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Research article

On the low Mach number limit for 2D Navier–Stokes–Korteweg systems[†]

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Abstract: This paper addresses the low Mach number limit for two-dimensional Navier–Stokes–Korteweg systems. The primary purpose is to investigate the relevance of the capillarity tensor for the analysis. For the sake of a concise exposition, our considerations focus on the case of the quantum Navier-Stokes (QNS) equations. An outline for a subsequent generalization to general viscosity and capillarity tensors is provided. Our main result proves the convergence of finite energy weak solutions of QNS to the unique Leray-Hopf weak solutions of the incompressible Navier-Stokes equations, for general initial data without additional smallness or regularity assumptions. We rely on the compactness properties stemming from energy and BD-entropy estimates. Strong convergence of acoustic waves is proven by means of refined Strichartz estimates that take into account the alteration of the dispersion relation due to the capillarity tensor. For both steps, the presence of a suitable capillarity tensor is pivotal.

Keywords: Navier–Stokes–Korteweg equation; incompressible Navier–Stokes equation; capillarity; quantum fluids; low Mach number limit; acoustic waves; Strichartz estimates; energy estimates; BD-entropy estimates

1. Introduction

The class of Navier-Stokes-Korteweg equations arises in the modelling of capillary fluid flow as it occurs for instance in physical phenomena such as diffuse interfaces [27, 40]. Capillarity effects are mathematically described by a dispersive stress tensor depending on the density and its derivatives. In their general form, these systems read

$$\begin{aligned}&\partial_t\rho + \operatorname{div}(\rho u) = 0,\\ &\partial_t(\rho u) + \operatorname{div}(\rho u\otimes u) + \nabla P(\rho) = 2\nu \operatorname{div}\mathbb{S} + \kappa^2 \operatorname{div}\mathbb{K}.\end{aligned} \tag{1.1}$$

The unknowns are the density ρ and the velocity field u. We consider the isentropic pressure law $P(\rho) = \frac{1}{\gamma}\rho^\gamma$ with $\gamma > 1$. The parameters $\nu, \kappa > 0$ denote the viscosity and capillarity coefficients respectively. The viscous stress tensor $\mathbb{S} = \mathbb{S}(\nabla u)$ equals

$$\mathbb{S} = \mu(\rho)\mathbf{D}u + \lambda(\rho)\operatorname{div}(u)\mathbf{I},$$

where μ, λ denote the shear and bulk viscosity coefficients respectively and satisfy $\mu(\rho) + 2\lambda(\rho) \geq 0$. The capillary term $\mathbb{K} = \mathbb{K}(\rho, \nabla\rho)$ amounts to

$$\mathbb{K} = \left(\rho \operatorname{div}(k(\rho)\nabla\rho) - \frac{1}{2}(\rho k'(\rho) - k(\rho)|\nabla\rho|^2)\right)\mathbb{I} - k(\rho)\nabla\rho\otimes\nabla\rho. \tag{1.2}$$

The capillary tensor is referred to as Korteweg tensor [40], see also [54, 55]. The family of Navier-Stokes-Korteweg equations has rigorously been derived in [27] and more recently in [33]. A prominent example of (1.1) are the quantum Navier-Stokes (QNS) equations that will mainly be considered in this paper. The QNS equations are obtained from (1.1) by choosing the shear viscosity to depend linearly on the density, namely $\mu(\rho) = \rho$, vanishing bulk viscosity $\lambda(\rho) = 0$, and $k(\rho) = 1/\rho$. Its inviscid counterpart (considering $\lambda(\rho) = \mu(\rho) = 0$) is the Quantum Hydrodynamic system (QHD) [7, 9, 10] which has a strong analogy with Gross-Pitaevskii type equations describing for instance the effective dynamics in terms of a macroscopic order parameter of superfluid helium [39] or Bose-Einstein condensation [50]. This close link to NLS type equations highlights the quantum mechanical nature of the model, see e.g., [7, 34]. Beyond that the QHD system also serves as model for semi-conductor devices [30]. In this regard, (1.3) can be interpreted as a viscous regularization, but can also be derived as the moment closure method with a BGK type collision term [23, 37], see also [36] for an overview of dissipative quantum fluid models and their utility for numerical simulations. The class of systems (1.1) with capillarity tensor $\mathbb{K}$ but with $\nu = 0$, namely inviscid systems such as QHD, goes under the name of Euler-Korteweg system [14, 20]. The choice $k(\rho) = \text{const.}$, $\mu(\rho) = \rho$, $\lambda(\rho) = 0$ constitutes a second example that is extensively studied in literature and is commonly referred to as Navier-Stokes-Korteweg (NSK) system [13, 18, 19]. Finally, we mention that for $\kappa = 0$, namely (1.1) without capillarity term, one recovers the compressible Navier-Stokes equations with density-dependent viscosity [18, 45, 56].

The aim of this paper is to investigate the low Mach number limit of (1.1) posed on $[0,\infty)\times\mathbb{R}^2$ in the class of weak solutions and for general ill-prepared data. To that end, we focus on the analysis of the (QNS) equations, namely

$$\begin{aligned}&\partial_t\rho + \operatorname{div}(\rho u) = 0,\\ &\partial_t(\rho u) + \operatorname{div}(\rho u\otimes u) + \nabla P(\rho) = 2\nu\operatorname{div}(\rho\mathbf{D}u) + 2\kappa^2\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right),\end{aligned} \tag{1.3}$$

being complemented with the non-trivial far-field behavior

$$\rho(x) \to 1, \qquad |x| \to \infty. \tag{1.4}$$

For (1.3) the capillarity tensor div $\mathbb{K}$, defined in (1.2), can formally be rewritten as

$$\frac{1}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right)=\frac{1}{4}\nabla\Delta\rho-\operatorname{div}(\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho})=\operatorname{div}\left(\sqrt{\rho}(\nabla^2\sqrt{\rho}-4\nabla\rho^{\frac{1}{4}}\otimes\nabla\rho^{\frac{1}{4}})\right). \tag{1.5}$$

The total energy associated to (1.3) reads

$$E(\rho,u)=\int_{\mathbb{R}^2}\frac{1}{2}\rho|u|^2+2\kappa^2|\nabla\sqrt{\rho}|^2+F(\rho)\mathrm{d}x,\qquad F(\rho)=\frac{\rho^\gamma-\gamma(\rho-1)}{\gamma(\gamma-1)}. \tag{1.6}$$

Note that the assumption of finite energy $E(\rho,u)<+\infty$ enforces the far-field behavior (1.4) for the given choice of the internal energy $F(\rho)$.

The motivation to mainly study (1.3) is two-fold. First, our main purpose is to elucidate the relevance of the capillarity tensor $\mathbb{K}$ for the developed method. In this regard, the choice $k(\rho)=1/\rho$ allows for a more concise and straightforward exposition. Second, to the best of our knowledge, (1.3) is the only system within the class of (1.1) with density dependent viscosity and non-trivial far-field for which existence of finite energy weak solutions (FEWS) is known [8]. However, postulating existence of weak solutions we discuss how our results can be generalized, to the following set of capillarity and viscosity tensors satisfying the compatibility condition [18, 20, 21],

$$\lambda(\rho)=\rho\mu'(\rho)-\mu(\rho),\quad k(\rho)=\frac{(\mu'(\rho))^2}{\rho}, \tag{1.7}$$

see Remark 2.8 below. Assuming (1.7) describes a sufficient condition which leads to suitable *a priori* estimates required for our method. Note that e.g. NSK, namely $k(\rho)=$ const., $\mu(\rho)=\rho$, $\lambda(\rho)=0$ does not satisfy (1.7). Nevertheless, the BD-entropy estimates obtained in [13] enable us to include NSK in our considerations.

For the investigation of the low Mach number limit of (1.3), we consider a highly subsonic regime in which the *Mach* number $Ma=\varepsilon=U/c$ given by the ratio of the characteristic velocity U of the flow and the sound speed c goes to zero. One expects the flow to asymptotically behave like an incompressible one on large time scales and for small velocities. Given the dimensionless system (1.3), we introduce the scaling

$$t\mapsto\varepsilon t,\quad u\mapsto\varepsilon u,\quad \nu\mapsto\varepsilon\nu_\varepsilon\quad \kappa\mapsto\varepsilon\kappa_\varepsilon, \tag{1.8}$$

The scaled viscosity and capillarity coefficients are such that

$$\nu_\varepsilon\mapsto\tilde{\nu}>0,\quad \kappa_\varepsilon\to\tilde{\kappa}>0.$$

The scaled version of (1.3) then reads

$$\begin{aligned}&\partial_t\rho_\varepsilon+\operatorname{div}(\rho_\varepsilon u_\varepsilon)=0\\&\partial_t(\rho_\varepsilon u_\varepsilon)+\operatorname{div}(\rho_\varepsilon u_\varepsilon\otimes u_\varepsilon)+\frac{1}{\varepsilon^2}\nabla P(\rho_\varepsilon)=2\nu_\varepsilon\operatorname{div}(\rho_\varepsilon\mathbf{D}u_\varepsilon)+2\kappa_\varepsilon^2\rho_\varepsilon\nabla\left(\frac{\Delta\sqrt{\rho_\varepsilon}}{\sqrt{\rho_\varepsilon}}\right)\end{aligned} \tag{1.9}$$

For the sake of a concise notation, we suppress the ε-dependence of ν_ε and κ_ε. The scaled energy is given by

$$E(\rho_\varepsilon,u_\varepsilon)=\int_{\mathbb{R}^2}\frac{1}{2}\rho_\varepsilon|u_\varepsilon|^2+2\kappa^2|\nabla\sqrt{\rho_\varepsilon}|^2+F_\varepsilon(\rho_\varepsilon)\mathrm{d}x,\qquad F_\varepsilon(\rho_\varepsilon)=\frac{\rho_\varepsilon^\gamma-\gamma(\rho_\varepsilon-1)}{\varepsilon^2\gamma(\gamma-1)}. \tag{1.10}$$

We refer to [2, 28, 35] for details on the scaling analysis. Provided that the energy (1.10) is uniformly bounded, the heuristics suggests that $\rho_\varepsilon - 1$ converges to 0 as $\varepsilon \to 0$. Formally $\rho_\varepsilon u_\varepsilon \to u$ for which we infer from the continuity equation of (1.9) that $\operatorname{div} u = 0$. The limit function u is expected to solve the target system given by the incompressible Navier-Stokes equations

$$\partial_t u + (u \cdot \nabla)u + \nabla\pi = \nu\Delta u, \qquad \operatorname{div} u = 0. \tag{1.11}$$

Our main result states the following.

Theorem 1.1. *Let $T > 0$, $\{(\rho_\varepsilon^0, u_\varepsilon^0)\}_{\varepsilon>0}$ be a sequence of* general ill-prepared *initial data with uniformly bounded energy* (1.10) *such that $\sqrt{\rho_\varepsilon^0}u_\varepsilon^0$ converges weakly to some u^0 in L^2. Further, let $\{(\rho_\varepsilon, u_\varepsilon)\}_{\varepsilon>0}$ a sequence of (FEWS) to* (1.9) *with initial data $\{(\rho_\varepsilon^0, u_\varepsilon^0)\}_{\varepsilon>0}$. Then, $\rho_\varepsilon - 1$ converges strongly to zero and $\sqrt{\rho_\varepsilon}u_\varepsilon$ converges strongly to u, where u is the unique Leray-Hopf solution of* (1.11) *with initial data* $\mathbf{P}(u^0)$.

We refer to Theorem 2.6 below for a precise statement. Beyond its analytic scope, we mention that the low Mach number analysis is also motivated by the utility of (1.1) for the numerical purposes, such as investigation of diffuse interfaces [3]. For a general introduction to the mathematical low Mach number theory, we refer to the review papers [2, 35] and the monograph [28] and references therein.

In this introduction, we restrict ourselves to point out the key difficulties of the low Mach number theory for weak solutions of (1.9). A key issue in proving convergence towards the target systems consists in controlling the acoustic waves, carried by density fluctuations and the irrotational part of the momentum density. Unless the acoustic waves propagating with speed $\frac{1}{\varepsilon}$ are controlled in a suitable way one may only expect weak convergence of the sequence of momentum density $\rho_\varepsilon u_\varepsilon$. The latter is in particular insufficient to obtain compactness of the convective term $\operatorname{div}(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon)$ and for the passage to the limit in (1.9). The dispersion of acoustic waves can be exploited in order to infer the desired decay. When working on unbounded domains, Strichartz estimates for the wave equation provide an appropriate tool for such an analysis applied to the classical compressible Navier-Stokes equations, see [25] and the survey papers [2, 24, 35]. The dispersive tensor present in (1.9) alters the dispersion relation of acoustic waves in (1.9) that is no longer linear. We develop a refined dispersive analysis allowing for decay of acoustic waves at explicit improved convergence rates and under arbitrarily small loss of regularity. For that purpose, we adapt the analysis of acoustic oscillations initiated in [6] by the author in collaboration with P. Antonelli and P. Marcati, see Section 4 below. Refined Strichartz estimates taking into account the augmented dispersive relation are also used by the same authors [4] for the study of the low Mach number limit of (1.9) posed on $\mathbb{R}^3$. However, as the dispersion turns out to be weaker for $d = 2$ the estimates introduced in [4] do not yield the desired decay properties for $d = 2$. In [6], the authors complement the analysis of [4] for $d = 3$ with suitable refined Strichartz estimates for $d = 2$ and elucidate the link with the Bogoliubov dispersion relation [17] that governs the system of acoustic waves. These estimates can be considered a refinement of the Strichatz estimates in [16] and the ε-dependent version of [32] introduced in the framework of the Gross-Pitaevskii equation, see Section 4. Note, that the ε-dependent estimates do not follow from a direct scaling argument as the Bogoliubov dispersion relation is non-homogeneous.

Second, suitable *a priori* estimates are required in order to infer the compactness needed for the passage to the limit. At this stage, further difficulties related to the Cauchy Problem of (1.9) and its difference to the one for the classical compressible Navier-Stokes become apparent. The density dependence of the viscosity tensor $2\nu\,\mathrm{div}(\rho_\varepsilon \mathbf{D}u_\varepsilon)$ in (1.9) leads to a degeneracy close to vacuum regions. This prevents a suitable control of the velocity field u_ε which in general can not be defined a.e. on $[0,T)\times\mathbb{R}^2$. In addition, propagating regularity of ρ_ε is a difficult task due to presence of the highly nonlinear quantum correction term in (1.9). The lack of appropriate uniform estimates is compensated for by the Bresch-Desjardins (BD) entropy estimates [18, 19] which are available for (1.9) and more in general for (1.1) under specific conditions on μ, λ and k. While, in the case of (1.9), these provide bounds up to second order derivatives of $\sqrt{\rho_\varepsilon}$, they do not suffice to define u_ε a.e. on $\mathbb{R}^2$, see also (2.4) below. This distinguishes the present analysis from the incompressible limit for the classical compressible Navier-Stokes equations, see e.g., [25], for which only weaker information on the density in Orlicz spaces but on the other hand a uniform Sobolev bound for u_ε are available. This further motivates the need of an accurate dispersive analysis of the acoustic waves when dealing with weak solutions at low regularity. The presence of the capillarity tensor allows for both refined Strichartz estimates and additional uniform estimates on $\sqrt{\rho_\varepsilon}$ (compared to the case $\kappa = 0$), see also Remark 2.7.

Previously and to the best of our knowledge, the low Mach number limit for (1.9) has only been studied for $d = 3$. In the aforementioned paper [4], see also [5], the low Mach number limit for (1.3) posed on $\mathbb{R}^3$ is investigated. As detailed above, the dispersive analysis of the linearized system differs substantially from the present one due to the weaker dispersion for $d = 2$. Moreover, due to the uniqueness and regularity properties of weak solutions to (1.11) for $d = 2$, here we are able to infer additional information on the limit velocity field u, see Theorem 2.6. We also mention [41, 57] where the incompressible limit for (1.3) posed on $\mathbb{T}^3$ is considered. In these papers, the authors augment (1.3) by additional drag terms that allow for a direct control of the velocity field u_ε. In addition, [41, 57] consider local smooth solutions to the primitive system under further assumptions that are shown to converge to local strong solutions of (1.11) by means of a relative entropy method. In [41], the authors also study the limit of local smooth solutions to (1.3) posed on $\mathbb{R}^3$ including again additional drag terms and requiring the initial data to be smooth and well-prepared. Note that the class of weak solutions under consideration in this paper is not suitable for relative entropy methods. Finally, we mention that the low Mach number limit for the (QHD) system, the inviscid counterpart of (1.3), is investigated in [26] on $\mathbb{T}^d$ for $d = 2, 3$. Posed on $\mathbb{R}^d$, it will further be addressed by the author in a forthcoming paper including vortex solutions of infinite energy, see also [34].

The remaining part of this paper is organized as follows. Section 2 reviews the Cauchy Theory for the primitive system (1.3) and the target system (1.11) and provides a precise formulation of the main results of this paper. Subsequently, we collect the needed uniform estimates in Section 3. Strong convergence to zero of acoustic waves is proven in Section 4 while Section 5 completes the proof of the main theorem.

Notations

We list the notations of function spaces and operators used in the following. We denote

- the symmetric part of the gradient by $\mathbf{D}u = \frac{1}{2}(\nabla u + (\nabla u)^T)$ and the asymmetric part by $\mathbf{A}u = \frac{1}{2}(\nabla u - (\nabla u)^T)$,
- by $\mathcal{D}(\mathbb{R}_+\times\mathbb{R}^2)$ the space of test functions $C_c^\infty(\mathbb{R}_+\times\mathbb{R}^2)$ an by $\mathcal{D}'(\mathbb{R}_+\times\mathbb{R}^2)$ the space of distributions. The duality bracket between $\mathcal{D}$ and $\mathcal{D}'$ is denoted by $\langle\cdot,\cdot\rangle$,
- by $L^p(\mathbb{R}^2)$ for $1 \le p \le \infty$ the Lebesgue space with norm $\|\cdot\|_{L^p}$. We denote by p' the Hölder

conjugate exponent of p, i.e. $1 = \frac{1}{p} + \frac{1}{p'}$, and for $0 < T \leq \infty$ by $L^p(0,T;L^q(\mathbb{R}^2))$ the space of functions $u : (0,T) \times \mathbb{R}^2 \to \mathbb{R}^n$ with norm

$$\|u\|_{L^pL^q} = \left(\int_0^T \left| \int_{\mathbb{R}^2} |u(t,x)|^q \right|^{\frac{p}{q}} \mathrm{d}x\mathrm{d}t \right)^{\frac{1}{p}}.$$

By $L^{p-}(0,T;L^q(\mathbb{R}^2))$ we indicate the space of functions $f \in L^{p_0}(0,T;L^q(\mathbb{R}^2))$ for any $1 \leq p_0 < p$,

- The sum $L^{q_1}(\mathbb{R}^2) + L^{q_2}(\mathbb{R}^2)$ is a Banach space with norm $\|f\|_{L^{q_1}+L^{q_2}} = \inf\{\|g\|_{L^{q_1}} + \|h\|_{L^{q_2}} \ : \ f = g + h,\ g \in L^{q_1}(\mathbb{R}^2), h \in L^{q_2}(\mathbb{R}^2)\}$,
- by $L^p_2(\mathbb{R}^2)$ the Orlicz space defined as

$$L^p_2(\mathbb{R}^2) = \left\{ f \in L^1_{loc}(\mathbb{R}^2) \ : \ |f|\chi_{\{|f|\leq\frac{1}{2}\}} \in L^2(\mathbb{R}^2),\ |f|\chi_{\{|f|\geq\frac{1}{2}\}} \in L^p(\mathbb{R}^2) \right\},$$

  we refer to [1,48] for details.
- for $s \in \mathbb{R}$ and $p \in [1,\infty]$ the non-homogeneous Sobolev space by $W^{s,p}(\mathbb{R}^2) = (I - \Delta)^{-\frac{s}{2}} L^p(\mathbb{R}^2)$ and $H^s(\mathbb{R}^2) = W^{s,2}(\mathbb{R}^2)$. Its dual will be denoted by $W^{-s,p'}(\mathbb{R}^2)$ with p' being the Hölder conjugate of p. The homogeneous spaces are denoted by $\dot{W}^{s,p}(\mathbb{R}^2) = (-\Delta)^{-\frac{k}{s}} L^p(\mathbb{R}^2)$ and $\dot{W}^{s,2}(\mathbb{R}^2) = \dot{H}^s(\mathbb{R}^2)$, and the dual space $\dot{W}^{-s,p'}(\mathbb{R}^2)$. For $ps < 2$ we denote the critical Sobolev exponent by $p^* = \frac{2p}{2-ps}$. We refer to Theorem 6.4.5 and Theorem 6.5.1 in [15], see also Chapter 4 in [1], for the classical embedding results for Sobolev and Lebesgue spaces.
- by $\mathbf{Q}$ and $\mathbf{P}$ the Helmholtz–Leray projectors on irrotational and divergence-free vector fields, respectively:

$$\mathbf{Q} = \nabla\Delta^{-1}\operatorname{div}, \qquad \mathbf{P} = \mathbf{I} - \mathbf{Q}.$$

  For $f \in W^{k,p}(\mathbb{R}^2)$ with $1 < p < \infty$ and $s \in \mathbb{R}$ the operators $\mathbf{P}, \mathbf{Q}$ can be expressed as composition of Riesz multipliers and are bounded linear operators on $W^{s,p}(\mathbb{R}^2)$.

In what follows C will be any constant independent from ε.

For the convenience of the reader, we recall an interpolation result used several times throughout the paper.

Lemma 1.2 (Interpolation). *Let $T > 0$, $p_1, p_2, r \in (1,\infty)$ and $s_0 < s_1$ real numbers. Further, let $u \in L^{p_1}(0,T;W^{s_0,r}(\mathbb{R}^2)) \cap L^{p_2}(0,T;W^{s_1,r}(\mathbb{R}^2))$. Then, for all (p,s) such that there exists $\theta \in (0,1)$ with*

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad s = \theta s_0 + (1-\theta)s_1,$$

it holds $u \in L^p(0,T;W^{s,r}(\mathbb{R}^2))$ with

$$\|u\|_{L^p(0,T;W^{s,r}(\mathbb{R}^2))} \leq \|u\|^{\theta}_{L^{p_1}(0,T;W^{s_0,r}(\mathbb{R}^2))} \|u\|^{1-\theta}_{L^{p_2}(0,T;W^{s_1,r}(\mathbb{R}^2))}.$$

The Lemma is a simplified statement of Theorem 5.1.2 in [15] and can also be proven by standard interpolation of Sobolev spaces in the space variables, see e.g., Paragraph 7.53 in [1], followed by Hölder's inequality in the time variable.

2. Preliminary and main results

This section briefly reviews the Cauchy theory for both, the primitive system (1.3) and the target system (1.11). Subsequently, we state the main results of this paper characterising the incompressible limit of (1.9) in the class of weak solutions.

2.1. Cauchy theory

The mathematical analysis of (1.3), and more in general Navier-Stokes-Korteweg systems (1.1), encounters two major difficulties beyond the well-known ones arising in the study of classical compressible Navier-Stokes equations [48]: the density-dependence of the viscosity stress tensor and the presence of the highly non-linear dispersive stress tensor. For compressible fluid flow with constant coefficient viscosity, the energy bound yields $\sqrt{\rho}u \in L_t^\infty L_x^2$ and the energy dissipation provides a $L_{t,x}^2$-bound for ∇u. For the degenerate viscosity stress tensor considered in (1.3), the energy dissipation fails to provide suitable control on u. By consequence, the Lions-Feireisl theory [29, 48] which relies on a Sobolev bound for u can not be applied. Without further regularity assumptions, none of the quantities u, ∇u and $1/\sqrt{\rho}$ is defined a.e. on $\mathbb{R}^2$ due to the possible presence of vacuum $\{\rho = 0\}$. These difficulties are reminiscent of the ones encountered in the analysis of the QHD system [7, 10], the inviscid counter-part of (1.3), and also arise in the absence of a capillarity tensor, namely for the barotropic Navier-Stokes equations with density dependent viscosity [19]. It is hence pivotal for the development of the Cauchy Theory to obtain suitable control on the mass density, which turns out to be a difficult task given in particular the presence of the highly non-linear dispersive stress tensor. The lack of uniform bounds for the velocity field u is compensated for by the Bresch-Desjardins (BD)-entropy estimates [18, 19]. The mathematical theory for finite energy weak solutions is then developed in terms of the variables $(\sqrt{\rho}, \Lambda := \sqrt{\rho}u)$ which enjoy suitable bounds in the finite energy framework. Note that the mass is infinite in view of (1.4). Weak solutions are commonly constructed in terms of an approximation procedure [8, 11, 42]. This does in general not allow one to prove the energy inequality

$$E(t) + 2\nu \int_0^t \int_{\mathbb{R}^2} \rho|\mathbf{D}u|^2 \mathrm{d}x\mathrm{d}t \leq E(0). \tag{2.1}$$

The energy inequality is replaced by a weaker version by defining the tensor $\mathbf{T}_\nu \in L^2((0,T)\times\mathbb{R}^2)$ through

$$\sqrt{\nu}\sqrt{\rho}\mathbf{T}_\nu = \nu\nabla(\rho u) - 2\nu\Lambda \otimes \nabla\sqrt{\rho} \qquad \text{in} \quad \mathcal{D}'((0,T)\times\mathbb{R}^2)). \tag{2.2}$$

By denoting its symmetric part $\mathbf{S}_\nu = \mathbf{T}_\nu^{sym}$, we recover the identity $\sqrt{\nu}\sqrt{\rho}\mathbf{S}_\nu = \nu\rho\mathbf{D}u$ for smooth solutions. The energy inequality for (1.3) then reads

$$E(t) + 2\int_0^t \int_{\mathbb{R}^2} |\mathbf{S}_\nu|^2 \mathrm{d}x\mathrm{d}t \leq E(0). \tag{2.3}$$

The aforementioned (BD)-entropy estimates provide uniform bounds for the asymmetric part $\mathbf{A}_\nu = \mathbf{T}_\nu^{asym}$ and the second order derivatives of $\sqrt{\rho}$, see (2.4) below. We refer the reader to [4, 8] and references therein for a detailed discussion. Similarly, the capillary tensor given by (1.5) is well-defined in weak sense by virtue of the regularity properties stemming from the energy (2.3) and Bresch-Desjardins entropy inequality (2.4).

Concerning the far-field condition, we mention that the internal energy $F(\rho)$ and the pressure $P(\rho)$ are related through the identity $P(\rho) = \rho F'(\rho) - F(\rho)$. The particular choice for $F(\rho)$ in (2.3) enforces the desired far-field behavior.

Following [8] we introduce our notion of weak solutions to (1.3) with far-field behavior (1.4).

Definition 2.1. *A pair* (ρ, u) *with* $\rho \geq 0$ *is said to be a finite energy weak solution (FEWS) of the Cauchy Problem* (1.3) *posed on* $[0, T) \times \mathbb{R}^2$ *with initial data* (ρ^0, u^0) *if the following are satisfied*

1) *Integrability conditions*

$$\begin{aligned}
&\sqrt{\rho} \in L^2_{\mathrm{loc}}((0,T)\times\mathbb{R}^2); \quad \Lambda \in L^2_{\mathrm{loc}}((0,T)\times\mathbb{R}^2); \quad \nabla\sqrt{\rho} \in L^2_{\mathrm{loc}}((0,T)\times\mathbb{R}^2);\\
&\nabla\rho^{\frac{\gamma}{2}} \in L^2_{\mathrm{loc}}((0,T)\times\mathbb{R}^2); \quad \mathbf{T}_\nu \in L^2_{\mathrm{loc}}((0,T)\times\mathbb{R}^2); \quad \kappa\nabla^2\sqrt{\rho} \in L^2_{\mathrm{loc}}((0,T)\times\mathbb{R}^2);\\
&\sqrt{\kappa}\nabla\rho^{\frac{1}{4}} \in L^4_{\mathrm{loc}}((0,T)\times\mathbb{R}^2);
\end{aligned}$$

2) *Continuity equation*

$$\int_{\mathbb{R}^2} \rho_0\varphi(0)\mathrm{d}x + \int_0^T \int_{\mathbb{R}^2} \rho\varphi_t + \sqrt{\rho}\Lambda\cdot\nabla\varphi \mathrm{d}x\mathrm{d}t = 0,$$

   for any $\varphi \in C_c^\infty([0,T)\times\mathbb{R}^2)$.

3) *Equation for the momentum density*

$$\begin{aligned}
&\int_{\mathbb{R}^2} \rho_0 u_0\cdot\psi(0)\mathrm{d}x + \int_0^T \int_{\mathbb{R}^2} \sqrt{\rho}\Lambda\cdot\psi_t + (\Lambda\otimes\Lambda) : \nabla\psi + \rho^\gamma \operatorname{div}\psi \mathrm{d}x\mathrm{d}t\\
&- 2\nu \int_0^T \int_{\mathbb{R}^2} (\Lambda\otimes\nabla\sqrt{\rho}) : \nabla\psi - 2\nu\int_0^T\int_{\mathbb{R}^2} (\nabla\sqrt{\rho}\otimes\Lambda) : \nabla\psi\mathrm{d}x\mathrm{d}t\\
&+ \nu\int_0^T\int_{\mathbb{R}^2} \sqrt{\rho}\Lambda\Delta\psi + \nu\int_0^T\int_{\mathbb{R}^2}\sqrt{\rho}\Lambda\cdot\nabla\operatorname{div}\psi\mathrm{d}x\mathrm{d}t\\
&- 4\kappa^2\int_0^T\int_{\mathbb{R}^2}(\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}) : \nabla\psi + 2\kappa^2\int_0^T\int_{\mathbb{R}^2}\sqrt{\rho}\nabla\sqrt{\rho}\cdot\nabla\operatorname{div}\psi\mathrm{d}x\mathrm{d}t = 0,
\end{aligned}$$

   for any $\psi \in C_c^\infty([0,T)\times\mathbb{R}^2;\mathbb{R}^2)$.

4) *Energy inequality. There exists a tensor* $\mathbf{T}_\nu \in L^2((0,T)\times\mathbb{R}^2)$ *satisfying identity* (2.2) *such that the energy inequality* (2.3) *holds for a.e.* $t \in [0,T]$.

5) *BD-entropy inequality. Let*

$$B(t) = \int_{\mathbb{R}^2} \frac{1}{2}|\Lambda|^2 + (2\kappa^2 + 4\nu^2)\left|\nabla\sqrt{\rho}\right|^2 + F(\rho)\mathrm{d}x.$$

   Then for a.e. $t \in [0,T]$,

$$\begin{aligned}
&B(t) + \int_0^t\int_{\mathbb{R}^2}\frac{1}{2}|\mathbf{A}_\nu|^2\,\mathrm{d}x\mathrm{d}s + \nu\kappa^2\int_0^t\int_{\mathbb{R}^2}\left|\nabla^2\sqrt{\rho}\right|^2 + |\nabla\rho^{\frac{1}{4}}|^4\mathrm{d}x\mathrm{d}s + \nu\int_0^t\int_{\mathbb{R}^2}\left|\nabla\rho^{\frac{\gamma}{2}}\right|^2\mathrm{d}x\mathrm{d}s\\
&\leq CB(0).
\end{aligned} \tag{2.4}$$

   where $\mathbf{A}_\nu = \mathbf{T}_\nu^{asym}$, *with* $\mathbf{T}_\nu$ *defined as in the previous point.*

Global existence of (FEWS) to (1.3) posed on $\mathbb{T}^d$ for $d = 2, 3$ is proven in [11] and [42] following different approaches. In collaboration with P. Antonelli and S. Spirito [8], the author proves global existence of (FEWS) to (1.3) posed on $\mathbb{R}^d$ for $d = 2, 3$ with or without non-trivial far-field (1.4) and initial data of finite energy.

Theorem 2.2 ([8]). *Let $\gamma > 1$ and $\nu, \kappa > 0$. Given initial data (ρ^0, u^0) of finite energy, there exists a global FEWS (ρ, u) to* (1.3) *with far-field* (1.4).

In particular, vacuum regions are included in the weak formulation of the equations. The method of [8] consists in a invading domains approach. More precisely, by a suitable truncation argument a sequence of approximate solutions is constructed. To that end, the authors rely on the existence result [42] on periodic domains. The compactness properties provided by the energy and BD-entropy bounds allow for the passage to the limit in the truncated formulation yielding finally a global (FEWS) to (1.3). Further, the weak solutions constructed in [8] are such that (2.3) and (2.4) are satisfied. The validity of the energy inequality (2.1) for general weak solutions to (1.3) is at present not clear. In addition, the minimal assumptions on weak solutions such that (2.3) and (2.4) are fulfilled remain to determine. For a more detailed discussion of these issues, see e.g., [8], [4, Section 2 and Appendix A] and [49].

Concerning the Cauchy Theory of (1.11), we recall the following well-known result.

Definition 2.3. *A vector-field u is a weak solution to* (1.11) *posed on $[0, T) \times \mathbb{R}^2$ with initial data u^0 if $u \in L^2_{\text{loc}}([0, T) \times \mathbb{R}^2)$, $\operatorname{div} u = 0$ in distributional sense and for all $\psi \in C_c^\infty([0, \infty) \times \mathbb{R}^2)$ such that $\operatorname{div} \psi = 0$ it holds*

$$\int_{\mathbb{R}^2} u_0 \cdot \psi(0) \mathrm{d}x + \int_0^T \int_{\mathbb{R}^2} u \cdot \partial_t \psi + (u \otimes u) : \nabla\psi - \nu u \Delta\psi \mathrm{d}x\mathrm{d}t = 0.$$

A weak solution u is called a Leray-Hopf weak solution to (1.11) *if the energy equality*

$$E(t) + \nu \int_0^t \int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x\mathrm{d}t = E(0) \tag{2.5}$$

is satisfied for a.e. $t \in [0, T)$, where the kinetic energy is defined as

$$E(t) = \int_{\mathbb{R}^2} \frac{1}{2} |u|^2(t) \mathrm{d}x.$$

Existence and uniqueness of Leray-Hopf weak solutions to (1.11) for initial data of finite kinetic energy is due to [44].

Theorem 2.4 ([44]). *Let $\nu > 0$. Given $u^0 \in L^2(\mathbb{R}^2)$ such that $\operatorname{div} u^0 = 0$, there exists a unique global Leray-Hopf weak solution $u \in L^\infty(0, \infty; L^2(\mathbb{R}^2)) \cap L^2(0, \infty; \dot{H}^1(\mathbb{R}^2))$ to* (1.11).

We refer to the monograph [43] for the analysis of (1.11) and we limit ourselves to the following comments. The space $L^2(\mathbb{R}^2)$ corresponds to the energy space of (1.11), namely the space of velocity fields of finite kinetic energy and enjoys scaling invariance. Moreover, due to the Ladyzhenskaya-Prodi-Serrin regularity criterion [43, 51, 52], see also [46], any weak solution u such that $u \in L^4(0, T; L^4(\mathbb{R}^2))$ satisfies (2.5).

2.2. Main results

We specify the assumptions on the sequence of initial data $(\rho_\varepsilon^0, u_\varepsilon^0)$ that we consider to be *general and ill-prepared* and without further regularity or smallness assumptions. The assumptions are stated in therms of the hydrodynamic states $(\sqrt{\rho_\varepsilon^0}, \Lambda_\varepsilon^0)$.

Assumption 2.5. Let $(\rho_\varepsilon^0, u_\varepsilon^0)$ be a sequence of initial data such that

1) there exists $C > 0$ such that $E(\rho_\varepsilon^0, u_\varepsilon^0) \leq C$ for all $\varepsilon > 0$, where the scaled energy functional is defined in (1.10),
2) there exists $u^0 \in L^2(\mathbb{R}^2)$ such that $\Lambda_\varepsilon^0 \rightharpoonup u^0$ weakly in $L^2(\mathbb{R}^2)$.

Note that Theorem 2.2 guarantees the global existence of a sequence of (FEWS) to (1.9) with initial data satisfying Assumption 2.5. Our main result then characterises the low Mach number limit of (FEWS) to (1.9) with such initial data.

Theorem 2.6. *Let $\gamma > 1$, $\kappa, \nu > 0$, $T > 0$ and $\{(\rho_\varepsilon, u_\varepsilon)\}_{\varepsilon>0}$ be a sequence of (FEWS) to* (1.9) *on $[0,T) \times \mathbb{R}^2$ with initial data $\{(\rho_\varepsilon^0, u_\varepsilon^0)\}_{\varepsilon>0}$ satisfying Assumption 2.5. Then,*

1) *$\rho_\varepsilon - 1$ converges strongly to 0 in $L^\infty(0,T;L^2(\mathbb{R}^2)) \cap L^{\frac{4}{s}-}(0,T;H^s(\mathbb{R}^2))$ for $s \in (0,2)$,*
2) *Λ_ε converges weakly-$*$ in $L^\infty(0,\infty;L^2(\mathbb{R}^2))$ and strongly in $L^2_{\mathrm{loc}}([0,T) \times \mathbb{R}^2)$ to u, where the limit velocity field u is the unique global Leray-Hopf weak solution to* (1.11) *with initial data $u\big|_{t=0} = \mathbf{P}(u^0)$, in particular $u \in L^\infty(0,T;L^2(\mathbb{R}^2)) \cap L^2(0,T;\dot{H}^1(\mathbb{R}^2))$,*
3) *the irrotational part of the momentum density $\mathbf{Q}(\rho_\varepsilon u_\varepsilon)$ converges strongly to 0 in $L^2(0,T;L^r(\mathbb{R}^2))$ for any $r \in (2,\infty)$.*

Even though only the weak form of the energy inequality (2.3) is available for $\varepsilon > 0$, we recover unique Leray-Hopf weak solutions in the limit. Note that for the general ill-prepared data for the primitive system, the possible formation of an initial layer can not be ruled out. In particular, $E(\rho_\varepsilon^0, u_\varepsilon^0)$ does not converge to $E_{INS}(\mathbf{P}(u^0))$ and one may not infer the energy inequality for u by passing to the limit in (2.3). However, the validity of the energy equality follows from Ladyzenskaya-Prodi-Serrin regularity criterion [43, 51, 52] as we prove that $u \in L^4(0,T;L^4(\mathbb{R}^2))$. Moreover, by virtue of the uniqueness of Leray-Hopf weak solutions, we conclude that the sequence Λ_ε converges without requiring any extraction of a subsequences. The regularity properties stemming from the energy and BD-entropy estimates are essential for that purpose.

This is in contrast to the low Mach number limit for (1.3) posed on $\mathbb{R}^3$ considered in [4]. In $3D$, the regularity properties of the limit velocity field u do not suffice in order to infer the validity of the energy inequality in the limit, see [4, Theorem 2.4 and Remark 2.5]. By consequence, one recovers a global weak Leray-Hopf solution only for well prepared initial data, namely data such that $E(\rho_\varepsilon^0, u_\varepsilon^0)$ does converge to $E_{INS}(\mathbf{P}(u^0))$. In addition, for $d = 3$ convergence does hold up to subsequences only.

On the other hand, we can not rely on the dispersive estimates providing suitable decay of the acoustic waves for $d = 3$, see [4, Proposition 4.2] due to the weaker dispersion for $d = 2$, see Section 4 below.

Remark 2.7. The presence of the capillarity tensor $\mathbb{K}$ in (1.9) is essential for both the uniform estimates (Section 3) and the acoustic analysis (Section 4). Regarding the former, the respective BD-entropy inequality (2.4) allows for uniform bounds of second order derivatives of $\sqrt{\rho_\varepsilon} - 1$ which enable us to infer a suitable Sobolev bound on $\rho_\varepsilon u_\varepsilon$, see Lemma 3.3 and Remark 3.4 below. For the latter, it leads to improved decay rates for the acoustic waves through an alteration of the dispersion relation, see (4.1). Both are in general no longer available without capillarity tensor, namely for $\kappa = 0$ corresponding to the degenerate compressible Navier–Stokes equations; the low Mach number limit of which will be subject of future investigation.

Remark 2.8. The presented theory generalizes to systems (1.1) provided that the capillarity tensor is chosen in a suitable way so that the respective BD-entropy inequality (2.4) entails bounds on second order derivatives of $\sqrt{\rho_\varepsilon}$. This is in particular the case provided that the BD relation (1.7) is satisfied [20, 21], see also Remark 3.4. We stress that even though the NSK equations do not satisfy (1.7) suitable estimate can be shown, see [12,13]. In addition, the linearized system for acoustic waves turns out to be still governed by the dispersion relation obtained for (1.9), see Remark 4.10. Finally, this allows one to infer the required compactness properties for $\{(\rho_\varepsilon, \sqrt{\rho_\varepsilon}u_\varepsilon)\}_{\varepsilon>0}$ and to prove convergence of FEWS towards Leray-Hopf weak solutions of (1.11).

3. Uniform estimates

We collect the necessary *a priori* estimates for finite energy weak solutions (FEWS) to (1.3) with uniformly bounded (in ε) energy and BD-entropy functionals. We denote the sequence of momenta $m_\varepsilon := \sqrt{\rho_\varepsilon}\Lambda_\varepsilon$ and $m_\varepsilon^0 := \sqrt{\rho_\varepsilon^0}\Lambda_\varepsilon^0$. The next Lemma only relies on uniform bounds implied by (2.3).

Lemma 3.1. *Let $T > 0$ and the initial data $(\rho_\varepsilon^0, u_\varepsilon^0)$ such that there exists $C > 0$ (independent of ε) and $E(\rho_\varepsilon^0, u_\varepsilon^0) \leq C$, then*

1) $\sqrt{\rho_\varepsilon^0} - 1 \in H^1(\mathbb{R}^2)$ is uniformly bounded with $\| \sqrt{\rho_\varepsilon^0} - 1\|_{L^2} \leq C\varepsilon$,
2) $\rho_\varepsilon^0 - 1 \in L_2^\gamma(\mathbb{R}^2)$ and $\rho_\varepsilon^0 - 1 \in L^2(\mathbb{R}^2)$ are uniformly bounded with $\|\rho_\varepsilon^0 - 1\|_{L^2} \leq C\varepsilon$ for $\gamma \geq 2$ and $\|\rho_\varepsilon^0 - 1\|_{L^2} \leq C\sqrt{\varepsilon}$ for $\gamma \in (1, 2)$,
3) $\Lambda_\varepsilon^0 \in L^2(\mathbb{R}^2)$ is uniformly bounded,
4) $m_\varepsilon^0 \in H^{-\delta}(\mathbb{R}^2)$ is uniformly bounded for $\delta > 0$ arbitrarily small.

If further $(\rho_\varepsilon, u_\varepsilon)$ is a sequence of FEWS to (1.3) with initial data $(\rho_\varepsilon^0, u_\varepsilon^0)$, then

5) $\sqrt{\rho_\varepsilon} - 1 \in L^\infty(0, \infty; H^1(\mathbb{R}^2))$ is uniformly bounded with $\| \sqrt{\rho_\varepsilon} - 1\|_{L^\infty L^2} \leq C\varepsilon$,
6) $\rho_\varepsilon - 1 \in L^\infty(0, \infty; L_2^\gamma(\mathbb{R}^2))$ and $\rho_\varepsilon - 1 \in L^\infty(0, \infty; L^2(\mathbb{R}^2))$ are uniformly bounded with $\|\rho_\varepsilon - 1\|_{L^\infty L^2} \leq C\varepsilon$ for $\gamma \geq 2$ and $\|\rho_\varepsilon - 1\|_{L^\infty L^2} \leq C\sqrt{\varepsilon}$ for $\gamma \in (1, 2)$.
7) $\Lambda_\varepsilon \in L^\infty(0, \infty; L^2(\mathbb{R}^2))$ is uniformly bounded,
8) $m_\varepsilon \in L^\infty(0, T; L^2(\mathbb{R}^2) + L^r(\mathbb{R}^2))$ is uniformly bounded for all $r \in [1, 2)$. In particular, $m_\varepsilon^0 \in L^\infty(0, T; H^{-\delta}(\mathbb{R}^2))$ is uniformly bounded for $\delta > 0$ arbitrarily small,
9) $\mathbf{S}_{\nu,\varepsilon} \in L^2(0, T; L^2(\mathbb{R}^2))$ is uniformly bounded.

Proof. It follows from the energy inequality (2.3) and the fact that F is non-negative that the internal energy $F(\rho_\varepsilon) \in L^\infty(0, \infty; L^1(\mathbb{R}^2))$ is uniformly bounded. Exploiting the convexity of F, see e.g., [47, p. 590] or [4, Lemma 3.2], we obtain $\rho_\varepsilon - 1 \in L^\infty(0, \infty; L_2^\gamma(\mathbb{R}^2))$, where L_2^γ denotes the Orlicz space. More precisely,

$$\int_{\mathbb{R}^2} |\rho_\varepsilon(t) - 1|^2 \, 1_{\{|\rho_\varepsilon - 1| \leq \frac{1}{2}\}} + |\rho_\varepsilon(t) - 1|^\gamma \, 1_{\{|\rho_\varepsilon - 1| > \frac{1}{2}\}} \mathrm{d}x \leq C\varepsilon^2 \tag{3.1}$$

for a.e. $t \in \mathbb{R}_+$. If $\gamma \geq 2$, then (3.1) yields

$$\|\rho_\varepsilon - 1\|_{L^\infty(0,\infty;L^2(\mathbb{R}^2))} \leq C\varepsilon,$$

which proves (6) for $\gamma \geq 2$. We postpone the proof of (6) for general $\gamma > 1$ and proceed to show (5). We observe that $\nabla\sqrt{\rho_\varepsilon} \in L^\infty(0, \infty; L^2(\mathbb{R}^2))$ bounded uniformly from (2.3) and since $|\sqrt{\rho_\varepsilon} - 1| \leq |\rho_\varepsilon - 1|$, it holds

$$\int_{\mathbb{R}^2} \left| \sqrt{\rho_\varepsilon(t)} - 1 \right|^2 1_{\{|\rho_\varepsilon - 1| \le \frac{1}{2}\}} + \left| \sqrt{\rho_\varepsilon(t)} - 1 \right|^\gamma 1_{\{|\rho_\varepsilon - 1| > \frac{1}{2}\}} \mathrm{d}x \le C\varepsilon^2, \tag{3.2}$$

for a.e. $t \in \mathbb{R}_+$. Hence, in order to conclude that $\sqrt{\rho_\varepsilon} - 1 \in L^\infty(0,\infty; H^1(\mathbb{R}^2))$ uniformly bounded it remains to show that

$$\int_{\mathbb{R}^2} \left| \sqrt{\rho_\varepsilon(t)} - 1 \right|^2 1_{\{|\rho_\varepsilon - 1| > \frac{1}{2}\}} \mathrm{d}x \le C$$

For $\gamma \ge 2$, it follows from (3.2) that

$$\int_{\mathbb{R}^2} \left| \sqrt{\rho_\varepsilon(t)} - 1 \right|^2 1_{\{|\rho_\varepsilon - 1| > \frac{1}{2}\}} \mathrm{d}x \le C\varepsilon^2.$$

For all $\gamma > 1$, it follows from (3.1) and the Chebychev inequality that

$$\mathcal{L}^2(\{|\rho_\varepsilon - 1| > c\}) \le \frac{1}{c^\gamma} \int_{\mathbb{R}^2} |\rho_\varepsilon(t) - 1|^\gamma 1_{\{|\rho_\varepsilon - 1| > c\}} \mathrm{d}x \le C\varepsilon^2,$$

where $\mathcal{L}^2$ denotes the Lebesgue measure. Note that if f is a measurable function such that $\nabla f \in L^2(\mathbb{R}^2)$ and $\mathrm{supp}(f)$ is of finite Lebesgue measure then

$$\|f\|_{L^p(\mathbb{R}^2)} \le \|\nabla f\|_{L^2(\mathbb{R}^2)} \mathcal{L}^2\left(\mathrm{supp}(f)\right)^{\frac{1}{p}}. \tag{3.3}$$

For a proof of (3.3) see for instance [22, Inequality (3.10)]. Let $\chi \in C_c^\infty(\mathbb{R})$ such that $\mathbf{1}_{[3/4,5/4]}(r) \le \chi(r) \le \mathbf{1}_{[1/2,3/2]}(r)$. Applying inequality to $(\sqrt{\rho_\varepsilon} - 1)(1 - \chi(\rho_\varepsilon))$ yields

$$\begin{aligned}
\left\| (\sqrt{\rho_\varepsilon} - 1) 1_{\{|\rho_\varepsilon - 1| > \frac{1}{2}\}} \right\|_{L^\infty(0,\infty;L^2(\mathbb{R}^2))} &\le \left\| (\sqrt{\rho_\varepsilon} - 1)(1 - \chi(\rho_\varepsilon)) \right\|_{L^\infty(0,\infty;L^2(\mathbb{R}^2))} \\
&\le \|\nabla \sqrt{\rho_\varepsilon}\|_{L^\infty(0,\infty;L^2(\mathbb{R}^2))} \mathcal{L}^2\left(\mathrm{supp}(1 - \chi(\rho_\varepsilon))\right)^{\frac{1}{2}} \le C\varepsilon \|\nabla \sqrt{\rho_\varepsilon}\|_{L^\infty(0,\infty;L^2(\mathbb{R}^2))} \le C\varepsilon.
\end{aligned}$$

We obtain that

$$\|\sqrt{\rho_\varepsilon} - 1\|_{L^\infty(0,\infty;L^2(\mathbb{R}^2))} \le C\varepsilon, \quad \|\nabla(\sqrt{\rho_\varepsilon} - 1)\|_{L^\infty(0,\infty;L^2(\mathbb{R}^2))} \le C,$$

which completes the proof of (5). The proof of (1) follows verbatim the one of (5) and is omitted. The first statement of (6) has already been shown for $\gamma \ge 2$. For $\gamma \in (1,2)$, we observe that $|\rho_\varepsilon - 1| \le C|\sqrt{\rho_\varepsilon} - 1|^2$ on the set $\{|\rho_\varepsilon - 1| > \frac{1}{2}\}$ so that

$$\begin{aligned}
\int_{\mathbb{R}^2} |\rho_\varepsilon(t) - 1|^2 1_{\{|\rho_\varepsilon - 1| > \frac{1}{2}\}} \mathrm{d}x &\le C \int_{\mathbb{R}^2} \left| \sqrt{\rho_\varepsilon(t)} - 1 \right|^4 1_{\{|\rho_\varepsilon - 1| > \frac{1}{2}\}} \mathrm{d}x \\
&\le C \|\nabla(\sqrt{\rho_\varepsilon} - 1)\|^2_{L^\infty(0,\infty;L^2(\mathbb{R}^2))} \|\sqrt{\rho_\varepsilon} - 1\|^2_{L^\infty(0,\infty;L^2(\mathbb{R}^2))} \le C\varepsilon^2,
\end{aligned}$$

statement (6) then follows together with (3.1). Statement (2) is proven analogously to (6). Note that (3) and (7) and (9) are immediate consequences of (2.3). Finally, we show (8), statement (4) following verbatim. It suffices to decompose $m_\varepsilon = \Lambda_\varepsilon + (\sqrt{\rho_\varepsilon} - 1)\Lambda_\varepsilon$ and to exploit (5). Indeed, one has

$$m_\varepsilon = \Lambda_\varepsilon + (\sqrt{\rho_\varepsilon} - 1)\Lambda_\varepsilon \in L^\infty(0,\infty; L^2(\mathbb{R}^2) + L^p(\mathbb{R}^2)),$$

for all $p \in [1,2)$. □

Remark 3.2. Note that in contrast to compressible fluid flow with constant viscosity coefficients [47] the assumption for the initial data to be of uniformly bounded energy and (2.3) only yield a bound on the symmetric part $\mathbf{S}_{\nu,\varepsilon}$ of $\mathbf{T}_{\nu,\varepsilon}$, see (6) of Lemma 3.1. In particular, no L^2 or Sobolev bound for u_ε is available. On the other hand, the control of $\nabla\sqrt{\rho_\varepsilon}$ allows one to prove that $\sqrt{\rho_\varepsilon}-1$ converges to 0 in $L^\infty(0,\infty;H^s(\mathbb{R}^2))$ for any $s\in[0,1)$ by virtue of (1) Lemma 3.1 while in the constant viscosity coefficient case such bounds are available in Orlicz spaces only [25, 47].

Additional uniform bounds can be obtained from (2.4). Note that the scaled BD-entropy functional reads

$$B(\rho_\varepsilon,u_\varepsilon)(t)=\int_{\mathbb{R}^2}\frac{1}{2}|\Lambda_\varepsilon|^2+(2\kappa^2+4\nu^2)\left|\nabla\sqrt{\rho_\varepsilon}\right|^2+F_\varepsilon(\rho_\varepsilon)\mathrm{d}x. \tag{3.4}$$

As the initial data $(\rho_\varepsilon^0,u_\varepsilon^0)$ is of uniformly bounded energy it follows that $B(\rho_\varepsilon^0,u_\varepsilon^0)\le C$ for some $C>0$. In particular, this allows one to infer a L^2-bound on $\mathbf{T}_{\nu,\varepsilon}$. Similarly, it provides Sobolev bounds of second order for $\sqrt{\rho_\varepsilon}-1$.

Lemma 3.3. *Under the assumptions of Lemma 3.1, the following hold true,*

1) $\sqrt{\rho_\varepsilon}-1\in L^{\frac{2}{s-1}}(0,T;H^s(\mathbb{R}^2))$ *is uniformly bounded for all* $s\in[1,2]$ *with* $\sqrt{\rho_\varepsilon}-1\in L^p(0,T;L^\infty(\mathbb{R}^2))$ *uniformly bounded for all* $p\in[2,\infty)$. *Moreover,*

$$\|\sqrt{\rho_\varepsilon}-1\|_{L^4(0,T;H^1(\mathbb{R}^2))}\le C\sqrt{\varepsilon},\quad \|\sqrt{\rho_\varepsilon}-1\|_{L^4(0,T;L^\infty(\mathbb{R}^2))}\le C\sqrt{\varepsilon}; \tag{3.5}$$

2) $m_\varepsilon\in L^p(0,T;L^2(\mathbb{R}^2))$ *is uniformly bounded for all* $p\in[2,\infty)$ *and* $m_\varepsilon\in L^{\frac{2}{s}-}(0,T;H^s(\mathbb{R}^2))$ *is uniformly bounded for* $s\in(0,1)$.

Proof. To prove (1), we note that $\sqrt{\rho_\varepsilon}-1\in L^2(0,T;H^2(\mathbb{R}^2))$ is uniformly bounded for any $T>0$ by combining $\nabla^2\sqrt{\rho_\varepsilon}\in L^2(0,T;L^2(\mathbb{R}^2))$ from (2.4) and $\sqrt{\rho_\varepsilon}-1\in L^\infty(0,\infty;H^1(\mathbb{R}^2))$ from (5) Lemma 3.1. Applying the interpolation Lemma 1.2 leads to

$$\|\sqrt{\rho_\varepsilon}-1\|_{L^p(0,T;H^s(\mathbb{R}^2))}\le\|\sqrt{\rho_\varepsilon}-1\|^{\theta}_{L^2(0,T;H^2(\mathbb{R}^2))}\|\sqrt{\rho_\varepsilon}-1\|^{1-\theta}_{L^\infty(0,T;H^1(\mathbb{R}^2))},$$

where $\theta\in(0,1]$, $s=1+\theta$ and $p=2/\theta$. In particular, the Sobolev embedding $H^s(\mathbb{R}^2)\hookrightarrow L^\infty(\mathbb{R}^2)$ for any $s>1$ yields that $\sqrt{\rho_\varepsilon}-1\in L^p(0,T;L^\infty(\mathbb{R}^2))$ for all $p\in[2,\infty)$. Again by interpolation, one has

$$\|\sqrt{\rho_\varepsilon}-1\|_{L^p(0,T;H^s(\mathbb{R}^2))}\le\|\sqrt{\rho_\varepsilon}-1\|^{\theta}_{L^2(0,T;H^2(\mathbb{R}^2))}\|\sqrt{\rho_\varepsilon}-1\|^{1-\theta}_{L^\infty(0,T;L^2(\mathbb{R}^2))}\le C\varepsilon^{1-\theta}.$$

In particular, it follows from the Gagliardo-Nirenberg-Sobolev inequality that $\sqrt{\rho_\varepsilon}-1$ converges strongly to 0 in $L^4(0,T;L^\infty(\mathbb{R}^2))$ at convergence rate $\sqrt{\varepsilon}$. Inequalities (3.5) follow.

We show (2). It suffices to decompose $m_\varepsilon=\Lambda_\varepsilon+(\sqrt{\rho_\varepsilon}-1)\Lambda_\varepsilon$ and to exploit (1) of Lemma 3.3 in order to verify that for any $T>0$ it holds $m_\varepsilon\in L^\infty(0,T;L^2(\mathbb{R}^2))+L^p(0,T;L^2(\mathbb{R}^2))\subset L^p(0,T;L^2(\mathbb{R}^2))$ uniformly bounded for any $p\in[2,\infty)$. By definition of $\mathbf{T}_{\nu,\varepsilon}$, see (2.2), it holds

$$\nabla m_\varepsilon=\sqrt{\rho_\varepsilon}\mathbf{T}_{\nu,\varepsilon}+2\Lambda_\varepsilon\otimes\nabla\sqrt{\rho_\varepsilon}=\mathbf{T}_{\nu,\varepsilon}+(\sqrt{\rho_\varepsilon}-1)\mathbf{T}_{\nu,\varepsilon}+2\Lambda_\varepsilon\otimes\nabla\sqrt{\rho_\varepsilon}. \tag{3.6}$$

One has $\mathbf{T}_{\nu,\varepsilon}\in L^2(0,T;L^2(\mathbb{R}^2))$ uniformly bounded and $(\sqrt{\rho_\varepsilon}-1)\mathbf{T}_{\nu,\varepsilon}\in L^2(0,T;L^{r_1}(\mathbb{R}^2))$ uniformly bounded for all $r_1\in[1,2)$ upon using (5) of Lemma 3.1. Further, as $\nabla\sqrt{\rho_\varepsilon}\in L^2(0,T;L^{r_2}(\mathbb{R}^2))$ for any

$r_2 \in [2,\infty)$ from $\nabla^2\sqrt{\rho_\varepsilon} \in L^2(0,T;L^2(\mathbb{R}^2))$, we recover that $2\Lambda_\varepsilon \otimes \nabla\sqrt{\rho_\varepsilon} \in L^2(0,T;L^r(\mathbb{R}^2))$ uniformly bounded for all $r \in [1,2)$. Hence,

$$\nabla m_\varepsilon \in L^2(0,T;L^2(\mathbb{R}^2)) + L^2(0,T;L^r(\mathbb{R}^2)) \hookrightarrow L^2(0,T;L^2(\mathbb{R}^2)) + L^2(0,T;\dot{H}^{-\delta}(\mathbb{R}^2))$$

for arbitrary small $\delta > 0$. Thus,

$$m_\varepsilon \in \left(L^2(0,T;\dot{H}^1(\mathbb{R}^2)) + L^2(0,T;\dot{H}^{1-\delta}(\mathbb{R}^2))\right) \cap L^p(0,T;L^2(\mathbb{R}^2))$$

with $p \in [2,\infty)$. Finally, $m_\varepsilon \in L^2(0,T;H^{1-\delta}(\mathbb{R}^2))$ uniformly bounded for arbitrary small $\delta > 0$. Interpolating with the bound $m_\varepsilon \in L^p(0,T;L^2(\mathbb{R}^2))$ yields that

$$\|m_\varepsilon\|_{L^{p_0}(0,T;H^s(\mathbb{R}^2))} \le \|m_\varepsilon\|^{\theta}_{L^2(0,T;H^{1-\delta}(\mathbb{R}^2))}\|m_\varepsilon\|^{1-\theta}_{L^p(0,T;L^2(\mathbb{R}^2))},$$

where

$$s = \theta(1-\delta), \qquad \frac{1}{p_0} = \frac{\theta}{2} + \frac{1-\theta}{p}.$$

As $p \in [2,\infty)$ can be chosen arbitrarily large but finite and $\theta > 0$ arbitrarily close to 1, it follows that for $s \in (0,1)$ it holds $m_\varepsilon \in L^{\frac{2}{s}-}(0,T;H^s(\mathbb{R}^2))$ uniformly bounded. □

Remark 3.4. We emphasize that both statements of Lemma 3.3 rely on the uniform bound for $\nabla^2\sqrt{\rho_\varepsilon}$ stemming from (2.4) which is not available for $\kappa = 0$. In particular, if $\kappa = 0$ then the third term on the right-hand side of (3.6) is merely bounded in $L^\infty(0,T;L^1(\mathbb{R}^2))$. In turn, we are no longer able to state that $m_\varepsilon \in L^p(0,T;W^{s,r}(\mathbb{R}^2))$ for some $s > 0$, $r \ge 2$ and $p \in [1,\infty)$.

On the other hand, for $\kappa > 0$ and different choices of viscosity $\mu(\rho)$, $\lambda(\rho)$ and capillarity coefficients $k(\rho)$ the essential uniform bound for $\nabla^2\sqrt{\rho_\varepsilon}$ can be inferred provided that (1.7) is satisfied [21] or $k(\rho) = \text{const.}$, $\mu(\rho) = \rho$, $\lambda(\rho) = 0$ see [12, 13].

4. Control of acoustic oscillations

The aim of this section is to provide suitable control of fast-propagating acoustic waves, namely the density fluctuations $\sigma_\varepsilon := \varepsilon^{-1}(\rho_\varepsilon - 1)$ and the irrotational part of the momentum density $\mathbf{Q}(m_\varepsilon)$. In general, for ill-prepared data these fast oscillations may prevent the sequence $\mathbf{Q}(m_\varepsilon)$ from converging strongly to the incompressible limit velocity field u and only allow for weak convergence. However, when the problem is posed on the whole space, the dispersion at infinity can be exploited to prove strong convergence to zero of the acoustic waves as $\varepsilon \to 0$ in suitable space-time norms at an explicit convergence rate. We refer to the monograph [28, Chapter 7] and the survey paper [24] for the analysis on bounded domains. The acoustic equations are obtained by linearizing (1.9) around the constant solution $(\rho_\varepsilon = 1, u_\varepsilon = 0)$, see also the scaling in (1.9), and applying the Leray-Helmholtz projection onto curl-free vector fields to the moment equation. More precisely,

$$\begin{aligned} &\partial_t\sigma_\varepsilon + \frac{1}{\varepsilon}\operatorname{div}\mathbf{Q}(m_\varepsilon) = 0, \\ &\partial_t\mathbf{Q}(m_\varepsilon) + \frac{1}{\varepsilon}\nabla\sigma_\varepsilon - \varepsilon\kappa^2\nabla\Delta\sigma_\varepsilon = \mathbf{Q}(G_\varepsilon), \end{aligned} \tag{4.1}$$

where the Leray-Helmholtz projections are defined by $\mathbf{Q} := \nabla\Delta^{-1}\operatorname{div}$ and $\mathbf{P} := \mathbb{I} - \mathbf{Q}$ respectively and

$$G_\varepsilon = -\nabla F(\rho_\varepsilon) + 2\nu \operatorname{div}(\sqrt{\rho_\varepsilon}\mathbf{S}_{\nu,\varepsilon}) - 4\kappa^2 \operatorname{div}(\nabla\sqrt{\rho_\varepsilon} \otimes \nabla\sqrt{\rho_\varepsilon}) - \operatorname{div}(\Lambda_\varepsilon \otimes \Lambda_\varepsilon). \tag{4.2}$$

Formally, the density fluctuations σ_ε satisfy the Boussinesq-type equation

$$\partial_{tt}^2\sigma_\varepsilon - \frac{1}{\varepsilon^2}\Delta(1 - \varepsilon^2\kappa^2\Delta)\sigma_\varepsilon = -\operatorname{div}\mathbf{Q}(G_\varepsilon). \tag{4.3}$$

The fourth-order term stems from the dispersive stress tensor $\operatorname{div}\mathbf{K}$ in the equation for the momentum density upon using identity (1.5) and alters the dispersion relation for the acoustic equations. In the absence of capillary effects, namely for $\kappa = 0$, (4.3) reduces to the wave equation with sound speed $1/\varepsilon$ which is known to govern the evolution of acoustic waves for a classical compressible fluid. For $\kappa > 0$, the dispersion relation for high frequencies (above the threshold $1/\varepsilon$) is no longer linear but quadratic. For a discussion of the physical background and the link to the Bogoliubov dispersion relation [17] appearing in the microscopic theory for Bose-Einstein condensation we refer to [6]. Moreover, by an accurate dispersive analysis of (4.1) it is proven in [6], see also [4, 16], that the presence of the quantum correction term leads to improved decay rates of acoustic waves on $\mathbb{R}^d$ with $d \geq 2$ compared to compressible fluids without capillarity effects. For that purpose, (4.1) is symmetrized by means of the transformation

$$\widetilde{\sigma_\varepsilon} := (1 - \varepsilon^2\kappa^2\Delta)^{\frac{1}{2}}\sigma_\varepsilon, \quad \widetilde{m_\varepsilon} = (-\Delta)^{-\frac{1}{2}}\operatorname{div} m_\varepsilon, \tag{4.4}$$

so that the system reads

$$\begin{aligned} &\partial_t\widetilde{\sigma}_\varepsilon + \frac{1}{\varepsilon}(-\Delta)^{\frac{1}{2}}(1 - \varepsilon^2\kappa^2\Delta)^{\frac{1}{2}}\widetilde{m}_\varepsilon = 0, \\ &\partial_t\widetilde{m}_\varepsilon - \frac{1}{\varepsilon}(-\Delta)^{\frac{1}{2}}(1 - \varepsilon^2\kappa^2\Delta)^{\frac{1}{2}}\widetilde{\sigma}_\varepsilon = \widetilde{G}_\varepsilon, \end{aligned} \tag{4.5}$$

where $\widetilde{G}_\varepsilon = (-\Delta)^{-\frac{1}{2}}\operatorname{div} G_\varepsilon$. Upon controlling $(\sigma_\varepsilon, \mathbf{Q}(m_\varepsilon))$ in terms of $(\widetilde{\sigma}_\varepsilon, \widetilde{m}_\varepsilon)$, it suffices to investigate (4.5). System (4.5) can be characterised by means of the linear semigroup operator e^{itH_ε} where H_ε is defined via the Fourier multiplier

$$\phi_\varepsilon(|\xi|) = \frac{|\xi|}{\varepsilon}\sqrt{1 + \varepsilon^2\kappa^2|\xi|^2}. \tag{4.6}$$

A stationary phase argument leads to the following dispersive estimate for the semigroup operator e^{itH_ε}, see [4, Corollary B.6] and also [6, Corollary 4.3].

Lemma 4.1. *Let $d \geq 2$, ϕ_ε as in* (4.6)*, $R > 0$ and let $\chi(r) \in C_c(0, \infty)$ be a smooth cut-off frequency cut-off localizing in frequencies of order R. Then there exists a constant $C > 0$ such that*

$$\sup_{x\in\mathbb{R}^d}\left|\int_{\mathbb{R}^d} e^{it\phi_\varepsilon(|\xi|)+ix\cdot\xi}\chi(|\xi|)\,\mathrm{d}\xi\right| \leq \frac{C}{(\kappa t)^{\frac{d}{2}}}\left(\frac{\varepsilon\kappa R}{\sqrt{1 + (\varepsilon\kappa)^2R^2}}\right)^{\delta}, \tag{4.7}$$

for any $\delta \in [0, \frac{d-2}{2}]$.

For $\varepsilon = 1$, the dispersive estimate (4.7) is proven in [32] to investigate the large time behavior of solution to the Gross-Pitaevskii (GP) equation. Here, we only mention that the (GP) equation is formally equivalent to the QHD system, the inviscid counter part of (1.3), see [6]. Note that the right-hand side of (4.7) blows up for $\kappa \to 0$. Indeed, the acoustic dispersion is then governed by the wave equation while (4.7) is a Schrödinger-like dispersive estimate. Here, we consider $\kappa > 0$ to be fixed.

The symbol ϕ_ε is non-homogeneous and does not allow for a separation of scales. Hence the ε-dependent version can not be obtained by a simple scaling argument. For $\delta = 0$, the dispersive estimate (4.7) reduces to the one for the free Schrödinger propagator $e^{it\Delta}$. In addition, (4.7) yields a regularizing effect for low frequencies for $d > 2$ that provides decay of order ε^δ at the expense of a factor R^δ for $\delta > 0$ arbitrarily small. This is related to the curvature of the hypersurface $\tau = \phi_\varepsilon(|\xi|)$ which depends on the spatial dimension d. For $d = 2$, (4.7) does not yield any decay in ε. It is shown in [6, Proposition 3.8] that the desired decay for $d = 2$ can be obtained by separating the regimes of frequencies above and below the threshold $1/\varepsilon$. The symbol ϕ_ε is well approximated by $\frac{|\xi|}{\varepsilon}$, namely the wave operator with speed $1/\varepsilon$ for frequencies below the threshold $1/\varepsilon$ and by $|\xi|^2$, i.e., the free Schrödinger operator for frequencies larger than $1/\varepsilon$. The desired decay then follows from the wave-like estimate for low frequencies and Sobolev embedding for high frequencies. However, this leads to a loss of the aforementioned regularizing effect. Interpolating in the low frequency regime between the wave-type estimate and (4.7) allows one to obtain Strichartz estimates with arbitrarily small loss of regularity.

Definition 4.2. *The exponents (q, r) are said to be μ-*admissible *if $2 \leq q, r \leq \infty$, $(q, r, \mu) \neq (2, \infty, 1)$ and*

$$\frac{1}{q} + \frac{\mu}{r} = \frac{\mu}{2}.$$

We say that a pair is Schrödinger or wave admissible if $\mu = \frac{d}{2}$ or $\mu = \frac{d-1}{2}$ respectively. Further, we denote $\beta = \beta(r) := \frac{1}{2} - \frac{1}{r}$.

Proposition 4.3 ([6]). *Let $\varepsilon > 0$ and $\theta \in [0, 1)$. Then, for any $\frac{2-\theta}{2}$-admissible pair (q, r) and $s_0 = 3\beta(r)\theta$, it holds*

$$\|e^{itH_\varepsilon} f\|_{L^q(0,T;L^r(\mathbb{R}^2))} \leq C\varepsilon^{\frac{s}{3}} \|f\|_{\dot{H}^s(\mathbb{R}^2)}. \tag{4.8}$$

For $\theta = 0$, the Strichartz estimate reduces to the one for the Schrödinger group and does not provide any decay in ε. For $\theta = 1$, we recover a wave-like estimate with sound speed $1/\varepsilon$. In [16, Corollary B.1], the authors prove estimate (4.8) for $\theta = 1$ and $d \geq 2$ and low frequencies in the framework of the (GP)-equation. For high frequencies, a Schrödinger type estimate is obtained. In this regard, (4.8) can be considered as a refinement of [16, Corollary B.1].

Remark 4.4. In [6], Corollary 4.3 is stated in terms of Besov spaces which is slightly more precise but not needed for our purpose.

Abstract arguments [31, 38] then also yield the non-homogeneous Strichartz estimate for e^{itH_ε}.

Corollary 4.5. *Let $\varepsilon > 0$ and $\theta \in [0, 1)$. Then, for any $\frac{2-\theta}{2}$-admissible pair (q, r) and $s_0 = 3\beta(r)\theta$, it holds*

$$\|e^{itH_\varepsilon} f\|_{L^q(0,T;L^r(\mathbb{R}^2))} \leq C\varepsilon^{\frac{s}{3}} \|f\|_{\dot{H}^s(\mathbb{R}^2)}.$$

Further, for any $\frac{2-\theta}{2}$-admissible pairs (q, r) and (q_1, r_1) it holds

$$\left\| \int_0^t e^{i(t-\tau)H_\varepsilon} F(\tau) \mathrm{d}\tau \right\|_{L^q(0,T;L^r(\mathbb{R}^2))} \leq C\varepsilon^{\frac{s_0+s_1}{3}} \|F\|_{L^{q_1'}(0,T;W^{s_0+s_1,r_1'}(\mathbb{R}^2))}, \tag{4.9}$$

with $s_1 = 3\theta\beta(r_1)$.

In order to obtain suitable bounds on $(\widetilde{\sigma}_\varepsilon, \widetilde{m}_\varepsilon)$ by means of the Strichartz estimates, we start by proving uniform bounds for $(\sigma_\varepsilon^0, m_\varepsilon^0)$.

Lemma 4.6. *Under the Assumptions 2.5, the initial data* $(\sigma_\varepsilon^0, m_\varepsilon^0)$ *satisfy the following bounds:*

1) $\sigma_\varepsilon^0 \in L^2(\mathbb{R}^2)$ *uniformly bounded,*
2) $\varepsilon\nabla\sigma_\varepsilon^0 \in H^{-\delta}(\mathbb{R}^2)$ *uniformly bounded for* $\delta > 0$ *arbitrarily small,*
3) $m_\varepsilon^0 \in H^{-\delta}(\mathbb{R}^2)$ *uniformly bounded for* $\delta > 0$ *arbitrarily small.*

Proof. It follows from (2) Lemma 3.1 that $\sigma_\varepsilon^0 = \varepsilon^{-1}(\rho_\varepsilon^0 - 1) \in L^2(\mathbb{R}^2)$ uniformly bounded. Statement (2) follows from the identity

$$\varepsilon\nabla\sigma_\varepsilon^0 = \nabla(\rho_\varepsilon^0 - 1) = 2\nabla\sqrt{\rho_\varepsilon^0} + 2\left(\sqrt{\rho_\varepsilon^0} - 1\right)\nabla\sqrt{\rho_\varepsilon^0}$$

and the uniform bound for $\sqrt{\rho_\varepsilon^0} - 1 \in H^1(\mathbb{R}^2) \hookrightarrow L^r(\mathbb{R}^2)$ for $r \in [2, \infty)$ provided by (1) Lemma 3.1. To prove statement (3), we recall that $\Lambda_\varepsilon^0 \in L^2(\mathbb{R}^2)$ uniformly bounded and $\sqrt{\rho_\varepsilon} - 1 \in H^1(\mathbb{R}^2) \hookrightarrow L^{r_1}$ with $r_1 \in [2, \infty)$ uniformly bounded. It follows $m_\varepsilon = \Lambda_\varepsilon + (\sqrt{\rho_\varepsilon^0} - 1)\Lambda_\varepsilon \in L^2(\mathbb{R}^2) + L^r(\mathbb{R}^2) \hookrightarrow H^{-\delta}(\mathbb{R}^2)$ uniformly bounded with $\delta > 0$ arbitrarily small.

□

To infer a bound on the nonlinearity G_ε defined in (4.2), G_ε is decomposed as $G_\varepsilon = -G_\varepsilon^1 + G_\varepsilon^2$ where

$$G_\varepsilon^1 := \operatorname{div}(\Lambda_\varepsilon \otimes \Lambda_\varepsilon) + 4\kappa^2 \operatorname{div}(\nabla\sqrt{\rho_\varepsilon} \otimes \nabla\sqrt{\rho_\varepsilon}) + \nabla F_\varepsilon(\rho_\varepsilon), \qquad G_\varepsilon^2 := 2\nu \operatorname{div}(\sqrt{\rho_\varepsilon}\mathbf{S}_{\nu,\varepsilon}).$$

Exploiting the a priori estimates stemming from inequalities (2.3) and (2.4) we obtain the following uniform bounds.

Lemma 4.7. *Let* $T > 0$, $(\rho_\varepsilon^0, u_\varepsilon^0)$ *be initial data satisfying Assumptions 2.5 and* $(\rho_\varepsilon, u_\varepsilon)$ *a FEWS with initial data* $(\rho_\varepsilon^0, u_\varepsilon^0)$*. Then,*

1) $G_\varepsilon^1 \in L^\infty(0, T; W^{-2,r'}(\mathbb{R}^2))$ *uniformly bounded for all* $r' \in [1, 2)$*;*
2) $G_\varepsilon^2 \in L^p(0, T; H^{-1}(\mathbb{R}^2))$ *uniformly bounded for all* $p \in [1, 2)$*.*

Proof. It follows from (2.3) and Lemma 3.1 that

$$\Lambda_\varepsilon \otimes \Lambda_\varepsilon \in L^\infty(0, \infty; L^1(\mathbb{R}^2)), \quad \nabla\sqrt{\rho_\varepsilon} \otimes \nabla\sqrt{\rho_\varepsilon} \in L^\infty(0, \infty; L^1(\mathbb{R}^2)), \quad F(\rho_\varepsilon) \in L^\infty(0, \infty; L^1(\mathbb{R}^2)),$$

uniformly bounded. As $W^{1,r}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$ for all $r > 2$ and by duality $L^1 \hookrightarrow W^{-1,r'}(\mathbb{R}^2)$, one obtains,

$$G_\varepsilon^1 = \operatorname{div}\left(\Lambda_\varepsilon \otimes \Lambda_\varepsilon + 4\kappa^2\nabla\sqrt{\rho_\varepsilon} \otimes \nabla\sqrt{\rho_\varepsilon} + (\gamma - 1)F(\rho_\varepsilon)\mathbf{Id}\right) \in L^\infty(0, \infty; W^{-2,r'}(\mathbb{R}^2))$$

uniformly bounded for all $r' \in [1, 2)$. To bound G_ε^2, we observe that

$$\sqrt{\rho_\varepsilon}\mathbf{S}_{\nu,\varepsilon} = \mathbf{S}_{\nu,\varepsilon} + (\sqrt{\rho_\varepsilon} - 1)\mathbf{S}_{\nu,\varepsilon} \in L^2(0, T; L^2(\mathbb{R}^2)) + L^p(0, T; L^2(\mathbb{R}^2)) \subset L^p(0, T; L^2(\mathbb{R}^2))$$

uniformly bounded for all $p \in [1, 2)$. Indeed, (9) Lemma 3.1 yields that $\mathbf{S}_{\nu,\varepsilon} \in L^2(0, T; L^2(\mathbb{R}^2))$ uniformly bounded and $\sqrt{\rho_\varepsilon} - 1 \in L^{p_1}(0, T; L^\infty(\mathbb{R}^2))$ uniformly bounded for all $p_1 \in [2, \infty)$ from (1) Lemma 3.3. Hence,

$$G_\varepsilon^2 \in L^p(0, T; H^{-1}(\mathbb{R}^2))$$

uniformly bounded for all $p \in [1, 2)$. □

The uniform estimates for $(\sigma_\varepsilon^0, m_\varepsilon^0)$ and G_ε together with the Strichartz estimates allow one to infer strong convergence of $(\widetilde{\sigma}_\varepsilon, \widetilde{m}_\varepsilon)$ to 0 as $\varepsilon \to 0$ in space-time norms.

Proposition 4.8. *Let $T > 0$, $(\rho_\varepsilon^0, u_\varepsilon^0)$ be initial data satisfying Assumptions 2.5 and $(\rho_\varepsilon, u_\varepsilon)$ a FEWS with initial data $(\rho_\varepsilon^0, u_\varepsilon^0)$. Further, let $\theta \in (0,1)$ and (q,r) be a $\frac{2-\theta}{2}$-admissible pair, then,*

$$\|\widetilde{\sigma}_\varepsilon\|_{L^q(0,T;W^{-2(1+s),r}(\mathbb{R}^2))} + \|\widetilde{m}_\varepsilon\|_{L^q(0,T;W^{-2(1+s),r}(\mathbb{R}^2))} \leq C\varepsilon^{\frac{s}{3}}$$

where $s = 3\beta(r)\theta$ with $\beta(r)$ as in Definition 4.2.

Note that $s > 0$ provided that $r > 2$ and that s can be made arbitrarily small by choosing $\theta > 0$ sufficiently small.

Proof. First, we show that the initial data satisfy $\widetilde{\sigma}_\varepsilon^0, \widetilde{m}_\varepsilon^0 \in H^{-\frac{1}{2}}(\mathbb{R}^2)$ uniformly bounded. Lemma 4.6 states that $\sigma_\varepsilon^0, m_\varepsilon^0 \in H^{-\frac{1}{2}}(\mathbb{R}^2)$ uniformly bounded. The transformation $T_1 : \sigma_\varepsilon^0 \mapsto \widetilde{\sigma}_\varepsilon^0$ is defined through the Fourier-multiplier $\varphi_\varepsilon(|\xi|) := (1+\varepsilon^2\kappa^2|\xi|^2)^{\frac{1}{2}}$ which entails a loss of derivatives for high frequencies only, namely

$$\varphi_\varepsilon(|\xi|) \leq \begin{cases} C_\kappa & |\xi| \leq \varepsilon^{-1}, \\ C_\kappa \varepsilon|\xi| & |\xi| > \varepsilon^{-1}. \end{cases}$$

It then follows from (1) and (2) of Lemma 4.6 that

$$\|\widetilde{\sigma}_\varepsilon^0\|_{H^{-\frac{1}{2}}(\mathbb{R}^2)} \leq C_\kappa \left(\|P_{\leq \varepsilon^{-1}}(\sigma_\varepsilon^0)\|_{H^{-\frac{1}{2}}(\mathbb{R}^2)} + \|P_{>\varepsilon^{-1}}(\varepsilon\nabla\sigma_\varepsilon^0)\|_{H^{-\frac{1}{2}}(\mathbb{R}^2)} \right) \leq C_\kappa', \tag{4.10}$$

where $P_{\leq\varepsilon^{-1}}$ and $P_{>\varepsilon^{-1}}$ denote the projections on frequencies of order below or above ε^{-1} respectively. Similarly, the transformation $T_2 : \mathbf{Q}(m_\varepsilon^0) \mapsto \widetilde{m}_\varepsilon^0 = \Delta^{-\frac{1}{2}} \operatorname{div} \mathbf{Q}(m_\varepsilon)$ is given by the Riesz transform which is bounded on $L^2(\mathbb{R}^2)$. One has

$$\|\widetilde{m}_\varepsilon^0\|_{H^{-\frac{1}{2}}(\mathbb{R}^2)} \leq C\|\mathbf{Q}(m_\varepsilon^0)\|_{H^{-\frac{1}{2}}(\mathbb{R}^2)} \leq C\|m_\varepsilon^0\|_{H^{-\frac{1}{2}}(\mathbb{R}^2)}. \tag{4.11}$$

Analogously, it follows that for any $r_1', q_2' \in [1,2)$ that

$$\begin{aligned} \|\widetilde{G}_\varepsilon^1\|_{L^\infty(0,T;W^{-2,r_1'}(\mathbb{R}^2))} &\leq \|G_\varepsilon^1\|_{L^\infty(0,T;W^{-2,r_1'}(\mathbb{R}^2))}, \\ \|\widetilde{G}_\varepsilon^2\|_{L^{q_2}(0,T;H^{-1}(\mathbb{R}^2))} &\leq \|G_\varepsilon^2\|_{L^{q_2}(0,T;H^{-1}(\mathbb{R}^2))} \end{aligned} \tag{4.12}$$

from Lemma 4.7. Second, for $\theta \in (0,1)$ and any $\frac{2-\theta}{2}$ admissible pair (q,r), the Strichartz estimate (4.8) applied to $(1-\Delta)^{-\frac{1}{2}-s}\widetilde{\sigma}_\varepsilon^0$ with $s = 3\theta\beta(r)$ yields

$$\begin{aligned} \|e^{itH_\varepsilon}\widetilde{\sigma}_\varepsilon^0\|_{L^q(0,T;W^{-\frac{1}{2}-s,r}(\mathbb{R}^2))} &\leq C\varepsilon^{\frac{s}{3}}\|\widetilde{\sigma}_\varepsilon^0\|_{H^{-\frac{1}{2}}(\mathbb{R}^2))} \leq C\varepsilon^{\frac{s}{3}}, \\ \|e^{itH_\varepsilon}\widetilde{m}_\varepsilon^0\|_{L^q(0,T;W^{-\frac{1}{2}-s,r}(\mathbb{R}^2))} &\leq C\varepsilon^{\frac{s}{3}}\|\widetilde{m}_\varepsilon^0\|_{H^{-\frac{1}{2}}(\mathbb{R}^2))} \leq C\varepsilon^{\frac{s}{3}}, \end{aligned}$$

where we exploited (4.10) and (4.11) respectively. Similarly, for any $\theta \in (0,1)$ and any $\frac{2-\theta}{2}$ admissible pair (q,r), the non-homogeneous Strichartz estimate (4.9) applied to $(1-\Delta)^{-2-(s+s_1)}\widetilde{G}_\varepsilon^1$ yields

$$\left\| \int_0^t e^{i(t-\tau)H_\varepsilon}\widetilde{G}_\varepsilon^1(\tau)\mathrm{d}\tau \right\|_{L^q(0,T;W^{-(2+s+s_1),r}(\mathbb{R}^2))} \leq C\varepsilon^{\frac{s+s_1}{3}} T^{\frac{1}{q_1'}} \|G_\varepsilon^1\|_{L^\infty(0,T;W^{-2,r_1'}(\mathbb{R}^2))} \leq C(T)\varepsilon^{\frac{s+s_1}{3}},$$

where we used the first inequality of (4.12) and (1) Lemma 4.7 in the last step. Similarly, for any $\theta \in (0,1)$ and any $\frac{2-\theta}{2}$ admissible pair (q,r) and $q_2' \in [1,2)$ it holds that

$$\left\| \int_0^t e^{i(t-\tau)H_\varepsilon} \widetilde{G}_\varepsilon^2(\tau) \mathrm{d}\tau \right\|_{L^q(0,T;W^{-(2+s),r}(\mathbb{R}^2))} \leq C\varepsilon^{\frac{s}{3}} T^{\frac{1}{q_2}} \|G_\varepsilon^2\|_{L^{q_2'}(0,T;H^{-1}(\mathbb{R}^2))} \leq C(T)\varepsilon^{\frac{s}{3}},$$

upon using the second inequality of (4.12) and (2) Lemma 4.7. We are now in position to infer the desired decay of $(\widetilde{\sigma_\varepsilon}, \widetilde{m_\varepsilon})$. The Duhamel formula associated to (4.5) is given by

$$\begin{pmatrix} \widetilde{\sigma}_\varepsilon(t) \\ \widetilde{m}_\varepsilon(t) \end{pmatrix} = \begin{pmatrix} \mathrm{Re}(e^{itH_\varepsilon}) & -\mathrm{Im}(e^{itH_\varepsilon}) \\ \mathrm{Im}(e^{itH_\varepsilon}) & \mathrm{Re}(e^{itH_\varepsilon}) \end{pmatrix} \begin{pmatrix} \widetilde{\sigma}_\varepsilon^0 \\ \widetilde{m}_\varepsilon^0 \end{pmatrix} + \int_0^t \begin{pmatrix} -\mathrm{Im}(e^{i(t-s)H_\varepsilon})\widetilde{G}_\varepsilon(s) \\ \mathrm{Re}(e^{i(t-s)H_\varepsilon})\widetilde{G}_\varepsilon(s) \end{pmatrix} \mathrm{d}s.$$

Summarizing it follows that for any $\theta \in (0,1)$ and $\frac{2-\theta}{2}$ admissible pairs (q,r) and (q_1,r_1) one has

$$\begin{aligned} \|(\widetilde{\sigma}_\varepsilon, \widetilde{m}_\varepsilon)\|_{L^q(0,T;W^{-2(1+s),r}(\mathbb{R}^2)} \leq C\varepsilon^{\frac{s}{3}} \Big(\|\sigma_\varepsilon^0\|_{H^{-\frac{1}{2}}(\mathbb{R}^2)} + \|\varepsilon\nabla\sigma_\varepsilon^0\|_{H^{-\frac{1}{2}}(\mathbb{R}^2)} + \|m_\varepsilon^0\|_{H^{-\frac{1}{2}}(\mathbb{R}^2)} \\ + \varepsilon^{\frac{s_1}{3}} T^{\frac{1}{q_1'}} \|G_\varepsilon^1\|_{L^\infty(0,T;W^{-2,r_1'}(\mathbb{R}^2))} + T^{\frac{1}{q_2}} \|G_\varepsilon^1\|_{L^{q_2'}(0,T;H^{-1}(\mathbb{R}^2))} \Big) \leq C(T)\varepsilon^{\frac{s}{3}}. \end{aligned}$$

Note that $s > 0$ provided that $r > 2$. □

The desired decay of $\mathbf{Q}(m_\varepsilon)$ follows upon observing that $\mathbf{Q}(m_\varepsilon)$ can be controlled in terms of $\widetilde{m_\varepsilon}$ and interpolation with the uniform bound of m_ε at Sobolev regularity from (2) Lemma 3.3.

Corollary 4.9. *Let $T > 0$, $(\rho_\varepsilon^0, u_\varepsilon^0)$ be initial data satisfying Assumptions 2.5 and $(\rho_\varepsilon, u_\varepsilon)$ a FEWS with initial data $(\rho_\varepsilon^0, u_\varepsilon^0)$. Then, for all $r \in (2,\infty)$, there exists $\alpha > 0$ such that*

$$\|\mathbf{Q}(m_\varepsilon)\|_{L^2(0,T;W^{\alpha,r}(\mathbb{R}^2))} \leq C(T)\varepsilon^\alpha. \tag{4.13}$$

Proof. First, from the uniform bound $m_\varepsilon \in L^{\frac{2}{s_1}-}(0,T;H^{s_1}(\mathbb{R}^2))$ for all $s_1 \in (0,1)$ provided by (2) Lemma 3.3, we conclude that

$$m_\varepsilon \in L^{\frac{2}{s_1}-}(0,T;W^{s_1-2\beta(r),r}(\mathbb{R}^2))$$

for any $r \in [2,\infty)$ from the Sobolev embedding. In particular, given $r \in [2,\infty)$ one may choose s_1 sufficiently close to 1 such that $s_1 - 2\beta(r) > 0$. Second, it follows from Proposition 4.8 that for any $\theta \in [0,1)$ and $\frac{2-\theta}{2}$-admissible pair (q,r) one has

$$\|\mathbf{Q}(m_\varepsilon)\|_{L^q(0,T;W^{-2(1+s_2),r}(\mathbb{R}^2))} \leq C\|\widetilde{m}_\varepsilon\|_{L^q(0,T;W^{-2(1+s_2),r}(\mathbb{R}^2))} \leq C\varepsilon^{\frac{s_2}{3}}$$

where $s_2 = 3\theta\beta(r)$ with $\beta(r)$ as in Definition 4.2. For $\theta \in (0,1)$ to be fixed later, let (q,r) be a $\frac{2-\theta}{2}$-admissible pair. Then, the interpolation Lemma 1.2 yields that

$$\|\mathbf{Q}(m_\varepsilon)\|_{L^{q_0}(0,T;W^{s_0,r}(\mathbb{R}^2))} \leq \|\mathbf{Q}(m_\varepsilon)\|^{\theta'}_{L^{\frac{2}{s_1}-}(0,T;W^{s_1-2\beta(r),r}(\mathbb{R}^2))} \|\mathbf{Q}(m_\varepsilon)\|^{1-\theta'}_{L^q(0,T;W^{-2(1+s_2),r}(\mathbb{R}^2))} \leq C\varepsilon^{\frac{s_2}{3}(1-\theta')},$$

where

$$s_0 = \theta'(s_1 - 2\beta(r)) + (1-\theta')(-2)(1+s_2) = \theta'(s_1 - 2\beta(r)) - (1-\theta')(2+6\beta(r)\theta), \qquad \frac{1}{q_0} > \frac{\theta' s_1}{2} + \frac{1-\theta'}{q}.$$

For $\theta \in (0,1)$ sufficiently small, s_1 sufficiently close to 1 and any $r \in (2,\infty)$ one may choose $\theta' \in (0,1)$ sufficiently close to 1 so that $s_0 > 0$. Moreover, since (q,r) is $\frac{2-\theta}{2}$-admissible with $r > 2$ one has that $q \in (2,\infty)$ and therefore $q_0 \in [2,\infty)$. It follows that for any $r \in (2,\infty)$ there exists $\alpha > 0$ such that

$$\|\mathbf{Q}(m_\varepsilon)\|_{L^2(0,T;W^{\alpha,r}(\mathbb{R}^2))} \leq C(T)\varepsilon^\alpha.$$

□

Remark 4.10. If (1.1) is considered with a general capillarity tensor $\mathbb{K}$, as defined in (1.2), and in the scaling (1.8), then the respective linearized system amounts to (4.1) at leading order. More precisely, we wish to linearize (1.2) for $\rho_\varepsilon = 1 + \varepsilon\sigma_\varepsilon$ and note that only the first term $\rho\,\mathrm{div}(k(\rho)\nabla\rho)\mathbb{I}$ of $\mathbb{K}$ yields a contribution of order $O(\varepsilon)$ while the second and third term contribute with terms of order at least $O(\varepsilon^2)$. Those may be discharged into G_ε on the right-hand side of (4.1) and bounded in appropriate Sobolev spaces at negative regularity. We recover the Bogoliubov dispersion relation as in (4.3) and the dispersive analysis then follows the same lines.

Note that if $\kappa = 0$, one may still prove that $\mathbf{Q}(m_\varepsilon)$ strongly converges to zero in $L^q(0,T;W^{-s,r}(\mathbb{R}^2))$ for some $s > 2$ and wave-admissible exponents (q,r), though with increased loss of regularity and worse decay rate as (4.7) is no longer available. However, under the light of Remarks 2.7 and 3.4 we lack an appropriate uniform estimate to perform the interpolation argument of Corollary 4.9.

5. Proof of the main theorem

This section provides the proof of Theorem 2.6. First, we show strong convergence of Λ_ε and m_ε in $L^2_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^2)$. Second, we pass to the limit in (1.9) to show that the limit function is the unique Leray weak solution of (1.11).

In order to show strong convergence of the momentum density $\{m_\varepsilon\}_{\varepsilon>0}$, it remains to prove compactness of the solenoidal part $\{\mathbf{P}(m_\varepsilon)\}_{\varepsilon>0}$ of the momentum density m_ε.

Lemma 5.1. *Under the assumptions of Theorem 2.6, there exists $u \in L^\infty(0,\infty;L^2(\mathbb{R}^2))$ such that up to extracting a subsequence (not relabeled)*

1) *the sequence of momentum density $\{m_\varepsilon\}_{\varepsilon>0}$ converges strongly to u in $L^2_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^2)$,*
2) *$\{\Lambda_\varepsilon\}_{\varepsilon>0}$ converges weakly-$*$ in $L^\infty(0,\infty;L^2(\mathbb{R}^2))$ and strongly in $L^2_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^2)$ to u.*

Proof. The momentum density is decomposed in the irrotational and the solenoidal part $m_\varepsilon = \mathbf{Q}(m_\varepsilon) + \mathbf{P}(m_\varepsilon)$ by applying the Leray-Helmholtz projections. It follows from Corollary 4.9 that for all $q \in [2,\infty)$ there exists $\alpha > 0$ such that $\mathbf{Q}(m_\varepsilon)$ converges strongly to 0 in $L^2(0,T;W^{\alpha,r}(\mathbb{R}^2))$. In particular, $\mathbf{Q}(m_\varepsilon) \to 0$ in $L^2(0,T;L^r(\mathbb{R}^2))$ for all $r \in [2,\infty)$. By consequence, one also has $\mathbf{Q}(m_\varepsilon) \to 0$ in $L^2_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^2)$. Note that no extraction of subsequences is required. Second, since $\Lambda_\varepsilon \in L^\infty(0,\infty;L^2(\mathbb{R}^2))$ uniformly bounded, there exists $u \in L^\infty(0,\infty;L^2(\mathbb{R}^2))$ such that $\Lambda_\varepsilon \rightharpoonup u$ in $L^\infty(0,\infty;L^2(\mathbb{R}^2))$ up to extraction of a subsequence (not relabeled). We conclude that also $m_\varepsilon \rightharpoonup u$ in $L^{\frac{2}{s}-}(0,T;H^s(\mathbb{R}^2))$ for any $T > 0$ und $s \in (0,1)$ up to extraction of a further subsequence (not relabeled) by virtue of (2) Lemma 3.3. We show that $\mathbf{P}(m_\varepsilon)$ converges strongly to u in $L^2(0,T;L^2_{\mathrm{loc}}(\mathbb{R}^2))$. We recall that $\mathbf{P}(m_\varepsilon) \in L^{\frac{2}{s}-}(0,T;H^s(\mathbb{R}^2))$ uniformly bounded for all $s \in (0,1)$ from Lemma 3.3. Further, by projecting the moment equation of (1.3) on solenoidal vector fields we obtain

$$\partial_t\mathbf{P}(m_\varepsilon) = -\mathbf{P}\left(\mathrm{div}(\Lambda_\varepsilon \otimes \Lambda_\varepsilon + \kappa^2\nabla\sqrt{\rho_\varepsilon} \otimes \sqrt{\rho_\varepsilon}\right) + 2\nu\mathbf{P}\left(\mathrm{div}(\sqrt{\rho_\varepsilon}\mathbf{S}_{\nu,\varepsilon})\right),$$

where $\mathbf{S}_{\nu,\varepsilon}$ denotes the symmetric part of $\mathbf{T}_{\nu,\varepsilon}$ defined by (2.2). Exploiting the uniform bounds of Lemma 3.1 and Lemma 3.3 and arguing as in the proof of Lemma 4.7, we conclude that $\partial_t \mathbf{P}(m_\varepsilon) \in L^2(0,T;H^{-s_1}(\mathbb{R}^2))$ uniformly bounded for all $s_1 > 2$. Hence,

$$\mathbf{P}(m_\varepsilon) \in L^2(0,T;H^s(\mathbb{R}^2)), \quad \partial_t \mathbf{P}(m_\varepsilon) \in L^2(0,T;H^{-s_1}(\mathbb{R}^2)),$$

with $s \in (0,1)$ and $s_1 > 2$. The Aubin-Lions Lemma, see e.g., [53], then implies that $\mathbf{P}(m_\varepsilon)$ converges strongly to u in $L^2(0,T;L^2_{\mathrm{loc}}(\mathbb{R}^2))$. By consequence,

$$m_\varepsilon \to u \quad \text{strongly in } L^2_{\mathrm{loc}}(([0,\infty) \times \mathbb{R}^2)).$$

As Λ_ε in $L^\infty(0,T;L^2(\mathbb{R}^2))$ uniformly bounded and $\sqrt{\rho_\varepsilon} - 1$ converges strongly to 0 in $L^2(0,T;L^\infty(\mathbb{R}^2))$ by virtue of Lemma 3.3, it follows from Corollary 4.9 that there exists $\alpha > 0$ such that

$$\begin{aligned}\|\Lambda_\varepsilon - u\|_{L^2_{\mathrm{loc}}(([0,\infty)\times\mathbb{R}^2))} &\le \|\mathbf{Q}(m_\varepsilon)\|_{L^2_{\mathrm{loc}}(([0,\infty)\times\mathbb{R}^2))} + \|\mathbf{P}(m_\varepsilon) - u\|_{L^2_{\mathrm{loc}}(([0,\infty)\times\mathbb{R}^2))} + \|(\sqrt{\rho_\varepsilon} - 1)\Lambda_\varepsilon\|_{L^2_{\mathrm{loc}}(([0,\infty)\times\mathbb{R}^2))} \\ &\le C(T)\varepsilon^\alpha + \|\mathbf{P}(m_\varepsilon) - u\|_{L^2_{\mathrm{loc}}(([0,\infty)\times\mathbb{R}^2))} + \|\Lambda_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}^2))}\|\sqrt{\rho_\varepsilon} - 1\|_{L^2(0,T;L^\infty(\mathbb{R}^2))}.\end{aligned}$$

Therefore, Λ_ε converges strongly to u in $L^2_{\mathrm{loc}}([0,\infty) \times \mathbb{R}^2)$. □

We are now in position to prove the main result Theorem 2.6.

Proof of Theorem 2.6. To show the first statement, we note that (6) Lemma 3.1 yields $\|\rho_\varepsilon - 1\|_{L^\infty L^2} \le C\sqrt{\varepsilon}$. On the other hand,

$$\nabla^2(\rho_\varepsilon - 1) = 2\nabla^2\sqrt{\rho_\varepsilon} + 2(\sqrt{\rho_\varepsilon} - 1)\nabla^2\sqrt{\rho_\varepsilon} + 2\nabla\sqrt{\rho_\varepsilon} \otimes \nabla\sqrt{\rho_\varepsilon}.$$

As $\sqrt{\rho_\varepsilon} - 1 \in L^p(0,T;L^\infty(\mathbb{R}^2))$ and $\nabla\sqrt{\rho_\varepsilon} \in L^4(0,\infty;L^4(\mathbb{R}^2))$, it follows that $\nabla^2(\rho_\varepsilon - 1) \in L^{p_2}(0,T;L^2(\mathbb{R}^2))$ for all $p_2 \in [1,2)$. Upon applying Lemma 1.2 we infer that for all $\theta \in (0,1)$

$$\|\rho_\varepsilon - 1\|_{L^p(0,T;H^s(\mathbb{R}^2)} \le \|\rho_\varepsilon - 1\|^\theta_{L^2(0,T;H^2(\mathbb{R}^2))}\|\rho_\varepsilon - 1\|^{1-\theta}_{L^\infty(0,T;L^2(\mathbb{R}^2))} \le C\varepsilon^{\frac{1-\theta}{2}},$$

where $s = 2\theta$ and $p = 2/\theta$. Statement (3) follows from Corollary 4.9. To show (2), we first prove that $u \in L^\infty(0,\infty;L^2(\mathbb{R}^2) \cap L^2(0,T;\dot{H}^1(\mathbb{R}^2))$ for arbitrary $T > 0$. To that end, we identify the weak limit of $\mathbf{T}_{\nu,\varepsilon}$. By Definition, see (2.2), it holds

$$\nu\nabla m_\varepsilon = \sqrt{\nu}\sqrt{\rho_\varepsilon}\mathbf{T}_{\nu,\varepsilon} - 2\nu\Lambda_\varepsilon \otimes \nabla\sqrt{\rho_\varepsilon} = \sqrt{\nu}\mathbf{T}_{\nu,\varepsilon} + \sqrt{\nu}(\sqrt{\rho_\varepsilon} - 1)\mathbf{T}_{\nu,\varepsilon} - 2\nu\Lambda_\varepsilon \otimes \nabla\sqrt{\rho_\varepsilon},$$

in distributional sense. One has that ∇m_ε converges to ∇u in the sense of distributions as $m_\varepsilon \to u$ in $L^2_{\mathrm{loc}}([0,\infty) \times \mathbb{R}^2))$ from Lemma 5.1. Further, $\mathbf{T}_{\nu,\varepsilon} \in L^2(0,T;L^2(\mathbb{R}^2))$ is uniformly bounded. Thus, there exists $T \in L^2(0,T;L^2(\mathbb{R}^2))$ such that $\mathbf{T}_{\nu,\varepsilon} \rightharpoonup T$ in $L^2(0,T;L^2(\mathbb{R}^2))$ up to passing to a subsequence. The term $\nu(\sqrt{\rho_\varepsilon} - 1)\mathbf{T}_{\nu,\varepsilon}$ converges to 0 in the sense of distributions as $\sqrt{\rho_\varepsilon} - 1$ converges strongly to 0 in $L^\infty(0,\infty;L^r(\mathbb{R}^2))$ for all $r \in [2,\infty)$. The third term $2\nu\Lambda_\varepsilon \otimes \nabla\sqrt{\rho_\varepsilon}$ converges to 0 in $\mathcal{D}'([0,\infty) \times \mathbb{R}^2)$ as $\nabla\sqrt{\rho_\varepsilon}$ converges strongly to 0 in $L^4(0,T;L^2(\mathbb{R}^2))$ from Lemma 3.3 and $\Lambda_\varepsilon \in L^\infty(0,\infty;L^2(\mathbb{R}^2))$ is uniformly bounded. Hence, by uniqueness of weak limits, it holds $\nu\nabla u = \sqrt{\nu}T$ in $\mathcal{D}'([0,\infty) \times \mathbb{R}^2)$. Moreover, $\nu\mathbf{D}u = \sqrt{\nu}S$ where $S := T^{sym}$. It follows that

$$\nu\int_0^T\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x\mathrm{d}t \le \liminf_{\varepsilon\to 0}\int_0^T\int_{\mathbb{R}^2}\left|\mathbf{T}_{\nu,\varepsilon}\right|^2\mathrm{d}x\mathrm{d}t, \quad \nu\int_0^T\int_{\mathbb{R}^2}|\mathbf{D}u|^2\,\mathrm{d}x\mathrm{d}t \le \liminf_{\varepsilon\to 0}\int_0^T\int_{\mathbb{R}^2}\left|\mathbf{S}_{\nu,\varepsilon}\right|^2\mathrm{d}x\mathrm{d}t.$$

It is straightforward to check that for $u \in L^2(0,T;H^1(\mathbb{R}^2))$ such that $\operatorname{div} u = 0$ it holds

$$2\nu \int_0^T \int_{\mathbb{R}^2} |\mathbf{D}u|^2 \,\mathrm{d}x\mathrm{d}t = \nu \int_0^T \int_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}x\mathrm{d}t.$$

One may hence pass to the limit in (2.3) in order to obtain

$$\begin{aligned}&\int_{\mathbb{R}^2} \frac{1}{2}|u|^2 \mathrm{d}x + \nu \int_0^T \int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x\mathrm{d}t \\ &\quad \le \liminf_{\varepsilon\to 0} \left(\int_{\mathbb{R}^2} \frac{1}{2}|\Lambda_\varepsilon|^2 + \kappa^2 |\nabla \sqrt{\rho_\varepsilon}|^2 + \frac{1}{\varepsilon^2} F(\rho_\varepsilon)\mathrm{d}x + 2\int_0^T \int_{\mathbb{R}^2} |\mathbf{S}_{\nu,\varepsilon}|^2 \mathrm{d}x\mathrm{d}t \right) \le \limsup_{\varepsilon\to 0} E(\rho_\varepsilon^0, u_\varepsilon^0) \le C.\end{aligned}$$

As the initial data $(\rho_\varepsilon^0, u_\varepsilon^0)$ are ill-prepared, the right-hand side of the previous inequality does in general not converge to $\int_{\mathbb{R}^2} \frac{1}{2}|u_0|^2 \mathrm{d}x$. However, note that $u \in L^4(0,T;L^4(\mathbb{R}^2))$ by interpolation and the validity of the energy equality (2.5) will follow from the Ladyzhenskaya-Prodi-Serrin regularity criterion, see e.g., [43, Chapter 6] and also [51, 52] once we have shown that u is indeed a weak solution to (1.11) with initial data $\mathbf{P}(u^0) \in L^2(\mathbb{R}^2)$.

Second, we show that $u \in L^\infty(0,\infty;L^2(\mathbb{R}^2)) \cap L^2(0,T;\dot{H}^1(\mathbb{R}^2))$ is a weak solution to the incompressible Navier-Stokes equations (1.11) with initial data $\mathbf{P}(u_0)$. Note that the strong convergence of $\sqrt{\rho_\varepsilon} - 1$ to 0 in $L^\infty(0,\infty;L^2(\mathbb{R}^2))$ from (5) Lemma 3.1, the analogue statement for $\sqrt{\rho_\varepsilon^0} - 1$ from (1) Lemma 3.1 and the weak-$*$ convergence of Λ_ε to u in $L^\infty(0,\infty;L^2(\mathbb{R}^2))$ suffice to pass to the limit in the continuity equation of (1.3). Namely, let $\varphi \in C_c^\infty([0,\infty) \times \mathbb{R}^2)$, then the weak formulation reads

$$\int_{\mathbb{R}^2} \sqrt{\rho_\varepsilon^0}\sqrt{\rho_\varepsilon^0}\varphi(0,x)\mathrm{d}x + \int_0^T \int_{\mathbb{R}^2} \sqrt{\rho_\varepsilon}\sqrt{\rho_\varepsilon}\partial_t\varphi + \sqrt{\rho_\varepsilon}\Lambda_\varepsilon \cdot \nabla\varphi \mathrm{d}x\mathrm{d}t = 0,$$

and it follows that $\operatorname{div} u = 0$ as $\varepsilon \to 0$. Next, we pass to the limit in the equation for the momentum density. Let $\psi \in C_c^\infty([0,T) \times \mathbb{R}^2)$ such that $\operatorname{div} \psi = 0$, then for all $\varepsilon > 0$ it holds

$$\begin{aligned}&\int_{\mathbb{R}^2} \sqrt{\rho_\varepsilon}^0 \Lambda_\varepsilon^0 \psi(0)\mathrm{d}x + \int_0^T \int_{\mathbb{R}^2} \sqrt{\rho_\varepsilon}\Lambda_\varepsilon\psi_t + (\Lambda_\varepsilon \otimes \Lambda_\varepsilon) : \nabla\psi \mathrm{d}x\mathrm{d}t \\ &= \int_0^T \int_{\mathbb{R}^2} 2\nu\left(\Lambda_\varepsilon \otimes \nabla\sqrt{\rho_\varepsilon} + \nabla\sqrt{\rho_\varepsilon} \otimes \Lambda_\varepsilon\right)\nabla\psi - \nu\sqrt{\rho_\varepsilon}\Lambda_\varepsilon\Delta\psi + 4\kappa^2\left(\nabla\sqrt{\rho_\varepsilon} \otimes \nabla\sqrt{\rho_\varepsilon}\right) : \nabla\psi \mathrm{d}x\mathrm{d}t.\end{aligned} \tag{5.1}$$

Note that the weak L^2-convergence of Λ_ε^0 by Assumption 2.5 and the strong L^2-convergence of $\sqrt{\rho_\varepsilon^0} - 1$ to 0 from (2) Lemma 3.1 yield

$$\int_{\mathbb{R}^2} \sqrt{\rho_\varepsilon}^0 \Lambda_\varepsilon^0 \psi(0)\mathrm{d}x = \int_{\mathbb{R}^2} \left((\sqrt{\rho_\varepsilon}^0 - 1)\Lambda_\varepsilon^0 + \Lambda_\varepsilon \right)\psi(0)\mathrm{d}x \to \int_{\mathbb{R}^2} u_0\psi(0)\mathrm{d}x = \int_{\mathbb{R}^2} \mathbf{P}(u_0)\psi(0)\mathrm{d}x.$$

Since $\Lambda_\varepsilon \to u$ in $L^2_{\mathrm{loc}}([0,\infty) \times \mathbb{R}^2)$ strongly and $\sqrt{\rho_\varepsilon} - 1 \to 0$ in $L^\infty(0,\infty;L^r(\mathbb{R}^2))$ strongly for all $r \in [2,\infty)$, we may pass to the limit on the left-hand-side of (5.1) to obtain

$$\int_0^T \int_{\mathbb{R}^2} \sqrt{\rho_\varepsilon}\Lambda_\varepsilon\psi_t + (\Lambda_\varepsilon \otimes \Lambda_\varepsilon) : \nabla\psi \mathrm{d}x\mathrm{d}t \to \int_0^T \int_{\mathbb{R}^2} u\psi_t + (u \otimes u) : \nabla\psi \mathrm{d}x\mathrm{d}t.$$

In order to pass to the limit on the right-hand side of (5.1) we note that (3.5) yields strong convergence of $\nabla\sqrt{\rho_\varepsilon}$ to 0 in $L^4(0,T;L^2(\mathbb{R}^2))$. It follows

$$4\kappa^2\int_0^T\int_{\mathbb{R}^2}(\nabla\sqrt{\rho_\varepsilon}\otimes\nabla\sqrt{\rho_\varepsilon}):\nabla\psi\mathrm{d}x\mathrm{d}t\to 0,$$

as $\varepsilon\to 0$. Taking into account the aforementioned convergence properties of Λ_ε and $\nabla\sqrt{\rho_\varepsilon}$ we conclude that

$$\int_0^T\int_{\mathbb{R}^2}2\nu\left(\Lambda_\varepsilon\otimes\nabla\sqrt{\rho_\varepsilon}+\nabla\sqrt{\rho_\varepsilon}\otimes\Lambda_\varepsilon\right)\nabla\psi-\nu\sqrt{\rho_\varepsilon}\Lambda_\varepsilon\Delta\psi\mathrm{d}x\mathrm{d}t\to-\nu\int_0^T\int_{\mathbb{R}^2}u\Delta\psi\mathrm{d}x\mathrm{d}t.$$

Summarizing, we have shown that $u\in L^\infty(0,\infty;L^2(\mathbb{R}^2))\cap L^2(0,T;\dot{H}^1(\mathbb{R}^2))$ is a weak solution to (1.11). By virtue of the Ladyzhenskaya-Prodi-Serrin [43, 51, 52] regularity criteria, the velocity field u is the unique Leray-Hopf weak solution to (1.11) with initial data $\mathbf{P}(u^0)$. The uniqueness result then implies convergence of the whole sequence Λ_ε. □

Remark 5.2. Note that Lemma 5.1 and the proof of Theorem 2.6 can be developed along the same lines when dealing with general viscosity and capillarity coefficients satisfying (1.7) or for NSK by adapting carefully the respective uniform estimates stemming from the energy and BD-entropy estimates, see Remark 3.4. In particular, the compactness of $\mathbf{P}(m_\varepsilon)$ can be inferred in the same manner.

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Conflict of interest

The author declares no conflict of interest.

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