http://www.aimspress.com/journal/mine

Mathematics in Engineering, 5(2): 1–5.
DOI:10.3934/mine.2023033
Received: 27 April 2022
Revised: 27 April 2022
Accepted: 27 April 2022
Published: 12 May 2022

Editorial

Fluid instabilities, waves and non-equilibrium dynamics of interacting particles: a short overview†

Roberta Bianchini[1,*] **and Chiara Saffirio**[2]

[1] Istituto per le Applicazioni del Calcolo "Mauro Picone", CNR, via dei Taurini 19, I-00185, Rome, Italy

[2] Department of Mathematics and Computer Science, University of Basel, Spiegelgasse 1, 4051 Basel, Switzerland

† **This contribution is part of the Special Issue:** Fluid instabilities, waves and non-equilibrium dynamics of interacting particles
Guest Editors: Roberta Bianchini; Chiara Saffirio
Link: www.aimspress.com/mine/article/5859/special-articles

* **Correspondence:** Email: roberta.bianchini@cnr.it.

1. Introduction

The study of physical systems in regimes at which turbulence arises has always been a focal topic in physics. However, a rigorous mathematical approach has been lacking, as pointed out by Feynman ([16]):

"there is a physical problem that is common to many fields, that is very old, and that has not been solved. It is not the problem of finding new fundamental particles, but something left over from a long time ago–over a hundred years. Nobody in physics has really been able to analyze it mathematically satisfactorily in spite of its importance to the sister sciences. It is the analysis of circulating or turbulent fluids."

In the last decades new mathematical tools have been developed to tackle some questions related to turbulence in a rigorous way and considerable results have been obtained. Notably, the works [5, 10] resolved the celebrated Onsager conjecture, intimately related to the solution of the Euler equation in the turbulent regime. Furthermore, in the context of phase-mixing problems, the seminal work [3] represents a great advancement in the understanding of the Sommerfeld paradox with the mathematical description of high-to-low frequency cascades and the formations of special resonances called *echoes*. More recently, the rigorous range of validity of the kinetic wave turbulence equation, arising as an effective theory for physical systems describing a large number of interacting waves, has been understood in the homogeneous and inhomogeneous settings ([6, 11, 19]). This equation provides an accurate description of a broad class of physical phenomena, varying from the dynamics of waves in the ocean to Bose-Einstein condensates, thus its study deepens our understanding on the long time behavior of the solutions to the Schrödinger equation as well as on the emergency of nonlinearities and instabilities due to the interactions among waves.

Turbulence theory aims at describing the dynamics of strongly excited states of dissipative systems with a large number of degrees of freedom and far from thermodynamic equilibrium. Mathematically, this is by no means straightforward. Thus, step-by-step approaches result more feasible by considering systems that are - strictly speaking - not *in turbulence*. These are still very challenging from a mathematical point of view and shed light on the mathematical framework and technical tools to be developed in order to tackle more complicated situations in which turbulence fully appears.

In this volume, we collect some examples of mathematical techniques that have been developed to deal with problems which are related to turbulence in different ways and in various contexts. Two main research lines can be identified among the contributions: *fluids* on one side and *waves* on the other one. It is not by chance that this duality is also present in turbulence theory, where one historically distinguishes between *turbulence in incompressible fluids* and *weak wave turbulence*. These are instances in which the physical natures of the mechanisms describing and detecting turbulence are different, as well as the mathematical techniques involved. Fal'kovich & Shafarenko ([13]) states that *"few would not argue that the problem of turbulence stretches beyond the framework of hydrodynamics of ideal incompressible fluids"*. On one side, the classical turbulence of incompressible fluids is generated by strongly nonlinear equations however small the level of excitation of the involved states may be. Weak wave turbulence, on the other side, describes the dynamics of small-amplitude waves with a given dispersion relation. In this case, a consistent approximation to the leading dynamics is represented by the linearization: being the amplitude of waves very small, linear effects dominate the dynamics of the involved states, so that dispersion of phase velocity has a stronger impact than interaction between waves.

Even though they are two different phenomena, some features are shared by fluid and wave turbulence. As conceptual parallels often lead to the possibility of merging mathematical tools, this is a key point of the present collection. Among others, two fundamental shared properties of fluid and wave turbulence are the appearance of *energy cascades* and the convergence - in a proper sense and under suitable conditions - to some *steady states* with a power-law behavior. In fluid turbulence, this is part of the *cascade hypothesis* by Richardson-Kolmogorov-Obukhov and the Onsager conjecture. Later on, Zakharov found steady-state spectra as exact solutions of wave kinetic equations (after applying a specific conformal transform), which are known by now as *Kolmogorov-Zakharov spectra of weak wave turbulence* ([20]). Although mathematical breakthroughs in both contexts have been reached in recent times (the aforementioned results on the Osager conjecture on the one hand, and the full derivation of the wave kinetic equation from the nonlinear Schrödinger equation on the other hand), the range of mathematically open questions in this matter is incredibly wide. First, the wave kinetic equation still lacks a rigorous mathematical theory, starting from basic questions like well-posedness and long-time behavior. In this view, because of the leading *linear and dispersive* behavior of waves in weak turbulence, the mathematical tools taking roots from the community of *dispersive equations* and *dynamical systems* have a strong potential appeal. In this context, we refer to the techniques based on Strichartz estimates as the one developed in this volume by Federico & Staffilani ([14]), Duerinckx ([12]) and Hientzsch ([18]) on one side, and the analysis of discrete, exact resonances and small divisors in the work by Feola, Iandoli & Murgante ([15]). For the modeling of turbulent fluids reaching stationary states for long times we mention the work by Flandoli & Luongo ([17]). Transfer of information throughout different frequency scales and in particular *mixing phenomena* are investigated by Crippa & Schulze ([7]). Multiscale limits of fluid systems in different regimes of the physical parameters are treated in the contribution by Del Santo, Fanelli, Sbaiz & Wróblewska-Kamińska ([9]).

Going further in the direction of big mathematical challenges in turbulence theory, the formation of the stationary cascade spectra of Kolmogorov type in wave turbulence is predicted to happen in a self-similar manner ([13]). A rigorous mathematical proof of this conjecture for the WKE of weak wave turbulence is still lacking, because of several difficulties essentially related to the universality of Kolmogorov-Zakharov spectra, which are out-of-equilibrium steady states whose power-law behavior is supposed to be independent from the external forcing. For this reason, self-similar regimes of evolution are of high interest especially in the context of kinetic equations. In this regard, we mention the contribution in this volume on homoenergetic solutions to the Boltzmann equation by Nota & Velázquez ([22]), where self-similarity is also taken into account.

A related question is to understand the emergence of these models featuring turbulence from the microscopic laws of physics. A contribution of this volume in this direction is due to Basile, Benedetto, Caglioti & Bertini ([2]), where the discrete homogeneous Boltzmann equation is considered.

Although important progresses have been done, many questions remain open in turbulence theory. Two main directions that we would like to highlight in this volume are the work by Apolinário & Chevillard ([1]) and the review paper on instabilities in stratified fluids by Varma, Mathur & Dauxois ([23]). In Apolinário & Chevillard, some features of turbulence are obtained from a *linear* model. The intuition behind the possibility of a linear modeling of turbulence originates from [8] and is somewhat surprising, breaking the conceptual link between turbulence and the nonlinearity of the equations. A crucial role in generating turbulent effects is played by the continuous spectrum of the linear operator, so highlighting the importance of the geometry of the boundary and related boundary conditions. A complete review on the role of boundary conditions in turbulent fluids is provided in this collection by Nobili ([21]). In this regard, and in the context of turbulent effects originated by thermal convection, we mention the contribution in this volume by Bevilacqua ([4]).

The review by Varma, Mathur & Dauxois is an accurate description of the state of the art in instabilities in stratified fluids, a field where various quite interesting mathematical problems connected to turbulence arise. Among them, we mention the description of the Triadic Resonant Instability, which is the spontaneous creation of higher harmonic waves from disturbances in stratified fluids and the recent observation in laboratory experiments of the group of Dauxois of statistically steady states with a power-law behavior satisfying the prediction of Kolmogorov.

Acknowledgments

We would like to warmly thank Enrico Valdinoci and Antonio De Simone for their support and every single author who accepted to contribute in this project. C. S. acknowledges the NCCR SwissMAP and the support of the SNSF through the project PCEFP2_181153.

Conflict of interest

The authors declare no conflict of interest.

References

1. G. B. Apolinário, L. Chevillard, Space-time statistics of a linear dynamical energy cascade model, *Mathematics in Engineering*, **5** (2023), 1–23. http://doi.org/10.3934/mine.2023025
2. G. Basile, D. Benedetto, E. Caglioti, L. Bertini, Large deviations for a binary collision model: energy evaporation, *Mathematics in Engineering*, **5** (2023), 1–12. http://doi.org/10.3934/mine.2023001
3. J. Bedrossian, N. Masmoudi, Inviscid damping and the asymptotic stability of planar shear flows in the 2D Euler equations, *Publ. math. IHES*, **122** (2015), 195–300. http://doi.org/10.1007/s10240-015-0070-4
4. G. Bevilacqua, Symmetry break in the eight bubble compaction, *Mathematics in Engineering*, **4** (2022), 1–24. http://doi.org/10.3934/mine.2022010
5. T. Buckmaster, C. De Lellis, L. Székelyhidi Jr., Dissipative Euler flows with Onsager-critical spatial regularity, *Commun. Pure Appl. Math.*, **69** (2016), 1613–1670. http://doi.org/10.1002/cpa.21586
6. C. Collot, P. Germain, Derivation of the kinetic wave equation for quadratic dispersive problems in the inhomogeneous setting, 2021, arXiv:2107.11819.
7. G. Crippa, C. Schulze, Sub-exponential mixing of generalized cellular flows with bounded palenstrophy, *Mathematics in Engineering*, **5** (2023), 1–12. http://doi.org/10.3934/mine.2023006
8. Y. C. de Verdière, L. Saint-Raymond, Attractors for two-dimensional waves with homogeneous Hamiltonians of degree 0, *Commun. Pure Appl. Math.*, **73** (2020), 421–462. http://doi.org/10.1002/cpa.21845
9. D. Del Santo, F. Fanelli, G. Sbaiz, A. Wróblewska-Kamińska, On the influence of gravity in the dynamics of geophysical flows, *Mathematics in Engineering*, **5** (2023), 1–33. http://doi.org/10.3934/mine.2023008
10. C. De Lellis, L. Székelyhidi Jr., Dissipative Euler flows and Onsager's conjecture, *J. Eur. Math. Soc.*, **16** (2014), 1467–1505. http://doi.org/10.4171/JEMS/466
11. Y. Deng, Z. Hani, Full derivation of the wave kinetic equation, 2021, arXiv:2104.11204.
12. M. Duerinckx, On nonlinear Schrödinger equations with random initial data, *Mathematics in Engineering*, **4** (2022),1–14. http://doi.org/10.3934/mine.2022030
13. G. E. Fal'kovich, A. V. Shafarenko, Nonstationary wave turbulence, *J. Nonlinear Sci.*, **1** (1991), 457–480.
14. S. Federico, G. Staffilani, Sharp Strichartz estimates for some variable coefficient Schrödinger operators on $\mathbb{R} \times \mathbb{T}^2$, *Mathematics in Engineering*, **4** (2022), 1–23. http://doi.org/10.3934/mine.2022033

15. R. Feola, F. Iandoli, F. Murgante, Long-time stability of the quantum hydrodynamic system on irrational tori, *Mathematics in Engineering*, **4** (2022), 1–24. http://doi.org/10.3934/mine.2022023

16. R. P. Feynman, R. Leighton, M. Sands, *The Feynman lectures on physics, Volume I*, 2015.

17. F. Flandoli, E. Luongo, Heat diffusion in a channel under white noise modeling of turbulence, *Mathematics in Engineering*, **4** (2022), 1–21. http://doi.org/10.3934/mine.2022034

18. L. E. Hientzsch, On the low Mach number limit for 2D Navier–Stokes–Korteweg systems, *Mathematics in Engineering*, **5** (2023), 1–26. http://doi.org/10.3934/mine.2023023

19. J. Lukkarinen, H. Spohn, Weakly nonlinear Schrödinger equation with random initial data, *Invent. Math.*, **183** (2011), 79–188. http://doi.org/10.1007/s00222-010-0276-5

20. S. Nazarenko, *Wave turbulence*, Berlin, Heidelberg: Springer, 2011. http://doi.org/10.1007/978-3-642-15942-8

21. C. Nobili, The role of boundary conditions in scaling laws for turbulent heat transport, *Mathematics in Engineering*, **5** (2023), 1–41. http://doi.org/10.3934/mine.2023013

22. A. Nota, J. J. L. Velázquez, Homoenergetic solutions of the Boltzmann equation: the case of simple-shear deformations, *Mathematics in Engineering*, **5** (2023), 1–25. http://doi.org/10.3934/mine.2023019

23. D. Varma, M. Mathur, T. Dauxois, Instabilities in internal gravity waves, *Mathematics in Engineering*, **5** (2023), 1–34. http://doi.org/10.3934/mine.2023016

AIMS Press