http://www.aimspress.com/journal/mine

Mathematics in Engineering, 5(2): 1–22.
DOI:10.3934/mine.2023029
Received: 03 January 2022
Revised: 27 March 2022
Accepted: 27 March 2022
Published: 05 May 2022

Research article

Stable anisotropic capillary hypersurfaces in a wedge†

Miyuki Koiso*

Institute of Mathematics for Industry, Kyushu University, 744, Motooka, Nishi-ku, Fukuoka, 819-0395, Japan

† **This contribution is part of the Special Issue:** Geometric Partial Differential Equations in Engineering
Guest Editor: James McCoy
Link: www.aimspress.com/mine/article/5820/special-articles

* **Correspondence:** Email: koiso@imi.kyushu-u.ac.jp.

Abstract: We study a variational problem for hypersurfaces in a wedge in the Euclidean space. Our wedge is bounded by a finitely many hyperplanes passing a common point. The total energy of each hypersurface is the sum of its anisotropic surface energy and the wetting energy of the planar domain bounded by the boundary of the considered hypersurface. An anisotropic surface energy is a generalization of the surface area which was introduced to model the surface tension of a small crystal. We show an existence and uniqueness result of local minimizers of the total energy among hypersurfaces enclosing the same volume. Our result is new even when the special case where the surface energy is the surface area.

Keywords: Wulff shape; capillary surface; anisotropic surface energy; constant anisotropic mean curvature; stable; wetting energy

1. Introduction

We prove an existence and uniqueness result of stable equilibrium hypersurfaces in wedge-like domains in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ for anisotropic surface energy, which serve as a mathematical model of small crystals and small liquid crystals with anisotropy. Our result is new even when the special case where the surface energy is merely the surface area.

Let $\gamma : S^n \to \mathbb{R}_{>0}$ be a positive continuous function on the unit sphere $S^n = \{\nu \in \mathbb{R}^{n+1} \ ; \ \|\nu\| = 1\}$ in $\mathbb{R}^{n+1}$. Let X be a closed piecewise-smooth hypersurface in $\mathbb{R}^{n+1}$. X will be represented as a mapping $X : M \to \mathbb{R}^{n+1}$ from an n-dimensional oriented connected compact C^∞ manifold M into $\mathbb{R}^{n+1}$. Let ν be the unit normal vector field along $X|_{M\setminus S[X]}$, where $S[X]$ is the set of singular points of X. The anisotropic energy $\mathcal{F}_\gamma(X)$ of X is defined as $\mathcal{F}_\gamma(X) := \int_{M\setminus S[X]} \gamma(\nu)\, dA$, where dA is the n-dimensional volume form of M induced by X. Such an energy was introduced by Gibbs (1839–1903) in order to model the shape of small crystals, and it is used as a mathematical model of anisotropic surface energy [19, 20]. It is known that, for any positive number $V > 0$, among all closed piecewise-smooth hypersurfaces as above enclosing the same $(n+1)$-dimensional volume V, there exists a unique (up to translation in $\mathbb{R}^{n+1}$) minimizer $W_\gamma(V)$ of $\mathcal{F}_\gamma$ [17]. Each $W_\gamma(V)$ is homothetic to the so-called Wulff shape for γ (the definition of the Wulff shape will be given in §2), which we will denote by W_γ. When $\gamma \equiv 1$, $\mathcal{F}_\gamma(X)$ is the usual n-dimensional volume of the hypersurface X and W_γ is the unit sphere S^n.

The Wulff shape W_γ is not smooth in general. However, in this paper we assume that W_γ is a smooth strictly convex hypersurface like the previous works that studied variational problems of anisotropic surface energies in differential geometry (cf. [1, 4, 5, 8–12, 14, 15]).

Each equilibrium hypersurface X of $\mathcal{F}_\gamma$ for variations that preserve the enclosed $(n+1)$-dimensional volume (we will call such a variation a volume-preserving variation) has constant anisotropic mean curvature. Here, the anisotropic mean curvature Λ of a piecewise-C^2 hypersurface X is defined at each regular point of X as (cf. [8, 15]) $\Lambda := (1/n)(-\mathrm{div}_M D\gamma + nH\gamma)$, where $D\gamma$ is the gradient of γ and H is the mean curvature of X. If $\gamma \equiv 1$, $\Lambda = H$ holds.

Let Ω be a wedge-shaped domain in $\mathbb{R}^{n+1}$ bounded by k hyperplanes $\Pi_1, \cdots, \Pi_k$ $(k \le n+1)$ such that the intersection $\Pi_1 \cap \cdots \cap \Pi_k$ includes the origin 0 of $\mathbb{R}^{n+1}$ (Figure 1). Denote by $\tilde{N}_j$ the unit normal to Π_j which points outward from Ω. We assume that $\tilde{N}_1, \cdots, \tilde{N}_k$ are linearly independent. We call each $\Pi_i \cap \Pi_j$ $(i \neq j)$ an edge of Ω. Let ω_j $(j = 1, \cdots, k)$ be non-negative constants. Let M be an n-dimensional oriented connected compact C^∞ manifold with boundary $\partial M = \sigma_1 \cup \cdots \cup \sigma_k$, where each σ_j is topologically S^{n-1}. Consider any C^∞-immersion $X : (M, \sigma_1, \cdots, \sigma_k) \to (\overline{\Omega}, \Pi_1, \cdots, \Pi_k)$ of which the restriction $X|_{\partial M}$ to ∂M is an embedding. Set $C_j = X(\sigma_j)$, and let $D_j = D_j(X) \subset \Pi_j$ be the n-dimensional domain bounded by C_j $(j = 1, \cdots, k)$. We assume that the unit normal ν to X points outward from the $(n+1)$-dimensional domain bounded by $X(M) \cup (\cup_{j=1}^k D_j)$ near each C_j. We define the wetting energy $\mathcal{W}(X)$ of X as

$$\mathcal{W}(X) = \sum_{j=1}^{k} \omega_j \mathcal{H}^n(D_j),$$

where $\mathcal{H}^n(D_j)$ is the n-dimensional Hausdorff measure of D_j. Then, we define the total energy $E(X) = E_\gamma(X)$ of X by

$$E(X) = \mathcal{F}_\gamma(X) + \mathcal{W}(X).$$

Note that $X(M) \cup D_1 \cup \cdots \cup D_k$ is an oriented closed piecewise smooth hypersurface without boundary (possibly with self-intersection). We denote by $V(X)$ the oriented volume enclosed by $X(M) \cup D_1 \cup \cdots \cup D_k$ which is represented as

$$V(X) = \frac{1}{n+1} \int_M \langle X, \nu \rangle \, dA.$$

We call a critical point of E for volume-preserving variations an anisotropic capillary hypersurface (or, simply, a capillary hypersurface). A capillary hypersurface is said to be stable if the second variation of E is nonnegative for all volume-preserving variations of X. Otherwise, it is said to be unstable. In this paper, we prove the following two results on the uniqueness and the existence of stable capillary hypersurfaces.

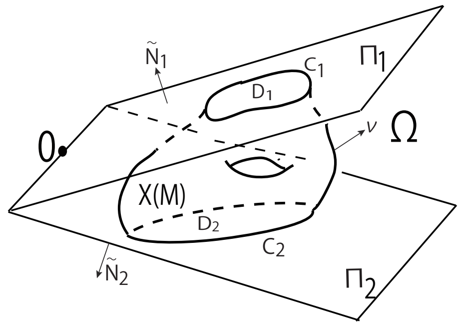

Figure 1. The wedge Ω and an admissible surface M for $k = 2$.

Theorem 1. *Let $X : (M, \sigma_1, \cdots, \sigma_k) \to (\overline{\Omega}, \Pi_1, \cdots, \Pi_k)$ be a compact oriented immersed anisotropic capillary hypersurface that is disjoint from any edge of Ω, having embedded boundary and satisfying $X(\partial M) \cap \Pi_j = \partial D_j$ for a nonempty bounded domain D_j in Π_j. If X is stable (and all $D_1, \cdots, D_k$ are convex if $n \geq 3$), then $X(M)$ is (up to translation and homothety) a part of the Wulff shape W_γ. Conversely, if X is an embedding onto a part of W_γ (up to translation and homothety), then it is stable.*

Theorem 2. *There exists an anisotropic capillary hypersurface X that is a part of the Wulff shape (up to translation and homothety) and that intersects Π_j with more than two points if and only if $\omega_j < \gamma(\tilde{N}_j)$ holds.*

As for previous works which are closely related to Theorem 1, we have the followings. First, Theorem 1 is a generalization of the main result of [3], where we proved the uniqueness result similar to Theorem 1 for isotropic capillary hypersurfaces in a wedge in $\mathbb{R}^{n+1}$ with $k = 2$; here, isotropic means that $\gamma \equiv 1$. [13] also studies the isotropic case for general k, but it does not prove that parts of the Wulff shape are stable. As for the anisotropic case, we studied the existence and uniqueness of stable anisotropic capillary surfaces between two parallel planes Π_1 and Π_2 in $\mathbb{R}^3$ [9–11]. There, stable anisotropic capillary surfaces are not necessarily parts of the Wulff shape (up to translation and homothety). This suggests that the assumption of the linear independence of $\tilde{N}_1, \cdots, \tilde{N}_k$ cannot be removed in our Theorem 1. Finally we mention that Theorem 1 was announced in [7]. Moreover there an outline of the proof of Theorem 1 for $n = 2$ and $k = 2$ was given.

This article is organized as follows. In Section 2 we give preliminary contents. We give the definition and a representation of the Wulff shape. We also give the definitions of various anisotropic curvatures for hypersurfaces. Moreover we recall the definition of anisotropic parallel hypersurfaces and a Steiner-type integral formula for these hypersurfaces. In Section 3, we give the first variation formulas and the Euler–Lagrange equations for our variational problems. Also the proof of Theorem 2 is given. Sections 4, 6 and 7 will be devoted to proving the uniqueness part of Theorem 1. The existence part that is the last statement of Theorem 1 will be proved in Section 5.

2. Preliminaries

In this paper, we call the boundary W_γ of the convex set $\tilde{W}[\gamma] := \cap_{\nu \in S^n}\{X \in \mathbb{R}^{n+1} \ ; \ \langle X, \nu \rangle \leq \gamma(\nu)\}$ the Wulff shape for γ, where $\langle \ , \ \rangle$ means the standard inner product in $\mathbb{R}^{n+1}$. In other literatures, $\tilde{W}[\gamma]$ is often called the Wulff shape.

From now on, any parallel translation of the Wulff shape W_γ will be also called the Wulff shape, and it will be denoted also by W_γ, if it does not cause any confusion.

From now on, we assume that, for simplicity, γ is of class C^∞. We also assume that the homogeneous extension $\overline{\gamma} : \mathbb{R}^{n+1} \to \mathbb{R}_{\geq 0}$ of γ that is defined as $\overline{\gamma}(rX) := r\gamma(X)$ $(\forall X \in S^n,\ \forall r \geq 0)$ is a strictly convex function. In this case, we say that γ is strictly convex, which is equivalent to the $n \times n$ matrix $D^2\gamma + \gamma \cdot I_n$ is positive definite at any point in S^n. Here, $D^2\gamma$ is the Hessian of γ on S^n and I_n is the identity matrix of size n. The Wulff shape W_γ is smooth and strictly convex (that is, each principal curvature of W_γ with respect to the inward-pointing normal is positive at each point of W_γ) if and only if γ is of class C^2 and strictly convex.

The Cahn–Hoffman map $\xi_\gamma : S^n \to \mathbb{R}^{n+1}$ for γ is defined as $\xi_\gamma(\nu) = D\gamma|_\nu + \gamma(\nu)\nu$, $(\nu \in S^n)$. Here, the tangent space $T_\nu(S^n)$ of S^n at ν is naturally identified with the hyperplane in $\mathbb{R}^{n+1}$ which is tangent to S^n at ν. Because γ is strictly convex, ξ_γ gives an embedding onto W_γ. Moreover, the outward-pointing unit normal at a point $\xi_\gamma(\nu)$ to W_γ coincides with ν (cf. [8]).

The Cahn-Hoffman field $\tilde{\xi}$ along X for γ is defined as $\tilde{\xi} := \xi_\gamma \circ \nu : M \to \mathbb{R}^{n+1}$. Since the unit normal $\nu(p)$ of X at $p \in M$ coincides with the unit normal of ξ_γ at the point $\nu(p)$, we can identify T_pM with $T_{\tilde{\xi}(p)}\xi_\gamma(S^n)$.

The linear map $S^\gamma_p : T_pM \to T_pM$ $(p \in M)$ given by the $n \times n$ matrix $S^\gamma := -d\tilde{\xi}$ is called the anisotropic shape operator of X. Various anisotropic curvatures of X are defined as follows.

Definition 1 (anisotropic curvatures; cf. [5, 15])**.** (i) The eigenvalues of S^γ are called the anisotropic principal curvatures of X. We denote them by $k^\gamma_1, \cdots, k^\gamma_n$.

(ii) Let σ^γ_r be the elementary symmetric functions of $k^\gamma_1, \cdots, k^\gamma_n$:

$$\sigma^\gamma_r := \sum_{1 \leq l_1 < \cdots < l_r \leq n} k^\gamma_{l_1} \cdots k^\gamma_{l_r}, \quad r = 1, \cdots, n. \tag{2.1}$$

Set $\sigma^\gamma_0 := 1$. $H^\gamma_r := \sigma^\gamma_r / {}_nC_r$ is called the rth anisotropic mean curvature of X, where ${}_nC_r = \dfrac{n!}{k!(n-k)!}$.

(iii) H^γ_1 is called also the anisotropic mean curvature of X, and we often denote it by Λ; that is, $\Lambda = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} k^\gamma_i = -\dfrac{1}{n}\mathrm{trace}_M(d\tilde{\xi})$.

As we mentioned above, for the Cahn-Hoffman map $\xi_\gamma : S^n \to \mathbb{R}^{n+1}$, it is shown that the unit normal vector field ν_{ξ_γ} is given by ξ_γ^{-1}. Hence, the anisotropic shape operator of ξ_γ is $S^\gamma = -d(\xi_\gamma \circ \nu_{\xi_\gamma}) = -d(\mathrm{id}_{S^n}) = -I_n$. Therefore, the anisotropic principal curvatures of ξ_γ are -1, and hence, each rth anisotropic mean curvature of ξ_γ is $(-1)^r$. Particularly, the anisotropic mean curvature of ξ_γ for the normal ν and that of W_γ for the outward-pointing unit normal is -1 at any point. More generally, the anisotropic mean curvature of an immersion $X : M \to \mathbb{R}^{n+1}$ is the mean value of the ratios of the principal curvatures of the Wulff shape and the corresponding curvatures of X which is explained as follows.

Remark 1 (cf. [8])**.** Let $X : M \to \mathbb{R}^{n+1}$ be an immersion. Take any point $p \in M$. We compute the anisotropic mean curvature $\Lambda(p)$ of X at p. Let $\{e_1, \cdots, e_n\}$ be a locally defined frame on S^n such that $(D^2\gamma + \gamma \cdot I_n)e_i = (1/\mu_i)e_i$, where μ_i are the principal curvatures of ξ_γ with respect to ν. Note that the basis $\{e_1, \cdots, e_n\}$ at $\nu(p)$ also serves as an orthogonal basis for the tangent hyperplane of X at p. Let

$(-w_{ij})$ be the matrix representing $d\nu$ with respect to this basis. Then

$$-S^{\gamma} = d\xi_{\gamma} \circ d\nu = (D^2\gamma + \gamma \cdot I_n)d\nu = \begin{pmatrix} -w_{11}/\mu_1 & \cdots & -w_{1n}/\mu_1 \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ -w_{n1}/\mu_n & \cdots & -w_{nn}/\mu_n \end{pmatrix}.$$

This with the definition of Λ gives

$$\Lambda = -\frac{1}{n}\mathrm{trace}_M(d\tilde{\xi}) = (1/n)(w_{11}/\mu_1 + \cdots + w_{nn}/\mu_n). \tag{2.2}$$

S^{γ} is not symmetric in general. However, we have the following good properties of the anisotropic curvatures.

Remark 2. (i) If $d\xi_{\gamma} = D^2\gamma + \gamma \cdot I_n$ is positive definite at a point $\nu(p)$ $(p \in M)$, then all of the anisotropic principal curvatures of X at p are real [4].

(ii) k_i^{γ} is not a real value in general. However, each H_r^{γ} is always a real valued function on M [6].

At the end of this section, we recall a useful concept that is "anisotropic parallel hypersurface" and an important integral formula that is a generalization of the classical Steiner's formula. Anisotropic parallel hypersurface is a generalization of parallel hypersurface and is defined as follows.

Definition 2 (Anisotropic parallel hypersurface, cf. [15])**.** Let $X : M \to \mathbb{R}^{n+1}$ be an immersion. For any real number t, we call the map $X_t := X + t\tilde{\xi} : M \to \mathbb{R}^{n+1}$ the anisotropic parallel deformation of X of height t. If X_t is an immersion, then we call it the anisotropic parallel hypersurface of X of height t.

The anisotropic energy $\mathcal{F}_{\gamma}(X_t)$ of the anisotropic parallel hypersurface $X_t := X + t\tilde{\xi}$ is a polynomial of t with degree at the most n as follows.

Fact 1 (Steiner-type formula [4])**.** *Assume that $\gamma : S^n \to \mathbb{R}_{>0}$ is of class C^{∞}. Let $X : M \to \mathbb{R}^{n+1}$ be an immersion. Consider anisotropic parallel hypersurfaces $X_t = X + t\tilde{\xi} : M \to \mathbb{R}^{n+1}$. Then, the n-dimensional volume form dA_{X_t} and the anisotropic energy $\mathcal{F}_{\gamma}(X_t)$ of X_t have the following representations.*

$$\begin{aligned} dA_{X_t} &= (1 - tk_1^{\gamma}) \cdots (1 - tk_n^{\gamma})\, dA \\ &= \sum_{r=0}^{n} (-1)^r t^r ({}_nC_r) H_r^{\gamma}\, dA, \end{aligned} \tag{2.3}$$

$$\mathcal{F}_{\gamma}(X_t) = \int_M \gamma(\nu) \sum_{r=0}^{n} (-1)^r t^r ({}_nC_r) H_r^{\gamma}\, dA. \tag{2.4}$$

The isotropic version of Fact 1 is known as the Weyl's tube formula [18]. The isotropic 2-dimensional version is the well-known Steiner's formula.

3. Euler–Lagrange equations

In order to obtain the Euler–Lagrange equations for our capillary problem, first we recall the first variation formula for the anisotropic surface energy $\mathcal{F}_\gamma$.

Proposition 1 ([6]). *Let $X_\epsilon : M \to \mathbb{R}^{n+1}$ ($\epsilon \in J := (-\epsilon_0, \epsilon_0)$), be a smooth variation of X; that is, $\epsilon_0 > 0$ and $X_0 = X$. Set*

$$\delta X := \frac{\partial X_\epsilon}{\partial \epsilon}\Big|_{\epsilon=0}, \quad \psi := \langle \delta X, \nu \rangle.$$

Then, the first variation of the anisotropic energy $\mathcal{F}_\gamma$ is given as follows.

$$\delta \mathcal{F}_\gamma := \frac{d\mathcal{F}_\gamma(X_\epsilon)}{d\epsilon}\Big|_{\epsilon=0} = -\int_M n\Lambda\psi \, dA - \oint_{\partial M} \langle \delta X, R(p(\tilde{\xi})) \rangle \, ds, \tag{3.1}$$

where ds is the $(n-1)$-dimensional volume form of ∂M induced by X, N is the outward-pointing unit conormal along ∂M, R is the $\pi/2$-rotation on the (N, ν)-plane, and p is the projection from $\mathbb{R}^{n+1}$ to the (N, ν)-plane.

On the other hand, the first variation of the $(n+1)$-dimensional volume enclosed by the region between X and X_ϵ is given by ([2]) as

$$\delta V = \int_M \langle \delta X, \nu \rangle \, dA. \tag{3.2}$$

Similarly, the first variation of $\mathcal{H}^n(D_j)$ is given as follows.

$$\delta \mathcal{H}^n(D_j) = \int_{\sigma_j} \langle \delta X, \rho \rangle \, ds, \tag{3.3}$$

where ρ is the outward-pointing unit normal along $X|_{\sigma_j} : \sigma_j \to \Pi_j$, and ds is the $(n-1)$-dimensional volume form of $X|_{\sigma_j}$.

The Eq (3.1) with (3.2), (3.3) gives the following Euler–Lagrange equations.

Proposition 2. *An immersion $X : (M, \sigma_1, \cdots, \sigma_k) \to (\overline{\Omega}, \Pi_1, \cdots, \Pi_k)$ is a capillary hypersurface if and only if both of the following conditions (i) and (ii) hold.*

(i) The anisotropic mean curvature Λ of X is constant on M.

(ii) $\langle \tilde{\xi}, \tilde{N}_j \rangle = \omega_j$ on σ_j ($j = 1, \cdots, k$), where $\tilde{\xi}$ is the Cahn–Hoffman field along X.

Proof. Note that X is a capillary hypersurface if and only if, for all volume-preserving variations $X_\epsilon : (M, \sigma_1, \cdots, \sigma_k) \to (\overline{\Omega}, \Pi_1, \cdots, \Pi_k)$, $(-\epsilon_0 < \epsilon < \epsilon_0)$, $\delta E = 0$ holds. This is equivalent to the fact that, there exists a constant Λ_0 such that

$$\begin{aligned} &\delta(E + n\Lambda_0 V) \\ &= \delta\mathcal{F}_\gamma + \delta\mathcal{W} + n\Lambda_0 \delta V \\ &= -n\int_M (\Lambda - \Lambda_0)\langle \delta X, \nu \rangle \, dA - \sum_{j=1}^k \oint_{\sigma_j} \Big\langle \delta X, \Big(R(p(\tilde{\xi})) - \omega_j \rho\Big) \Big\rangle \, ds \qquad (3.4) \\ &= 0 \qquad (3.5) \end{aligned}$$

holds for all variations $X_\epsilon : (M, \sigma_1, \cdots, \sigma_k) \to (\overline{\Omega}, \Pi_1, \cdots, \Pi_k)$, $(-\epsilon_0 < \epsilon < \epsilon_0,\ \epsilon_0 > 0)$ of X.

First we assume that X is a capillary hypersurface. By taking variations that preserve the boundary values $X|_{\partial M}$, we have from (3.4), (3.5) that

$$\int_M (\Lambda - \Lambda_0)\psi \, dA = 0, \quad \forall \psi \in C_0^\infty(\Sigma), \tag{3.6}$$

which implies

$$\Lambda - \Lambda_0 = 0, \quad \text{on } M. \tag{3.7}$$

This proves (i). Next, take variations that preserve the boundary values $X|_{\sigma_2 \cup \cdots \cup \sigma_k}$. Then we have from (3.4), (3.5), (3.6) that

$$\oint_{\sigma_1} \left\langle \delta X, \left(R(p(\tilde{\xi})) - \omega_1 \rho \right) \right\rangle ds = 0 \tag{3.8}$$

holds for all variations X_ϵ satisfying $X_\epsilon(\sigma_1) \subset \Pi_1$. This means that

$$\left(R(p(\tilde{\xi})) - \omega_1 \rho \right) \parallel \tilde{N}_1, \quad \text{on } \sigma_1 \tag{3.9}$$

holds. Note that the (N, ν)-plane is the same as the $(\tilde{N}_1, \rho)$-plane because both of them are the orthogonal compliment of the $(n-1)$-dimensional tangent space of $X(\sigma_1)$ in $\mathbb{R}^{n+1}$ at each point in $X(\sigma_1)$. Therefore, (3.9) is equivalent to

$$\langle p(\tilde{\xi}) - \omega_1 \tilde{N}_1, \tilde{N}_1 \rangle = 0, \quad \text{on } \sigma_1. \tag{3.10}$$

And (3.10) is equivalent to

$$\langle \tilde{\xi} - \omega_1 \tilde{N}_1, \tilde{N}_1 \rangle = 0, \quad \text{on } \sigma_1, \tag{3.11}$$

which means that

$$\langle \tilde{\xi}, \tilde{N}_1 \rangle = \omega_1, \quad \text{on } \sigma_1. \tag{3.12}$$

Similarly, we have

$$\langle \tilde{\xi}, \tilde{N}_j \rangle = \omega_j, \quad \text{on } \sigma_j, \quad j = 1, \cdots, k, \tag{3.13}$$

which proves (ii).

Next we assume that (i) and (ii) are satisfied. Then, by using the computations above, we know that X is a capillary hypersurface. □

Theorem 2 is given by Proposition 2.

Here we pose a new variational problem that will be used in the proof of a balancing formula (Lemma 2) in §4. Consider a more general class of surfaces than above:

$$\begin{aligned} \mathcal{S} \ := \ & \{X : (M, \sigma_1, \cdots, \sigma_k) \to (\mathbb{R}^{n+1}, \tilde{\Pi}_1, \cdots, \tilde{\Pi}_k)\ ;\ X \text{ is an immersion,} \\ & \text{each } \tilde{\Pi}_j \text{ is any hyperplane that is parallel to } \Pi_j, \text{ and} \\ & \text{the restriction } X|_{\partial M} \text{ is an embedding onto the disjoint union of} \\ & k \text{ topological } S^{n-1}.\ \}. \end{aligned}$$

For $X \in \mathcal{S}$, set $C_j := X(\sigma_j)$ and denote by D_j the n-dimensional domain bounded by C_j in $\tilde{\Pi}_j$. Moreover, set

$$S_X := X(M) \cup D_1 \cup \cdots \cup D_k.$$

Denote by dA the n-dimensional standard volume form on S_X. Note that each $\tilde{N}_j$ is a unit normal to $\tilde{\Pi}_j$. Define the algebraic volume $\overline{V}(X)$ enclosed by S_X as

$$\overline{V}(X) := \frac{1}{n+1}\int_M \langle X, \nu\rangle\, dA + \frac{1}{n+1}\sum_{j=1}^{k}\int_{D_j}\langle x, \tilde{N}_j\rangle\, dA,$$

where we denoted the variable point in D_j by x. And define the energy $\overline{\mathcal{F}}_\gamma(X)$ of X as

$$\overline{\mathcal{F}}_\gamma(X) = \mathcal{F}_\gamma(X) + \sum_{j=1}^{k}\omega_j \mathcal{H}^n(D_j).$$

Then, similarly to our capillary problem, we obtain the following first variation formulas.

Lemma 1.

$$\delta\overline{V} = \int_M \langle \delta X, \nu\rangle\, dA + \sum_{j=1}^{k}\int_{D_j}\langle \delta X, \tilde{N}_j\rangle\, dA, \tag{3.14}$$

$$\delta\mathcal{F}_\gamma = -\int_M n\Lambda\langle \delta X, \nu\rangle - \oint_{\partial M}\langle \delta X, R(p(\tilde{\xi}))\rangle\, ds, \tag{3.15}$$

$$\delta\mathcal{H}^n(D_j) = \int_{\sigma_j}\langle \delta X, \rho\rangle\, ds. \tag{3.16}$$

Proof. (3.14) is a standard formula (cf. [2]). (3.15) is given in proposition 1. We will prove (3.16). Consider the orthogonal projection $p_j : \tilde{\Pi}_j \to \Pi_j$. Then,

$$p_j(\delta X) = \delta X - \langle \delta X, \tilde{N}_j\rangle \tilde{N}_j.$$

Since

$$\mathcal{H}((D_j)_\epsilon) = \frac{1}{n}\int_{\sigma_j}\langle X_\epsilon, \rho_\epsilon\rangle\, ds_\epsilon = \frac{1}{n}\int_{\sigma_j}\langle p_j(X_\epsilon), \rho_\epsilon\rangle\, ds_\epsilon,$$

we obtain

$$\begin{aligned}\delta\mathcal{H}^n(D_j) &= \int_{\sigma_j}\langle \delta(p_j(X)), \rho\rangle\, ds = \int_{\sigma_j}\langle p_j(\delta X), \rho\rangle\, ds\\ &= \int_{\sigma_j}\langle \delta X - \langle \delta X, \tilde{N}_j\rangle\tilde{N}_j, \rho\rangle\, ds = \int_{\sigma_j}\langle \delta X, \rho\rangle\, ds,\end{aligned}$$

which proves (3.16). □

4. Proof of the first half of Theorem 1: the uniqueness of the stable solution

In order to prove the uniqueness part of Theorem 1, we will show that any capillary hypersurface $X : (M, \sigma_1, \cdots, \sigma_k) \to (\overline{\Omega}, \Pi_1, \cdots, \Pi_k)$ is unstable unless it is a part of a rescaling of the Wulff shape.

In view of the condition (ii) in Proposition 2, it is useful to consider the anisotropic energy for $(n-1)$-dimensional closed hypersurfaces in Π_j. First, define hyperplanes P_j in $\mathbb{R}^{n+1}$ ($j = 1, \cdots, k$) by

$$P_j := \{x \in \mathbb{R}^{n+1}\ ;\ \langle x, \tilde{N}_j\rangle = \omega_j\}.$$

Then, set the followings:

$$\hat{W}_j := W_\gamma \cap P_j, \quad \hat{O}_j := \omega_j \tilde{N}_j.$$

Assume that $\omega_j \geq 0$ is sufficiently small so that $\hat{W}_j$ includes at least two distinct points. Then, $\hat{W}_j$ is a strictly convex closed $(n-1)$-dimensional C^∞ hypersurface in the n-dimensional linear space P_j. We regard the point $\hat{O}_j$ as the origin of P_j. Denote by $\hat{\gamma}_j : S^{n-1} \to \mathbb{R}_{>0}$ the support function of $\hat{W}_j$, that is, for any $\rho \in S^{n-1}$, $\hat{\gamma}_j(\rho)$ is the distance between the origin $\hat{O}_j$ and the tangent space of $\hat{W}_j$ at the uniquely-determined point $w \in \hat{W}_j$ such that the outward-pointing unit normal to $\hat{W}_j$ at p coincides with ρ. Then, $\hat{W}_j$ is the Wulff shape for $\hat{\gamma}_j$. For later use, we denote by $\hat{\xi}_j$ the Cahn–Hoffman map for $\hat{\gamma}_j$.

Now, let $\chi : S^{n-1} \to \Pi_j$ be a C^∞ embedding with outward unit normal $\rho = \rho_j$. Define the anisotropic energy of χ by

$$\hat{\mathcal{F}}_j(\chi) := \int_{S^{n-1}} \hat{\gamma}_j(\rho)\, ds, \tag{4.1}$$

where ds is the $(n-1)$-dimensional volume form of χ.

From now on, we assume that

$$X : (M, \sigma_1, \cdots, \sigma_k) \to (\overline{\Omega}, \Pi_1, \cdots, \Pi_k)$$

is a capillary hypersurface. Set

$$\chi_j := X|_{\sigma_j}.$$

Denote by ρ the outward-pointing unit normal to χ_j in the hyperplane Π_j. X has the following property, which we call the balancing formula that is a generalization of the balancing formula for the isotropic case [3].

Lemma 2.

$$\hat{\mathcal{F}}_j(\chi_j) = -n\Lambda \mathcal{H}^n(D_j), \quad j = 1, \cdots, k. \tag{4.2}$$

Proof. Let u be a constant vector in $\mathbb{R}^{n+1}$. Under parallel translations:

$$X_t = X + tu,$$

$\overline{V}$, $\mathcal{F}_\gamma$, $\mathcal{H}^n(D_j)$ are invariant. Hence, using the first variation formulas we gave above, we compute

$$\begin{aligned}
0 &= \frac{d}{dt}\Big|_{t=0} \overline{\mathcal{F}}_\gamma(X_t) = \frac{d}{dt}\Big|_{t=0} (\overline{\mathcal{F}}_\gamma(X_t) + n\Lambda \overline{V}(X_t)) \\
&= -n \int_M \Lambda \langle u, \nu \rangle\, dA - \oint_{\partial M} \langle u, R(p(\tilde{\xi})) \rangle\, ds \\
&\quad + \sum_{j=1}^k \omega_j \oint_{\sigma_j} \langle u, \rho \rangle\, ds + n\Lambda \int_M \langle u, \nu \rangle\, dA + n\Lambda \sum_{j=1}^k \int_{D_j} \langle u, \tilde{N}_j \rangle\, dA \\
&= -\oint_{\partial M} \langle u, R(p(\tilde{\xi})) \rangle\, ds + n\Lambda \sum_{j=1}^k \int_{D_j} \langle u, \tilde{N}_j \rangle\, dA.
\end{aligned}$$

By setting $u = (1, 0, \cdots, 0), (0, 1, \cdots, 0), \cdots, (0, \cdots, 0, 1)$, we have

$$-n\Lambda \sum_{j=1}^k \mathcal{H}^n(D_j) \tilde{N}_j = -\sum_{j=1}^k \oint_{\sigma_j} R(p(\tilde{\xi}))\, ds. \tag{4.3}$$

On σ_j, since $\langle\tilde{\xi}, \tilde{N}_j\rangle = \omega_j$ and $\langle\tilde{\xi}, \rho_j\rangle = \langle\hat{\xi}_j, \rho_j\rangle = \hat{\gamma}_j$ hold, we can write

$$\tilde{\xi} = \omega_j\tilde{N}_j + \hat{\gamma}_j\rho_j + \tau,$$

where τ is tangent to C_j. Then, we have

$$R(p(\tilde{\xi})) = R(\omega_j\tilde{N}_j + \hat{\gamma}_j\rho_j) = \omega_j\rho_j - \hat{\gamma}_j\tilde{N}_j. \tag{4.4}$$

Note that, by the divergence theorem, it holds that

$$\oint_{\sigma_j} \rho_j \, ds = 0.$$

Hence, substituting Eq (4.4) into Eq (4.3), we obtain

$$-n\Lambda \sum_{j=1}^{k} \mathcal{H}^n(D_j)\tilde{N}_j = \sum_{j=1}^{k} \oint_{\sigma_j} \hat{\gamma}_j\tilde{N}_j \, ds = \sum_{j=1}^{k} \hat{\mathcal{F}}_j(\chi_j)\tilde{N}_j. \tag{4.5}$$

Because $\tilde{N}_1, \cdots, \tilde{N}_k$ are linearly independent, Eq (4.5) implies Eq (4.2). □

Now, consider the anisotropic parallel hypersurfaces $X_t := X + t\tilde{\xi}$ ($t \in \mathbb{R}$, $|t| << 1$) of X (Figure 2, upper left). If $\omega_j > 0$ for some $j \in \{1, \cdots, k\}$, then X_t does not satisfy the boundary condition, that is, the boundary $X_t(\partial M)$ of the hypersurface may not be included in the boundary $\partial\Omega \subset \Pi_1 \cup \cdots \cup \Pi_k$ of the wedge-shaped domain Ω. We will prove that, by taking a suitable parallel translation $Z_t = X_t + t\vec{v}$ of X_t, Z_t satisfies the boundary condition (Figure 2, upper right).

For any $a \in \mathbb{R}$, define the hyperplane Π_j^a in $\mathbb{R}^{n+1}$ that is parallel to the hyperplane Π_j as follows.

$$\Pi_j^a := \Pi_j + a\tilde{N}_j = \{P + a\tilde{N}_j \,|\, P \in \Pi_j\}. \tag{4.6}$$

Then, we show:

Lemma 3. *$X_t(\sigma_j) \subset \Pi_j^{t\omega_j}$ holds.*

Proof. From Proposition 2 (ii), we have

$$\langle\tilde{\xi}, \tilde{N}_j\rangle = \omega_j, \quad \text{on } \sigma_j, \ (j = 1, \cdots, k). \tag{4.7}$$

Since $X(\sigma_j) \subset \Pi_j$, (4.7) gives the desired result. □

By using Lemma 3, we will show the following.

Lemma 4. *There exists some $\vec{v} \in \mathbb{R}^{n+1}$ such that*

$$\langle\vec{v}, \tilde{N}_j\rangle = -\omega_j, \quad j = 1, \cdots, k \tag{4.8}$$

is satisfied, and the parallel translation $Z_t = X_t + t\vec{v}$ of X_t satisfies the boundary condition, that is,

$$Z_t(\sigma_j) \subset \Pi_j, \quad j = 1, \cdots, k \tag{4.9}$$

holds.

Proof. From Lemma 3,

$$X_t(\sigma_j) \subset \Pi_j + t\omega_j \tilde{N}_j$$

holds. Hence, for any $\vec{v} \in \mathbb{R}^{n+1}$,

$$X_t(\sigma_j) + t\vec{v} \subset \Pi_j + t(\omega_j \tilde{N}_j + \vec{v})$$

holds. Therefore, $X_t(\sigma_j) + t\vec{v} \subset \Pi_j$ if and only if $\omega_j \tilde{N}_j + \vec{v} \in \Pi_j$, which is equivalent to

$$\langle \omega_j \tilde{N}_j + \vec{v}, \tilde{N}_j \rangle = 0, \tag{4.10}$$

that is,

$$\langle \vec{v}, \tilde{N}_j \rangle = -\omega_j. \tag{4.11}$$

Now set $\vec{v} = (v_1, \cdots, v_{n+1})$. Then,

$$\langle \vec{v}, \tilde{N}_j \rangle = -\omega_j, \quad j = 1, \cdots, k \tag{4.12}$$

is a system of linear equations in the $(n+1)$ variables $v_1, \cdots, v_{n+1}$ with k equations satisfying $k \leq n+1$. Hence, (4.12) has at least one solution $\vec{v}$, which proves the desired result. □

We have proved that $Z_t = X_t + t\vec{v}$ satisfies the boundary condition. However, it is not a volume-preserving variation in general. We can take suitable homotheties

$$Y_t := \mu(t) Z_t = \mu(t)(X + t(\tilde{\xi} + \vec{v})), \ \mu(t) \geq 0, \ \mu(0) = 1,$$

of Z_t if necessary so that $Y_t : (M, \sigma_1, \cdots, \sigma_k) \to (\overline{\Omega}, \Pi_1, \cdots, \Pi_k)$ is a volume-preserving variation of X (Figure 2).

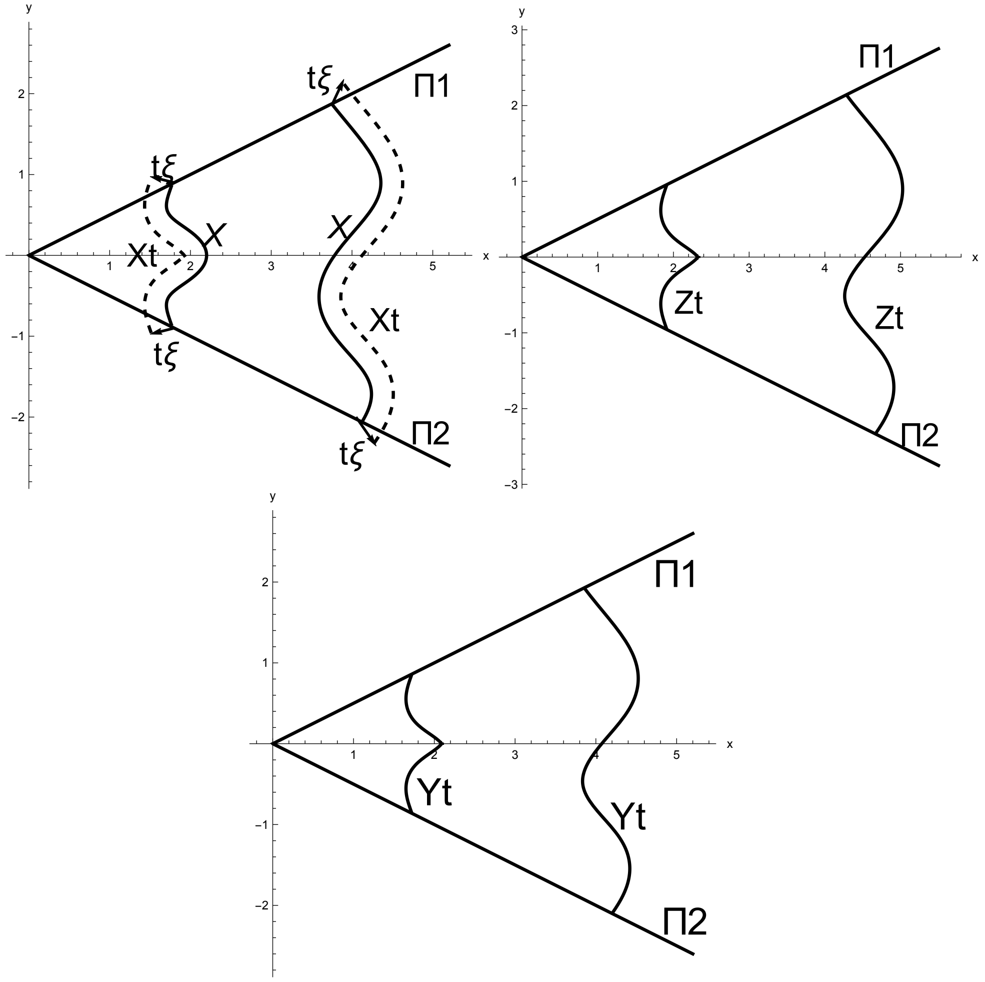

Figure 2. Construction of volume-preserving variation Y_t using anisotropic parallel hypersurfaces X_t of X. Upper left: A capillary hypersurface X and its anisotropic parallel hypersurface X_t. Upper right: A parallel translation Z_t of X_t that satisfies the boundary condition. Bottom: A homothety Y_t of Z_t that satisfies both of the boundary and the volume conditions.

Denote by $e(t)$ the total energy $E(Y_t)$ of Y_t. Then we obtain

Claim 1.

$$e''(0) = \frac{-1}{n}\int_M \gamma(\nu) \sum_{1\le i<j\le n} (k_i^\gamma - k_j^\gamma)^2 \, dA \tag{4.13}$$

$$-\frac{n-1}{n}\sum_{j=1}^{k} \omega_j \left(n\int_{\sigma_j} \hat{\gamma}_j(\rho)\hat{\Lambda}\, ds + \frac{\left(\int_{\sigma_j} \hat{\gamma}_j(\rho)\, ds\right)^2}{\mathcal{H}^n(D_j)} \right), \tag{4.14}$$

where $\hat{\Lambda}$ is the anisotropic curvature of χ_j for $\hat{\gamma}_j$.

Claim 1 will be proved in §6. Note that, from Remark 2(i), each k_j^γ is real. Since X has constant anisotropic mean curvature Λ, the first term of the right hand side of Eq (4.13) is nonnegative if and only if $k_1^\gamma = \cdots = k_n^\gamma = \Lambda/n \neq 0$. Hence, by Corollary 1 in [15], $X(M) \subset (1/|\Lambda|)W_\gamma$ holds. Here we used $\Lambda \neq 0$ which is true because of Lemma 2.

Let us study the second term of the right hand side of the equation of $e''(0)$ in Claim 1. Set

$$B_j := \int_{\sigma_j} \hat{\gamma}_j(\rho)\hat{\Lambda}\, ds + \frac{\left(\int_{\sigma_j} \hat{\gamma}(\rho)\, ds\right)^2}{n\mathcal{H}^n(D_j)}. \tag{4.15}$$

Then we can prove the following statement (see §7 for its proof).

Claim 2. *$B_j \geq 0$ holds and that the equality holds if and only if $\chi_j(\sigma_j) = r\hat{W}_j$ for some $r > 0$.*

Now we are in the final position to complete the proof of the first half of Theorem 1. If the capillary hypersurface X is stable, then $e''(0) \geq 0$ must hold. Hence, by the above observations, $X(M) \subset (1/|\Lambda|)W_\gamma$ holds.

5. Proof of the second half of Theorem 1: the existence of the stable solution

Let $X : (M, \sigma_1, \cdots, \sigma_k) \to (\overline{\Omega}, \Pi_1, \cdots, \Pi_k)$ be an anisotropic capillary hypersurface, and we assume that X is an embedding onto a part of W_γ (up to translation and homothety). We will prove that X is stable. It is sufficient to prove the stability for the case where X is an embedding onto a part of W_γ. Then, there exists a point $Q \in \mathbb{R}^{n+1}$ such that

$$X(M) = (W_\gamma + Q) \cap \overline{\Omega}$$

holds. Set

$$\Sigma := X(M) \cup D_1 \cup \cdots \cup D_k = \big((W_\gamma + Q) \cap \overline{\Omega}\big) \cup D_1 \cup \cdots \cup D_k. \tag{5.1}$$

Then, Σ is a closed convex piecewise-smooth hypersurface in $\mathbb{R}^{n+1}$. We will derive the support function of Σ ([16]). For any $x \in S^n$, we define a hyperplane $P(x)$ that is orthogonal to x as follows.

$$P(x) := \{y \in \mathbb{R}^{n+1} \mid \langle x, y\rangle = 0\}. \tag{5.2}$$

Define a continuous function $\varphi : S^n \to \mathbb{R}_{>0}$ as follows.

$$\varphi(x) := \max\{t \in \mathbb{R} \mid (Q + tx + P(x)) \cap \Sigma \neq \emptyset\}. \tag{5.3}$$

Then, the homogeneous extension $\overline{\varphi} : \mathbb{R}^{n+1} \to \mathbb{R}$ of φ is the support function of Σ. Then, Σ is the Wulff shape for φ ([17]), that is, $\Sigma = W_\varphi$ holds. Note that

$$\begin{aligned} \gamma(x) &:= \max\{t \in \mathbb{R} \mid (tx + P(x)) \cap W_\gamma \neq \emptyset\} \\ &= \max\{t \in \mathbb{R} \mid (Q + tx + P(x)) \cap (W_\gamma + Q) \neq \emptyset\} \end{aligned} \tag{5.4}$$

holds. Since $\gamma(x) \geq \varphi(x)$ holds, we have $\tilde{W}[\gamma] \supset \tilde{W}[\varphi]$. Hence we have the followings:

(i) If $x \in \nu(M)$, then $\varphi(x) = \gamma(x)$.
(ii) If $x = \tilde{N}_j$, then $\varphi(x) = \omega_j < \gamma(x)$, $(j = 1, \cdots, k)$.
(iii) If $x \in S^n \setminus (\nu(M) \cup \{\tilde{N}_1, \cdots, \tilde{N}_k\})$, then $\varphi(x) < \gamma(x)$.

Now let $X_t : (M, \sigma_1, \cdots, \sigma_k) \to (\overline{\Omega}, \Pi_1, \cdots, \Pi_k)$ be a volume-preserving variation of X. Denote by $D_j(X_t)$ the domain bounded by $X_t(\sigma_j)$ in Π_j. Set

$$\Sigma_t := X_t(M) \cup D_1(X_t) \cup \cdots, \cup D_k(X_t).$$

Then, because $\gamma \geq \varphi$ and Σ is the minimizer of $\mathcal{F}_\varphi$ among all closed piecewise-smooth hypersurfaces enclosing the same $(n+1)$-dimensional volume, we obtain

$$\begin{aligned} E_\gamma(X) &= \mathcal{F}_\varphi(\Sigma) \leq \mathcal{F}_\varphi(\Sigma_t) = \mathcal{F}_\varphi(X_t) + \mathcal{W}(X_t) \\ &\leq \mathcal{F}_\gamma(X_t) + \mathcal{W}(X_t) = E_\gamma(X_t) \end{aligned}$$

holds. Therefore, X is stable.

6. Proof of Claim 1

Recall

$$Y_t = \mu(t) Z_t, \ \mu(t) \geq 0, \ \mu(0) = 1, \tag{6.1}$$
$$Z_t = X_t + t\vec{v}, \tag{6.2}$$
$$X_t = X + t\tilde{\xi}, \tag{6.3}$$

$Y_t : (M, \sigma_1, \cdots, \sigma_k) \to (\overline{\Omega}, \Pi_1, \cdots, \Pi_k)$ is a volume-preserving variation of X that satisfies the boundary condition. Note that the unit normal vector field along X_t, Z_t, and Y_t coincide with $\nu : M \to S^n$ that is the unit normal vector field along X. Hence,

$$e(t) := E(Y_t) = \mathcal{F}_\gamma(Y_t) + \mathcal{W}(Y_t) = \int_M \gamma(\nu)\, dA_{Y_t} + \sum_{j=1}^{k} \omega_j \mathcal{H}^n(D_j(Y_t))$$

holds, where dA_{Y_t} is the n-dimensional volume form of Y_t.

Set

$$V_0 := V(X), \quad E_0 := E(X), \quad F_0 := \mathcal{F}_\gamma(X), \quad W_0 := \mathcal{W}_\gamma(X), \tag{6.4}$$

and

$$v(t) := V(Y_t), \quad f(t) := \mathcal{F}_\gamma(X_t), \quad w(t) := \mathcal{W}_\gamma(X_t). \tag{6.5}$$

Then

$$e(t) = (\mu(t))^n (f(t) + w(t)), \tag{6.6}$$
$$v(t) = (\mu(t))^{n+1} V(Z_t), \tag{6.7}$$

$$e'(t) = n(\mu(t))^{n-1}\mu'(t)(f(t) + w(t)) + (\mu(t))^n (f'(t) + w'(t)), \tag{6.8}$$
$$\begin{aligned} e''(t) &= n(n-1)(\mu(t))^{n-2}(\mu'(t))^2 (f(t) + w(t)) \\ &\quad + 2n(\mu(t))^{n-1}\mu'(t)(f'(t) + w'(t)) \\ &\quad + n(\mu(t))^{n-1}\mu''(t)(f(t) + w(t)) \end{aligned}$$

$$
\begin{aligned}
&+(\mu(t))^n(f''(t)+w''(t)), &(6.9)\\
v'(t) &= (n+1)(\mu(t))^n\mu'(t)V(Z_t)+(\mu(t))^{n+1}\frac{d}{dt}V(Z_t), &(6.10)\\
v''(t) &= (n+1)n(\mu(t))^{n-1}(\mu'(t))^2V(Z_t)\\
&+(n+1)(\mu(t))^n\mu''(t)V(Z_t)\\
&+2(n+1)(\mu(t))^n\mu'(t)\frac{d}{dt}V(Z_t)\\
&+(\mu(t))^{n+1}\frac{d^2}{dt^2}V(Z_t). &(6.11)
\end{aligned}
$$

In order to compute $e''(0)$, we need to compute $\mu'(0), \mu''(0), f'(0)+w'(0), f''(0)$, and $w''(0)$. First, using the Steiner-type formula (2.4), we obtain the followings.

$$
f'(0) = -n\int_M \Lambda\gamma(\nu)\,dA = -n\Lambda F_0, \tag{6.12}
$$

$$
\begin{aligned}
f''(0) &= \int_M 2({}_nC_2)\gamma(\nu)H_2^\gamma\,dA = \int_M 2\gamma(\nu)\sigma_2^\gamma\,dA\\
&= 2\int_M \gamma(\nu)\sum_{1\le i<j\le n} k_i^\gamma k_j^\gamma\,dA. &(6.13)
\end{aligned}
$$

Before computing the other derivatives, we prepare two useful lemmas.

Lemma 5. *We have the following equalities.*

(i)

$$
\frac{d}{dt}V(Z_t) = E(Z_t). \tag{6.14}
$$

(ii) In the special case where $k=0$, that is $\partial M=\emptyset$, we have

$$
\frac{d}{dt}V(X_t) = \mathcal{F}_\gamma(X_t). \tag{6.15}
$$

Proof. Since

$$
dA_{Z_t} = dA_{X_t}, \quad \tilde{\xi} = D\gamma|_\nu + \gamma(\nu)\nu,
$$

by using the first variation formula (3.2) for the volume, we have

$$
\begin{aligned}
\frac{d}{dt}V(Z_t) &= \int_M \Big\langle \frac{\partial Z_t}{\partial t}, \nu\Big\rangle\,dA_{X_t}\\
&= \int_M (\gamma(\nu)+\langle\vec{v},\nu\rangle)\,dA_{X_t}\\
&= \mathcal{F}_\gamma(X_t) + \int_M \langle\vec{v},\nu\rangle\,dA_{X_t}. &(6.16)
\end{aligned}
$$

We compute the last term of (6.16). Denote by $dA(\Pi_j^{t\omega_j})$ the standard volume form on the n-plane $\Pi_j^{t\omega_j}$. Using the divergence formula and the equality $\langle\vec{v},\tilde{N}_j\rangle = -\omega_j$ (see Lemma 4), we have

$$
\int_M \langle\vec{v},\nu\rangle\,dA_{X_t} = -\sum_{j=1}^k \int_{D_j(X_t)} \langle\vec{v},\tilde{N}_j\rangle\,dA(\Pi_j^{t\omega_j})
$$

$$
\begin{aligned}
&= \sum_{j=1}^{k} \omega_j \mathcal{H}^n(D_j(X_t)) \\
&= \mathcal{W}(Z_t). \qquad (6.17)
\end{aligned}
$$

(6.16) with (6.17) gives the desired equality (6.14). The proof of (6.15) is similar. □

Lemma 6. *We have the following equality.*

$$E_0 = -(n+1)\Lambda V_0. \qquad (6.18)$$

Proof. Consider the variation $\hat{X}_\epsilon := (1+\epsilon)X$ of X, and set $F(\epsilon) := \mathcal{F}_\gamma(\hat{X}_\epsilon)$. Then,

$$F(\epsilon) = (1+\epsilon)^n F_0.$$

Hence,

$$F'(0) = nF_0. \qquad (6.19)$$

On the other hand, the first variation formula (3.1) of $\mathcal{F}_\gamma$ gives

$$
\begin{aligned}
F'(0) &= -n \int_M \Lambda \langle X, \nu \rangle \, dA - \int_{\partial M} \langle X, R(p(\tilde{\xi})) \rangle \, ds \\
&= -n(n+1)\Lambda V_0 - \sum_{j=1}^{k} \int_{\sigma_j} \langle X, R(p(\tilde{\xi})) \rangle \, ds. \qquad (6.20)
\end{aligned}
$$

We compute the integrand of the second term of (6.20) on σ_j. Proposition 2 (ii) gives $\langle \tilde{\xi}, \tilde{N}_j \rangle = \omega_j$ on σ_j. Hence,

$$p(\tilde{\xi}) = \omega_j \tilde{N}_j + \langle p(\tilde{\xi}), \rho \rangle \rho. \qquad (6.21)$$

Using (6.21) and the equality $\langle X, \tilde{N}_j \rangle = 0$, we have

$$\langle X, R(p(\tilde{\xi})) \rangle = \langle X, \omega_j \rho - \langle p(\tilde{\xi}), \rho \rangle \tilde{N}_j \rangle = \omega_j \langle X, \rho \rangle. \qquad (6.22)$$

Inserting (6.22) to (6.20), we obtain

$$
\begin{aligned}
F'(0) &= -n(n+1)\Lambda V_0 - \sum_{j=1}^{k} \omega_j \int_{\sigma_j} \langle X, \rho \rangle \, ds \\
&= -n(n+1)\Lambda V_0 - n \sum_{j=1}^{k} \omega_j \mathcal{H}^n(D_j) \\
&= -n(n+1)\Lambda V_0 - nW_0. \qquad (6.23)
\end{aligned}
$$

(6.19) with (6.23) gives the desired equality (6.18). □

Let us continue the proof of Claim 1. Using the equalities (6.1), (6.10), and (6.14), we have

$$0 = \nu'(0) = (n+1)\mu'(0)V_0 + E_0. \qquad (6.24)$$

Hence we have

$$\mu'(0) = \frac{-E_0}{(n+1)V_0}. \tag{6.25}$$

Next we compute $f'(0) + w'(0)$. Note that $e'(0) = 0$ because X is a capillary hypersurface. Using (6.8) and (6.25), we obtain

$$0 = e'(0) = n\mu'(0)E_0 + f'(0) + w'(0) = \frac{-nE_0^2}{(n+1)V_0} + f'(0) + w'(0), \tag{6.26}$$

which implies that

$$f'(0) + w'(0) = \frac{nE_0^2}{(n+1)V_0} \tag{6.27}$$

holds.

Now we compute $w''(0)$. Using the first variation formula (3.2) for volume, we obtain

$$w'(t) = \sum_{j=1}^{k} \omega_j \int_{\sigma_j} \langle \tilde{\xi}, \rho_t \rangle \, ds_t = \sum_{j=1}^{k} \omega_j \hat{\mathcal{F}}_j(X_t|_{\sigma_j}), \tag{6.28}$$

where ρ_t is the outward-pointing unit normal vector field along $X_t|_{\sigma_j} : \sigma_j \to \Pi_j^{t\omega_j}$, and ds_t is the $(n-1)$-dimensional volume form of $X_t|_{\sigma_j}$. Using (6.28) and the first variation formula (3.1) of the anisotropic energy, we obtain

$$w''(0) = -(n-1) \sum_{j=1}^{k} \omega_j \int_{\sigma_j} \hat{\Lambda} \hat{\gamma}_j(\rho) \, ds. \tag{6.29}$$

Finally we compute $\mu''(0)$ by using $v''(0) = 0$ and (6.11). From (6.14), we have

$$\frac{d}{dt} V(Z_t)|_{t=0} = E_0, \tag{6.30}$$

$$\frac{d^2}{dt^2} V(Z_t)|_{t=0} = \frac{d}{dt} E(Z_t)|_{t=0} = \frac{nE_0^2}{(n+1)V_0}, \tag{6.31}$$

here in the last equality we used (6.27). Inserting (6.25), (6.30), (6.31) to (6.11), we compute to obtain

$$\mu''(0) = \frac{2E_0^2}{(n+1)^2 V_0^2}. \tag{6.32}$$

Inserting (6.13), (6.25), (6.27), (6.29), and (6.32) to (6.9) at $t = 0$, we obtain

$$\begin{aligned} e''(0) \quad = \quad & \frac{-n(n-1)E_0^3}{(n+1)^2 V_0^2} + 2 \int_M \gamma(\nu) \sum_{1 \le i < j \le n} k_i^\gamma k_j^\gamma \, dA \\ & -(n-1) \sum_{j=1}^{k} \omega_j \int_{\sigma_j} \hat{\Lambda} \hat{\gamma}_j(\rho) \, ds. \end{aligned} \tag{6.33}$$

From Lemma 6, we have

$$\frac{E_0}{V_0} = -(n+1)\Lambda.$$

Inserting this to (6.33), we obtain

$$\begin{aligned} e''(0) &= -n(n-1)\Lambda^2 E_0 + 2\int_M \gamma(\nu) \sum_{1\le i<j\le n} k_i^\gamma k_j^\gamma \, dA \\ &\quad -(n-1)\sum_{j=1}^k \omega_j \int_{\sigma_j} \hat{\Lambda}\hat{\gamma}_j(\rho)\, ds \\ &= \frac{-1}{n}\int_M \gamma(\nu) \sum_{1\le i<j\le n} (k_i^\gamma - k_j^\gamma)^2 \, dA - n(n-1)\Lambda^2 W_0 \\ &\quad -(n-1)\sum_{j=1}^k \omega_j \int_{\sigma_j} \hat{\Lambda}\hat{\gamma}_j(\rho)\, ds. \end{aligned} \tag{6.34}$$

Now we are in the final stage to prove Claim 1. The balancing formula (4.2) implies

$$\Lambda = \frac{-\int_{\sigma_j} \hat{\gamma}_j(\rho)\, ds}{n\mathcal{H}^n(D_j)}, \quad j = 1, \cdots, k.$$

Hence

$$-n(n-1)\Lambda^2 W_0 = -n(n-1)\sum_{j=1}^k \Lambda^2 \omega_j \mathcal{H}^n(D_j) = -\frac{n-1}{n}\sum_{j=1}^k \omega_j \frac{\left(\int_{\sigma_j} \hat{\gamma}_j(\rho)\, ds\right)^2}{\mathcal{H}^n(D_j)}$$

holds. This with (6.34) proves Claim 1.

7. Proof of Claim 2

In this section, we examine the sign of

$$B_j := \int_{\sigma_j} \hat{\gamma}_j(\rho)\hat{\Lambda}\, ds + \frac{(\hat{\mathcal{F}}_j(\chi_j))^2}{n\mathcal{H}^n(D_j)}, \tag{7.1}$$

where

$$\hat{\mathcal{F}}_j(\chi_j) = \int_{\sigma_j} \hat{\gamma}(\rho)\, ds.$$

First, we recall two known useful propositions, which we prove for completeness.

Proposition 3. *Let $\gamma : S^m \to \mathbb{R}_{>0}$ be a positive strictly convex function of class at least C^2, and $W \subset \mathbb{R}^{m+1}$ be its Wulff shape. Denote by $\mathcal{F}_\gamma(W)$ the anisotropic energy of W for γ, and by $\mathcal{H}^{m+1}(W)$ the $(m+1)$-dimensional Hausdorff measure of the domain bounded by W in $\mathbb{R}^{m+1}$. Then,*

$$\mathcal{F}_\gamma(W) = (m+1)\mathcal{H}^{m+1}(W) \tag{7.2}$$

holds.

Proof. Let $\xi : S^m \to \mathbb{R}^{m+1}$ be the Cahn-Hoffman map of γ defined by $\xi(\nu) = D\gamma|_\nu + \gamma(\nu)\nu$. Then ξ is an embedding onto W. Hence, denoting by dA_ξ the volume form of ξ, we have

$$\mathcal{F}_\gamma(W) = \int_{S^m} \gamma(\nu)\, dA_\xi = \int_{S^m} \langle \xi(\nu), \nu \rangle\, dA_\xi = (m+1)\mathcal{H}^{m+1}(W),$$

which proves (7.2). □

Proposition 4 (Anisotropic isoperimetric inequality)**.** *Let γ, W be the same as in Proposition 3. We also use the same notation as in Proposition 3. Then, for any closed embedded piecewise-smooth hypersurface M in $\mathbb{R}^{m+1}$, it holds that*

$$(\mathcal{F}_\gamma(M))^{m+1} \geq (m+1)^{m+1}\mathcal{H}^{m+1}(W)(\mathcal{H}^{m+1}(M))^m, \tag{7.3}$$

where the equality holds if and only if M coincides with W up to homothety and translation in $\mathbb{R}^{m+1}$.

Proof. Recall that the Wulff shape W is the minimizer of $\mathcal{F}_\gamma$ among closed hypersurfaces enclosing the same $(m+1)$-dimensional Hausdorff measure. Set

$$c = \left(\frac{\mathcal{H}^{m+1}(W)}{\mathcal{H}^{m+1}(M)}\right)^{\frac{1}{m+1}}.$$

Then, the hypersurface

$$M_c := cM$$

similar to M encloses the same $(m+1)$-dimensional Hausdorff measure as W. Hence,

$$\mathcal{F}_\gamma(M_c) \geq \mathcal{F}_\gamma(W) \tag{7.4}$$

holds, where the equality holds if and only if M coincides with W up to homothety and translation. On the other hand,

$$\mathcal{F}_\gamma(M_c) = c^m \mathcal{F}_\gamma(M) = \left(\frac{\mathcal{H}^{m+1}(W)}{\mathcal{H}^{m+1}(M)}\right)^{\frac{m}{m+1}} \mathcal{F}_\gamma(M). \tag{7.5}$$

The inequality (7.4) combined with the equalities (7.2) and (7.5) gives (7.3). □

Now we examine B_j. From now on, for simplicity, we identify an embedded closed hypersurface in an euclidean space with the domain bounded by this hypersurface. We also identify an embedded closed hypersurface with its representation mapping.

Using Proposition 4, we have

$$\begin{aligned} B_j &= \int_{\sigma_j} \hat{\gamma}_j(\rho)\hat{\Lambda}\, ds + \frac{(\hat{\mathcal{F}}_j(\chi_j))^2}{n\mathcal{H}^n(D_j)} \\ &\geq \int_{\sigma_j} \hat{\gamma}_j(\rho)\hat{\Lambda}\, ds + n(\mathcal{H}^n(\hat{W}_\gamma))^{\frac{2}{n}}(\mathcal{H}^n(D_j))^{\frac{n-2}{n}}. \end{aligned} \tag{7.6}$$

First we study the special case where $n = 2$. In this case, using (7.6) with Proposition 3, we have

$$B_j \geq \int_{\sigma_j} \hat{\gamma}_j(\rho)\hat{\Lambda}\, ds + 2\mathcal{H}^2(\hat{W}_j) = \int_{\sigma_j} \hat{\gamma}_j(\rho)\hat{\Lambda}\, ds + \hat{\mathcal{F}}_j(\hat{W}_j). \tag{7.7}$$

We use the representation (2.2) of the anisotropic mean curvature in Remark 1. Note that σ_j is topologically S^1. Denote by $G : \hat{W}_j \to S^1$ the outward-pointing unit normal vector field on $\hat{W}_j$ and by $\hat{s}$ the arc-length parameter of $\hat{W}_j$. Then $G = \hat{\xi}_j^{-1}$, where $\hat{\xi}_j : S^1 \to \hat{W}_j$ is the Cahn-Hoffman map of $\hat{W}_j$. Hence, if we take $\rho \in S^1$ as the parameter of $\hat{W}_j$ through $\hat{\xi}_j$, G is the identity mapping on S^1. We also denote by κ_j, $\hat{\kappa}_j$ the curvatures of χ_j, $\hat{W}_j$ with respect to the outward-pointing unit normal, respectively. Then, using (2.2), we have

$$\begin{aligned}\int_{\sigma_j} \hat{\gamma}_j(\rho)\hat{\Lambda}\, ds &= -\int_{\sigma_j} \hat{\gamma}_j(\rho)\frac{\kappa_j}{\hat{\kappa}_j}\, ds = -\int_{S^1} \hat{\gamma}_j(\rho)\frac{\frac{d\rho}{ds}}{\frac{dG}{d\hat{s}}}\, ds \\ &= -\int_{S^1} \hat{\gamma}_j(\rho)\frac{d\rho}{dG}\, d\hat{s} = -\int_{S^1} \hat{\gamma}_j(\rho)\, d\hat{s} \\ &= -\hat{\mathcal{F}}_j(\hat{W}_j). \end{aligned} \tag{7.8}$$

Inserting (7.8) into (7.7), we have $B_j \geq 0$.

Next we assume $n \geq 3$. We assume that D_j are convex. Below, for simplicity, we omit the subscript j, that is, we write D instead of D_j, for instance. On the Minkowski sum $D + t\hat{W}$, there holds

$$\mathcal{H}^n(D + t\hat{W}) = \sum_{i=0}^{n} ({}_nC_i)t^i v(\overbrace{D, \ldots, D}^{(n-i)\text{ times}}, \overbrace{\hat{W}, \ldots, \hat{W}}^{i\text{ times}}), \tag{7.9}$$

where $v(K_1, \cdots, K_n)$ is the so-called the mixed volume of convex bodies $K_1, \cdots, K_n$ in $\mathbb{R}^n$ ([16, Theorem 5.1.7]). On the other hand, from Lemma 5 and the Steiner-type formula (2.4), we have

$$\begin{aligned}\mathcal{H}^n(D + t\hat{W}) &= \mathcal{H}^n(\chi + t\hat{\xi}) \\ &= \mathcal{H}^n(\chi) + \int_0^t \hat{\mathcal{F}}(\chi + t\hat{\xi})\, dt \\ &= \mathcal{H}^n(D) + \int_0^t \Big(\int_\sigma \hat{\gamma}(\rho) \sum_{r=0}^{n-1} (-1)^r ({}_{n-1}C_r)t^r \hat{H}_r^{\hat{\gamma}}\, ds\Big)\, dt \\ &= \mathcal{H}^n(D) + \sum_{r=0}^{n-1} \frac{(-1)^r}{r+1} ({}_{n-1}C_r)t^{r+1} \int_\sigma \hat{\gamma}(\rho)\hat{H}_r^{\hat{\gamma}}\, ds. \end{aligned} \tag{7.10}$$

Comparing (7.9) with (7.10), we obtain

$$v(\overbrace{D, \ldots, D}^{n\text{ times}}) = \mathcal{H}^n(D), \tag{7.11}$$

$$v(\overbrace{D, \ldots, D}^{(n-1)\text{ times}}, \hat{W}) = \frac{1}{n}\int_\sigma \hat{\gamma}(\rho)\, ds, \tag{7.12}$$

$$v(\overbrace{D, \ldots, D}^{(n-2)\text{ times}}, \hat{W}, \hat{W}) = -\frac{1}{n}\int_\sigma \hat{\gamma}(\rho)\hat{\Lambda}\, ds. \tag{7.13}$$

The Minkowski's second inequality ([16, Theorem 7.2.1]) gives

$$\Big(v(\overbrace{D, \ldots, D}^{(n-1)\text{ times}}, \hat{W})\Big)^2 \geq v(\overbrace{D, \ldots, D}^{n\text{ times}}) \cdot v(\overbrace{D, \ldots, D}^{(n-2)\text{ times}}, \hat{W}, \hat{W}), \tag{7.14}$$

where the equality holds if and only if D is homothetic to $\hat{W}$. Combining (7.11)–(7.14), we obtain

$$\Big(\int_\sigma \hat{\gamma}(\rho)\, ds\Big)^2 \geq -n\mathcal{H}^n(D) \int_\sigma \hat{\gamma}(\rho)\hat{\Lambda}\, ds, \tag{7.15}$$

here the equality holds if and only if D is homothetic to $\hat{W}$. This completes the proof of Claim 2.

Acknowledgments

This work was partially supported by JSPS KAKENHI Grant Number JP20H01801, JSPS Grant-in-Aid for Scientific Research on Innovative Areas "Discrete Geometric Analysis for Materials Design": Grant Numbers JP18H04487 and JP20H04642, JST CREST GrantNumber JPMJCR1911, and JSPS(Japan)-FWF(Austria) Joint Research Project (2018-2019, Project leader (Japan): Miyuki Koiso).

Conflict of interest

The authors declare no conflict of interest.

References

1. N. Ando, Hartman-Wintner's theorem and its applications, *Calc. Var.*, **43** (2012), 389–402. http://doi.org/10.1007/s00526-011-0415-x
2. J. L. Barbosa, M. Do Carmo, J. Eschenburg, Stability of hypersurfaces of constant mean curvature in Riemannian manifolds, *Math. Z.*, **197** (1988), 123–138. https://doi.org/10.1007/BF01161634
3. J. Choe, M. Koiso, Stable capillary hypersurfaces in a wedge, *Pac. J. Math.*, **280** (2016), 1–15. http://doi.org/10.2140/pjm.2016.280.1
4. Y. He, H. Li, Integral formula of Minkowski type and new characterization of the Wulff shape, *Acta. Math. Sin.-English Ser.*, **24** (2008), 697–704. https://doi.org/10.1007/s10114-007-7116-6
5. Y. He, H. Li, H. Ma, J. Ge, Compact embedded hypersurfaces with constant higher order anisotropic mean curvatures, *Indiana Univ. Math. J.*, **58** (2009), 853–868. http://doi.org/10.1512/iumj.2009.58.3515
6. M. Koiso, Uniqueness of stable closed non-smooth hypersurfaces with constant anisotropic mean curvature, arXiv.1903.03951.
7. M. Koiso, Uniqueness of closed equilibrium hypersurfaces for anisotropic surface energy and application to a capillary problem, *Math. Comput. Appl.*, **24** (2019), 88. https://doi.org/10.3390/mca24040088
8. M. Koiso, B. Palmer, Geometry and stability of surfaces with constant anisotropic mean curvature, *Indiana Univ. Math. J.*, **54** (2005), 1817–1852. http://doi.org/10.1512/iumj.2005.54.2613
9. M. Koiso, B. Palmer, Stability of anisotropic capillary surfaces between two parallel planes, *Calc. Var.*, **25** (2006), 275–298. http://doi.org/10.1007/s00526-005-0336-7

10. M. Koiso, B. Palmer, Anisotropic capillary surfaces with wetting energy, *Calc. Var.*, **29** (2007), 295–345. http://doi.org/10.1007/s00526-006-0066-5
11. M. Koiso, B. Palmer, Uniqueness theorems for stable anisotropic capillary surfaces, *SIAM J. Math. Anal.*, **39** (2007), 721–741. https://doi.org/10.1137/060657297
12. M. Koiso, B. Palmer, Anisotropic umbilic points and Hopf's theorem for surfaces with constant anisotropic mean curvature, *Indiana Univ. Math. J.*, **59** (2010), 79–90. http://doi.org/10.1512/iumj.2010.59.4164
13. H. Li, C. Xiong, Stability of capillary hypersurfaces with planar boundaries, *J. Geom. Anal.*, **27** (2017), 79–94. https://doi.org/10.1007/s12220-015-9674-7
14. B. Palmer, Stability of the Wulff shape, *Proc. Am. Math. Soc.*, **126** (1998), 3661–3667. http://doi.org/10.1090/S0002-9939-98-04641-3
15. R. C. Reilly, The relative differential geometry of nonparametric hypersurfaces, *Duke Math. J.*, **43** (1976), 705–721. http://doi.org/10.1215/S0012-7094-76-04355-6
16. R. Schneider, Convex bodies: the Brunn-Minkowski theory, 2 Eds., New York: Cambridge University Press, 2014. https://doi.org/10.1017/CBO9781139003858
17. J. E. Taylor, Crystalline variational problems, *Bull. Amer. Math. Soc.*, **84** (1978), 568–588. https://doi.org/10.1090/S0002-9904-1978-14499-1
18. H. Weyl, On the volume of tubes, *Amer. J. Math.*, **61** (1939), 461–472. https://doi.org/10.2307/2371513
19. W. L. Winterbottom, Equilibrium shape of a small particle in contact with a foreign substrate, *Acta Metal.*, **15** (1967), 303–310. https://doi.org/10.1016/0001-6160(67)90206-4
20. G. Wulff, Zur Frage der Geschwindigkeit des Wachsthums und der Auflösung der Krystallflächen, *Zeitschrift für Krystallographie und Mineralogie*, **34** (1901), 449–530. https://doi.org/10.1524/zkri.1901.34.1.449