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Research article

Spacelike translating solitons of the mean curvature flow in Lorentzian product spaces with density†

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Abstract: By applying suitable Liouville-type results, an appropriate parabolicity criterion, and a version of the Omori-Yau's maximum principle for the drift Laplacian, we infer the uniqueness and nonexistence of complete spacelike translating solitons of the mean curvature flow in a Lorentzian product space $\mathbb{R}_1 \times \mathbb{P}^n_f$ endowed with a weight function f and whose Riemannian base $\mathbb{P}^n$ is supposed to be complete and with nonnegative Bakry-Émery-Ricci tensor. When the ambient space is either $\mathbb{R}_1 \times \mathbb{G}^n$, where $\mathbb{G}^n$ stands for the so-called n-dimensional Gaussian space (which is the Euclidean space $\mathbb{R}^n$ endowed with the Gaussian probability measure) or $\mathbb{R}_1 \times \mathbb{H}^n_f$, where $\mathbb{H}^n$ denotes the standard n-dimensional hyperbolic space and f is the square of the distance function to a fixed point of $\mathbb{H}^n$, we derive some interesting consequences of our uniqueness and nonexistence results. In particular, we obtain nonexistence results concerning entire spacelike translating graphs constructed over $\mathbb{P}^n$.

Keywords: weighted Lorentzian product spaces; Bakry-Émery-Ricci tensor; drift Laplacian; f-mean curvature; mean curvature flow; spacelike translating solitons; Gaussian space

This paper is dedicated with great esteem and admiration to Rosario Mingione, on the occasion of his 50th birthday.

1. Introduction

Let Σ^n be an n-dimensional connected manifold and let $\overline{M}^{n+1}$ be a $(n+1)$-dimensional Lorentzian manifold. Furthermore, let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a spacelike immersion, that is Σ^n, endowed by the metric induced by $\langle\cdot,\cdot\rangle$ via the map ψ, is a Riemannian manifold. In such a case the map $\psi : \Sigma^n \to \overline{M}^{n+1}$ is said to be a spacelike hypersurface.

The mean curvature flow $\Psi : [0,T) \times \Sigma^n \to \overline{M}^{n+1}$ of the spacelike immersion $\psi : \Sigma^n \to \overline{M}^{n+1}$, satisfies $\Psi(0,\cdot) = \psi(\cdot)$ and the evolution equation

$$\frac{\partial \Psi}{\partial t} = \vec{H},$$

where $\vec{H}(t,\cdot)$ is the (non-normalized) mean curvature vector field of the spacelike hypersurface $\Sigma^n_t = \Psi(t,\Sigma^n)$ for every $t \in [0,T)$. Roughly speaking, the family of hypersurfaces $\Sigma^n_t = \Psi(t,\Sigma^n)$ evolves by mean curvature flow Ψ if the velocity $\dfrac{\partial \Psi}{\partial t}$ coincide with the mean curvature vector $\vec{H}$ at every point of $[0,T) \times \Sigma^n$.

Mean curvature flow in a Lorentzian manifold is an important thematic in the scope of Geometric Analysis and it has been extensively studied by several authors; see, among others, the papers [1, 18–21, 23, 27] as well as [29–34]. This wide interest in the current literature is mainly due to the fact that *spacelike translating solitons* can be regarded as a natural way of foliating spacetimes by almost null-like hypersurfaces; for more details, see [19]. For the sake of clarity, we recall here that spacelike translating solitons are spacelike hypersurfaces $\psi : \Sigma^n \to \overline{M}^{n+1}$ such that $\vec{H} = cV^\perp$ for some constant c, where V stands for a suitable timelike vector field globally defined on the $(n+1)$-dimensional Lorentzian manifold $\overline{M}^{n+1}$.

Particular examples may give insight into the structure of certain spacetimes at null infinity and have possible applications in General Relativity; see [19] for comments and details.

In the Riemannian setting, de Lira and Martín [17] have investigated solitons invariant with respect to the flow generated by a complete parallel vector field in a Riemannian manifold. A special case occurs when the ambient manifold is the Riemannian product space $\mathbb{R} \times \mathbb{P}^n$ and the complete parallel vector field is just the coordinate vector field ∂_t. In such a case, in analogy with the Euclidean framework, they preserve the term translating solitons. Moreover, when the metric of the base $\mathbb{P}^n$ is rotationally invariant and its sectional curvature is nonpositive, the authors characterize all the rotationally invariant translating solitons deducing several nonexistence results by using careful geometric analysis of these new families of barriers.

On the other hand, it is well known that many problems lead us to consider Riemannian manifolds endowed with a measure that has a smooth positive density with respect to the Riemannian one. This turns out to be compatible with the metric structure of the manifold and the resulting spaces are the *weighted manifolds*, which are also called manifolds with density or smooth metric measure spaces in the literature. More precisely, given a complete n-dimensional Riemannian manifold $(\mathbb{P}^n, g)$ and a smooth function $f \in C^\infty(\mathbb{P}^n)$, the weighted manifold $\mathbb{P}^n_f$ associated to $\mathbb{P}^n$ and f is the triple $(\mathbb{P}^n, g, d\mu = e^{-f}d\mathbb{P})$, where $d\mathbb{P}$ denotes the standard volume element of $\mathbb{P}^n$.

Appearing naturally in the study of self-shrinkers, Ricci solitons, harmonic heat flows and many others, weighted manifolds are proved to be important nontrivial generalizations of Riemannian manifolds and, nowadays, there are several geometric investigations concerning them. For a brief overview of results in this scope, we refer the articles of Morgan [37] and Wei-Wylie [42].

We point out that a theory of Ricci curvature for weighted manifolds goes back to Lichnerowicz [35, 36] and it was later developed by Bakry and Émery in the seminal work [10], where they introduced the *Bakry-Émery-Ricci tensor* Ric_f of a weighted manifold $\mathbb{P}^n_f$ as being the following extension of the standard Ricci tensor Ric of $\mathbb{P}^n$:

$$\mathrm{Ric}_f = \mathrm{Ric} + \mathrm{Hess} f.$$

The Bakry-Émery-Ricci curvature tensor arises in scalar-tensor gravitation theories in the conformal gauge known as the Jordan frame; see [22, 43] for more details. It is also worth mentioning that Case [13] has shown that a sign condition on timelike components of the Bakry-Émery-Ricci tensor, the so-called f-timelike convergence condition, will, in an analogous fashion to the Riemannian case, imply that singularity theorems and the timelike splitting theorem hold; see Remark 1.

Motivated by this previous digression and adapting the concept of translating soliton established in [17] and [9], here we investigate the uniqueness and nonexistence of complete spacelike translating solitons of the mean curvature flow in a Lorentzian product space $\mathbb{R}_1 \times \mathbb{P}^n_f$ endowed with a weight function f and whose Riemannian base $\mathbb{P}^n$ is supposed to be complete and with nonnegative Bakry-Émery-Ricci tensor. This is made through the applications of suitable Liouville-type results, an appropriate parabolicity criterion, and a version of the Omori-Yau's maximum principle for the drift Laplacian (see Theorems 1, 2, 3, 4 and 5). Applications to $\mathbb{R}_1 \times \mathbb{G}^n$ are also given, where $\mathbb{G}^n$ stands for the so-called n-dimensional Gaussian space (see Example 1 and Corollaries 1 and 4), as well as applications to $\mathbb{R}_1 \times \mathbb{H}^n_f$, where $\mathbb{H}^n$ denotes the standard n-dimensional hyperbolic space and f is the square of the distance function to a fixed point of $\mathbb{H}^n$; see Examples 2 and 3, as well as Corollaries 2, 3 and 6. Furthermore, we also infer the nonexistence of entire spacelike translating graphs constructed over the Riemannian base $\mathbb{P}^n$; see Theorems 6 and 7, as well as Corollaries 7, 8 and 9.

2. Preliminaries

Throughout this paper, let us consider an $(n+1)$-dimensional Lorentzian product space $\overline{M}^{n+1}$ of the form $\mathbb{R}_1 \times \mathbb{P}^n$, where $(\mathbb{P}^n, \langle \cdot, \cdot \rangle_{\mathbb{P}^n})$ is an n-dimensional connected Riemannian manifold and $\overline{M}^{n+1}$ is endowed with the standard product metric

$$\langle \cdot, \cdot \rangle = -\pi^*_{\mathbb{R}}(dt^2) + \pi^*_{\mathbb{P}^n}(\langle \cdot, \cdot \rangle_{\mathbb{P}^n}),$$

where $\pi_{\mathbb{R}}$ and $\pi_{\mathbb{P}^n}$ denote the canonical projections from $\mathbb{R}_1 \times \mathbb{P}^n$ onto each factor. For a fixed $t_0 \in \mathbb{R}$, we say that $\mathbb{P}^n_{t_0} = \{t_0\} \times \mathbb{P}^n$ is a *slice* of $\overline{M}^{n+1}$.

Given an n-dimensional connected manifold Σ^n, a smooth immersion $\psi : \Sigma^n \to \overline{M}^{n+1}$ is said to be a *spacelike hypersurface* if Σ^n, furnished with the metric induced by $\langle \cdot, \cdot \rangle$ via ψ, is a Riemannian manifold. If this is so, we shall always assume that the metric on Σ^n is the induced one, which will also be denoted by $\langle \cdot, \cdot \rangle$. In this setting, it follows from the connectedness of Σ^n that one can uniquely choose a globally defined timelike unit vector field $N \in \mathfrak{X}(\Sigma)^{\perp}$, having the same time-orientation of ∂_t, i.e., such that $\langle N, \partial_t \rangle \leq -1$. One then says that N is the *future-pointing Gauss map* of Σ^n.

In this setting, we will consider its Weingarten operator $A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$, which is given by $A(X) = -\overline{\nabla}_X N$, where $\overline{\nabla}$ stands for the Levi-Civita connection of $\overline{M}^{n+1}$. So, the (non-normalized) *future mean curvature function* of Σ^n is defined as been $H = -\mathrm{tr}(A)$. Moreover, we will also denote by $\overline{\nabla}$ and ∇ the gradients with respect to the metrics of $\overline{M}^{n+1}$ and Σ^n, respectively. A simple computation shows that the gradient of $\pi_{\mathbb{R}}$ on $\overline{M}^{n+1}$ is given by

$$\overline{\nabla}\pi_{\mathbb{R}} = -\langle\overline{\nabla}\pi_{\mathbb{R}}, \partial_t\rangle\partial_t = -\partial_t. \tag{2.1}$$

So, from (2.1) we conclude that the gradient of the (vertical) height function $h = (\pi_{\mathbb{R}})|_{\Sigma}$ of Σ^n is given by

$$\nabla h = (\overline{\nabla}\pi_{\mathbb{R}})^{\top} = -\partial_t^{\top} = -\partial_t - \Theta N, \tag{2.2}$$

where $(\cdot)^{\top}$ denotes the tangential component of a vector field in $\mathfrak{X}(\overline{M}^{n+1})$ along Σ^n and $\Theta = \langle N, \partial_t\rangle$ stands for the hyperbolic angle function of Σ^n. Thus, we get the following relation

$$|\nabla h|^2 = \Theta^2 - 1, \tag{2.3}$$

where $|\cdot|$ denotes the norm of a vector field on Σ^n. Moreover, from (2.2) we deduce that the Hessian of h on Σ, $\mathrm{Hess}\, h : \mathfrak{X}(\Sigma)\times\mathfrak{X}(\Sigma) \to C^{\infty}(\Sigma)$, is given by

$$\mathrm{Hess}\, h(X, Y) = -\langle\nabla_X\partial_t^{\top}, Y\rangle = -\langle\overline{\nabla}_X(\partial_t + \Theta N), Y\rangle = \langle AX, Y\rangle\Theta. \tag{2.4}$$

Hence, from (2.4) we obtain that the Laplacian of h is

$$\Delta h = -H\Theta. \tag{2.5}$$

Now, we consider a smooth function f defined on $\mathbb{R}_1 \times \mathbb{P}^n$. The triple $(\mathbb{R}_1 \times \mathbb{P}^n, \langle\cdot,\cdot\rangle, d\mu = e^{-f}d\sigma)$, where $d\sigma$ denotes the canonical volume element associated to the metric $\langle\cdot,\cdot\rangle$, will be called a *weighted Lorentzian product space* and f its *weight function*. According to [10], the *Bakry-Émery-Ricci tensor* $\overline{\mathrm{Ric}}_f$ of such a weighted manifold is defined as being the following extension of the standard Ricci tensor Ric

$$\overline{\mathrm{Ric}}_f = \overline{\mathrm{Ric}} + \overline{\mathrm{Hess}} f, \tag{2.6}$$

where $\overline{\mathrm{Hess}}$ denotes the Hessian of a function defined on $\overline{M}$.

For a spacelike hypersurface Σ^n immersed in a weighted Lorentzian product space, the *f-divergence operator* on Σ^n is defined by

$$\mathrm{div}_f(X) = e^f \mathrm{div}(e^{-f}X), \tag{2.7}$$

where X is a tangent vector field on Σ^n and div stands for the standard divergence operator of Σ^n. From this, for all smooth function $u : \Sigma^n \to \mathbb{R}$, we define the *drift Laplacian* of u by

$$\Delta_f u = \mathrm{div}_f(\nabla u) = \Delta u - \langle\nabla u, \nabla f\rangle. \tag{2.8}$$

Following the ideas of Gromov [24, Section 9.4.E] and considering the future-pointing Gauss map N of Σ^n, its (non-normalized) *future f-mean curvature H_f* is defined by

$$H_f = H - \langle\overline{\nabla} f, N\rangle, \tag{2.9}$$

where H denotes the future mean curvature of Σ^n.

Remark 1. As a consequence of a splitting theorem due to Case [13, Theorem 1.2], if a weighted Lorentzian product space $\mathbb{R}_1 \times \mathbb{P}$ is endowed with a bounded weight function f and if its Bakry-Émery-Ricci tensor is such that $\overline{\mathrm{Ric}}_f(V, V) \geq 0$, for all timelike vector field V on $\mathbb{R}_1 \times \mathbb{P}$, then f must be constant along $\mathbb{R}$. Motivated by this result, along this work we will always consider that the ambient space is a weighted Lorentzian product space $\mathbb{R}_1 \times \mathbb{P}^n$ whose weight function f does not depend on the parameter $t \in \mathbb{R}$, that is, $\langle \overline{\nabla} f, \partial_t \rangle = 0$ and, for sake of simplicity, we will denote it by $\mathbb{R}_1 \times \mathbb{P}^n_f$. In this setting, we get from (2.9) that the slices $\mathbb{P}^n_{t_0}$ are *f-maximal*, which means that H_f is identically zero.

3. Uniqueness and nonexistence of spacelike translating solitons

Considering the observations done in Remark 1, hereafter we study hypersurfaces in manifolds of the kind $\mathbb{R}_1 \times \mathbb{P}^n_f$. In such setting and in a similar spirit of [17, Definition 2] or [9, Eq (2.6)], we say that a spacelike hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$ immersed in a Lorentzian product space $\overline{M}^{n+1} = \mathbb{R}_1 \times \mathbb{P}^n$ is a *spacelike translating soliton* of the mean curvature flow with respect to ∂_t and with *soliton constant* $c \in \mathbb{R}$ if its future mean curvature function satisfies

$$H = c\,\Theta. \tag{3.1}$$

So, we observe that the slices $\{t\} \times \mathbb{P}^n$ are spacelike translating solitons of the mean curvature flow with respect to ∂_t and with soliton constant $c = 0$.

In order to establish our first result, we quote a Liouville-type result due to Pigola, Rigoli and Setti, which is a consequence of [40, Theorem 1.1]. For this, we will consider the following set

$$\mathcal{L}^p_f(M) := \left\{ u : M^n \to \mathbb{R} : \int_M |u|^p(x) e^{-f(x)} dM < +\infty \right\}.$$

Before introduce some useful results, recall that a smooth function u is f-subharmonic if $\Delta_f u \leq 0$ and u is semi-bounded whether u is bounded from above or from below.

The first useful Liouville-type result reads as follows:

Lemma 1. *Let u be a nonnegative smooth f-subharmonic function on a complete Riemannian manifold M^n. If $u \in \mathcal{L}^p_f(M)$, for some $p > 1$, then u is constant.*

The next lemma is a consequence of an extension of another Liouville-type result due to Yau in [45].

Lemma 2. *The only harmonic semi-bounded functions defined on an n-dimensional complete Riemannian manifold whose Ricci tensor is nonnegative are the constant ones.*

Before presenting our first uniqueness result concerning spacelike translating solitons, we recall that a spacelike hypersurface is said *maximal* when its mean curvature is identically zero.

Theorem 1. *Let $\mathbb{R}_1 \times \mathbb{P}^n_f$ be a weighted Lorentzian product space such that its Riemannian base $\mathbb{P}^n$ is complete, with nonnegative Bakry-Émery-Ricci tensor. Let $\psi : \Sigma^n \to \mathbb{R}_1 \times \mathbb{P}^n_f$ be a complete spacelike translating soliton of the mean curvature flow with respect to ∂_t, having soliton constant c and constant future f-mean curvature. If $H \in \mathcal{L}^q_f(\Sigma)$, for some $q > 2$, then Σ^n is maximal. Moreover, if in addition $\mathbb{P}^n$ has nonnegative sectional curvature and Σ^n lies in a vertical half-space of $\mathbb{R}_1 \times \mathbb{P}^n_f$, then Σ^n must be a slice $\mathbb{P}^n_{t_0}$ for some $t_0 \in \mathbb{R}$.*

Proof. Let $\psi : \Sigma^n \to \mathbb{R}_1 \times \mathbb{P}^n_f$ be such a spacelike translating soliton. If $c = 0$ the first conlusion is immediate. So, assume that $c \neq 0$. From [5, Corollary 8.2] we have the following key formula

$$\Delta\Theta = (\widetilde{\mathrm{Ric}}(N^*, N^*) + |A|^2)\Theta + \langle \nabla H, \partial_t \rangle. \tag{3.2}$$

where $\widetilde{\mathrm{Ric}}$ is the standard Ricci tensor of $\mathbb{P}^n$ and $N^* = N + \Theta\partial_t$ denotes the orthonormal projection of N onto $\mathbb{P}^n$.

Thus, since H_f is constant, from (2.9) and (3.2) we obtain

$$\begin{aligned} \frac{1}{2}\Delta\Theta^2 &= (\widetilde{\mathrm{Ric}}(N^*, N^*) + |A|^2)\Theta^2 + \langle \nabla H, \partial_t \rangle\Theta + |\nabla\Theta|^2 \\ &= (\widetilde{\mathrm{Ric}}(N^*, N^*) + |A|^2)\Theta^2 + \partial_t^\top(\langle \overline{\nabla} f, N \rangle)\Theta + |\nabla\Theta|^2. \end{aligned} \tag{3.3}$$

On the other hand, [39, Proposition 7.35] gives that

$$\overline{\nabla}_X \partial_t = 0, \tag{3.4}$$

for every tangent vector field X on the ambient space. Then, from (2.2) and (3.4) we have that

$$X(\Theta) = \langle \overline{\nabla}_X N, \partial_t \rangle = \langle A(\nabla h), X \rangle$$

and, consequently,

$$\nabla\Theta = A(\nabla h). \tag{3.5}$$

Thus, using once more (3.4), from (2.2) and (3.5) we get that

$$\begin{aligned} \partial_t^\top(\langle \overline{\nabla} f, N \rangle) &= \langle \overline{\nabla}_{\partial_t^\top} \overline{\nabla} f, N \rangle + \langle \overline{\nabla} f, \overline{\nabla}_{\partial_t^\top} N \rangle \\ &= \overline{\mathrm{Hess}} f(N, N)\Theta + \langle \overline{\nabla} f, \nabla\Theta \rangle. \end{aligned} \tag{3.6}$$

Substituting (3.6) in (3.3), we obtain

$$\frac{1}{2}\Delta\Theta^2 = (\widetilde{\mathrm{Ric}}(N^*, N^*) + |A|^2)\Theta^2 + \overline{\mathrm{Hess}} f(N, N)\Theta^2 + \frac{1}{2}\langle \overline{\nabla} f, \nabla(\Theta^2) \rangle + |\nabla\Theta|^2. \tag{3.7}$$

Consequently, since $\overline{\mathrm{Hess}} f(N, N) = \widetilde{\mathrm{Hess}} f(N^*, N^*)$, where $\widetilde{\mathrm{Hess}}$ stands for the Hessian computed in the metric of $\mathbb{P}^n$, using (2.6) and (2.8) in (3.7) we reach at the following equation

$$\frac{1}{2}\Delta_f\Theta^2 = (\widetilde{\mathrm{Ric}}_f(N^*, N^*) + |A|^2)\Theta^2 + |\nabla\Theta|^2, \tag{3.8}$$

where $\widetilde{\mathrm{Ric}}_f$ stands for the Bakry-Émery-Ricci tensor of $\mathbb{P}^n$.

Hence, since we are assuming that the soliton constant c is nonzero, from (3.1) and (3.8) we get the following inequality

$$\frac{1}{2}\Delta_f H^2 \geq (\widetilde{\mathrm{Ric}}_f(N^*, N^*) + |A|^2)H^2. \tag{3.9}$$

Since we are also supposing that $\widetilde{\mathrm{Ric}}_f$ is nonnegative, from (3.9) we obtain that $\Delta_f H^2 \geq 0$. So, since $H \in \mathcal{L}^q_f(\Sigma)$, for some $q > 2$, we can apply Lemma 1 for $p = q/2$ to conclude that H is constant.

Returning to (3.9), we also get that $|A|$ vanishes identically on Σ^n, which means that Σ^n is totally geodesic. Consequently, H must be identically zero and thus Σ^n is maximal and the first conclusion follows.

Moreover, considering $X \in \mathfrak{X}(\Sigma)$ and $\alpha = \inf_{\mathbb{P}^n} K_{\mathbb{P}^n} \geq 0$, where $K_{\mathbb{P}^n}$ stands for the sectional curvature of $\mathbb{P}^n$, from the proof of [8, Lemma 3.1] we obtain the following suitable lower bound for the Ricci tensor of Σ^n

$$\mathrm{Ric}(X, X) \geq (n-1)\alpha|X|^2 + \alpha|\nabla h|^2|X|^2 + (n-2)\alpha\langle X, \nabla h\rangle^2 + |AX|^2. \tag{3.10}$$

In particular, from (3.10) we get that the Ricci tensor of Σ^n is nonnegative.

Therefore, since from (2.5) we have that h is an harmonic function on Σ^n, and if we also assume that Σ^n lies in vertical half-space of $\mathbb{R}_1 \times \mathbb{P}^n_f$ (which means that h is semi-bounded), we can apply Lemma 2 conclude that h is constant, that is, Σ^n is a slice $\mathbb{P}^n_{t_0}$ for some $t_0 \in \mathbb{R}$. □

Since the drift Laplacian is an elliptic operator and taking into account inequality (3.9), we can apply the classical strong maximum principle due to Hopf [26] obtaining the following result; see also the classical book [41] due to Pucci and Serrin for related topics.

Theorem 2. *Let $\mathbb{R}_1 \times \mathbb{P}^n_f$ be a weighted Lorentzian product space such that its Riemannian base $\mathbb{P}^n$ is complete, with nonnegative Bakry-Émery-Ricci tensor. Let $\psi : \Sigma^n \to \mathbb{R}_1 \times \mathbb{P}^n_f$ be a complete spacelike translating soliton of the mean curvature flow with respect to ∂_t, having soliton constant c and constant future f-mean curvature. If H^2 attains its maximum on Σ^n, then Σ^n is maximal. Moreover, if in addition $\mathbb{P}^n$ has nonnegative sectional curvature and Σ^n lies in vertical half-space of $\mathbb{R}_1 \times \mathbb{P}^n_f$, then Σ^n must be a slice $\mathbb{P}^n_{t_0}$ for some $t_0 \in \mathbb{R}$.*

Example 1. An important example of weighted Riemannian manifold is the so-called *Gaussian space* $\mathbb{G}^n$, which corresponds to the Euclidean space $\mathbb{R}^n$ endowed with the Gaussian probability measure

$$e^{-f}dx^2 = (2\pi)^{-\frac{n}{2}}e^{-\frac{|x|^2}{2}}dx^2. \tag{3.11}$$

Concerned with the weighted Lorentzian product space $\mathbb{R}_1 \times \mathbb{G}^n$, An et al extended the classical Bernstein's theorem [11] showing that the only entire f-maximal graphs $\Sigma^n(u)$ of functions $u(x_2, \cdots, x_{n+1}) = x_1$ defined over $\mathbb{G}^n$, with $\sup_{\Sigma(u)} |Du|_{\mathbb{R}^n} < 1$ (where $|\cdot|_{\mathbb{R}^n}$ is the standard norm of $\mathbb{R}^n$), are the hyperplanes $x_1 =$ constant; see [7, Theorem 4].

Since the Bakry-Émery-Ricci tensor of $\mathbb{G}^n$ is positive, from Theorem 1 and Example 1 we obtain the following consequence.

Corollary 1. *The only complete spacelike translating solitons $\psi : \Sigma^n \to \mathbb{R}_1 \times \mathbb{G}^n$ of the mean curvature flow with respect to ∂_t, having constant future f-mean curvature* (*where f is the Gaussian probability measure defined in* (3.11)), *lying in a vertical half-space of $\mathbb{R}_1 \times \mathbb{G}^n$ and such that either H^2 attains its maximum on Σ^n or $H \in \mathcal{L}^q_f(\Sigma)$, for some $q > 2$, are the slices $\{t\} \times \mathbb{G}^n$.*

Example 2. Let $\mathbb{H}^n = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_n > 0\}$ be the n-dimensional hyperbolic space endowed with its standard complete metric

$$\langle \cdot, \cdot \rangle_{\mathbb{H}^n} = \frac{1}{x_n^2}(dx_1^2 + \cdots + dx_n^2)$$

and let $f : \mathbb{H}^n \to \mathbb{R}$ be the weight function given by

$$f(x) = (\mathrm{d}(x, x_0))^2, \tag{3.12}$$

where $\mathrm{d}(x, x_0)$ denotes the distance in $\mathbb{H}^n$ between x and a fixed point $x_0 \in \mathbb{H}^n$. According to [42, Example 7.2], we have that the Bakry-Émery-Ricci tensor of $\mathbb{H}^n_f$ satisfies $\mathrm{Ric}_f \geq (n-1)$. Thus, the weighted Lorentzian product space $\mathbb{R}_1 \times \mathbb{H}^n_f$ is such that its base $\mathbb{H}^n_f$ has nonnegative Bakry-Émery-Ricci tensor.

So, from Theorem 1 and Example 2 we also get the following result.

Corollary 2. *Let $\mathbb{R}_1 \times \mathbb{H}^n_f$ be the weighted Lorentzian product space with weight function f defined by* (3.12)*. Let $\psi : \Sigma^n \to \mathbb{R}_1 \times \mathbb{H}^n_f$ be a complete spacelike translating soliton of the mean curvature flow with respect to ∂_t, having soliton constant c and constant future f-mean curvature. If either H^2 attains its maximum on Σ^n or $H \in \mathcal{L}^q_f(\Sigma)$, for some $q > 2$, then Σ^n is maximal.*

In [45], Yau established the following version of Stokes' Theorem on an n-dimensional complete noncompact Riemannian manifold Σ^n: if $\omega \in \Omega^{n-1}(\Sigma^n)$ is an integrable $(n-1)$-differential form on Σ^n, then there exists a sequence B_i of domains on Σ^n such that

$$B_i \subset B_{i+1}, \quad \Sigma^n = \bigcup_{i \geq 1} B_i, \quad \text{and} \quad \lim_i \int_{B_i} d\omega = 0.$$

Later on, supposing that Σ^n is oriented by the volume element $d\Sigma$ and denoting by $\omega = \iota_X d\Sigma$ the contraction of $d\Sigma$ in the direction of a smooth vector field X on Σ^n, Caminha extended this Yau's result showing that if the divergence of X, $\mathrm{div}_\Sigma X$, does not change sign and that $|X|$ is Lebesgue integrable on Σ^n, then $\mathrm{div}_\Sigma X$ must be identically zero on Σ^n; see [12, Proposition 2.1].

Taking into account (2.7), from [12, Proposition 2.1] above mentioned we obtain our next auxiliary lemma.

Lemma 3. *Let u be a smooth function on a complete Riemannian manifold Σ^n endowed with a weight function $f : \Sigma^n \to \mathbb{R}$, such that $\Delta_f u$ does not change sign on Σ^n. If $|\nabla u| \in \mathcal{L}^1_f(\Sigma)$, then $\Delta_f u$ vanishes identically on Σ^n.*

We will use Lemma 3 to establish our next uniqueness result.

Theorem 3. *Let $\mathbb{R}_1 \times \mathbb{P}^n_f$ be a weighted Lorentzian product space such that its Riemannian base $\mathbb{P}^n$ is complete, with nonnegative Bakry-Émery-Ricci tensor. Let $\psi : \Sigma^n \to \mathbb{R}_1 \times \mathbb{P}^n_f$ be a complete spacelike translating soliton of the mean curvature flow with respect to ∂_t, having soliton constant c and constant future f-mean curvature. If H is bounded and $|\nabla H| \in \mathcal{L}^1_f(\Sigma)$, then Σ^n is maximal. Moreover, if in addition $\mathbb{P}^n$ has nonnegative sectional curvature and Σ^n lies in vertical half-space of $\mathbb{R}_1 \times \mathbb{P}^n_f$, then Σ^n must be a slice $\mathbb{P}^n_{t_0}$ for some $t_0 \in \mathbb{R}$.*

Proof. Since we are assuming that H is bounded and $|\nabla H| \in \mathcal{L}^1_f(\Sigma)$, we obtain

$$|\nabla(H^2)| = 2|H||\nabla H| \in \mathcal{L}^1_f(\Sigma).$$

Therefore, we can reason as in the proof of Theorem 1 applying Lemma 3 instead of Lemma 1 to obtain the proof of Theorem 3. □

From Theorem 3 we obtain the following consequence.

Corollary 3. *Let $\mathbb{R}_1 \times \mathbb{H}^n_f$ be the weighted Lorentzian product space with weight function f defined by* (3.12)*. Let $\psi : \Sigma^n \to \mathbb{R}_1 \times \mathbb{H}^n_f$ be a complete spacelike translating soliton of the mean curvature flow with respect to ∂_t, having soliton constant c and constant future f-mean curvature. If H is bounded and $|\nabla H| \in \mathcal{L}^1_f(\Sigma)$, then Σ^n is maximal.*

Example 3. Considering the same context of Example 2 and fixing a constant $c \in \mathbb{R}$ with $0 < |c| < 1$, from [15, Example 4.4] we have that

$$\Sigma^n = \{(c \ln x_n, x_1, ..., x_n) : x_n > 0\} \subset \mathbb{R}_1 \times \mathbb{H}^n$$

is a complete spacelike translating soliton of the mean curvature flow with respect to ∂_t, having soliton constant c and constant future mean curvature

$$H = \frac{c}{\sqrt{1 - c^2}} = c\,\Theta.$$

Consequently, we have that the assumption that H_f being constant in Theorem 3 is a necessary hypothesis to conclude that the spacelike translating soliton is maximal.

We recall that (according to the classical terminology in linear potential theory) a weighted manifold Σ^n endowed with a weight function f is said to be *f-parabolic* if there does not exist a nonconstant, nonnegative, f-superharmonic function defined on Σ^n. In this context, from [4, Corollary 2] we have the following f-parabolicity criterion.

Lemma 4. *Let $\mathbb{R}_1 \times \mathbb{P}^n_f$ be a weighted Lorentzian product space whose Riemannian base $\mathbb{P}^n$ is complete with f-parabolic universal Riemannian covering and let $\psi : \Sigma^n \to \mathbb{R}_1 \times \mathbb{P}^n_f$ be a complete spacelike hypersurface. If the hyperbolic angle function Θ is bounded, then Σ^n is f-parabolic.*

Using this previous parabolicity criterion, we obtain the following result.

Theorem 4. *Let $\mathbb{R}_1 \times \mathbb{P}^n_f$ be a weighted Lorentzian product space such that its Riemannian base $\mathbb{P}^n$ is complete, with nonnegative Bakry-Émery-Ricci tensor and f-parabolic universal Riemannian covering. Let $\psi : \Sigma^n \to \mathbb{R}_1 \times \mathbb{P}^n_f$ be a complete spacelike translating soliton of the mean curvature flow with respect to ∂_t, having soliton constant c and constant future f-mean curvature. If H is bounded, then Σ^n is maximal. Moreover, if in addition $\mathbb{P}^n$ has nonnegative sectional curvature and Σ^n lies in vertical half-space of $\mathbb{R}_1 \times \mathbb{P}^n_f$, then Σ^n must be a slice $\mathbb{P}^n_{t_0}$ for some $t_0 \in \mathbb{R}$.*

Proof. We argue as the proof of Theorem 1. If $c = 0$, the there is nothing to do. So assume that $c \neq 0$. Thus, since we are assuming that H is bounded, we define the smooth function φ on Σ^n by

$$\varphi := \frac{1}{2}\left\{\left(\sup_{\Sigma} H^2\right) - H^2\right\}.$$

Since $\widetilde{\mathrm{Ric}}_f$ is supposed to be nonnegative, from (3.9) we get

$$\Delta_f \varphi \leq -(\widetilde{\mathrm{Ric}}_f(N^*, N^*) + |A|^2)H^2 \leq 0.$$

Consequently, φ is a nonnegative, f-superharmonic function defined on Σ^n.

On the other hand, from (3.1) we have that the boundedness of H also implies the boundedness of the hyperbolic angle function Θ. So, Lemma 4 guarantees that Σ^n is f-parabolic.

Hence, we conclude that H must be constant on Σ^n. At this point, we can reason as in the proof of Theorem 1 to conclude our result. □

From [25, Corollary 3] we have that the Gaussian space $\mathbb{G}^n$ has finite f-volume, where f is the Gaussian probability measure defined in (3.11). Consequently, taking into account [28, Remark 3.8], we conclude that $\mathbb{G}^n$ is f-parabolic. This fact enable us to state the following application of Theorem 4.

Corollary 4. *The only complete spacelike translating solitons $\psi : \Sigma^n \to \mathbb{R}_1 \times \mathbb{G}^n$ of the mean curvature flow with respect to ∂_t, having constant future f-mean curvature (where f is the Gaussian probability measure defined in (3.11)), lying in a vertical half-space of $\mathbb{R}_1 \times \mathbb{G}^n$ and such that H is bounded, are the slices $\{t\} \times \mathbb{G}^n$.*

For our next results we need the following definition, which first appeared with Omori [38] for the Hessian and, afterwards, with Yau [44] for the Laplacian. For the drift Laplacian, we can find some general versions in [6].

Definition 1. *Let $(\Sigma^n, \langle\cdot,\cdot\rangle)$ be a Riemannian manifold and let $f : \Sigma^n \to \mathbb{R}$ be a smooth function. We say that the drift Laplacian Δ_f satisfies the Omori-Yau's maximum principle on Σ if, for every $u \in C^2$ with $u^* = \sup_\Sigma u < \infty$, there exists a sequence $\{x_k\} \subset \Sigma$ such that*

$$u(x_k) > u^* - \frac{1}{k}, \quad |\nabla u(x_k)| < \frac{1}{k}, \quad \text{and} \quad \Delta_f u(x_k) < \frac{1}{k},$$

for every $k \in \mathbb{N}$.

The proposition below gives sufficient conditions to guarantee that the drift Laplacian on a spacelike translating soliton immersed in a Lorentzian product space satisfies the Omori-Yau maximum principle for a weight function f whose Hessian is assumed to be bounded from below.

Proposition 1. *Let $\overline{M}^{n+1} = \mathbb{R}_1 \times \mathbb{P}^n_f$ be a weighted Lorentzian product space, whose Riemannian base $\mathbb{P}^n$ is complete with sectional curvature $K_{\mathbb{P}^n}$ such that $K_{\mathbb{P}^n} \geq -\kappa$ for some positive constant κ. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike translating soliton with soliton constant $c \neq 0$, bounded mean curvature and the Hessian of $f + ch$ along Σ^n is bounded from below. Then, the drift Laplacian Δ_f on Σ^n satisfies the Omori-Yau's maximum principle.*

Proof. We recall that the curvature tensor R of a spacelike hypersurface $\psi : \Sigma^n \to \mathbb{R}_1 \times \mathbb{P}^n$ can be described in terms of the corresponding shape operator A and the curvature tensor $\overline{R}$ of $\mathbb{R}_1 \times \mathbb{P}^n$ by the so-called Gauss equation given by

$$R(X,Y)Z = (\overline{R}(X,Y)Z)^\top - \langle AX, Z\rangle AY + \langle AY, Z\rangle AX, \tag{3.13}$$

for every tangent vector fields $X, Y, Z \in \mathfrak{X}(\Sigma)$. Here, as in [39], the curvature tensor R is given by

$$R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z,$$

where $[\cdot,\cdot]$ denotes the Lie bracket and $X, Y, Z \in \mathfrak{X}(\Sigma)$.

Let us consider $X \in \mathfrak{X}(\Sigma)$ and a local orthonormal frame $\{E_1, ..., E_n\}$ of $\mathfrak{X}(\Sigma)$. Then, it follows from the Gauss equation (3.13) that

$$\mathrm{Ric}(X, X) = \sum_{i=1}^{n} \langle \overline{R}(X, E_i)X, E_i\rangle - H\langle AX, X\rangle + |AX|^2. \tag{3.14}$$

Moreover, we have that

$$\begin{aligned}\langle \overline{R}(X, E_i)X, E_i\rangle =& \langle R(X^*, E_i^*)X^*, E_i^*\rangle_{\mathbb{P}^n} \\ =& K_{\mathbb{P}^n}(X^*, E_i^*)(\langle X^*, X^*\rangle_{\mathbb{P}^n}\langle E_i^*, E_i^*\rangle_{\mathbb{P}^n} - \langle X^*, E_i^*\rangle^2_{\mathbb{P}^n}).\end{aligned} \tag{3.15}$$

On the other hand, since $X^* = X + \langle X, \partial_t\rangle\partial_t$, $E_i^* = E_i + \langle E_i, \partial_t\rangle\partial_t$ and $\nabla h = -\partial_t^\top$, with a straightforward computation we see that

$$\langle X^*, X^*\rangle_{\mathbb{P}^n}\langle E_i^*, E_i^*\rangle_{\mathbb{P}^n} = (1 + \langle E_i, \nabla h\rangle^2)(|X|^2 + \langle X, \nabla h\rangle^2) \tag{3.16}$$

and

$$\langle X^*, E_i^*\rangle^2_{\mathbb{P}^n} = \langle X, E_i\rangle^2 + 2\langle X, \nabla h\rangle\langle E_i, \nabla h\rangle\langle X, E_i\rangle + \langle X, \nabla h\rangle^2\langle E_i, \nabla h\rangle^2. \tag{3.17}$$

Then, since we are supposing that $K_{\mathbb{P}^n} \geq -\kappa$ for some positive constant κ, inserting (3.16) and (3.17) into (3.15) we obtain

$$\sum_{i=1}^{n} \langle \overline{R}(X, E_i)X, E_i\rangle \geq -\kappa\left((n-1)|X|^2 + (n-2)\langle X, \nabla h\rangle^2 + |X|^2|\nabla h|^2\right). \tag{3.18}$$

Consequently, from (2.3) and (3.18) we get

$$\sum_{i=1}^{n} \langle \overline{R}(X, E_i)X, E_i\rangle \geq -(n-1)\kappa\Theta^2|X|^2. \tag{3.19}$$

Thus, from (3.14), (3.19), and taking into account that $|H| \leq \sqrt{n}|A|$, we obtain

$$\mathrm{Ric}(\mathrm{X}, \mathrm{X}) \geq -(n-1)\kappa\Theta^2|X|^2 - H\langle AX, X\rangle + |AX|^2 \geq -\left\{(n-1)\kappa\Theta^2\right\}|X|^2 + \mathrm{Hess}(ch)(X, X). \tag{3.20}$$

Hence, since $\mathrm{Hess}(f + ch) \geq -\beta$, H is bounded and using (2.6), (3.1) and (3.20) we get

$$\mathrm{Ric}_f(X, X) \geq -((n-1)\kappa \sup_{\Sigma} \Theta^2 + \beta)|X|^2.$$

Therefore, we can apply [14, Theorem 1] to conclude our result. □

As a direct consequence of the previous result we have the following corollary.

Corollary 5. *Let $\overline{M}^{n+1} = \mathbb{R}_1 \times \mathbb{P}^n_f$ be a weighted Lorentzian product space, whose Riemannian base $\mathbb{P}^n$ is complete with sectional curvature $K_{\mathbb{P}^n}$ such that $K_{\mathbb{P}^n} \geq -\kappa$ for some positive constant κ, and Hessian of f bounded from below. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike translating soliton with soliton constant $c \neq 0$, H_f constant and bounded second fundamental form. Then, the drift Laplacian Δ_f on Σ^n satisfies the Omori-Yau's maximum principle.*

Proof. We notice that $\overline{\mathrm{Hess}} f(X,Y) = \mathrm{Hess} f(X,Y) + f_N\langle AX, Y\rangle$, for all $X, Y \in \mathfrak{X}(\Sigma)$, where f_N stands for $\langle N, \overline{\nabla} f\rangle$. Thus, since $H_f = H - f_N$ is a constant e and $\mathrm{Hess}(ch) = HA$, we deduce that:

$$\overline{\mathrm{Hess}} f = \mathrm{Hess}(f + ch) - e \cdot A.$$

From this expression and our hypothesis, we obtain that

$$\mathrm{Hess}(f + ch) \geq -\beta \text{ and } H \text{ is bounded},$$

and the result follows from Proposition 1. □

Next, we apply Proposition 1 to establish the following nonexistence result.

Theorem 5. *Let $\mathbb{R}_1 \times \mathbb{P}^n_f$ be a weighted Lorentzian product space such that its Riemannian base $\mathbb{P}^n$ is complete, with sectional curvature $K_{\mathbb{P}^n}$ such that $K_{\mathbb{P}^n} \geq -\kappa$ for some positive constant κ, and nonnegative Bakry-Émery-Ricci tensor. There does not exist a complete spacelike translating soliton $\psi : \Sigma^n \to \mathbb{R}_1 \times \mathbb{P}^n_f$ of the mean curvature flow with respect to ∂_t, having soliton constant $c \neq 0$, bounded mean curvature and Hessian of $f + ch$ is bounded from below.*

Proof. Let us suppose by contradiction the existence of such a spacelike translating soliton Σ^n. Since $H^2 \leq n|A|^2$, from our assumptions jointly with inequality (3.9), we can apply Proposition 1 obtaining a sequence of points $\{p_k\}$ in Σ^n such that

$$0 \geq \frac{1}{2} \limsup_k \Delta_f H^2(p_k) \geq \limsup_k (\widetilde{\mathrm{Ric}}_f(N^*, N^*) + |A|^2)(p_k) \sup_\Sigma H^2 \geq 0. \tag{3.21}$$

Therefore, since our hypothesis that $c \neq 0$ implies that $\sup_\Sigma H^2 > 0$ and using once more that $H^2 \leq n|A|^2$, from (3.21) we get

$$0 < \sup_\Sigma H^2 \leq n \lim_k |A|^2(p_k) = 0,$$

reaching at a contradiction. □

As a consequence of the proof of Theorem 5 and using Corollary 5 we obtain the following consequence.

Corollary 6. *There does not exist a complete spacelike translating soliton immersed in either $\mathbb{R}_1 \times \mathbb{G}^n$ or $\mathbb{R}_1 \times \mathbb{H}^n_f$ (where the weight function f is defined by* (3.12)*) of the mean curvature flow with respect to ∂_t, having soliton constant $c \neq 0$, constant future f-mean curvature and bounded second fundamental form.*

4. Nonexistence of entire spacelike translating graphs

We recall that an entire graph over the Riemannian base $\mathbb{P}^n$ is determined by a smooth function $u \in C^\infty(\mathbb{P}^n)$ and it is given by

$$\Sigma(u) = \{(u(x), x); x \in \mathbb{P}^n\} \subset \mathbb{R}_1 \times \mathbb{P}^n_f.$$

The metric induced on $\mathbb{P}^n$ from the Lorentzian metric on the ambient space via $\Sigma(u)$ is

$$\langle \cdot, \cdot \rangle = -du^2 + \langle \cdot, \cdot \rangle_{\mathbb{P}^n}. \tag{4.1}$$

It can be easily seen from (4.1) that a graph $\Sigma(u)$ is a spacelike hypersurface if, and only if, $|Du|^2_{\mathbb{P}^n} < 1$, Du being the gradient of u in $\mathbb{P}^n$ and $|Du|_{\mathbb{P}^n}$ its norm, both with respect to the metric of $\mathbb{P}^n$. It is well known that in the case where $\mathbb{P}^n$ is a simply connected manifold, every complete spacelike hypersurface Σ^n immersed in $\mathbb{R}_1 \times \mathbb{P}^n$ is an entire spacelike graph over $\mathbb{P}^n$; see, for instance, [3, Lemma 3.1]. It is interesting to observe that, according to the examples of non-complete entire maximal graphs in $\mathbb{R}_1 \times \mathbb{H}^2$ due to Albujer in [2, Section 3], an entire spacelike graph $\Sigma(u)$ in $\mathbb{R}_1 \times \mathbb{P}^n$ is not necessarily complete, in the sense that the induced Riemannian metric is not necessarily complete on $\mathbb{P}^n$. However, it was proven in the beginning of [16, Corollary 1] that if $\mathbb{P}^n$ is complete and $|Du|_{\mathbb{P}^n} \leq \alpha$ for certain positive constant $\alpha < 1$, then $\Sigma(u)$ must be complete.

The future-pointing Gauss map of an entire spacelike graph $\Sigma(u)$ constructed over the Riemannian fiber $\mathbb{P}^n$ is given by the vector field

$$N(x) = \frac{1}{\sqrt{1 - |Du(x)|^2_{\mathbb{P}^n}}} \left(\partial_t|_{(u(x),x)} + Du(x) \right), \quad x \in \mathbb{P}^n. \tag{4.2}$$

Moreover, the second fundamental form A of $\Sigma(u)$ with respect to its orientation (4.2) is given by

$$AX = -\frac{1}{\sqrt{1 - |Du|^2_{\mathbb{P}^n}}} D_X Du - \frac{\langle D_X Du, Du \rangle_{\mathbb{P}^n}}{(1 - |Du|^2_{\mathbb{P}^n})^{3/2}} Du, \tag{4.3}$$

for any smooth vector field X tangent to $\mathbb{P}^n$. Consequently, if $\Sigma(u)$ is a spacelike entire graph over the Riemannian fiber $\mathbb{P}^n$ of a weighted Lorentzian product space $\mathbb{R}_1 \times \mathbb{P}^n$, it is not difficult to verify from (2.9) and (4.3) that the future f-mean curvature function $H_f(u)$ of $\Sigma(u)$ is given by

$$H_f(u) = \operatorname{div}_f \left(\frac{Du}{\sqrt{1 - |Du|^2_{\mathbb{P}^n}}} \right).$$

When an entire spacelike graph $\Sigma(u) \subset \mathbb{R}_1 \times \mathbb{P}^n_f$ is a spacelike translating soliton of the mean curvature flow with respect to ∂_t, it is called an *entire spacelike translating graph*. In this context, it is not difficult to verify that from Theorem 4 we obtain the following nonexistence result.

Theorem 6. *Let $\mathbb{R}_1 \times \mathbb{P}^n_f$ be a weighted Lorentzian product space such that its Riemannian base $\mathbb{P}^n$ is complete, with nonnegative sectional curvature, nonnegative Bakry-Émery-Ricci tensor and f-parabolic universal Riemannian covering. For any constants $C \neq 0$ and $0 < \alpha < 1$, there does not exist an entire spacelike translating graph $\Sigma(u) \subset \mathbb{R}_1 \times \mathbb{P}^n_f$ such that the corresponding smooth function $u \in C^\infty(\mathbb{P}^n)$ is a semi-bounded solution of the system*

$$\begin{cases} \operatorname{div}_f \left(\dfrac{Du}{\sqrt{1 - |Du|^2_{\mathbb{P}^n}}} \right) = C \\ |Du|_{\mathbb{P}^n} \leq \alpha. \end{cases}$$

Remark 2. Taking into account Theorem 2 and since (4.2) implies that the mean curvature H of an entire spacelike translating graph $\Sigma(u) \subset \mathbb{R}_1 \times \mathbb{P}^n_f$ is such that

$$H^2 = \frac{c^2}{1 - |Du(x)|^2_{\mathbb{P}^n}},$$

we can replace the assumption that the Riemannian base $\mathbb{P}^n$ has f-parabolic universal Riemannian covering in Theorem 6, by the hypothesis that $|Du|_{\mathbb{P}^n}$ attains its maximum on $\Sigma(u)$.

When the ambient space is $\mathbb{R}_1 \times \mathbb{G}^n$, Theorem 6 reads as follows.

Corollary 7. *For any constants $C \neq 0$ and $0 < \alpha < 1$, there does not exist an entire spacelike translating graph $\Sigma(u) \subset \mathbb{R}_1 \times \mathbb{G}^n$ such that the corresponding smooth function $u \in C^\infty(\mathbb{R}^n)$ is a semi-bounded solution of the system*

$$\begin{cases} \operatorname{div}_f \left(\dfrac{Du}{\sqrt{1 - |Du|^2_{\mathbb{R}^n}}} \right) = C \\ |Du|_{\mathbb{R}^n} \leq \alpha, \end{cases}$$

where f is the Gaussian probability measure defined in (3.11).

We say that $u \in C^\infty(\mathbb{P}^n)$ has finite C^2 norm when

$$||u||_{C^2(\mathbb{P}^n)} := \sup_{|\gamma| \leq 2} |D^\gamma u|_{L^\infty(\mathbb{P}^n)} < \infty.$$

In this context, observing that (4.3) guarantees that a spacelike translating graph $\Sigma(u) \subset \mathbb{R}_1 \times \mathbb{P}^n_f$ whose corresponding smooth function $u \in C^\infty(\mathbb{P}^n)$ has finite C^2 norm has bounded second fundamental form, from the proof of Theorem 5 and Corollary 5 we obtain the following nonexistence result.

Theorem 7. *Let $\mathbb{R}_1 \times \mathbb{P}^n_f$ be a weighted Lorentzian product space such that its Riemannian base $\mathbb{P}^n$ is complete, with sectional curvature $K_{\mathbb{P}^n}$ such that $K_{\mathbb{P}^n} \geq -\kappa$ for some positive constant κ, and the Hessian of the weight function f bounded from below, and nonnegative Bakry-Émery-Ricci tensor. For any constants $C \neq 0$ and $0 < \alpha < 1$, there does not exist an entire spacelike translating graph $\Sigma(u) \subset \mathbb{R}_1 \times \mathbb{P}^n_f$ such that the corresponding smooth function $u \in C^\infty(\mathbb{P}^n)$ has finite C^2 norm and it is a solution of the system*

$$\begin{cases} \operatorname{div}_f \left(\dfrac{Du}{\sqrt{1 - |Du|^2_{\mathbb{P}^n}}} \right) = C \\ |Du|_{\mathbb{P}^n} \leq \alpha. \end{cases}$$

We close our paper with the following consequences of Theorem 7.

Corollary 8. *For any constants $C \neq 0$ and $0 < \alpha < 1$, there does not exist an entire spacelike translating graph $\Sigma(u) \subset \mathbb{R}_1 \times \mathbb{G}^n$ such that the corresponding smooth function $u \in C^\infty(\mathbb{R}^n)$ has finite C^2 norm and it is a solution of the system*

$$\begin{cases} \operatorname{div}_f \left(\dfrac{Du}{\sqrt{1 - |Du|^2_{\mathbb{R}^n}}} \right) = C \\ |Du|_{\mathbb{R}^n} \leq \alpha, \end{cases}$$

where f is the Gaussian probability measure defined in (3.11).

Corollary 9. *For any constants $C \neq 0$ and $0 < \alpha < 1$, there does not exist an entire spacelike translating graph $\Sigma(u) \subset \mathbb{R}_1 \times \mathbb{H}^n_f$ such that the corresponding smooth function $u \in C^\infty(\mathbb{H}^n)$ has finite C^2 norm and it is a solution of the system*

$$\begin{cases} \operatorname{div}_f\left(\dfrac{Du}{\sqrt{1-|Du|^2_{\mathbb{H}^n}}}\right) = C \\ |Du|_{\mathbb{H}^n} \leq \alpha, \end{cases}$$

where f is the weight function defined in (3.12).

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Conflict of interest

The authors declare no conflict of interest.

References

1. M. A. S. Aarons, Mean curvature flow with a forcing term in Minkowski space, *Calc. Var.*, **25** (2005), 205–246. http://doi.org/10.1007/S00526-005-0351-8
2. A. L. Albujer, New examples of entire maximal graphs in $\mathbb{H}^2 \times \mathbb{R}_1$, *Differ. Geom. Appl.*, **26** (2008), 456–462. https://doi.org/10.1016/j.difgeo.2007.11.035
3. A. L. Albujer, L. J. Alías, Calabi-Bernstein results for maximal surfaces in Lorentzian product spaces, *J. Geom. Phys.*, **59** (2009), 620–631. https://doi.org/10.1016/j.geomphys.2009.01.008
4. A. L. Albujer, H. F. de Lima, A. M. Oliveira, M. A. L. Velásquez, ϕ-Parabolicity and the uniqueness of spacelike hypersurfaces immersed in a spatially weighted GRW spacetime, *Mediterr. J. Math.*, **15** (2018), 84. https://doi.org/10.1007/s00009-018-1134-8
5. L. J. Alías, A. G. Colares, Uniqueness of spacelike hypersurfaces with constant higher order mean curvature in generalized Robertson-Walker spacetimes, *Math. Proc. Cambridge*, **143** (2007), 703–729. http://doi.org/10.1017/S0305004107000576
6. L. J. Alías, P. Mastrolia, M. Rigoli, *Maximum principles and geometric applications*, Cham: Springer, 2016. https://doi.org/10.1007/978-3-319-24337-5
7. H. V. Q. An, D. V. Cuong, N. T. M. Duyen, D. T. Hieu, T. L. Nam, On entire f-maximal graphs in the Lorentzian product $\mathbb{G}^n \times \mathbb{R}_1$, *J. Geom. Phys.*, **114** (2017), 587–592. http://doi.org/10.1016/j.geomphys.2016.12.023

8. C. P. Aquino, H. I. Baltazar, H. F. de Lima, A new Calabi-Bernstein type result in spatially closed generalized Robertson-Walker spacetimes, *Milan J. Math.*, **85** (2017), 235–245. https://doi.org/10.1007/s00032-017-0271-z

9. M. Batista, H. F. de Lima, Spacelike translating solitons in Lorentzian product spaces: Nonexistence, Calabi-Bernstein type results and examples, *Commun. Contemp. Math.*, **24** (2022), 2150034. http://doi.org/10.1142/S0219199721500346

10. D. Bakry, M. Émery, Diffusions hypercontractives, In: *Séminaire de Probabilités XIX 1983/84*, Berlin: Springer, 1985, 177–206. https://doi.org/10.1007/BFb0075847

11. S. Bernstein, Sur les surfaces d'efinies au moyen de leur courboure moyenne ou totale, *Annales scientifiques de l'École Normale Supérieure, Serie 3*, **27** (1910), 233–256. https://doi.org/10.24033/asens.621

12. A. Caminha, The geometry of closed conformal vector fields on Riemannian spaces, *Bull. Braz. Math. Soc.*, **42** (2011), 277–300. https://doi.org/10.1007/s00574-011-0015-6

13. J. S. Case, Singularity theorems and the Lorentzian splitting theorem for the Bakry-Émery-Ricci tensor, *J. Geom. Phys.*, **60** (2010), 477–490. https://doi.org/10.1016/j.geomphys.2009.11.001

14. Q. Chen, H. Qiu, Rigidity of self-shrinkers and translating solitons of mean curvature flows, *Adv. Math.*, **294** (2016), 517–531. https://doi.org/10.1016/j.aim.2016.03.004

15. H. F. de Lima, E. A. Lima Jr, Generalized maximum principles and the unicity of complete spacelike hypersurfaces immersed in a Lorentzian product space, *Beitr. Algebra Geom.*, **55** (2013), 59–75. http://doi.org/10.1007/s13366-013-0137-7

16. H. F. de Lima, A. M. Oliveira, M. S. Santos, Rigidity of complete spacelike hypersurfaces with constant weighted mean curvature, *Beitr. Algebra Geom.*, **57** (2016), 623–635. http://doi.org/10.1007/s13366-015-0253-7

17. J. H. S. de Lira, F. Martín, Translating solitons in Riemannian products, *J. Differ. Equations*, **266** (2019), 7780–7812. https://doi.org/10.1016/j.jde.2018.12.015

18. K. Ecker, On mean curvature flow of spacelike hypersurfaces in asymptotically flat spacetime, *J. Aust. Math. Soc.*, **55** (1993), 41–59. https://doi.org/10.1017/S1446788700031918

19. K. Ecker, Interior estimates and longtime solutions for mean curvature flow of noncompact spacelike hypersurfaces in Minkowski space, *J. Differ. Geom.*, **46** (1997), 481–498. http://doi.org/10.4310/jdg/1214459975

20. K. Ecker, Mean curvature flow of spacelike hypersurfaces near null initial data, *Commun. Anal. Geom.*, **11** (2003), 181–205. https://doi.org/10.4310/CAG.2003.v11.n2.a1

21. K. Ecker, G. Huisken, Parabolic methods for the construction of spacelike slices of prescribed mean curvature in cosmological spacetimes, *Commun. Math. Phys.*, **135** (1991), 595–613. http://doi.org/https://doi.org/10.1007/BF02104123

22. G. J. Galloway, E. Woolgar, Cosmological singularities in Bakry-Émery spacetimes, *J. Geom. Phys.*, **86** (2014), 359–369. https://doi.org/10.1016/j.geomphys.2014.08.016

23. S. Gao, G. Li, C. Wu, Translating spacelike graphs by mean curvature flow with prescribed contact angle, *Arch. Math.*, **103** (2014), 499–508. https://doi.org/10.1007/s00013-014-0699-0

24. M. Gromov, Isoperimetry of waists and concentration of maps, *Geom. Funct. Anal.*, **13** (2003), 178–215. http://doi.org/10.1007/s000390300004

25. D. T. Hieu, T. L. Nam, Bernstein type theorem for entire weighted minimal graphs in $\mathbb{G}^n \times \mathbb{R}$, *J. Geom. Phys.*, **81** (2014), 87–91. http://doi.org/10.1016/j.geomphys.2014.03.011

26. E. Hopf, *Elementare Bemerkungen űber die Lősungen partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus*, Akad. Wiss. Berlin: S.-B. Preuss, 1927: 147–152.

27. G. Huisken, S. T. Yau, Definition of center of mass for isolated physical system and unique foliations by stable spheres with constant curvature, *Invent. Math.*, **124** (1996), 281–311. http://doi.org/10.1007/s002220050054

28. D. Impera, M. Rimoldi, Stability properties and topology at infinity of f-minimal hypersurfaces, *Geom. Dedicata*, **178** (2015), 21–47. http://doi.org/10.1007/s10711-014-9999-6

29. H. Jian, Translating solitons of mean curvature flow of noncompact spacelike hypersurfaces in Minkowski space, *J. Differ. Equations*, **220** (2006), 147–162. https://doi.org/10.1016/j.jde.2005.08.005

30. H. Ju, J. Lu, H. Jian, Translating solutions to mean curvature flow with a forcing term in Minkowski space, *Commun. Pure Appl. Anal.*, **9** (2010), 963–973. https://doi.org/10.3934/cpaa.2010.9.963

31. B. Lambert, A note on the oblique derivative problem for graphical mean curvature flow in Minkowski space, *Abh. Math. Semin. Univ. Hambg.*, **82** (2012), 115–120. https://doi.org/10.1007/s12188-012-0066-7

32. B. Lambert, The perpendicular Neumann problem for mean curvature flow with a timelike cone boundary condition, *Trans. Amer. Math. Soc.*, **366** (2014), 3373–3388. https://www.jstor.org/stable/23813865

33. B. Lambert, J. D. Lotay, Spacelike mean curvature flow, *J. Geom. Anal.*, **31** (2021), 1291–1359. https://doi.org/10.1007/s12220-019-00266-4

34. G. Li, I. Salavessa, Mean curvature flow of spacelike graphs, *Math. Z.*, **269** (2011), 697–719. https://doi.org/10.1007/s00209-010-0768-4

35. A. Lichnerowicz, Variétés Riemanniennes à tenseur C non négatif, *C. R. Acad. Sci. Paris Sér. A*, **271** (1970), 650–653.

36. A. Lichnerowicz, Variétés Kählériennes à première classe de Chern non negative et variétés Riemanniennes à courbure de Ricci généralisée non negative, *J. Differ. Geom.*, **6** (1971), 47–94. https://doi.org/10.4310/jdg/1214428089

37. F. Morgan, Manifolds with density, *Notices of the American Mathematical Society*, **52** (2005), 853–858.

38. H. Omori, Isometric immersions of Riemannian manifolds, *J. Math. Soc. Japan*, **19** (1967), 205–214. https://doi.org/10.2969/jmsj/01920205

39. B. O'Neill, *Semi-Riemannian geometry with applications to relativity*, London: Academic Press, 1983.

40. S. Pigola, M. Rigoli, A. G. Setti, Vanishing theorems on Riemannian manifolds, and geometric applications, *J. Funct. Anal.*, **229** (2005), 424–461. https://doi.org/10.1016/j.jfa.2005.05.007

41. P. Pucci, J. Serrin, The strong maximum principle revisited, *J. Differ. Equations*, **196** (2004), 1–66. https://doi.org/10.1016/j.jde.2003.05.001
42. G. Wei, W. Willie, Comparison geometry for the Bakry-Émery Ricci tensor, *J. Differ. Geom.*, **83** (2009), 377–405. https://doi.org/10.4310/jdg/1261495336
43. E. Woolgar, Scalar-tensor gravitation and the Bakry-Émery-Ricci tensor, *Class. Quantum Grav.*, **30** (2013), 085007. https://doi.org/10.1088/0264-9381/30/8/085007
44. S. T. Yau, Harmonic functions on complete Riemannian manifolds, *Commun. Pure Appl. Math.*, **28** (1975), 201–228. https://doi.org/10.1002/cpa.3160280203
45. S. T. Yau, Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry, *Indiana Univ. Math. J.*, **25** (1976), 659–670. https://doi.org/10.1512/iumj.1976.25.25051

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