$ in (1.13), see [48] for general N) and are critical equations not allowing for simple regularity by bootstrap techniques[¶]. Indeed, in [48], we even establish a "curvature-like" formulation similar to (1.13) in general, but the structure is most easily recognizable in the case $N = S^{n-1}$.

Regularity properties of half-harmonic maps on S^1 have been studied extensively and we know that these maps are smooth. The theory of half-harmonic maps was started by Da Lio-Rivière [10, 11] and

¶Similar to the harmonic map equation, where $u|\nabla u|^2 \in L^1$ only, if $u \in H^1(B_1(0); N)$, the half-harmonic map equation leads to $u|d_{1/2}u|^2 \in L^1$ for $u \in H^{1/2}(S^1; N)$.

later expanded by other authors, in particular Schikorra [35] and Mazowiecka-Schikorra [25]. There are essentially two approaches to the regularity theory: By three-term commutators based on improved regularity of certain linear combinations of terms and non-local Wente estimates building upon the theory of fractional gradients and fractional divergences. Three-term commutator estimates are in some sense various incarnations of estimates of operators such as

$$\mathcal{T} : L^2(\mathbb{R};\mathbb{R}^n) \times \dot{H}^{1/2}(\mathbb{R};\mathbb{R}^{n\times n}) \to \dot{H}^{-1/2}(\mathbb{R};\mathbb{R}^n),$$

defined by

$$\mathcal{T}(v, Q) := (-\Delta)^{1/4}(Qv) - Q(-\Delta)^{1/4}v + (-\Delta)^{1/4}Q \cdot v.$$

This operator quantifies the failure of Leibniz' rule for the 1/4-Laplacian. It is proven in Da Lio-Rivière [10] that

$$\|\mathcal{T}(v, Q)\|_{\dot{H}^{-1/2}} \lesssim \|Q\|_{\dot{H}^{1/2}}\|v\|_{L^2}.$$

One should keep in mind that a-priori, each summand in $\mathcal{T}(v, Q)$ on its own does not belong to $\dot{H}^{-1/2}$.

To draw similarities with the harmonic map equation, we focus on the non-local Wente/Coifman-Lions-Meyer-Semmes estimate found in Mazowiecka-Schikorra [25]:

Proposition 1.4 ([25]). *Let $s \in (0, 1)$ and $p \in (1, \infty)$. For $F \in L^p_{od}(\mathbb{R} \times \mathbb{R})$ and $g \in \dot{W}^{s,p'}(\mathbb{R})$, where p' denotes the Hölder dual of p, we assume that $\operatorname{div}_s F = 0$. Then $F \cdot d_s g$ lies in the Hardy space $\mathcal{H}^1(\mathbb{R})$ and we have the estimate*

$$\|F \cdot d_s g\|_{\mathcal{H}^1(\mathbb{R})} \lesssim \|F\|_{L^p_{od}(\mathbb{R}\times\mathbb{R})}\|g\|_{\dot{W}^{s,p'}(\mathbb{R})}.$$

The general s-gradient is introduced in analogy to the 1/2-gradient and we say that $\operatorname{div}_s F = 0$, if for all $\varphi \in C^\infty_c(\mathbb{R})$

$$\int_{\mathbb{R}}\int_{\mathbb{R}} F(x, y)d_s\varphi(x, y)\frac{dydx}{|x - y|} = 0.$$

Lastly, we define

$$F \cdot d_s g(x) := \int_{\mathbb{R}} F(x, y)d_s g(x, y)\frac{dy}{|x - y|}.$$

Let us emphasise that there are several potential operations which $\cdot$ may refer to. It may refer, for example, to the real product on $\mathbb{R}$ and the non-local multiplication defined above. The precise meaning will usually be clear from the context.

These notions and results also apply to S^1 as the underlying domain instead of $\mathbb{R}$. Lastly, we have used the following spaces for $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$

$$F \in L^p_{od}(\mathbb{R} \times \mathbb{R}) \Leftrightarrow \|F\|_{L^p_{od}(\mathbb{R}\times\mathbb{R})} := \left(\int_{\mathbb{R}}\int_{\mathbb{R}} |F(x, y)|^p \frac{dydx}{|x - y|}\right)^{1/p} < +\infty, \tag{1.14}$$

and

$$g \in \dot{W}^{s,p'}(\mathbb{R}) \Leftrightarrow \|g\|_{\dot{W}^{s,p'}(\mathbb{R})} := \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \left|\frac{g(x) - g(y)}{|x - y|^s}\right|^{p'} \frac{dydx}{|x - y|}\right)^{1/p'} < +\infty, \tag{1.15}$$

equipped both with the natural (semi-)norms associated with each spaces. Regularity of fractional harmonic maps may be obtained from here using ad-hoc localisation arguments and weighted energy estimates as in Mazowiecka-Schikorra [25, p.15–19], building upon the non-local Wente estimate in Proposition 1.4 (see also Da Lio-Pigati [9], Da Lio-Rivière [10]).

Next we are going to introduce the harmonic and half-harmonic gradient flow. We consider functions $u : [0,T[\times M \to N$, where M, N are embedded submanifolds with the induced metrics and $T \in \mathbb{R} \cup \{+\infty\}$ and would like to solve the following equation

$$\partial_t u - \Delta_M u = A(u)(\nabla u, \nabla u), \tag{1.16}$$

with initial value $u(0,\cdot) = u_0(\cdot) \in H^1(M;N)$. This is the gradient flow associated with the Dirichlet energy $E(u)$. Its relevance derives from approximations of critical points of the Dirichlet energy functional (i.e., harmonic maps) and questions pertaining to homotopy of maps such as whether a given map in $H^1(M;N)$ is homotopic to a harmonic one. The latter question actually provided the motivation for the beginning of the study of the harmonic gradient flow in Eells-Sampson [15], culminating in an existence result for minimizers of the Dirichlet energy homotopic to any given initial datum, provided that the target manifold has non-positive sectional curvature. The first general result that applies independent of any geometric properties (such as sectional curvature) of the target space for two-dimensional domains and extended to arbitrary domains was proven in Struwe [41,42]. Unfortunately, homotopy properties related to harmonic maps as above don't hold in general, leading us to wonder what kind of convergence and regularity properties are to be expected in finite time as well as asymptotically. It was shown that bubbling may occur, i.e., the concentration of energy leading to specific rescalings (so-called *blow-ups*) converging in an appropriate sense to a harmonic map with domain S^{n-1}. Outside of the concentration points, the flow will remain smooth. For completeness' sake, let us state the result in Struwe [41] in the special case $N = S^{n-1}$:

Theorem 1.5 ([41]). *Let $u_0 \in H^1(\mathbb{T}^2; S^{n-1})$. Then there exists a solution $u \in H^1(]0,+\infty[; L^2(\mathbb{T}^2))$ of the harmonic gradient flow*

$$\partial_t u - \Delta u = u|\nabla u|^2 \quad \textit{in } \mathcal{D}'(]0,T[\times\mathbb{T}^2), \quad \forall T > 0, \tag{1.17}$$

together with the boundary conditions

$$u(0,x) = u_0(x), \quad \textit{for all } x \in \mathbb{T}^2, \tag{1.18}$$

and satisfying $E(u(t,\cdot)) \leq E(u_0)$ for all times $t \geq 0$. The solution u is smooth on $]0,+\infty[\times\mathbb{T}^2$, except in a finite number of points (t_k, x_k), $k = 1,\ldots,K$, for some $K \in \mathbb{N}$, where bubbling occurs. Additionally, u is unique in the class $\mathcal{E} \subset H^1_{loc}([0,+\infty[\times\mathbb{T}^2)$ which consists precisely of the $u \in H^1_{loc}([0,+\infty[\times\mathbb{T}^2)$, such that

$$\exists m \in \mathbb{N}, \exists T_0 = 0 < T_1 < \ldots < T_m < \infty : \quad u \in L^2([T_i, T_{i+1}[; W^{2,2}(\mathbb{T}^2)), \forall i \leq m-1. \tag{1.19}$$

Finally, there exists a constant $C > 0$ independent of u_0, such that

$$K \leq C \cdot E(u_0).$$

Theorem 1.5 immediately addresses existence, regularity and bubbling as well as providing a first uniqueness result (at least for so-called *strong solutions*, i.e., solutions which are once weakly L^2-differentiable with respect to the time-variable t and twice weakly L^2-differentiable with respect to the space-variables x). Interesting features include the fact that bubbling may occur, but only in a limited number of points. Thus, the energy $E(u(t))$ as a function of time is continuous up to the creation of bubbles[‖] which account for the jump down in energy completely. This also guarantees that the amount of bubbles that form is finite due to the lower bound for the energy of non-constant harmonic maps.

The proof of Theorem 1.5 relies on testing the harmonic gradient flow Eq (1.17) against derivatives of the solution u, which we shall briefly outline below. As a first simple application, let us test (1.17) against $\partial_t u$ (by which we mean we multiply the Eq (1.17) with $\partial_t u$ and then integrate) on $[t_1, t_2] \times \mathbb{T}^2$ which yields

$$\int_{t_1}^{t_2} \int_{\mathbb{T}^2} |\partial_t u|^2 dxdt + \frac{1}{2} E(u(t_2)) - \frac{1}{2} E(u(t_1)) = \int_{t_1}^{t_2} \int_{\mathbb{T}^2} |\partial_t u|^2 dxdt + \int_{t_1}^{t_2} \int_{\mathbb{T}^2} \partial_t \left(\frac{1}{2} |\nabla u|^2 \right) dxdt = 0,$$

since $\partial_t u$ is tangential to S^{n-1}, while u is perpendicular to the tangent space. This proves immediatly that the energy decreases along the half-harmonic gradient flow, because

$$\frac{1}{2} E(u(t_1)) = \frac{1}{2} E(u(t_2)) + \int_{t_1}^{t_2} \int_{\mathbb{T}^2} |\partial_t u|^2 dxdt \geq \frac{1}{2} E(u(t_2)), \tag{1.20}$$

a property expected as we follow the steepest descent of the Dirichlet energy $E(u)$, meaning that we strive to decrease the energy at the largest rate possible. Similarly, testing against $\varphi \partial_t u$ instead of $\partial_t u$, where φ is some cutoff-function, leads in a similar way to estimates for the localised energy and hence control of the energy concentration, a crucial tool to derive uniform estimates of approximating sequences as in the proof in Struwe [41]. Testing against $-\Delta u$ yields control for the second order derivatives

$$\begin{aligned}
&\int_0^T \int_{\mathbb{T}^2} \langle \partial_t u, -\Delta u \rangle dxdt + \int_0^T \int_{\mathbb{T}^2} |-\Delta u|^2 dxdt \\
&= \int_0^T \int_{\mathbb{T}^2} \langle u |\nabla u|^2, -\Delta u \rangle dxdt \\
&\leq \frac{1}{2} \int_0^T \int_{\mathbb{T}^2} |\nabla u|^4 dxdt + \frac{1}{2} \int_0^T \int_{\mathbb{T}^2} |-\Delta u|^2 dxdt,
\end{aligned} \tag{1.21}$$

which after absorption may be rewritten as

$$\begin{aligned}
\frac{1}{2} \int_0^T \int_{\mathbb{T}^2} |-\Delta u|^2 dxdt &\leq \frac{1}{2} \int_0^T \int_{\mathbb{T}^2} |\nabla u|^4 dxdt - \int_0^T \frac{d}{dt} \left(\frac{1}{2} \int_{\mathbb{T}^2} |\nabla u|^2 dx \right) dt \\
&\leq \frac{1}{2} \int_0^T \int_{\mathbb{T}^2} |\nabla u|^4 dxdt + E(u_0).
\end{aligned} \tag{1.22}$$

The L^4-norm may be estimated as in Struwe [41] using Ladyzhenskaya's estimate (Ladyzhenskaya [24]). As a result, such estimates allow for uniform control of higher-oder Sobolev norms in terms of the initial energy and the energy concentration. By an approximation process, one obtains existence of solutions for arbitrary initial values in $H^1(\mathbb{T}^2; S^{n-1})$. The rigorous treatment of the estimates is referred to Struwe [41].

[‖] A bubble is the limit of special rescalings (blow-ups) approaching a point of energy concentration (i.e., where a fixed amount of energy is accumulated in neighborhoods with radius converging to 0). Thus, a bubble $\tilde{u} : S^1 \to S^{n-1}$ is a non-constant half-harmonic map.

There are several natural questions to ask from here: Firstly, one may wonder whether bubbling in finite time actually occurs. This question has been answered by Chang, Ding and Ye [3]. They construct subsolutions blowing up in finite time and prove that appropriate boundary conditions exist to transfer the blow up to a solution of the harmonic map flow.

Another question pertains to whether the energy decay is necessary as an assumption. Indeed it is, without this no uniqueness statement is possible. For instance, in Topping [44], other kinds of blow ups violating the monotone decay of energy are constructed by using so-called reverse bubbling and prove existence of "non-physical" solutions. Furthermore, various types of blow ups may be considered using the inner-outer gluing scheme, as studied by Davila, del Pino and Wei [14] and other authors. It should be stated that a kind of non-uniqueness phenomena can already be observed for the linear heat equation in $\mathbb{R}^n$ where we need some decay at ∞ to ensure uniqueness, so these kinds of issues are not unexpected (see also John [22] for examples).

Lastly, one may pose the question whether uniqueness also holds among weak solutions. Again, this is true, at least for solutions with non-increasing energy (otherwise counterexamples via reverse bubbling exist) and has been shown by Rivière [31] and Freire [16]. Rivière's argument works for the small energy regime and for the target manifold S^{n-1} and employs an ingenious absorption argument that allows us to deduce that the solution is actually a strong solution. The key result may be stated as follows:

Lemma 1.6 ([31, p.99]). *Let $u \in H^1(\mathbb{T}^2; S^{n-1})$ and $f \in L^2(\mathbb{T}^2; \mathbb{R}^n)$ and assume that u solves the following non-linear quadratic PDE*

$$-\Delta u = u|\nabla u|^2 + f. \tag{1.23}$$

Then $u \in H^2(\mathbb{T}^2; S^{n-1})$.

It should be noted that Lemma 1.6 provides the maximal amount of regularity one may expect from a general u solving such an equation. In the context of the harmonic gradient flow, one would apply Lemma 1.6 to

$$-\Delta u(t) = u(t)|\nabla u(t)|^2 - \partial_t u(t),$$

at fixed times t, which holds for almost all times t. Thus, one is able to deduce that for almost all times t, the function $u(t)$ lies in $H^2(\mathbb{T}^2; S^{n-1})$. However, this does not suffice to prove uniqueness, as we need an L^2-bound on the H^2-norms when integrated over time. Fortunately, this follows by using a slight generalisation of Ladyzhenskaya's estimate (Ladyzhenskaya [24])

$$\|\nabla u(t)\|_{L^4} \lesssim \|\nabla u(t)\|_{L^2} \|\nabla u(t)\|_{H^1} \lesssim E(u_0)\|u(t)\|_{H^2},$$

so, by using elliptic regularity as well as $|u(t)| = 1$ almost everywhere, we arrive at

$$\|u(t)\|_{H^2} \lesssim \|u(t)\|_{L^2} + \left\| u(t)|\nabla u(t)|^2 - \partial_t u(t) \right\|_{L^2} \lesssim 1 + E(u_0)\|u(t)\|_{H^2} + \|\partial_t u(t)\|_{L^2}. \tag{1.24}$$

Now, if $E(u_0)$ is sufficiently small, the H^2-norm on the RHS of (1.24) may be absorbed in the LHS of (1.24) and by integration over time intervals, the desired local integrability follows and allows for the application of Theorem 1.5 to conclude.

The proof of Lemma 1.6 may be found in Rivière [31] and an adaption to the perturbed half-harmonic map equation in [47]. Indeed, the techniques discussed so far naturally generalise to the framework of the half-harmonic gradient flow. The motivation to study the fractional gradient flow stems once more from approximation of solutions to the half-harmonic map equation as well as the interest in expanding ideas from the local world to the fractional one. This is what the author has achieved in [47] for the case of the target manifold being a sphere and in [48] for the target being any closed manifold N. Since we shall restrict our considerations later on to the case $N = S^{n-1}$ anyways, let us focus on this special case, where the half-harmonic gradient flow equation takes the form

$$\partial_t u + (-\Delta)^{1/2} u = u|d_{1/2}u|^2, \tag{1.25}$$

with $u = u(t,x) \in H^1([0,T[;L^2(S^1;N)) \cap L^2([0,T[;H^{1/2}(S^1;N))$. We sometimes refer to such solutions as *energy class solutions*. It should be observed that $|d_{1/2}u|^2$ is to be taken only with respect to the space variable $x \in S^1$ to remain consistent with our previous definitions. The most natural formulation of the half-harmonic gradient flow equation in N is phrased as

$$\partial_t u + (-\Delta)^{1/2} u \perp T_u N \quad \text{in } \mathcal{D}'([0,T[\times S^1). \tag{1.26}$$

Of course, the Eqs (1.25) and (1.26) define the gradient flow associated with the half-Dirichlet energy $E_{1/2}(u)$. Now, one may wonder what has been known about the half-harmonic gradient flow before [47]. In Schikorra-Sire-Wang [36], the authors had already constructed solutions to the gradient flow of various non-local energies of similar type as the 1/2-Dirichlet energy by discretisation and approximation. Unfortunately, the result was limited to target spaces with inherent symmetry such as the sphere. In a different paper, Sire-Wei-Zheng [40] investigated the bubbling as $t \to +\infty$ by adapting the inner-outer gluing scheme to the non-local setup, proving that bubbling is possible for $N = S^1$ at time $t = +\infty$. The authors of Sire-Wei-Zheng [40, p.3] further conjectured that bubbling may actually only occur asymptotically, so no finite time bubbling should possible due to dimensional peculiarities of $\mathbb{R}$ and S^1. The conjecture is, to the author's knowledge, still open and under investigation.

Let us now turn to the main result as found in [47] that answered many questions about the half-harmonic gradient flow:

Theorem 1.7 ([47]). *Let $u_0 \in H^{1/2}(S^1;S^{n-1})$ be any initial data. There exists $\varepsilon > 0$, such that if*

$$E_{1/2}(u_0) \leq \varepsilon, \tag{1.27}$$

then there exists a unique energy class solution $u : \mathbb{R}_+ \times S^1 \to S^{n-1} \subset \mathbb{R}^n$ of the weak fractional harmonic gradient flow

$$\partial_t u + (-\Delta)^{1/2} u = u|d_{1/2}u|^2, \tag{1.28}$$

satisfying $u(0,\cdot) = u_0$ in the sense $u(t,\cdot) \to u_0$ in L^2, as $t \to 0$. Moreover, the solution fulfills the energy decay estimate

$$E_{1/2}(u(t)) \leq E_{1/2}(u(s)) \leq E_{1/2}(u_0), \quad \forall t \geq s \in [0,+\infty[.$$

In fact, $u \in C^\infty(]0,\infty[\times S^1)$ and for an appropriate subsequence $t_k \to \infty$, the sequence $u(t_k)$ converges weakly in $H^1(S^1)$ to a point.

If u_0 has arbitrary energy, then we still get the existence of a unique strong solution[**] *to* (1.28) *on some time interval $[0,T[$, where $T = T(u_0) > 0$ depends on the initial datum. This solution is actually*

[**]In this context, a strong solution is a function which is once weakly differentiable in time- and space-variable with L^2-derivative.

smooth and $T(u_0)$ is the first time such that

$$\limsup_{t\to T} \varepsilon(R;u,t) \geq \varepsilon_1, \quad \forall R \in]0, \frac{1}{2}[, \tag{1.29}$$

where

$$\varepsilon(R;u,t) := \sup_{x_0\in S^1, t'\in[0,t]} E_R(u;x_0,t') = \sup_{x_0\in S^1, t'\in[0,t]} \frac{1}{2}\int_{B_R(x_0)} |(-\Delta)^{1/4}u(t')|^2 dx, \tag{1.30}$$

and $\varepsilon_1 > 0$ is a quantity appearing in the proof of the result and is independent of u_0, R, T.

As already hinted at earlier, the entire result continues to hold true if we use an arbitrary closed manifold N instead of S^{n-1}. We are now going to present the main steps of the proof of Theorem 1.7. The main idea is once again that testing (1.28) against u and its fractional derivatives yields control over higher-order Sobolev norms and energy concentration and so one may deduce existence and regularity similar to Struwe [41] by means of approximation, once we have a local existence theory for smooth u_0. As a first example of the technique used, we deduce (by testing (1.28) against $\partial_t u$ and integrating over times $[t_1, t_2]$)

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_{S^1} |\partial_t u|^2 dxdt + \frac{1}{2}E_{1/2}(u(t_2)) - \frac{1}{2}E_{1/2}(u(t_1)) \\
&\quad = \int_{t_1}^{t_2}\int_{S^1} |\partial_t u|^2 dxdt + \int_{t_1}^{t_2}\int_{S^1} \partial_t\left(\frac{1}{2}\,|(-\Delta)^{1/4}u|^2\right) dxdt = 0,
\end{aligned} \tag{1.31}$$

again using the fact that $\partial_t u \in T_u S^{n-1} \perp u$, leading to a monotone decrease of the $1/2$-Dirichlet energy for smooth solutions of (1.28). If we were to test against $\varphi\partial_t u$ instead of $\partial_t u$, estimates governing the concentration of energy in points of S^1 can be found. To have uniform higher-order Sobolev estimates, we remember that in the case of the harmonic gradient flow we tested against $-\Delta u$, so it is natural to expect that a similar type of control can be derived by testing against $(-\Delta)^{1/2}u$

$$\begin{aligned}
&\int_0^T\int_{S^1} \langle \partial_t u, (-\Delta)^{1/2}u\rangle dxdt + \int_0^T\int_{S^1} |(-\Delta)^{1/2}u|^2 dxdt \\
&= \int_0^T\int_{S^1} \langle u|d_{1/2}u|^2, (-\Delta)^{1/2}\, u\rangle dxdt \\
&\leq \frac{1}{2}\int_0^T\int_{S^1} |d_{1/2}u|^4 dxdt + \frac{1}{2}\int_0^T\int_{S^1} |(-\Delta)^{1/2}\, u|^2 dxdt.
\end{aligned} \tag{1.32}$$

Now, one argues by employing a fractional version of Ladyzhenskaya's inequality as well as absorbing suitable terms. The uniform estimates in terms of initial energy and the concentration of energy are obtained by testing with further maps in much the same way, we refer to [47,48] for details. Thus, once we are able to establish existence of solutions to the half-harmonic gradient flow for small times and regular boundary data, we may deduce the general existence and regularity result by approximation. Meanwhile, local existence for the half-harmonic gradient flow can be derived by studying the non-linear operator

$$H : W^{1,p}([0,T]\times S^1) \to L^p([0,T]\times S^1), \quad H(u) := u_t + (-\Delta)^{1/2}u - u|d_{1/2}u|^2, \tag{1.33}$$

for $p > 2$. Notice that solutions u of the half-harmonic gradient flow satisfy $H(u) = 0$. Now, one may argue using the inverse function theorem as in [47, Prop. 3.12] and [48, Prop. 4.3] or apply a fixed-point argument as we shall explore in the upcoming sections to finally conclude the proof of Theorem 1.7.

Lastly, let us discuss uniqueness of solutions to the half-harmonic gradient flow in the energy class. As mentioned before, the argument in Rivière [31] generalises in the form of the following fractional version of Lemma 1.6:

Lemma 1.8 ([47]). *Let $f \in L^2(S^1;\mathbb{R}^n)$ and assume that $u \in H^{1/2}(S^1;S^{n-1})$ solves the following equation*

$$(-\Delta)^{1/2}u = u|d_{1/2}u|^2 + f. \tag{1.34}$$

Then, we have the following improved regularity property

$$u \in H^1(S^1;S^{n-1}).$$

The proof of uniqueness for weak solutions with sufficiently small energy now follows as in the local case by absorption and using fractional Ladyzhenskaya-type estimates. To be precise, one obtains

$$(-\Delta)^{1/2}u = u|d_{1/2}\, u|^2 - \partial_t u,$$

so that by applying Lemma 1.8 for almost all times t

$$u(t) \in H^1(S^1), \tag{1.35}$$

for almost every t. As a result, we have by using the definition of the H^1-norm

$$\begin{aligned}
\|u(t)\|_{H^1}^2 &\lesssim \|u(t)\|_{L^2}^2 + \|(-\Delta)^{1/2}u(t)\|_{L^2}^2 \\
&\lesssim 1 + \||d_{1/2}\, u(t)|^2\|_{L^2}^2 + \|\partial_t u(t)\|_{L^2}^2 \\
&\lesssim 1 + \|(-\Delta)^{1/4}u(t)\|_{L^2}\| \left(\|(-\Delta)^{1/4}u(t)\|_{L^2} + \|(-\Delta)^{1/2}u(t)\|_{L^2}\right) + \|\partial_t u(t)\|_{L^2}^2 \\
&\lesssim 1 + \|(-\Delta)^{1/4}u(t)\|_{L^2}^2 + \|(-\Delta)^{1/4}u(t)\|_{L^2}\|u(t)\|_{H^1}^2 + \|\partial_t u(t)\|_{L^2} \\
&\lesssim 1 + E_{1/2}(u_0)^2 + E_{1/2}(u_0)\|u(t)\|_{L^2} + \|\partial_t u(t)\|_{L^2}.
\end{aligned} \tag{1.36}$$

Details on the individual estimates may be found in [47, Proof of Lemma 3.7]. From here on, the considerations become the same as in the proof of Lemma 1.6, by absorbing the H^1-norm on the RHS of the Eq (1.36) into the LHS, provided $E_{1/2}(u_0)$ is sufficiently small. Integrating both sides of (1.36) in time yields the desired regularity properties to apply a uniqueness result ([47, Thm. 3.2], [48, Thm. 4.1]) for solutions of strong type, i.e., solutions in $H^1([0,+\infty[\times S^1;N)$.

Following the author's contributions to the field, the authors in Hyder-Segatti-Sire-Wang [21] have considered an alternative version of the half-harmonic heat flow not based on the L^2-gradient flow, but rather on ideas similar to the connection investigated by Caffarelli and Silvestre [2] between fractional Laplacians and Dirichlet-to-Neumann

$$(\partial_t - \Delta)^{1/2}u \perp T_u N,$$

which allows for a monotonicity formula not available in the case of our half-harmonic gradient flow (1.28). It is obvious that both flows allow for the same stationary solutions (i.e., time-independent solutions), namely half-harmonic maps. Nevertheless, the approaches are independent and provide different advantages and drawbacks. Moreover, in Struwe [43], an alternative approach to the existence of solutions to the Eq (1.28) is discussed using harmonic extensions to avoid technicalities due to the non-local quantities involved. However, this approach is restricted to target spaces N embedded in $\mathbb{R}^n$ with parallelisable normal bundle.

In the current paper, our goal is to understand the formation of singularities with respect to energy concentration better, which ultimately also helps us understand the importance of $T(u_0)$ in Theorem 1.7. To this end, we will employ the characterisation (1.29) above in order to state and prove results regarding the convergence properties of rescalings of $u(t, x)$ at fixed times, leading to the formation of bubbles. In fact, we shall show that at most finitely many points exist where bubbles form. This adds to the results in [47, 48] by determining what occurs at time $T(u_0)$ for a given initial datum u_0, relating $T(u_0) < +\infty$ to the concentration of energy and thus formation of half-harmonic bubbles. We may also show that there must localise a quantum of the $1/2$-Dirichlet energy at each point of energy concentration, implying that only finitely many bubbles may form. To summarise, we shall obtain the following new result:

Theorem 1.9. *Let u be a solution as in Theorem 1.7, the target space $N = S^{n-1}$ and $x_0 \in S^1$ be a point, such that*

$$\limsup_{t\to T} \int_{B_R(x_0)} |(-\Delta)^{1/4} u|^2 dx \geq \varepsilon_1, \quad \forall R > 0, \tag{1.37}$$

where $\varepsilon_1 > 0$ is as in [48, Lemma 4.10] and depends on the target sphere dimension $n-1$. Then there exists a half-harmonic map $v : \mathbb{R} \to S^{n-1}$, such that

$$u_m \to v \quad \text{weakly in } H^1(\mathbb{R}; S^{n-1}) \text{ and strongly in } H^{1/2}(\mathbb{R}; S^{n-1}), \tag{1.38}$$

where u_m is a suitable rescaling and translation of u.

Therefore, by using such considerations, we may conclude that the solution constructed in [48] can be extended by L^2-continuity and the extension will be smooth everywhere except at finitely many times $0 < T_1 < \ldots < T_l < +\infty$, which may be characterised by concentration identities as in Theorem 1.7:

Theorem 1.10. *Let $u_0 \in H^{1/2}(S^1; S^{n-1})$, then there exists a weak solution of* (1.28) *with non-increasing $1/2$-Dirichlet energy*

$$u : [0, +\infty[\, \times S^1 \to N,$$

with $u \in L^\infty([0, +\infty[; H^{1/2}(S^1; S^{n-1})) \cap H^1([0, +\infty[; L^2(S^1; S^{n-1}))$ such that, except for finitely many times $0 < T_1 < \ldots < T_l < T_{l+1} := +\infty$, the function u is smooth

$$u \in C^\infty(]T_k, T_{k+1}[; S^{n-1}), \quad \forall k = 1, \ldots l.$$

Moreover, we may bound the number $l = l(u_0)$ as follows

$$l(u_0) \leq \frac{E(u_0)}{\varepsilon_0},$$

where $\varepsilon_0 > 0$ is the minimum amount of $1/2$-energy a non-constant, half-harmonic map with values in S^{n-1} must possess.

Of course, both of the results in Theorem 1.9 and 1.10 for S^{n-1} carry over to the general manifold case N without much difficulty, once the additional technicalities are addressed.

In a future paper, we will investigate the smoothness at the critical times outside of bubbling points. This issue is quite delicate due to the non-local nature of the equation at hand and requires more care than in Struwe [41] where suitable localisations are immediately available. Additionally, the global existence result in Theorem 1.10 is still new in the case of arbitrary target manifolds, as previous papers such as Schikorra-Sire-Wang [36] have either only dealt with special target manifolds or with solutions for a potentially short amount of time as in the author's previous work [47, 48].

The paper is organised as follows: In Section 2, we will recall the some of the notions used throughout the paper, in particular fractional gradients and divergences, Triebel-Lizorkin spaces on S^1 and the fractional Laplacian. In Section 3, we provide proofs and precise statements for properties of the half-harmonic gradient flow regarding bubbling and global existence. Namely, in Section 3.1, we provide a new proof of local existence of solutions based on a fixed point argument. Afterwards, we investigate bubbling in finite time in Section 3.2, studying the concentration of energy, resulting in a proof of Theorem 1.9 which was previously not known. Lastly, Section 3.3 deals with extensions and other ideas to find global solutions to our main PDE, proving Theorem 1.10.

2. Preliminaries

In this brief preliminary section, we shall introduce some of the most important notions used throughout. In particular, we discuss Triebel-Lizorkin spaces on S^1, provide a short summary of fractional gradient and fractional divergences based on Mazowiecka-Schikorra [25] and finally recall some of the main results associated with the fractional heat flow. Most of the results are also discussed in [47] and the references provided throughout the upcoming subsections.

2.1. Triebel-Lizorkin spaces on the unit circle and fractional Laplacians

Firstly, we shall discuss Triebel-Lizorkin spaces on the unit circle $S^1 \subset \mathbb{R}^2$ and recall some of the most important properties of and formulas for the fractional Laplacian. Much of the current presentation is due to Prats [29], Prats-Saksman [30] and Schmeisser-Triebel [38]. Throughout, we shall use the distance

$$|x-y| = 2\left|\sin\left(\frac{x-y}{2}\right)\right|, \tag{2.1}$$

for all $x, y \in S^1 \simeq \mathbb{R} \mod 2\pi\mathbb{Z}$.

We define for any $f : S^1 \to \mathbb{R}$ the following quantity based on the fractional gradients $d_s f(x,y) = \frac{f(x)-f(y)}{|x-y|^s}$

$$\mathcal{D}_{s,q}(f)(x) := \left(\int_{S^1} |d_s f(x,y)|^q \frac{dy}{|x-y|}\right)^{1/q}, \tag{2.2}$$

for all $1 \leq q < \infty$ and $0 < s < 1$. We refer to the next subsection for some details on the fractional gradient $d_s f$. Then

$$\|f\|_{\dot{W}^{s,(p,q)}(S^1)} := \|\mathcal{D}_{s,q}(f)(x)\|_{L^p(S^1)}, \tag{2.3}$$

for every $1 \leq p \leq \infty$. If $p = q$, these spaces correspond to the usual homogeneous Gagliardo-Sobolev spaces $\dot{W}^{s,p}(S^1)$, see Prats [29], Prats-Saksman [30].

Furthermore, we shall denote, as usual, by $\mathcal{D}'(S^1)$ the set of all distributions on S^1 and occasionally use $\mathcal{D}(S^1)$ as an alternative notation for the space $C^\infty(S^1)$. Finally, $\hat{f}(k)$ shall always be the k-th Fourier coefficient of f, for all $f \in \mathcal{D}'(S^1)$. It is formally defined by

$$\hat{f}(k) := \frac{1}{2\pi}\langle f, e^{-ikx}\rangle = \frac{1}{2\pi} f\left(e^{-ikx}\right), \quad \forall k \in \mathbb{Z}. \tag{2.4}$$

In Schmeisser-Triebel [38], it is shown that one may define Triebel-Lizorkin spaces for S^1, denoted by $F^s_{p,q}(S^1)$, completely analogous to the usual space $F^s_{p,q}(\mathbb{R}^n)$ for any parameters $s \in \mathbb{R}$ and $p, q \in [1, \infty[$

$$F^s_{p,q}(S^1) := \{f \in \mathcal{D}'(S^1) \,\big|\, \|f\|_{F^s_{p,q}} < +\infty\}. \tag{2.5}$$

The norm is defined by

$$\|f\|_{F^s_{p,q}} := \left\| \left\| \left(\sum_{k\in\mathbb{N}} 2^{js} \varphi_j(k) \hat{f}(k) e^{ikx} \right)_{j\in\mathbb{N}} \right\|_{l^q} \right\|_{L^p(S^1)}, \tag{2.6}$$

for a suitable partition of unity $(\varphi_j)_{j\in\mathbb{N}}$ consisting of smooth, compactly supported functions on $\mathbb{R}$ with the properties

$$\operatorname{supp}\varphi_0 \subset B_2(0), \quad \operatorname{supp}\varphi_j \subset \{x \in \mathbb{R} \mid 2^{j-1} \leq |x| \leq 2^{j+1}\}, \forall j \geq 1,$$

as well as

$$\forall k \in \mathbb{N} : \sup_{j\in\mathbb{N}} 2^{jk} \|D^k \varphi_j\|_{L^\infty} \lesssim 1.$$

One may develop, as for example seen in [38, Chapter 3], a complete theory of Triebel-Lizorkin spaces on S^1 and more generally on the n-torus $\mathbb{T}^n$ by following the techniques of these spaces on $\mathbb{R}^n$. We refer for a detailed exposition of the periodic Triebel-Lizorkin spaces to Schmeisser-Triebel [38], but for now it suffices to be aware that all tools and results for Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$ are also available for $F^s_{p,q}(\mathbb{T}^n)$.

It turns out that the fractional gradients are exceptionally useful in studying non-local problems. As an example, the following result found in Prats-Saksman [30] is key to many of our arguments, allowing us to restrict our considerations to fractional gradients rather than fractional Laplacians:

Theorem 2.1 (Theorem 1.2, [30])**.** *Let $s \in (0, 1)$, $p, q \in]1, \infty[$ and $f \in L^p(\mathbb{R})$. Then*

(i) We know $\dot{W}^{s,(p,q)}(\mathbb{R}^n) \subset \dot{F}^s_{p,q}(\mathbb{R}^n)$ together with

$$\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{W}^{s,(p,q)}(\mathbb{R}^n)}. \tag{2.7}$$

(ii) If $p > \frac{nq}{n+sq}$, then we also have the converse inclusion together with

$$\|f\|_{\dot{W}^{s,(p,q)}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}. \tag{2.8}$$

The constants depend on s, p, q, n.

By using the properties in Schmeisser-Triebel [38] for periodic functions and employing Theorem 2.1, we can arrive at the following equivalence with Triebel-Lizorkin spaces for all $1 < q < \infty$ and $1 < p < \infty$

$$\dot{W}^{s,(p,q)}(S^1) = \dot{F}^s_{p,q}(S^1), \tag{2.9}$$

with equivalence of the corresponding seminorms, provided $p > \frac{q}{1+sq}$. As a simple, but important special case, let us observe that if $s = 1/2$ and $q = 2$, then $p > 1$ is the requirement in Theorem 2.1 for the equality of $\dot{F}^{1/2}_{p,2}$ and $\dot{W}^{1/2,(p,2)}$ to hold. Some more details and a proof of one direction of Theorem 2.1 can be found in the appendix of [47].

Finally, we would like to briefly address the fractional Laplacian. The simplest definition is based on the Fourier multiplier properties of the Laplacian itself, leading ultimately to the following definition for the fractional s-Laplacian on Fourier series on S^1

$$\widehat{(-\Delta)^s f}(k) = |k|^{2s} \hat{f}(k), \tag{2.10}$$

for every $k \in \mathbb{Z}$ and all $0 < s < 1$. There is an alternative formulation as a Cauchy principal value, which actually leads to the same operator and is often useful

$$(-\Delta)^s f(x) = C_s \cdot P.V. \int_{S^1} \frac{f(x) - f(y)}{|x-y|^{1+2s}} dy, \tag{2.11}$$

where $C_s > 0$ denotes a suitable constant depending on the parameter s. Additionally, it is of course possible to define the fractional Laplacians also on $\mathbb{R}^n$, leading again to two different characterisations (as a Fourier multiplier and Cauchy principal value) with the same type of formulas. We omit the details and refer to Nezza-Palatucci-Valdinocci [28].

An essential property of function spaces is their behaviour under Fourier multipliers, for example extending results such as Mikhlin's multiplier theorem for L^p-spaces. As the fractional Laplacian is obviously a Fourier multiplier operator, one expects characterisations of the spaces $F^s_{p,q}(S^1)$ based on these operators, compare with the Bessel potentials as in [17, Section 6.1.2][††]. Indeed, one easily sees (Schmeisser-Triebel [38])

$$(-\Delta)^s : \dot{F}^{t+2s}_{p,q} \to \dot{F}^t_{p,q},$$

for all $p, q \in (1, \infty)$ and $t, t + 2s \in \mathbb{R}$. This should not be surprising and follows along the same lines as in the case of Triebel-Lizorkin spaces on $\mathbb{R}^n$. Observe the use of $\dot{F}^s_{p,q}(S^1)$ rather than $F^s_{p,q}(S^1)$ indicating the use of homogeneous Triebel-Lizorkin spaces, which are again defined as usual, see also Schmeisser-Triebel [38].

2.2. *Fractional gradients and divergences*

Next, we would like to discuss in some depth the notion of fractional gradient and its derivatives, like the fractional divergence and certain weighted L^p-spaces. The presentation greatly draws from Mazowiecka-Schikorra [25]:

One may introduce $\mathcal{M}_{od}(\mathbb{R}\times\mathbb{R})$ as the set of all measurable functions $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ with respect to the weighted Lebesgue measure $\frac{dxdy}{|x-y|}$. In complete analogy, we do the same for S^1 instead of $\mathbb{R}$ as the

[††]The Bessel potential of order $z \in \mathbb{C}$ with positive real part is the operator $(Id - \Delta)^{-z/2}$. It is most easily understood in the context of its Fourier multiplier $(1 + |\xi|^2)^{-z/2}$. The Bessel potential space $H^{s,p}$ is the defined as the set of distributions f, such that $(Id - \Delta)^{s/2} f$ is in L^p and these actually coincide with the spaces $F^s_{p,2}$.

domain, denoting this space by $\mathcal{M}_{od}$ if both $\mathbb{R}$ or S^1 are possible as domains. Naturally, the associated L^p-spaces, denoted L^p_{od} are of interest and the defining (semi-)norms are given by

$$\|F\|_{L^p_{od}} := \left(\int \int |F(x,y)|^p \frac{dydx}{|x-y|} \right)^{1/p}, \tag{2.12}$$

for $1 \le p < \infty$. In the integral above, the domain over which the integral is taken could be either $S^1 \times S^1$ or $\mathbb{R} \times \mathbb{R}$. The space $L^\infty_{od}(S^1 \times S^1)$ and $L^\infty_{od}(\mathbb{R} \times \mathbb{R})$ as the sets of essentially bounded functions with the essential supremum as the (semi-)norm. Later on, the following quantity, defined in terms of $F, G \in \mathcal{M}_{od}$, will be useful

$$F \cdot G(x) := \int F(x,y)G(x,y) \frac{dy}{|x-y|}. \tag{2.13}$$

In the special case $F = G$, this becomes

$$F \cdot F(x) = |F|^2(x), \quad |F|(x) := \sqrt{F \cdot F(x)}. \tag{2.14}$$

Of course, this shows

$$\|F\|^2_{L^2_{od}} = \int |F|^2(x)dx.$$

Let us finally turn to the definition of fractional gradients. For a measurable function $f : \mathbb{R} \to \mathbb{R}$ or $f : S^1 \to \mathbb{R}$, we define for $0 \le s < 1$ the fractional s-gradient by

$$d_s f(x,y) = \frac{f(x) - f(y)}{|x-y|^s} \in \mathcal{M}_{od},$$

and the corresponding s-divergence by means of duality, i.e., for $F \in \mathcal{M}_{od}$

$$\langle \operatorname{div}_s F, \varphi \rangle = \int \int F(x,y) d_s\, \varphi(x,y) \frac{dydx}{|x-y|}, \quad \forall \varphi \in C^\infty_c(\mathbb{R}) \text{ or } C^\infty(S^1). \tag{2.15}$$

It is obvious that

$$d_s f(y,x) = -d_s f(x,y).$$

Also, a version of Leibniz' rule holds true

$$d_s\,(fg)\,(x,y) = d_s f(x,y)g(x) + f(y)d_s g(x,y).$$

Naturally, $\operatorname{div}_s F$ is only well-defined in a distributional sense.

Using the notions introduced for functions $F \in L^p_{od}$ and as we have already stated in the subsection before, we do now have

$$\left\| |d_s f|(\cdot) \right\|_{L^p(S^1)} = \|f\|_{\dot{W}^{s,(p,2)}(S^1)}, \tag{2.16}$$

We refer to Theorem 2.1 for the significance of this. Finally, the fractional Laplacian also has a place in the setting of fractional gradients and divergences, behaving much as expected from $\Delta = \operatorname{div} \circ \nabla$

$$(-\Delta)^s f = C_s \operatorname{div}_s d_s f, \tag{2.17}$$

for some constant $C_s > 0$ depending on s. Equation (2.17) has to be read as follows, testing against a smooth function g

$$C_s \int d_s f \cdot d_s g(x) dx = \int g(-\Delta)^s f dx = \int (-\Delta)^{s/2} f (-\Delta)^{s/2} g dx.$$

A key result to establish, for instance, regularity of fractional harmonic maps or the uniqueness of weak solutions to the half-harmonic gradient flow with small initial energy is the fractional Wente-type estimate in Proposition 1.4. In the case where $s = 1/2$ and $p = p' = 2$, we may immediately deduce $F \cdot d_{1/2} g \in H^{-1/2}(\mathbb{R})$ following the Sobolev embedding $\dot{H}^{1/2}(\mathbb{R}) \hookrightarrow BMO(\mathbb{R})$ with analogous estimates. Naturally, the result also remains valid in the case of the domain being S^1:

Lemma 2.2 ([25, 47]). *For $F \in L^2_{od}(S^1 \times S^1)$ and $g \in \dot{H}^{1/2}(S^1)$, we assume that $\operatorname{div}_{1/2} F = 0$. Then $F \cdot d_{1/2} g$ lies in the space $H^{-1/2}(S^1)$ and we have the estimate*

$$\|F \cdot d_{1/2} g\|_{H^{-1/2}(S^1)} \lesssim \|F\|_{L^2_{od}(S^1 \times S^1)} \cdot \|g\|_{\dot{H}^{1/2}(S^1)}. \tag{2.18}$$

We refer to Mazowiecka-Schikorra [25] (case of domain $\mathbb{R}$) or [47] (case of domain S^1) for some details of the proof.

3. Half-harmonic gradient flow

In the current section, we discuss in detail the local existence part of the proof of Theorem 1.7. To be precise, we shall supply the reader with an alternative and not previously published approach to the local existence result for the half-harmonic gradient flow with a smooth u_0 initial datum assuming values in our desired target manifold S^{n-1} for some brief interval of time based on Banach's fixed point theorem. Following this, we shall finally enter the discussion of finite-time bubbling based on concentration estimates in localised Gagliardo seminorms and rescaled versions of the solution and proof Theorem 1.9. This result is entirely new and complements Theorem 1.7. The approach is similar to Struwe [41], however the non-locality renders several steps more difficult and technical, ultimately forcing us to refine an estimate in [47, Lemmas 3.11–3.12] significantly. Only after having this refined estimate at our disposal, which possesses more suitable rescaling properties for the blow-up construction, we are in a position to investigate finite-time bubbling of the solution to the half-harmonic gradient flow. To conclude this section, we discuss global existence of solutions by using two distinct approaches, one producing a solution based on Theorem 1.7 with non-increasing energy, while the other proves existence based on variational arguments, but does not immediately exhibit monotonicity of energy.

3.1. A local existence and regularity result

In this first subsection, our goal is to prove the following Proposition:

Proposition 3.1. *Let $u_0 \in C^\infty(S^1; S^{n-1})$. Then there exists a solution $u : [0, T] \times S^1 \to S^{n-1}$ with $u(0, \cdot) = u_0$ of the equation* (1.28) *which is smooth on some time interval* $[0, T]$*, where* $T = T(u_0) > 0$.

This result was proven by different means in [47] and [48] by introducing the non-linear solution operator H as in (1.33) and applying the inverse function theorem for Banach spaces. Here, we will take a different approach and substitute the use of Fredholm theory by employing an argument based on Banach's fixed point theorem, leading to an alternative proof of the local existence of solutions to the half-harmonic gradient flow.

For the remainder of this section, we shall denote by

$$W^{1,p}_{u_0}([0,T]\times S^1) := \{u \in W^{1,p}([0,T]\times S^1) \big| u(0,\cdot) = u_0\},$$

where $u_0 \in C^\infty(S^1; S^{n-1})$ is a given boundary value. By slight abuse of notation, we shall also denote the extension $u_0(t,x) = u_0(x)$ by just u_0. Indeed, we shall prove:

Lemma 3.2. *Let $u_0 \in C^\infty(S^1; S^{n-1})$ and $p > 4$. Define the following operator*

$$S : W^{1,p}_{u_0}([0,T]\times S^1) \to W^{1,p}_{u_0}([0,T]\times S^1), \tag{3.1}$$

mapping any given $u \in W^{1,p}_{u_0}([0,T]\times S^1)$ to the unique solution $S(u) \in W^{1,p}_{u_0}([0,T]\times S^1)$ of the following system

$$\partial_t S(u) + (-\Delta)^{1/2} S(u) = u|d_{1/2}u|^2, \qquad \forall (t,x) \in [0,T]\times S^1 \tag{3.2}$$

$$S(u)(0,x) = u_0(x), \qquad \forall x \in S^1. \tag{3.3}$$

Given $R > 0$ sufficiently big and $T > 0$ sufficiently small, S is a contraction of the closed ball of radius R around u_0, denoted $B_R(u_0)$, onto itself and thus possesses a fixed point.

We remark at this point that by then employing the same kind of bootstrap procedure as in [47] and [48], we may conclude that every fixed point is smooth, thus Proposition 3.1 holds, once we have established Lemma 3.2. The reader should notice that we tacitly omit any assumption ensuring $u(t,x) \in S^{n-1}$ for $(t,x) \in [0,T]\times S^1$. This is no oversight, but relates to the fact that by employing the maximum principle for parabolic equations immediately proves this from the Eq (1.28), see [47, p.31].

Proof. First, one should observe that $u|d_{1/2}u|^2 \in L^\infty([0,T]\times S^1) \subset L^p([0,T]\times S^1)$. This follows by the Sobolev embedding of $W^{1,p}([0,T]\ \times S^1)$ into the Hölder space $C^\alpha([0,T]\ \times S^1)$ with $\alpha = 1 - \frac{2}{p} > \frac{1}{2}$. This also shows

$$|d_{1/2}u|^2 = \int_{S^1} \frac{|u(x)-u(y)|^2}{|x-y|^2} dy \leq \|u\|^2_{C^\alpha} \int_{S^1} \frac{|x-y|^{2\alpha}}{|x-y|^2} dy \lesssim \|u\|^2_{C^\alpha}. \tag{3.4}$$

Finiteness of the integral follows by $2 - 2\alpha = \frac{4}{p} < 1$. Therefore, the operator S is actually well-defined.

By abuse of notation, we denote by u_0 also its extension to $[0,T]\times S^1$ which is independent of time. Let us consider the following for arbitrary $u, v \in W^{1,p}_{u_0}([0,T]\times S^1)$

$$\begin{aligned}
\|S(u) - S(v)\|_{W^{1,p}} &\lesssim \|u|d_{1/2}u|^2 - v|d_{1/2}v|^2\|_{L^p} \\
&= \|(u-v)|d_{1/2}u|^2 + v d_{1/2}(u-v)\cdot d_{1/2}\, u + v d_{1/2}\, v \cdot d_{1/2}\,(u-v)\|_{L^p} \\
&\lesssim \|u-v\|_{L^p}\|u\|^2_{C^\alpha} + \|v\|_{L^\infty}\||d_{1/2}(u-v)|\|_{L^p}\,(\|u\|_{C^\alpha} + \|v\|_{C^\alpha}) \\
&\lesssim T^{1/p}\|u-v\|_{L^\infty}\|u\|_{W^{1,p}} + \|v\|_{W^{1,p}} \cdot T^{1/p}\|u-v\|_{C^\alpha} \cdot (\|u\|_{W^{1,p}} + \|v\|_{W^{1,p}}) \\
&\lesssim (R + \|u_0\|_{W^{1,p}})^2 \cdot T^{1/p} \cdot \|u-v\|_{W^{1,p}},
\end{aligned} \tag{3.5}$$

where we emphasise that all estimates have no further dependence on T and we applied (3.4) on the third line. Independence of T in the constants may be seen by mirror-extensions and applying the Sobolev embeddings on potentially larger sets. Thus, we may conclude, after fixing $R > 0$

$$S \text{ is a contraction, if } T \text{ is sufficently small in dependence on } R.$$

Thus, it remains to be seen that provided $R > 0$ is sufficiently large, then for every $u \in B_R(u_0)$, we also have

$$S(u) \in B_R(u_0).$$

To see this, we have to consider the difference

$$d := \|u_0 - S(u_0)\|_{W^{1,p}}.$$

We define for now $R = 2d$ and then choose $T > 0$ so small, that the Lipschitz constant in (3.5) is $1/2$ for the fixed value of R. Let us notice that for any $u \in B_R(u_0)$, we have

$$\begin{aligned}
\|u_0 - S(u)\|_{W^{1,p}} &\leq \|u_0 - S(u_0)\|_{W^{1,p}} + \|S(u_0) - S(u)\|_{W^{1,p}} \\
&\leq \frac{R}{2} + \frac{1}{2}\|u_0 - u\|_{W^{1,p}} \\
&\leq \frac{R}{2} + \frac{R}{2} = R,
\end{aligned} \tag{3.6}$$

thus

$$S(u) \in B_R(u_0).$$

This now enables us to apply Banach's fixed point theorem for contractions, as $W^{1,p}_{u_0}([0,T] \times S^1)$ is a complete metric space, and therefore concludes the proof of Lemma 3.2. □

3.2. Bubbling-analysis and concentration of energy

In the current section, we will first study the concentration of energy in greater detail and with more precise estimates. Two main result will come in two parts. Firstly, we shall improve Lemma 3.16 in [47] to Lemma 3.4, leading to rescaling properties consistent with the requirements for a blow-up construction. For the reader's convenience, we recall that $B_R(x_0)$ denotes the ball of radius $R > 0$ with respect to the natural metric (2.1) on S^1 around a given point $x_0 \in S^1$ and state here the result in Lemma 3.16 of [47]:

Lemma 3.3 ([47, Lemma 3.16]). *There exist $C > 0$ not depending on R, u, T, such that for any smooth u on $[0,T] \times S^1$ and $0 < R < 1$, the following estimate holds for all $x_0 \in S^1$*

$$\begin{aligned}
\int_0^T \int_{B_{\frac{3}{4}R}(x_0)} |(-\Delta)^{1/4} u|^4 dxdt \leq C \sup_{0 \leq t \leq T} \int_{B_R(x_0)} |(-\Delta)^{1/4} u(t)|^2 dx \\
\cdot \left(\int_0^T \int_{B_R(x_0)} |(-\Delta)^{1/2} u|^2 dxdt + \frac{1}{R^2} \int_0^T \int_{S^1} |(-\Delta)^{1/4} u|^2 dxdt \right),
\end{aligned} \tag{3.7}$$

by density the same result applies for all $u \in H^1([0,T] \times S^1)$, and all boundary terms $u_0 = u(0,\cdot) \in H^{1/2}(S^1)$, with bounded 1/2-Dirichlet energy. Similarly, we have

$$\int_0^T \int_{S^1} |(-\Delta)^{1/4} u|^4 dxdt \lesssim \sup_{0 \leq t \leq T, x_0 \in S^1} \int_{B_R(x_0)} |(-\Delta)^{1/4} u(t)|^2 dx \cdot \left(\int_0^T \int_{S^1} |(-\Delta)^{1/2} u|^2 dxdt + \frac{1}{R^3} \int_0^T \int_{S^1} |(-\Delta)^{1/4} u|^2 dxdt \right). \quad \square \tag{3.8}$$

The improvement will be in the order of power of R that occurs and this is indeed crucial for a non-local rescaling argument to work. Namely, we shall show that the factor R^{-3} in (3.8) may be replaced by R^{-2} which allows for suitable rescaling and a blow-up procedure. Secondly, we will connect the condition (1.29) to an analogous condition for the localised energy in balls, obtaining that the localised Gagliardo-seminorms are bounded from below if (1.29) holds. This is contained in Proposition 3.6. Observe that due to the non-local nature of the 1/4-Laplacian, $\varepsilon(R; u, t)$, defined in (1.30), takes into account not only value of $u(t,x)$ in a ball, but on the entire S^1. However, contributions "far away" are not as important (these are dealt with by enlarging the balls under consideration) and thus we may restrict our attention to the local Gagliardo seminorms on balls. Combining these two results will ultimately provide a proof of Theorem 3.7 and establishing a first bubbling property for the half-harmonic gradient flow.

3.2.1. Improving Lemma 3.3

In this subsection, we shall prove the following refinement of Lemma 3.16 in [47]:

Lemma 3.4. *There exist $C > 0$ not depending on R, u, T, such that for any solution $u \in H^1([0,T] \times S^1)$ of (1.28) on $[0,T] \times S^1$ with initial datum $u(0,\cdot) = u_0 \in H^{1/2}(S^1)$ and $0 < R < 1/2$, the following estimate holds fo all $x_0 \in S^1$*

$$\int_0^T \int_{B_{\frac{3R}{4}}(x_0)} |(-\Delta)^{1/4} u|^4 dxdt \leq C \sup_{0 \leq t \leq T} \int_{B_R(x_0)} |(-\Delta)^{1/4} u(t)|^2 dx \cdot \left(\int_0^T \int_{B_R(x_0)} |(-\Delta)^{1/2} u|^2 dxdt + \frac{1}{R^2} \int_0^T \int_{S^1} |(-\Delta)^{1/4} u|^2 dxdt \right). \tag{3.9}$$

Similarly, we have

$$\int_0^T \int_{S^1} |(-\Delta)^{1/4} u|^4 dxdt \lesssim \sup_{0 \leq t \leq T, x \in S^1} \int_{B_R(x)} |(-\Delta)^{1/4} u(t)|^2 dx \cdot \left(\int_0^T \int_{S^1} |(-\Delta)^{1/2} u|^2 dxdt + \frac{1}{R^2} \int_0^T \int_{S^1} |(-\Delta)^{1/4} u|^2 dxdt \right). \tag{3.10}$$

Proof. Let us suppose that $u \in C^\infty([0,T] \times S^1)$, the general case follows by a density argument. The key observation lies in the following estimate: In [47, Eq (87)], we used the rather crude estimate

$$\int_0^T \int_{S^1} \left| P.V. \int_{S^1} (-\Delta)^{1/4} u(y) \frac{\varphi(x) - \varphi(y)}{|x-y|^{3/2}} dy \right|^2 dxdt \tag{3.11}$$

$$\lesssim \int_0^T \int_{S^1} \left(\int_{S^1} |(-\Delta)^{1/4} u(y)\,|^2 \frac{1}{|x-y|^{1/2}} dy \cdot \int_{S^1} \frac{|\varphi(x) - \varphi(y)\,|^2}{|x-y|^{5/2}} dy \right) dxdt \tag{3.12}$$

$$\lesssim \frac{1}{R^2} \int_0^T \int_{S^1} |(-\Delta)^{1/4} u(y)\,|^2 dydt, \tag{3.13}$$

where φ is a cut-off function on some subset $B_R(x_0)$, $x_0 \in S^1$. In [47, p.36], we now chose a family of balls $B_{\frac{3}{4}R}(x_j)$, $j \in I$, covering the entire circle S^1 and such that only finitely many of the larger balls $B_j := B_R(x_j)$ contain any given point $x_0 \in S^1$, the number of balls containing an arbitrary x_0 being bounded independent of R. The estimate (3.8) is then obtained by summing the inequalities (3.13) over all $j \in I$ with φ_j instead of φ where φ_j is chosen such that

$$\operatorname{supp} \varphi_j = B_{\frac{7}{8}R}(x_j), \quad \varphi_j\big|_{B_{\frac{3}{4}R}(x_j)} = 1, \quad |\nabla \varphi_j| \le C \cdot \frac{1}{R}, \tag{3.14}$$

with C independent of R and j. In fact, the functions $\varphi_j(x) = \varphi((x - x_j)/R)$ can be realised as rescalings of a given cut-off φ with support in $B_{\frac{7}{8}}(0)$. It should be noted, that then for all $x \notin B_1(0)$ and $y \in \operatorname{supp} \varphi$ we have

$$|x - y| \ge \frac{1}{8},$$

and, by the formula for φ_j, also for $x \notin B_j$, $y \in \operatorname{supp} \varphi_j$, we similarly have

$$|x - y\,| \ge \frac{1}{8} R. \tag{3.15}$$

We may sometimes write φ_{B_j} for the cut-off φ_j.

Instead of using (3.13) as in [47] to deduce regularity, we will now continue with (3.12) and obtain a more refined estimate with more suitable behaviour under rescalings by lowering the power of $1/R$ appearing in (3.13). For each fixed value $x \in S^1$, we can consider, after summing over all covering balls, the following sum remains to be estimates in (3.12)

$$\sum_{j \in I} \frac{|\varphi_j(x) - \varphi_j(y)|^2}{|x-y|^{5/2}}. \tag{3.16}$$

Here, φ_j are the cut-offs to a covering I as in (3.14).

The covering I consists in taking balls of radius R around $\frac{8\pi}{3R}$ points which are evenly distributed along the circle. One can immediately notice that each point of S^1 lies in at most 3 of the balls included in the covering I. Let us now consider two cases.

Case 1: If $x \in B_j$, we utilise the Standard-Lipschitz estimate

$$\frac{|\varphi_j(x) - \varphi_j(y)|^2}{|x-y|^{5/2}} \le \|\nabla \varphi\|^2_{L^\infty} \frac{1}{|x-y|^{1/2}} \lesssim \frac{1}{R^2} \cdot \frac{1}{|x-y|^{1/2}} \tag{3.17}$$

Notice that $x \in B_j$ will only hold true for only finitely many of the $j \in I$, since we have seen that by evenly distributing centers, each point lies in only 3 of the balls included in I. So, by integrating over S^1 and exploiting the integrability of $1/|x-y|^{1/2}$ on S^1 in y, we deduce that the contribution of these terms in (3.16) may be bounded by $1/R^2$.

Case 2: Let us now consider all balls B_j with $x \notin B_j$. We shall denote this collection by $\mathcal{B}$. In this case, we have of course $\varphi_j(x) = 0$ due to the supports of our cut-off functions (3.14). By choice of the covering I as a collection of balls with evenly distributed centers and radius R around S^1, it is clear that then

$$|x - y| \geq \frac{1}{8}R, \tag{3.18}$$

using (3.15) for any y with $\varphi_j(y) \neq 0$ and thus $y \in \operatorname{supp}\varphi_j$. Clearly, only points y with this property may contribute to the summand, as due to $\varphi_j(x) = 0$, we find

$$\frac{|\varphi_j(x) - \varphi_j(y)|^2}{|x-y|^{5/2}} = \frac{\varphi_j(y)^2}{|x-y|^{5/2}},$$

for a given $j \in I$. So as $x \notin B_j$ and $y \in \operatorname{supp}\varphi_j$, we may deduce (3.18).

Now among all balls with $B_j \in \mathcal{B}$, there are exists one ball on each side of x (i.e., one each by going in clockwise and in counterclockwise direction along the circle S^1) which have minimal distance to the point x[‡‡]. We call these two balls $B_1^{(1)}, B_2^{(1)} \in \mathcal{B}$ the (next-)closest balls and denote by $\mathcal{B}_1$ the subset obtained by removing $B_1^{(1)}, B_2^{(1)}$ from $\mathcal{B}$. Every ball in $\tilde{B} \in \mathcal{B}_1$ must be further away from x than either $B_1^{(1)}$ or $B_2^{(1)}$, thus containing either the center of $B_1^{(1)}$ or the center of $B_2^{(1)}$ along the path between the center $\tilde{M}$ of $\tilde{B}$ and x. The length of the path between $\tilde{M}$ and x can be estimated by noticing that this path must pass through the center $M_1^{(1)}$ or $M_2^{(1)}$ of $B_1^{(1)}$ or $B_2^{(1)}$, then through the point $x_1^{(1)}$ or $x_2^{(1)}$ which denotes the point in the support of $\varphi_{B_1^{(1)}}$ or $\varphi_{B_2^{(1)}}$ respectively with minimal distance to x and finally through x. Therefore, the length of this path must be at least

$$|\tilde{M} - x| = |\tilde{M} - M_l^{(1)}| + |M_l^{(1)} - x_l^{(1)}| + |x_l^{(1)} - x| \geq \frac{3}{4}R + \frac{7}{8}R + \frac{1}{8}R, \tag{3.19}$$

where we used that all three path segments may be added to obtain a lower bound of the path length and that centers of balls are at least $3R/4$ apart by the uniform distribution along the circle S^1. Also, keep in mind that by (3.14), we have that the center of $B_l^{(1)}$ and $x_l^{(1)}$ will be $\frac{7}{8}R$ apart. In (3.19), l denotes the index among 1 and 2 belonging to the ball among $B_1^{(1)}$ and $B_2^{(1)}$ which has its center along the shortest path between $\tilde{M}$ and x. As a result, the distance between $\operatorname{supp}\varphi_{\tilde{B}}$ and x must be at least $R/8 + 3R/4$ due to the fact that along the shortest path between the support of $\varphi_{\tilde{B}}$ and x must at least be

$$|\tilde{x} - x\,| \geq |x - \tilde{M}| - |\tilde{M} - \tilde{x}| \geq \frac{3}{4}R + \frac{7}{8}R + \frac{3}{4}R - \frac{7}{8}R = \left(\frac{1}{8} + \frac{3}{4}\right)R, \tag{3.20}$$

where we denote by $\tilde{x} \in \operatorname{supp}\varphi_{\tilde{B}}$ the closest point to x. This implies that the distance between any ball $\tilde{B} \in \mathcal{B}_1$ and x must be bigger or equal to $(1/8 + 3/4)R$. As a result, for j associated with a ball in $\mathcal{B}_1$ and y with $\varphi_j(y) \neq 0$, we have

$$|x - y| \geq \left(\frac{1}{8} + \frac{3}{4}\right)R$$

Now we iterate the process: Among all balls in $\mathcal{B}_1$, choose again the balls in clockwise and counterclockwise direction which have minimal distance from x and denote them by $B_1^{(2)}, B_2^{(2)}$. Then,

[‡‡]Minimal distance refers here to minimizing $\inf_{z \in B_j} |x - z|$ among all balls $B_j \in \mathcal{B}$. Notice that as $x \notin B_j$ for each such ball and there is finitely many balls in $\mathcal{B}$, this minimal distance must be strictly positive.

$\mathcal{B}_2$ denotes the set of balls $\mathcal{B}_1$ after removing $B_1^{(2)}, B_2^{(2)}$. Arguing as above, for any $\tilde{B} \in \mathcal{B}_2$, let $\tilde{M}$ be its center and then we know

$$|\tilde{M} - x| = |\tilde{M} - M_l^{(2)}| + |M_l^{(2)} - x_l^{(2)}| + |x_l^{(2)} - x| \geq \frac{3}{4}R + \frac{7}{8}R + \left(\frac{1}{8} + \frac{3}{4}\right)R,$$

where again, $M_l^{(2)}$ denotes the center of $B_1^{(2)}$ and $B_2^{(2)}$ which lies on the shortest path between $\tilde{M}$ and x and $x_l^{(2)}$ denotes the point in $\operatorname{supp} \varphi_{B_l^{(2)}}$ which is closest to x. Again, the distance between the closest point $\tilde{x} \in \operatorname{supp} \varphi_{\tilde{B}}$ and x is then estimated from below as follows

$$|\tilde{x} - x\,| \geq |\tilde{M} - x| - |\tilde{x} - \tilde{M}| \geq \left(\frac{1}{8} + \frac{3}{4}\right)R + \frac{7}{8}R + \frac{3}{4}R - \frac{7}{8}R = \left(\frac{1}{8} + 2 \cdot \frac{3}{4}\right)R$$

Now repeat the argument, at each step identifying the closest balls $B_1^{(k)}, B_2^{(k)}$ in $\mathcal{B}_{k-1}$ and then defining $\mathcal{B}_k$ by excluding them from the set, until $\mathcal{B}_k = \emptyset$. Then, by integration of the sum (3.16) and estimating the distances $|x - y|$ for each $j \in I$ by using the corresponding lower bound as obtained by our iterative procedure above, we arrive at the following sum

$$\sum_{j \in \mathbb{N}_0} \frac{1}{R^{3/2}} \cdot \frac{2}{\left(\frac{1}{8} + j \cdot \frac{3}{4}\right)^{3/2}} \lesssim \frac{1}{R^{3/2}} \lesssim \frac{1}{R^2}, \quad \forall R \in]0, 1/2[.$$

Adding the estimates for Case 1 and Case 2 above, we obtain the improved estimate (3.10) by concluding just as in [47, Lemma 3.16]. □

Such a result also allows for a slightly more refined version of Lemma 3.19 in [47]:

Lemma 3.5. *There exists $\varepsilon_1 > 0$ such that for any $u \in H^1([0,T] \times S^1) \cap L^\infty([0,T]; H^{1/2}(S^1))$ solving*

$$\partial_t u + (-\Delta)^{1/2} u \perp T_u N \quad \text{in } \mathcal{D}'([0,T] \times S^1),$$

with values in N and any $R < 1/2$, there holds

$$\int_0^T \int_{S^1} |\nabla u|^2 dx dt \leq C E(u_0)\left(1 + \frac{T}{R^2}\right), \tag{3.21}$$

with C independent of u, T, R, provided $\varepsilon(R) < \varepsilon_1$. Here, $u(0, \cdot) = u_0 \in H^{1/2}(S^1; N)$ is the initial value.

The proof is as in Struwe [41, Lemma 3.7] or [47, Lemma 3.18] (where we used the weaker estimate [47, Lemma 3.16, Eq (83)]), the only change lies in the application of Lemma 3.4 instead of Lemma 3.16 in [47]. Therefore, the details are omitted. This improved version will be crucial in the blow-up procedure, as it will enable us to deduce that the H^1-energy of u on specific subsets is bounded and thus leads to a good solution after extracting a weakly convergent subsequence, since we have now an appropriate scaling-behaviour of time and space variable.

3.2.2. Lower bound for local Gagliardo seminorms

Next, we would like to establish a connection between the concentration condition (1.29) at blow-up points and the Gagliardo-seminorms at the same points. The intuition behind the estimate is that whenever $1/2$-Dirichlet energy concentrates close to a point, then also the localised Gagliardo seminorm around the same point should concentrate. Due to the non-local nature, however, contributions from further away may still be significant, forcing us to include a bigger domain in the estimate of the seminorm than in the $1/2$-energy to avoid concentration in "neck regions" that we would otherwise not account for. The key connection is the following:

Proposition 3.6. *Let $\varepsilon > 0$ be given and K big enough depending on ε. Assume that $u \in H^{1/2}(S^1)$ with $|u| \leq 1$, $x_0 \in S^1$ is a point on the unit circle and that*

$$\int_{B_R(x_0)} |(-\Delta)^{1/4} u|^2 dx \geq \varepsilon,$$

for some $R < 2^{-K-1}$. Then there is a $\delta > 0$ depending only on K and ε, such that

$$\int_{B_{2^K R}(x_0)} \int_{B_{2^K R}(x_0)} \frac{|u(x) - u(y)\,|^2}{|x - y|^2} dy dx \geq \delta.$$

In the proof below, we shall see what sufficiently big means for K, see (3.22). Also, the same proof continues to hold for arbitrary bounded u with δ depending also on $\|u\|_{L^\infty}$ which follows by a straightforward rescaling argument.

Proof. Firstly, we observe that the independence of R and x_0 of δ may be obtained by rescaling and rotations, possibly after using stereographic projection. So we do not have to worry about such dependencies.

Let us argue by contradiction. Assume the statement was wrong, then there exists a sequence $u_k \in H^{1/2}(S^1)$ of bounded functions, such that

$$\int_{B_R(x_0)} |(-\Delta)^{1/4} u_k|^2 dx \geq \varepsilon; \quad \int_{B_{2^K R}(x_0)} \int_{B_{2^K R}(x_0)} \frac{|u_k(x) - u_k(y)\,|^2}{|x - y|^2} dy dx < \frac{1}{k}.$$

In particular, we have (up to modifying the u_n by a constant and extracting a subsequence)

$$u_k \to u \quad \text{in } H^{1/2}(B_{2^K R}(x_0)).$$

Adapting the ideas in Nezza-Palatucci-Valdinocci [28, Lem. 5.2, Lem. 5.3, Thm. 5.4], we may extend the $u_k \in H^{1/2}(B_{2^K R}(x_0))$ to $v_k \in H^{1/2}(S^1)$ using reflection and a cut-off, resulting in v_k still being bounded by a common multiple of 1 and such that

$$\|v_k\|_{H^{1/2}(S^1)} \lesssim \|u_k\|_{H^{1/2}(B_{2^K R}(x_0))} \to 0,$$

which also shows

$$\lim_{k\to\infty} \int_{S^1} |(-\Delta)^{1/4} v_k|^2 dx = 0.$$

Thus, to arrive at a contradiction, we just need to show

$$\liminf_{k\to\infty}\int_{B_R(x_0)}|(-\Delta)^{1/4}(u_k-v_k)|^2dx<\varepsilon.$$

This can be easily obtained by observing

$$\begin{aligned}
&\int_{B_R(x_0)}|(-\Delta)^{1/4}(u_k-v_k)|^2dx\\
&\leq\int_{B_R(x_0)}\left(\int_{B_{2^KR}(x_0)^c}\frac{|u_k(y)-v_k(y)|}{|x-y|^{3/2}}dy\right)^2dx\\
&\lesssim\int_{B_R(x_0)}\left(\int_{B_{2^KR}(x_0)^c}\frac{1}{|x-y|^{3/2}}dy\right)^2dx\\
&\leq\int_{B_R(x_0)}\frac{1}{|x\mp 2^KR|}dx\\
&\lesssim|\log\left(1-2^{-K}\right)|\lesssim 2^{-K}<\varepsilon,
\end{aligned}\tag{3.22}$$

provided K was chosen sufficiently large at the beginning, depending on ε. Thus the required statement follows, as this contradicts our assumptions and thus provides the desired contradiction. □

The key feature of Proposition 3.6 lies in the fact that it connects the localised (but still non-local) Gagliardo-seminorms to the concentration of energy. The power of 2 that appears is due to the non-linearity and ensures that "not too much" energy is lost by restricting to balls. Ensuring that energy is stored in balls of sufficiently small radius is crucial to obtain half-harmonic maps in the limit.

3.2.3. Bubbling-analysis

Having proved Lemma 3.4 as well as Proposition 3.6, we are now able to study the bubbling process in points where energy accumulates. The analysis is inspired by Struwe [41], but has to take care of the non-local behaviour associated with the fractional Laplacian:

Theorem 3.7. *Let u be a solution to the half-harmonic gradient flow Eq* (1.28) *as in Theorem 1.7 and let $x_0\in S^1$ be a point, such that*

$$\limsup_{t\to T}\int_{B_R(x_0)}|(-\Delta)^{1/4}u|^2dx\geq\varepsilon_1,\quad\forall R>0,\tag{3.23}$$

where $\varepsilon_1>0$ is as in [48, Lemma 4.10]. Then there exists a half-harmonic map $v:\mathbb{R}\to S^{n-1}$, such that

$$u_k\to v\quad\text{weakly in } H^1(\mathbb{R})\text{ and strongly in } H^{1/2}(\mathbb{R}),\tag{3.24}$$

where u_k is a suitable rescaling and translation of the form $u_k(t,x)=u(t_k+R_k^2t,x_k+R_k^2x)$ with .

As stated in the introduction, an analogous result holds for any closed manifold N instead of S^{n-1} as target manifold, up to some technical changes in the formulas. Additionally, we highlight that (1.29) implies (3.23) at a suitable point by choosing subsequences. Therefore, Theorem 3.7 actually concerns the behaviour of functions at the critical time in Theorem 1.7. It should be noted that the number of points x_0 satisfying (3.23) is finite due to the limited amount of energy available, so these points may not accumulate.

Proof. Let us argue along the lines of [41, Theorem 4.3]. The key idea is to rescale u on subintervals of $[0,T[$ and apply the results in Lemma 3.4 and Proposition 3.6 to deduce convergence. Let us always assume that K is chosen large enough to allow for $\varepsilon = \varepsilon_1/2$ in Proposition 3.6 and take $\delta > 0$ to be the associated lower bound for the Gagliardo seminorms.

We now define rescalings as follows. For each $R > 0$, we have

$$\varphi_R : \mathbb{R} \to S^1 \simeq \mathbb{R}/\mathbb{Z} \simeq [-\pi;\pi[, \tag{3.25}$$

with the properties

$$\varphi_R(x) = R^2 x, \quad \forall x \in [-\frac{2^K}{R}, \frac{2^K}{R}]; \quad |\varphi_R'(x)| \leq R^2,$$

and

$$\lim_{x\to\pm\infty} \varphi_R(x) = \pm\pi.$$

The existence of such a function is clear.

Next, we may choose points $(t_k, x_k) \in [0,T[\times S^1$ as is done Struwe [41, Thm. 4.3, p.578], such that $t_k \to T, x_k \to x_0$ as well as

$$E_{R_k}(u(t_k,\cdot), x_k) = \varepsilon_1 = \sup_{0<t\leq t_k, x\in B_r(x_0)} E_{R_k}(u(t,\cdot), x),$$

where $R_k \to 0$ and $r > 0$ is chosen small enough that no other point with the property (3.23) is contained in $B_r(x_0)$. We shall now define

$$u_k : [-\gamma, 0] \times \mathbb{R} \to S^{n-1}, \quad u_k(t,x) := u(t_k + R_k^2 t, x_k + \varphi_{R_k}(x)). \tag{3.26}$$

Here, $\gamma > 0$ (using Lemma 3.5) is chosen in such a way to ensure

$$E_{2R_k}(u(t); x_k) \geq \varepsilon_1/2, \quad \forall t \in [t_k - \gamma R_k^2, t_k].$$

See also [48, Lemma 4.9] for a justification of this fact and compare this with Struwe [41]. To define $x_k + \varphi_{R_k}(x)$, we may use the periodicity of u in the space-variable. The key properties of these functions are their boundedness properties. For example, we have

$$\begin{aligned}
&\int_{-\gamma}^0 \int_{\mathbb{R}} |\nabla u_k(t,x)|^2 dxdt \\
&= \frac{1}{R_k^2} \int_{t_k-\gamma R_k^2}^{t_k} \int_{S^1} |\nabla u(s,y)|^2 |\, \varphi_{R_k}'(\varphi_{R_k}^{-1}(y))|^2 |(\varphi_{R_k}^{-1})'(y)| dyds \\
&= \frac{1}{R_k^2} \int_{t_k-\gamma R_k^2}^{t_k} \int_{S^1} |\nabla u(s,y)|^2 |\, \varphi_{R_k}'(\varphi_{R_k}^{-1}(y))| dyds \\
&\leq \int_{t_k-\gamma R_k^2}^{t_k} \int_{S^1} |\nabla u(s,y)|^2 dyds \lesssim E(u_0),
\end{aligned} \tag{3.27}$$

where we used Lemma 3.5 as well as the choice of points (t_k, x_k) as above. Notice that the chain rule is employed at one point to simplify the expression. Similarly

$$\int_{-\gamma}^0 \int_{B_{2^K/R_k}(0)} |\partial_t u_k(t,x)|^2 dxdt$$

$$
\begin{aligned}
&= R_k^2 \int_{t_k-\gamma R_k^2}^{t_n} \int_{B_{2KR_k}(x_k)} |\partial_t u(s,y)\,|^2 |(\varphi_{R_k}^{-1})'(y)\,| dy dt \\
&= \int_{t_k-\gamma R_k^2}^{t_k} \int_{B_{2KR_k}(x_k)} |\partial_t u(s,y)\,|^2 dy dt \\
&\leq \int_{t_k-\gamma R_k^2}^{t_k} \int_{S^1} |\partial_t u(s,y)\,|^2 dy dt \lesssim E(u_0).
\end{aligned} \tag{3.28}
$$

One may now extract convergent subsequences and times τ_k as in Struwe [41] satisfying

$$
\int_{\mathbb{R}} |\nabla u_k(\tau_k, x)|^2 dx \lesssim E(u_0), \quad \lim_{k\to+\infty} \int_{B_{2K/R_k}(0)} |\partial_t u_k(\tau_k, x)|^2 dx = 0
$$

Thus, choosing subsequences, we end up with $u_k(\tau_k, \cdot)$ converging weakly in $H^1(S^1)$ and strongly in $H^{1/2}(\mathbb{R})$ to a map $v \in H^1(\mathbb{R})$. Choosing the subsequence to be pointwise convergent a.e., we may even deduce

$$
v \in S^{n-1} \quad \text{a.e..}
$$

Furthermore, Proposition 3.6 shows, thanks to the concentration of energy, that

$$
\delta \leq \int_{B_{2KR_k}(x_k)} \int_{B_{2KR_k}(x_k)} \frac{|u(t,x) - u(t,y)|^2}{|x-y|^2} dy dx,
$$

for all $t \in [t_k - \gamma R_k^2, t_k]$. This also shows

$$
\delta \leq \int_{B_{2K/R_k}(x_k)} \int_{B_{2K/R_k}(x_k)} \frac{|u_k(\tau_k,x) - u_k(\tau_k,y)|^2}{|x-y|^2} dy dx.
$$

Thus, by passing to the limit as $k \to \infty$

$$
E_{1/2}(v) \geq \delta > 0,
$$

and so v may not be constant. It remains to check that v is actually half-harmonic. This is however an immediate consequence of the original equation

$$
\partial_t u + (-\Delta)^{1/2} u = u |d_{1/2} u|^2.
$$

Namely, since for τ_k, we have

$$
\partial_t u(\tau_k) \to 0,
$$

as $k \to \infty$ in $L^2_{loc}(\mathbb{R})$, it remains to prove convergence of the other terms. We have for any $\varphi \in C_c^\infty(\mathbb{R})$

$$
\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{R}} d_{1/2} v(x,y) d_{1/2}\varphi(x,y) \frac{dydx}{|x-y|} \\
&= \lim_{k\to\infty} \int_{\mathbb{R}} \int_{\mathbb{R}} d_{1/2} u_k(\tau_k)(x,y) d_{1/2}\varphi(x,y) \frac{dydx}{|x-y|} \\
&= \lim_{k\to\infty} \int_{B_{2K/R_k}} \int_{B_{2K/R_k}} d_{1/2} u_k(\tau_k)(x,y) d_{1/2}\varphi(x,y) \frac{dydx}{|x-y|}
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{k\to\infty} \int_{B_{2K_{R_k}}(x_n)} \int_{B_{2K_{R_k}}(x_k)} d_{1/2}u(\tau_k)(x,y) d_{1/2}\left(\varphi \circ \varphi_{R_k}^{-1}\right)(x,y) \frac{dydx}{|x-y|} \\
&= \lim_{k\to\infty} \int_{S^1} \int_{S^1} d_{1/2}u(\tau_k)(x,y) d_{1/2}\left(\varphi \circ \varphi_{R_k}^{-1}\right)(x,y) \frac{dydx}{|x-y|} \\
&= \lim_{k\to\infty} \left(\int_{S^1} -\partial_t u \cdot \varphi \circ \varphi_{R_k}^{-1} dx + \int_{S^1} u(\tau_k) |d_{1/2}u(\tau_k)|^2 \cdot \varphi \circ \varphi_{R_k}^{-1} dx \right) \\
&= \lim_{k\to\infty} \int_{S^1} u(\tau_k) |d_{1/2}u(\tau_k)|^2 \cdot \varphi \circ \varphi_{R_k}^{-1} dx \\
&= \lim_{k\to\infty} \int_{B_{2K_{R_k}}(x_k)} u(\tau_k) \int_{B_{2K_{R_k}}(x_k)} |d_{1/2}u(\tau_k)(x,y)\,|^2 \frac{dy}{|x-y|} \cdot \varphi \circ \varphi_{R_k}^{-1} dx \\
&= \lim_{k\to\infty} \int_{B_{2K/R_k}(x_k)} u_n(\tau_k) \int_{B_{2K/R_k}(x_k)} |d_{1/2}u_k(\tau_k)(x,y)\,|^2 \frac{dy}{|x-y|} \cdot \varphi dx \\
&= \int_{\mathbb{R}} v |d_{1/2}\, v|^2 \varphi,
\end{aligned}
\tag{3.29}
$$

which is the desired equation. Notice that throughout the computations, we used several times that appropriate terms may be omitted due to the boundedness of $u_k(\tau_k)$ and v, leading to omissions of parts of the domain of integration, switching between the distance function on S^1 and $\mathbb{R}$ and similar terms. A crucial observation is that φ is supported on a subdomain of B_{2^K/R_k} for R_k sufficiently small, so the estimates have good bounds everywhere, if k goes to ∞. So we are done, since v solves the half-harmonic map equation and thus is actually smooth, see Da Lio-Pigati [9]. In particular, v may be regarded as a 1/2-harmonic map after composition with the stereographic projection. □

3.3. *Existence of global weak solutions*

Finally, we tackle the global existence problem in full generality and prove Theorem 1.10. The main idea will be that one is easily able to extend solutions on a finite time-interval by using convergence properties as t goes to the critical time. A direct argument shows that the extension by gluing a solution at the critical time for appropriate initial data will give a global solution after at most finitely many such extensions.

3.3.1. Proof by "gluing" of weak solutions

Proof of Theorem 1.10: Let us show that we may extend a solution $u : [0,T[\times S^1 \to S^{n-1}$ to be a weak solution on a slightly bigger time interval. This may be done by first observing that due to the monotone decay of energy

$$
E_{1/2}(u(t)) \leq E_{1/2}(u_0) < +\infty. \tag{3.30}
$$

Therefore, we may deduce that for an appropriate sequence $u(t_k) \to v \in H^{1/2}(S^1)$ with $t_k \to T$. Moreover, since $u \in H^1([0,T[;L^2(S^1))$, we must have convergence

$$
\lim_{t\to T} u(t) = v \quad \text{in } L^2(S^1), \tag{3.31}
$$

due to a standard continuity argument. This also shows uniqueness of v independent of any choice of sequence $t_k \to T$.

Next, we want to estimate the 1/2-energy of v. To do this, let us assume that there is just one bubbling point x_0 at time T (the general case follows analogously, losing energy in finitely many points). Then we have

$$\begin{aligned} E_{1/2}(v) &= \int_{S^1}\int_{S^1} \frac{|v(x)-v(y)|^2}{|x-y|^2} dydx \\ &= \lim_{r\to 0} \int_{S^1\setminus B_r(x_0)}\int_{S^1\setminus B_r(x_0)} \frac{|v(x)-v(y)|^2}{|x-y|^2} dydx \\ &= \lim_{r\to 0} \liminf_{k\to\infty} \int_{S^1\setminus B_r(x_0)}\int_{S^1\setminus B_r(x_0)} \frac{|u(t_k,x)-u(t_k,y)|^2}{|x-y|^2} dydx \\ &\leq \liminf_{k\to\infty} E_{1/2}(u_k) - \varepsilon_0 = \lim_{t\to T} E_{1/2}(u(t)) - \varepsilon_0, \end{aligned} \tag{3.32}$$

where ε_0 denotes a quantum of energy that is concentrated close to x_0. As ε_0 is independent of u and T, we deduce that bubbling may only occur in finitely many points, as the 1/2-energy is decreasing and bounded from below by 0. Thus, we do not have to worry about accumulations of blow-up points.

One concludes now by extending the solution u after T by the main existence result in [48], Theorem 1.7. The fact that we have obtained a weak solution is easily verified by a direct computation based on the L^2-convergence of $u(t)$ as $t \to T$, thus establishing the desired global existence result. Indeed, we assume that $u : [0,+\infty[\times S^1 \to S^{n-1}$ bubbles at time $T = 1$, the general case with finitely many times in which bubbling occur follows completely analogously. Let $\varphi \in C_c^\infty(]0,\infty[\times S^1)$, since we know that the equation holds true for sufficiently small times. Then we have

$$\begin{aligned} &\int_0^\infty\int_{S^1} \partial_t u\cdot\varphi dxdt + \int_0^\infty\int_{S^1} (-\Delta)^{1/2}\, u\cdot\varphi dxdt \\ &= -\int_0^\infty\int_{S^1} u\cdot\partial_t\varphi dxdt + \int_0^\infty\int_{S^1} (-\Delta)^{1/4}u\cdot(-\Delta)^{1/4}\varphi dxdt \\ &= -\int_0^1\int_{S^1} u\cdot\partial_t\varphi dxdt + \int_0^1\int_{S^1} (-\Delta)^{1/4}u\cdot(-\Delta)^{1/4}\varphi dxdt \\ &- \int_1^\infty\int_{S^1} u\cdot\partial_t\varphi dxdt + \int_1^\infty\int_{S^1} (-\Delta)^{1/4}u\cdot(-\Delta)^{1/4}\varphi dxdt \\ &= \int_0^1\int_{S^1} \partial_t u\cdot\varphi dxdt + \int_0^1\int_{S^1} (-\Delta)^{1/4}u\cdot(-\Delta)^{1/4}\varphi dxdt - \int_{S^1} u(1,x)\varphi(1,x)dx \\ &+ \int_1^\infty\int_{S^1} \partial_t u\cdot\varphi dxdt + \int_1^\infty\int_{S^1} (-\Delta)^{1/4}u\cdot(-\Delta)^{1/4}\varphi dxdt + \int_{S^1} u(1,x)\varphi(1,x)dx \\ &= \int_0^1 u|d_{1/2}u\,|^2\varphi dxdt - \int_{S^1} u(1,x)\varphi(1,x)dx + \int_{S^1} u(1,x)\varphi(1,x)dx + \int_1^\infty u|d_{1/2}u\,|^2\varphi dxdt \\ &= \int_0^\infty u|d_{1/2}u\,|^2\varphi dxdt, \end{aligned} \tag{3.33}$$

which proves the fact that u extended as explained yields a global weak solution. The first line equation is just the distributional formulation, later on we use integration by parts on $[0,\tilde{t}]$ and taking limits $\tilde{t} \to T$. Naturally, similar limits are taken for $[\tilde{t},\infty[$. Observe that the boundary terms at time $T = 1$ appear due to the previous discussion of convergence in L^2 and by the boundary value properties of the extension, see Theorem 1.7. We highlight that $u(1,x)$ is defined for the extended solution to be that limit of the $u(t,x)$ in L^2 and weak limit in $H^{1/2}$, as $t \to 1$, see (3.31). Iterating this procedure finitely many times provides therefore a global weak solution. $\square$

A uniqueness statement may also be derived from the results in [47] and [48] for strong solutions, see also Theorem 1.7. However, it should be noted that uniqueness among energy class solution (weak solutions) cannot be proven by our previous arguments and thus requires further investigations. Finally, the existence of finite time bubbling remains open and subject to future investigations. Nevertheless, the result above could potentially provide a suitable regularity statement at bubbling points to understand obstructions to bubbling or build examples of finite-time bubbling in future work. Lastly, the proof could be generalised to arbitrary target manifolds by employing the formulation for the half-harmonic gradient flow in a general target manifold N as in [48].

3.3.2. Proof by variational arguments

In this section, we derive for completeness an alternative proof of the global weak existence of solutions to the half-harmonic gradient flow similar to Theorem 1.10 using techniques from Calculus of Variations similar to Audrito [1, Section 2]. This approach does lead to existence of solutions, however, it leaves open many questions regarding the properties of the solution, most importantly monotonicity of the 1/2-Dirichlet energy and regularity. In particular, if the solutions constructed do not necessarily have monotonically decaying energy, then the solution provides an example of non-uniqueness of solutions to the half-harmonic map equation.

The result to be proven is:

Theorem 3.8. *Let $u_0 \in H^{1/2}(S^1; S^{n-1})$, then there exists a weak solution of* (1.28)*, potentially with increasing* 1/2*-Dirichlet energy on subintervals of time*

$$u : [0, +\infty\,[\, \times S^1 \to S^{n-1},$$

with $u \in L^\infty([0,+\infty[; H^{1/2}(S^1; S^{n-1})) \cap H^1([0,+\infty[; L^2(S^1; S^{n-1}))$.

Proof of Theorem 3.8: The definition of the energy in (3.35) follows Audrito [1]. Let $\varepsilon > 0$ be any positive real number. We define the following space of functions for $s \in]0,1[$ and $1 < p < +\infty$

$$\mathcal{V}^{s,p} := H^1([0,+\infty[; L^2(S^1;\mathbb{R}^n)) \cap L^2_{loc}([0,+\infty[; W^{s,p}(S^1;\mathbb{R}^n)),$$

and use this definition to introduce for any $u_0 \in W^{s,p}(S^1; N)$, where N is a closed submanifold in $\mathbb{R}^n$

$$\mathcal{U}^{s,p}(u_0) := \{u \in \mathcal{V}^{s,p} \mid u(t,x) \in N \text{ a.e.}, u(0) = u_0\}. \tag{3.34}$$

Comparing with Schikorra-Sire-Wang [36], the space (3.34) actually coincides with space in which the solutions constructed there exist. Moreover, we define the following family of energies

$$\mathcal{E}^{s,p}_\varepsilon(u) := \int_0^{+\infty} \int_{S^1} e^{-t/\varepsilon} \left(\varepsilon \cdot |\partial_t u(t,x)|^2 + \frac{2}{p} \cdot \int_{S^1} \left| \frac{u(t,x) - u(t,y)}{|x-y|^s} \right|^p \frac{dy}{|x-y|} \right) dx dt, \tag{3.35}$$

for any $u \in \mathcal{U}^{s,p}(u_0)$. One notices that the energy is actually well-defined and finite in this case. An obvious member of $\mathcal{U}^{s,p}(u_0)$ is the following map

$$u(t,x) := u_0(x),$$

and this shows

$$\inf_{u\in\mathcal{U}^{s,p}(u_0)} \mathcal{E}_\varepsilon^{s,p}(u) \le 2E_{s,p}(u_0)\cdot\int_0^{+\infty} e^{-t/\varepsilon}dt = 2\varepsilon\cdot E_{s,p}(u_0), \tag{3.36}$$

where we use the definition of $E_{s,p}$ as in Schikorra-Sire-Wang [36, p.2]

$$E_{s,p}(u) := \frac{1}{p}\int_{S^1}\int_{S^1}\left|\frac{u(x)-u(y)}{|x-y|^s}\right|^p\frac{dydx}{|x-y|}, \qquad \forall s\in]0,1[, \forall 1<p<+\infty \tag{3.37}$$

Thus, we immeidately see that if $(u_\varepsilon)_{\varepsilon\in]0,1[}$ is a sequence of minimizers, then the energies will become arbitrarily small. Additionally, existence of minimizers can easily be proven by the direct method.

Defining $v(t,x) := u(\varepsilon t, x)$, we see

$$\begin{aligned}
\mathcal{E}_\varepsilon^{s,p}(u) &= \int_0^{+\infty}\int_{S^1} e^{-t/\varepsilon}\left(\varepsilon\cdot|\partial_t u(t,x)|^2 + \frac{2}{p}\cdot\int_{S^1}\left|\frac{u(t,x)-u(t,y)}{|x-y|^s}\right|^p\frac{dy}{|x-y|}\right)dxdt\\
&= \int_0^{+\infty}\int_{S^1} \varepsilon e^{-s}\left(\varepsilon\cdot|\partial_t u(\varepsilon s,x)|^2 + \frac{2}{p}\cdot\int_{S^1}\left|\frac{u(\varepsilon s,x)-u(\varepsilon s,y)}{|x-y|^s}\right|^p\frac{dy}{|x-y|}\right)dxds\\
&= \int_0^{+\infty}\int_{S^1} e^{-s}\left(|\partial_t v(s,x)|^2 + \frac{2\varepsilon}{p}\cdot\int_{S^1}\left|\frac{v(s,x)-v(s,y)}{|x-y|^s}\right|^p\frac{dy}{|x-y|}\right)dxds\\
&=: \mathcal{J}_\varepsilon^{s,p}(v).
\end{aligned} \tag{3.38}$$

Notice that v still lies in $\mathcal{U}^{s,p}(u_0)$ and that by computation above, we know that minimising $\mathcal{E}_\varepsilon$ and minimising $\mathcal{J}_\varepsilon$ is equivalent respecting the reparametrisation in time.

Let us now compute the Euler-Lagrange equation for $\mathcal{E}_\varepsilon^{s,p}$:

Lemma 3.9. *The Euler-Lagrange equation for minimisers $u\in\mathcal{U}^{s,p}(u_0)$ of $\mathcal{E}_\varepsilon^{s,p}$ can be stated as*

$$-\varepsilon\partial_t^2 u(t,x) + \partial_t u(t,x) + \mathrm{div}_s\left(|d_s u(t,x,y)|^{p-2}d_s u(t,x,y)\right) \perp T_uN, \quad \text{in } \mathcal{D}'(]0,+\infty[\times S^1). \tag{3.39}$$

Proof. We take the competitors

$$u_\delta(t,x) := \pi\left(u(t,x)+\delta\varphi(t,x)\right),$$

where $\delta\in\mathbb{R}$ and $\varphi\in C_c^\infty(]0,\infty[\times S^1;\mathbb{R}^n)$. Moreover, π denotes the closest point projection onto N.

If u is a minimizer, then

$$0 = \frac{d}{d\delta}\mathcal{E}_\varepsilon^{s,p}(u_\delta)\Big|_{\delta=0}.$$

Using the explicit formula (3.35) for the energy (observe that u_δ lies in the correct space for every $\delta\in\mathbb{R}$ sufficiently small), one can differentiate immediately (we use $d_0u(t,x,y) = u(t,x)-u(t,y)$ to simplify the terms)

$$0 = \int_0^{+\infty}\int_{S^1} e^{-t/\varepsilon}\left(2\varepsilon\partial_t u\cdot\partial_t\left(d\pi(u)\varphi\right) + 2\int_{S^1}\frac{|u(t,x)-u(t,y)\,|^{p-2}}{|x-y\,|^{1+sp}}d_0u(t,x,y)\cdot d_0\left(d\pi(u)\varphi\right)(t,x,y)dy\right)dxdt.$$

If we choose $\psi(t,x) = e^{-t/\varepsilon}\varphi(t,x)$, then

$$0 = \int_0^{+\infty}\int_{S^1}\varepsilon\partial_t u\cdot\partial_t\left(d\pi(u)\psi\right) + \partial_t u\cdot d\pi(u)\psi + \int_{S^1}\frac{|u(t,x)-u(t,y)\,|^{p-2}}{|x-y\,|^{1+sp}}d_0u(t,x,y)\cdot d_0\left(d\pi(u)\psi\right)(t,x,y)dydxdt.$$

So the Euler-Lagrange equation is equivalent to

$$-\varepsilon\partial_t^2 u(t,x) + \partial_t u(t,x) + \mathrm{div}_s\left(|d_s u(t,x,y)|^{p-2} d_s u(t,x,y)\right) \perp T_u N, \qquad \text{in } \mathcal{D}'(]0,+\infty[\times S^1),$$

i.e., up to the term involving the second derivative in time direction we recognise the fractional harmonic gradient flow. This proves (3.39). □

In particular, if $s = 1/2$, $p = 2$, we find the same equation as in [48], up to the second order derivative in t. This is also the case we shall restrict our attention to for now (writing $\mathcal{J}_\varepsilon$ instead of $\mathcal{J}_\varepsilon^{1/2,2}$), the general case for arbitrary fractional harmonic flows may be treated in a completely analogous way, also extending the existence result in Schikorra-Sire-Wang [36] in a wider setting.

The ideas to complete the proof of Theorem 3.8 then are very similar to Audrito [1]. Namely, one may define completely analogously

$$I(t) := \int_{S^1} |\partial_t v(t,x)|^2 dx \tag{3.40}$$

$$R(t) := \varepsilon \cdot \int_{S^1} |d_{1/2} v(t)|(x)^2 dx \tag{3.41}$$

$$E(t) := e^t \int_t^\infty e^{-s}\,(I(s) + R(s))\,ds. \tag{3.42}$$

It is easily observed that for miniizers v, we have $I, R \in L^1_{loc}([0,\infty[)$ and $e^{-s}(I(s)+R(s)) \in L^1([0,\infty[)$. Additionally, $E \in W^{1,1}_{loc}(]0,\infty[) \cap C^0([0,\infty[)$ as well as

$$E' = E - I - R \quad \text{ in } \mathcal{D}'(]0,\infty[).$$

The proof of the following lemma is an immediate adaption of the technique in Audrito [1]:

Lemma 3.10. *Assume v is a minimizer of $\mathcal{J}_\varepsilon$. Then*

$$E'(t) = -2I(t), \qquad \textit{in } \mathcal{D}'(]0,\infty[). \tag{3.43}$$

The proof relies on suitable choices of reparametrisations in time for v and then using minimality of v. Ultimately, this allows us to show:

Lemma 3.11. *For v a minimizer of $\mathcal{J}_\varepsilon$, we have*

$$\int_0^\infty |\partial_t v(t,x)|^2 dx dt \le C\varepsilon, \tag{3.44}$$

as well as for any $t \ge 0$

$$\int_t^{t+1} \int_{S^1} |d_{1/2} u|^2(t,x) dx dt \le C, \tag{3.45}$$

for some constant $C > 0$, depending on u_0, but not ε or v.

Proof. $E(t)$ is necessarily non-increasing due to $I(t) \ge 0$, therefore

$$E(t) \le E(0) = \mathcal{J}_\varepsilon(v), \forall t \ge 0.$$

Additionally, for any given t, we know

$$\int_0^t I(s)ds = \frac{1}{2}\int_0^t E'(s)ds = \frac{1}{2}\left(E(0) - E(t)\right) \leq \frac{1}{2}E(0) \leq \frac{C}{2}\varepsilon,$$

by using (3.36). Letting $t \to \infty$ proves (3.44) by using (3.40).

The remaining part of the proof requires us to use (3.41) as well as

$$\begin{aligned} \int_t^{t+1} R(s)ds &= e^{t+1}\int_t^{t+1} e^{-s}R(s)ds \\ &\leq e^{t+1}\int_t^{t+1} e^{-s}\left(I(s) + R(s)\right)ds \qquad (3.46) \\ &\leq e \cdot E(t) \leq Ce \cdot \varepsilon, \qquad (3.47) \end{aligned}$$

again relying on (3.36) and the bound on $E(t)$ established above. □

Thus, to obtain a solution of the half-harmonic gradient flow (which follows thanks to (3.39) after letting $\varepsilon \to 0$), one now just has to rescale the minimizer v back to u and use the following uniform bounds to extract weakly convergent subsequences. Thus, we are done, as we may extract further subsequences converging almost surely pointwise, ensuring that the limiting function assumes values only in N.

□

Acknowledgments

The author is grateful to the anonymous referee and Prof. Dr. Francesca Da Lio who both helped improve the presentation and flow of the paper significantly.

Conflict of interest

The author declares no conflict of interest.

References

1. A. Audrito, On the existence and Hölder regularity of solutions to some nonlinear Cauchy-Neumann problems, arXiv:2107.03308.
2. L. Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian, *Commun. Part. Diff. Eq.*, **32** (2007), 1245–1260. https://doi.org/10.1080/03605300600987306
3. K.-C. Chang, W. Y. Ding, R. Ye, Finite-time blow-up of the heat flow of harmonic maps from surfaces, *J. Differential Geom.*, **36** (1992), 507–515. https://doi.org/10.4310/jdg/1214448751
4. R. Coifman, P. Lions, Y. Meyer, S. Semmes, Compensated compactness and Hardy spaces, *Journal de mathématiques pures et appliquées*, **72** (1993), 247–286.

5. F. Da Lio, Compactness and bubbles analysis for half-harmonic maps into spheres, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **32** (2015), 201–224. https://doi.org/10.1016/j.anihpc.2013.11.003

6. F. Da Lio, Fractional harmonic maps into manifolds in odd dimensions > 1, *Calc. Var.*, **48** (2013), 421–445. https://doi.org/10.1007/s00526-012-0556-6

7. F. Da Lio, P. Laurain, T. Rivière, A Pohozaev-type formula and quantization of horizontal half-harmonic maps, arXiv:1607.05504.

8. F. Da Lio, Fractional harmonic maps, In: *Recent developments in nonlocal theory*, Warsaw, Poland: De Gruyter, 2018, 52–80. https://doi.org/10.1515/9783110571561-004

9. F. Da Lio, A. Pigati, Free boundary minimal surfaces: a nonlocal approach, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, **XX** (2020), 437–489. https://doi.org/10.2422/2036-2145.201801_008

10. F. Da Lio, T. Rivière, 3-Commutator estimates and the regularity of 1/2-harmonic maps into spheres, *Anal. PDE*, **4** (2011), 149–190. https://doi.org/10.2140/apde.2011.4.149

11. F. Da Lio, T. Rivière, Sub-criticality of non-local Schrödinger systems with antisymmetric potentials and applications to half-harmonic maps, *Adv. Math.*, **277** (2011), 1300–1348. https://doi.org/10.1016/j.aim.2011.03.011

12. F. Da Lio, A. Schikorra, n/p-Harmonic maps: regularity for the sphere case, *Adv. Calc. Var.*, **7** (2014), 1–26. https://doi.org/10.1515/acv-2012-0107

13. F. Da Lio, A. Schikorra, On regularity theory for n/p-harmonic maps into manifolds, *Nonlinear Anal.*, **165** (2017), 182–197. https://doi.org/10.1016/j.na.2017.10.001

14. J. Davila, M. Del Pino, J. Wei, Singularity formation for the two-dimensional harmonic map flow in S^2, *Invent. Math.*, **219** (2020), 345–466. https://doi.org/10.1007/s00222-019-00908-y

15. J. Eells, J. Sampson, Harmonic mappings of Riemannian manifolds, *Amer. J. Math.*, **86** (1964), 109–160.

16. A. Freire, Uniqueness for the harmonic map flow from surfaces into general targets, *Commentarii Mathematici Helvetici*, **70** (1995), 310–338. https://doi.org/10.1007/BF02566010

17. L. Grafakos, *Modern Fourier analysis*, 2 Eds., New York: Springer, 2009. https://doi.org/10.1007/978-0-387-09434-2

18. M. Grüter, Regularity of weak H-surfaces, *J. Reine Angew. Math.*, **1981** (1981), 1–15. https://doi.org/10.1515/crll.1981.329.1

19. F. Hélein, Régularité des applications faiblement harmoniques entre une surface et une varitée riemannienne, *C. R. Acad. Sci. Paris Sr. I Math.*, **311** (1990), 591–596.

20. F. Hélein, *Harmonic maps, conservation laws and moving frames*, 2 Eds., Cambridge: Cambridge University Press, 2002. https://doi.org/10.1017/CBO9780511543036

21. A. Hyder, A. Segatti, Y. Sire, C. Wang, Partial regularity of the heat flow of half-harmonic maps and applications to harmonic maps with free boundary, *Commun Part. Diff. Eq.*, **47** (2022), 1845–1882. https://doi.org/10.1080/03605302.2022.2091453

22. F. John, *Partial differential equations*, 3 Eds., New York: Springer, 1978. https://doi.org/10.1007/978-1-4684-0059-5

23. J. Jost, *Geometry and physics*, Heidelberg: Springer, 2009. https://doi.org/10.1007/978-3-642-00541-1

24. O. A. Ladyzhenskaya, Solutions “in the large” of the nonstationary boundary value problem for the Navier-Stokes system with two space variables, *Commun. Pure Appl. Math.*, **7** (1959), 427–433. https://doi.org/10.1002/cpa.3160120303

25. K. Mazowiecka, A. Schikorra, Fractional div-curl quantities and applications to nonlocal geometric equation, *J. Funct. Anal.*, **275** (2018), 1–44. https://doi.org/10.1016/j.jfa.2018.03.016

26. V. Millot, Y. Sire, On a fractional Ginzburg-Landau equation and $1/2$-harmonic maps into spheres, *Arch. Rational Mech. Anal.*, **215** (2015), 125–210. https://doi.org/10.1007/s00205-014-0776-3

27. C. Morrey, The problem of plateau on a Riemannian manifold, *Ann. Math.*, **49** (1948), 807–851.

28. E. Nezza, G. Palatucci, E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, *Bulletin des Sciences Mathématiques*, **136** (2012), 521–573. https://doi.org/10.1016/j.bulsci.2011.12.004

29. M. Prats, Measuring Triebel-Lizorkin fractional smoothness on domains in terms of first-order differences, *J. London Math. Soc.*, **100** (2019), 692–716. https://doi.org/10.1112/jlms.12225

30. M. Prats, E. Saksman, A $T(1)$ theorem for fractional Sobolev spaces on domains, *J. Geom. Anal.*, **27** (2017), 2490–2538. https://doi.org/10.1007/s12220-017-9770-y

31. T. Rivière, Le flot des applications faiblement harmoniques en dimension deux, PhD thesis, 1993.

32. T. Rivière, Conservation laws for conformally invariant variational problems, *Invent. Math.*, **168** (2007), 1–22. https://doi.org/10.1007/s00222-006-0023-0

33. T. Rivière, Conformally invariant variational problems, arXiv:1206.2116.

34. J. Sacks, K. Uhlenbeck, The existence of minimal immersions of 2-spheres, *Ann. Math.*, **113** (1981), 1–24. https://doi.org/10.2307/1971131

35. A. Schikorra, Regularity of n/2-harmonic maps into the sphere, *J. Differ. Equations*, **252** (2012), 1862–1911. https://doi.org/10.1016/j.jde.2011.08.021

36. A. Schikorra, Y. Sire, C. Wang, Weak solutions of geometric flows associated to integro-differential harmonic maps, *Manuscripta Math.*, **153** (2017), 389–402. https://doi.org/10.1007/s00229-016-0899-y

37. R. Schoen, S. Yau, Harmonic maps and the topology of stable hypersurfaces and manifolds with non-negative Ricci curvature, *Commentarii Mathematici Helvetici*, **51** (1976), 333–341. https://doi.org/10.1007/BF02568161

38. H. Schmeisser, H. Triebel, *Topics in Fourier analysis and function spaces*, Chichester: J. Wiley, 1987.

39. J. Shatah, Weak solutions and development of singularities of the $SU(2)$ σ-model, *Commun Pure Appl. Math.*, **41** (1988), 459–469. https://doi.org/10.1002/cpa.3160410405

40. Y. Sire, J. Wei, Y. Zheng, Infinite time blow-up for half-harmonic map flow from $\mathbb{R}$ into S^1, arXiv:1711.05387.

41. M. Struwe, On the evolution of harmonic mappings of Riemannian surfaces, *Commentarii Mathematici Helvetici*, **60** (1985), 558–581. https://doi.org/10.1007/BF02567432

42. M. Struwe, On the evolution of harmonic maps in higher dimension, *J. Differential Geom.*, **28** (1988), 485–502. https://doi.org/10.4310/jdg/1214442475

43. M. Struwe, Plateau flow or the heat flow for half-harmonic maps, arXiv:2202.02083.

44. P. Topping, Reverse bubbling and nonuniqueness in the harmonic map flow, *Int. Math. Res. Notices*, **10** (2002), 505–520. https://doi.org/10.1155/S1073792802105083

45. K. Uhlenbeck, Connections with L^p bounds on curvature, *Commun. Math. Phys.*, **83** (1982), 31–42. https://doi.org/10.1007/BF01947069

46. H. Wente, An existence theorem for surfaces of constant mean curvature, *J. Math. Anal. Appl.*, **26** (1969), 318–344. https://doi.org/10.1016/0022-247X(69)90156-5

47. J. Wettstein, Uniqueness and regularity of the fractional harmonic gradient flow in S^{n-1}, *Nonlinear Anal.*, **214** (2022), 112592. https://doi.org/10.1016/j.na.2021.112592

48. J. Wettstein, Existence, uniqueness and regularity of the fractional harmonic gradient flow in general target manifolds, arXiv:2109.11458.