http://www.aimspress.com/journal/mine

Mathematics in Engineering, 5(1): 1–25.
DOI:10.3934/mine.2023019
Received: 20 December 2021
Revised: 05 February 2022
Accepted: 05 February 2022
Published: 22 March 2022

Research article

Homoenergetic solutions of the Boltzmann equation: the case of simple-shear deformations†

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† **This contribution is part of the Special Issue:** Fluid instabilities, waves and non-equilibrium dynamics of interacting particles
Guest Editors: Roberta Bianchini; Chiara Saffirio
Link: www.aimspress.com/mine/article/5859/special-articles

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Abstract: In these notes we review some recent results on the homoenergetic solutions for the Boltzmann equation obtained in [4, 20–22]. These solutions are a particular class of non-equilibrium solutions of the Boltzmann equation which are useful to describe the dynamics of Boltzmann gases under shear, expansion or compression. Therefore, they do not behave asymptotically for long times as Maxwellian distributions, at least for all the choices of the collision kernels, and their behavior strongly depends on the homogeneity of the collision kernel and on the particular form of the hyperbolic terms which describe the deformation taking plance in the gas. We consider here the case of simple shear deformation and present different possible long-time asymptotics of these solutions. We discuss the current knowledge about the long-time behaviour of the homoenergetic solutions as well as some conjectures and open problems.

Keywords: Boltzmann equation; homoenergetic solutions; simple shear deformations; non-equilibrium; self-similar profiles; long-time asymptotics

1. Introduction

The classical Boltzmann equation reads as

$$\partial_t f + v\partial_x f = Qf(v) \quad , \quad f = f(t,x,v)$$

$$Qf(v) = \int_{\mathbb{R}^3} dv_* \int_{S^2} d\omega B(n\cdot\omega, |v-v_*|)\left[f'f'_* - f_*f\right], \tag{1.1}$$

where S^2 is the unit sphere in $\mathbb{R}^3$ and $n = n(v, v_*) = \frac{(v-v_*)}{|v-v_*|}$. Here (v, v_*) is a pair of velocities in incoming collision configuration (see Figure 1) and (v', v'_*) is the corresponding pair of outgoing velocities defined by the collision rule

$$v' = v + ((v_* - v)\cdot\omega)\,\omega, \tag{1.2}$$

$$v'_* = v_* - ((v_* - v)\cdot\omega)\,\omega. \tag{1.3}$$

The unit vector $\omega = \omega(v, V)$ bisects the angle between the incoming relative velocity $V = v_* - v$ and the outgoing relative velocity $V' = v'_* - v'$ as specified in Figure 1. The collision kernel $B(n\cdot\omega, |v-v_*|)$ is proportional to the cross section for the scattering problem associated to the collision between two particles. We use the conventional notation in kinetic theory, $f = f(t,x,v)$, $f_* = f(t,x,v_*)$, $f' = f(t,x,v')$, $f'_* = f(t,x,v'_*)$.

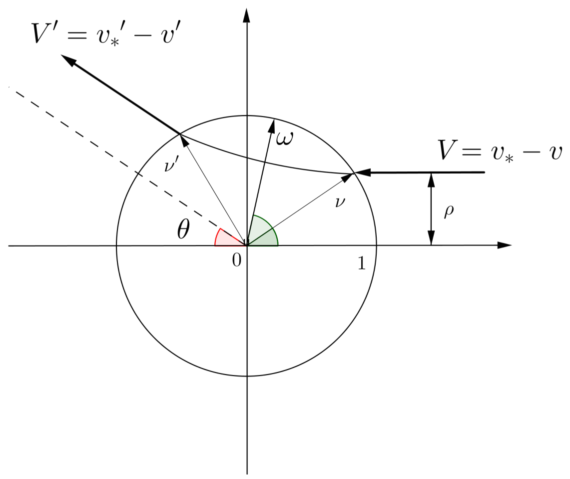

Figure 1. The two-body scattering.

We will assume that the kernel B is homogeneous with respect to the variable $|v - v_*|$ and we will denote its homogeneity by γ, i.e.,

$$B\left(\frac{(v-v_*)\cdot\omega}{|v-v_*|}, \lambda|v-v_*|\right) = \lambda^\gamma B\left(\frac{(v-v_*)\cdot\omega}{|v-v_*|}, |v-v_*|\right), \quad \lambda > 0. \tag{1.4}$$

The homogeneity γ is related to the properties of the interaction potential between particles. We recall that in the standard literature in kinetic theory (see [31]), interaction potentials with the form $V(x) = \frac{1}{|x|^{\nu-1}}$ have homogeneity $\gamma = \frac{\nu-5}{\nu-1}$ for the kernel B.

Given $f(t,x,v)$ we can compute the density ρ, the average velocity V and the internal energy ε at each point x and time t by means of

$$\rho(t,x) = \int_{\mathbb{R}^3} f(t,x,v)\,dv, \quad \rho(t,x)\,V(t,x) = \int_{\mathbb{R}^3} f(t,x,v)\,v\,dv. \tag{1.5}$$

The internal energy $\varepsilon(t,x)$ (or temperature) is given by

$$\rho(t,x)\,\varepsilon(t,x) = \int_{\mathbb{R}^3} f(t,x,v)\,(v - V(t,x))^2\,dv.$$

Ansatz: We look for solutions of the form

$$f(t,x,v) = g(t,w) \quad \text{with } w = v - \xi(t,x). \tag{1.6}$$

Under mild conditions of smoothness, this ansatz is found to reduce the Boltzmann equation if and only if $\xi(t,x) = A(I+tA)^{-1}x$. Then, if f is a solution of the Boltzmann equation (1.1) of the form (1.6) ("equidispersive solution"), the function g satisfies

$$\partial_t g - L(t)\, w \cdot \partial_w g = Qg(w) \tag{1.7}$$

where the collision operator Q is defined as in (1.1) and $L(t) = A(I+tA)^{-1}$. These solutions are called *homoenergetic solutions* and were introduced by Galkin [14] and Truesdell [30] and later considered both in the physical and mathematical literature. Most of the references in the physical literature focus on the evolution of the moments for the homoenergetic solutions, assuming that they exist, providing a large amount of information about quantities like the typical deviation of the velocity and similar quantities. For instance, we refer to [14–16, 28, 29], as well as to the books [18, 30]. On the mathematical side, the well-posedness of solutions to (1.7) in an L^1 setting has been considered for a particular choice of the deformation matrix $L(t)$ in [6, 7] or, in the class of Radon measures, in [20]. Invariance properties of solutions to the Boltzmann equation with respect to Galilean transformations have been considered in [3] in two dimensions.

Hence, homoenergetic solutions of (1.1) are solutions of the Boltzmann equation having the form $f(t,x,v) = g(t,w)$ with $w = v - \xi(t,x)$. The characteristic feature of these solutions is the fact that the dispersion of velocities is the same at any point $x \in \mathbb{R}^3$ for any given time. However, the mean value of the velocity depends on the position and changes also with time.

Notice that, under suitable integrability conditions, every solution of (1.1) with the form (1.6) yields only time-dependent internal energy and density

$$\varepsilon(t,x) = \varepsilon(t) \quad , \quad \rho(t,x) = \rho(t). \tag{1.8}$$

However, we have $V(t,x) = \xi(t,x)$ and therefore the average velocity depends also on the position.

A direct computation shows that in order to have solutions of (1.1) with the form (1.6) for a sufficiently large class of initial data we must have

$$\frac{\partial \xi_k}{\partial x_j} \text{ independent on } x \text{ and } \partial_t \xi + \xi \cdot \nabla \xi = 0. \tag{1.9}$$

The first condition implies that ξ is an affine function on x. However, we will restrict attention in these notes to the case in which ξ is a linear function of x, for simplicity, whence

$$\xi(t,x) = L(t)\, x, \tag{1.10}$$

where $L(t) \in M_{3\times 3}(\mathbb{R})$ is a 3×3 real matrix. The second condition in (1.9) then implies that

$$\frac{dL(t)}{dt} + (L(t))^2 = 0, \quad L(0) = A, \tag{1.11}$$

where we have added an initial condition.

Classical ODE theory shows that there is a unique continuous solution of (1.11),

$$L(t) = (I+tA)^{-1} A = A(I+tA)^{-1}, \tag{1.12}$$

defined on a maximal interval of existence $[0,a)$. On the interval $[0,a)$, $\det(I+tA) > 0$.

Remark 1.1 (Connection to molecular dynamics)**.** *The class of solutions under consideration (i.e., solutions of the form* (1.6)*) is motivated by an invariant manifold of solutions of the equations of classical molecular dynamics with certain symmetry properties (see for instance [9, 10, 19]). The molecular dynamic simulation method can be rephrased as an invariant manifold of the equations of molecular dynamics. The existence result we will present shows that this manifold is inherited faithfully by the Boltzmann equation.*

We recall that there exists a well developed theory of well posedness and a description of the long time behaviour for the solutions of (1.1) which are spatially homogeneous, i.e., when $f = f(v,t)$. In that case it can be proved, for a large class of collision kernels $B(v - v_*, \omega)$, that the solutions relax towards equilibrium, namely they approach asymptotically to a Maxwellian distribution

$$M(v) = \frac{\rho}{(2\pi T)^{\frac{3}{2}}} \exp\left(-\frac{|v-V|^2}{2T}\right) \ , \ \rho > 0,\ T > 0,\ V \in \mathbb{R}^3.$$

Maxwellian distributions describe particle distributions of gases in equilibrium situations. However, they cannot be expected to describe velocities in open systems which exchange matter, energy or momentum with the exterior. The mathematical theory of Boltzmann equation in open systems is much less understood than in the case of closed systems. This particular class of solutions of the Boltzmann equation we are considering here allows to provide some insight in the dynamics of open systems.

We also remark that, differently from the homogeneous Boltzmann equation, there is no H-Theorem for solutions to (1.7) since it is not possible to obtain a detailed balance condition associated to (1.7). This could be expected, because homoenergetic solutions to (1.1) can be thought as a model for dilute gases subject to boundary conditions at infinity yielding stress, compression or expansion. On physical grounds it is not reasonable to expect such open system to thermalize to a stationary state for long times.

The asymptotic behavior for long times of the solutions to (1.7) has been studied by the authors in a series of papers ([4, 20–22]). Our goal in this contribution is to present a summary of the possible long time asymptotics ($t \to \infty$) of the homoenergetic solutions of the Boltzmann equation, namely solutions to (1.7). More precisely, we discuss the problem of existence and uniqueness of a large class of homoenergetic solutions and we study their long time asymptotics. Their behavior strongly depends on the homogeneity of the collision kernel B and on the particular form of the hyperbolic terms, namely $L(t)\, w\cdot\partial_w g$. Indeed, we have different possible long time behaviors which are determined by the relative size for large times of the terms $L(t)\, w \cdot \partial_w g(w,t)$ and $Q[g](w,t)$. More precisely, one of the following situations takes place:

$$\begin{aligned} L(t)\, w \cdot \partial_w g(w,t) &\ll Q[g](w,t)\,, \\ L(t)\, w \cdot \partial_w g(w,t) &\simeq Q[g](w,t)\,, \\ L(t)\, w \cdot \partial_w g(w,t) &\gg Q[g](w,t)\,. \end{aligned}$$

In the first situation the collisions are the most relevant effect. The long time asymptotics of the solutions is then described approximately by Maxwellian distributions for which the temperature changes slowly in time as $t \to \infty$. The temperature can increase or decrease, depending on the specific form of the matrix $L(t)$. The second case is particularly interesting because there is a balance between collisions and deformations, the hyperbolic part of the equation and the collision term are of the same order of magnitude as $t \to \infty$. Specifically, in this case we will prove the existence of self-similar profiles for Maxwell molecules. These self-similar solutions are different from the Maxwellian distributions. Indeed, they reflect a nonequilibrium regime due to the balance between the hyperbolic part of the equation (which reflects effects like shear, dilatation) and the collision term. Finally, in the last situation the dominant terms are the hyperbolic ones and it will turn out that the particle distributions will be non Maxwellians and non self-similar.

1.1. Characterization of homoenergetic solutions

We will use the following norm in $M_{3\times 3}(\mathbb{R})$:

$$\|M\| = \max_{i,j} \left\| m_{i,j} \right\| \quad \text{with } M = \left(m_{i,j} \right)_{i,j=1,2,3}. \tag{1.13}$$

Homoenergetic flows defined for arbitrary large times can be characterized by a matrix $L(t)$ which describes the deformation taking place in the gas. We describe here the long time asymptotics of the function $\xi(t,x) = L(t)x = (I+tA)^{-1} Ax$ (cf. (1.10) and (1.12)). There are interesting choices of $A \in M_{3\times 3}(\mathbb{R})$ for which $L(t)$ blows up in finite time, but we will restrict the attention to the case in which the matrix $\det(I+tA) > 0$ for all $t \geq 0$.

The key idea is that one can study the form of the matrix $L(t)$ as $t \to \infty$ in a particular orthonormal basis using the Jordan normal form for real 3×3 matrices. The full classification of all the possible long-time asymptotics of the matrix $L(t)$ that one can obtain in the limit $t \to \infty$ has been obtained in [20] (see Theorem 3.1).

In this work we will restrict to the case of "simple shear deformation" (see Figure 2) since it represents a paradigmatic case.

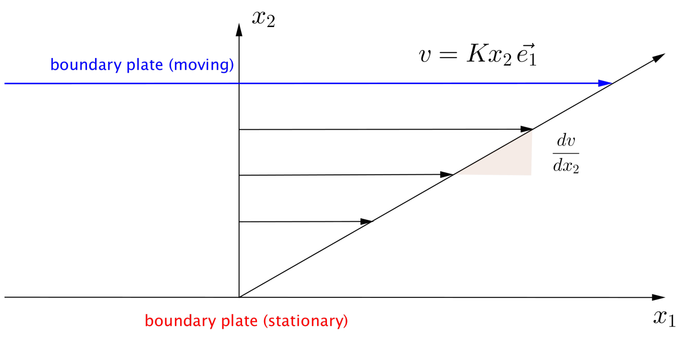

Figure 2. Simple shear deformation.

In this case, the matrix $L(t) = L$ and reads

$$L(t) = \begin{pmatrix} 0 & K & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K \neq 0. \tag{1.14}$$

With this choice of $L(t)$ (cf. (1.14)), (1.7) becomes

$$\partial_t g - K w_2 \partial_{w_1} g = Qg(w). \tag{1.15}$$

As we already discussed the properties of the solutions of (1.7) for large times t depend greatly on the homogeneity γ of the kernel yielding the cross section of the collision operator Qg. Some insights on the role of γ can be obtained in the following way. In systems subject to shear deformations (cf. (1.15)) the average velocity of the particles tends to increase (due to heating effects). Therefore, the role of the collisions becomes more relevant for larger times if the homogeneity of the kernel γ is positive and less relevant if γ is negative. On the contrary, expanding gases (under dilatations) cool down and thus the average velocity of the particles in the gas decreases. Then, for $\gamma < 0$ the role of collisions becomes more important and viceversa for $\gamma > 0$. This behaviour of the collision term must be weighted against the behaviour of the hyperbolic term $L(t)\, w \cdot \partial_w g = K w_2 \partial_{w_1} g$ in order to determine which is the dominant effect for long times.

1.2. Behavior of the density and internal energy for homoenergetic solutions

We recall that the equation describing homoenergetic flows is:

$$\partial_t g - L(t)\, w \cdot \partial_w g = Qg(w) \ . \tag{1.16}$$

We want to construct solutions of (1.16) with $L(t)$ as in (1.14). The solutions in which we are interested have some suitable scaling properties, and two quantities which play a crucial role determining how are these rescalings are the density $\rho(t)$ and the internal energy $\varepsilon(t)$. These are given by (cf. (1.5)):

$$\rho(t) = \int_{\mathbb{R}^3} g(t, dw) \quad , \quad \varepsilon(t) = \int_{\mathbb{R}^3} |w|^2 g(t, dw) \tag{1.17}$$

which will be assumed to be finite for each given t in all the solutions considered in these notes. Integrating (1.16) and using the conservation of mass property of the collision kernel, we obtain:

$$\partial_t \rho(t) + \mathrm{Tr}(L(t))\, \rho(t) = 0 \tag{1.18}$$

whence:

$$\rho(t) = \rho(0) \exp\left(- \int_0^t \mathrm{Tr}(L(s))\, ds \right). \tag{1.19}$$

Nevertheless it is not possible to derive a similarly simple equation for the internal energy $\varepsilon(t)$, because the term $-L(t)\, w \cdot \partial_w g$ on the left-hand side of (1.16) yields in general terms which cannot be written neither in terms of $\rho(t)$, $\varepsilon(t)$. Actually these terms have an interesting physical meaning, because they produce heating or cooling of the system and therefore they contribute to the change of $\varepsilon(t)$. To obtain the precise form of these terms we need to study the detailed form of the solutions of (1.16). The rate of growth or decay of $\varepsilon(t)$ would then typically appear as an eigenvalue of the corresponding PDE problem.

Notice that in the case in which the matrix $L(t)$ describes a simple shear deformation, namely $L(t)$ is as in (1.14), (1.19) becomes

$$\rho(t) = \rho(0)$$

since $\mathrm{Tr}(L(t)) = 0$. Hence the mass density ρ is constant in time.

2. Well posedness theory for homoenergetic flows

In this section we discuss the well-posedness theory for the Boltzmann equation for homoenergetic flows which shares some similarities with the corresponding theory for the spatially homogeneous Boltzmann equation. We will prove that homoenergetic flows with the form (1.6), (1.10), (1.11) exist for a large class of initial data $g_0(w)$ globally in time. This question has been originally considered by C. Cercignani in [6, 7], who developed an L^1 theory (cf. [5]) restricting to the case of simple shear (cf. (1.14)). More recently, in [20] we developed a well-posedness theory for homoenergetic flows for the Boltzmann equation in the class of Radon measures, a more suitable functional framework for the type of arguments needed. This theory, that we will briefly recall below, allows to study more general classes of homoenergetic flows for all the possible deformation matrices described in [20] (see Theorem 3.1).

In order to study the long time behavior of the solutions in the next sections, we will need to consider Boltzmann equations for homoenergetic flows with additional terms (due to possible rescalings). Therefore, the well-posedness results will be formulated for a family of Boltzmann equations with the degree of generality that we will require. More precisely, we are interestested in the following class of equations

$$\partial_t G - \partial_w \cdot ([\mathcal{L}(t)w]\, G) = QG(w) \tag{2.1}$$

$$QG(w) = \int_{\mathbb{R}^3} dw_* \int_{S^2} d\omega B(n\cdot\omega, |w-w_*|)\left[G'G'_* - G_*G\right] \ , \tag{2.2}$$

$$G(0,w) = G_0(w)\,, \tag{2.3}$$

where

$$\mathcal{L}(\cdot) \in C^1([0,\infty); M_{3\times 3}(\mathbb{R})) \ \ , \ \ \|\mathcal{L}(t)\| \leq c_1 + c_2 t, \text{ with } c_1 > 0, \ c_2 > 0 \tag{2.4}$$

with the matrix norm $\|\cdot\|$ defined as in (1.13). We assume that the function

$$\Lambda(w, w_*) = \int_{S^2} B(n\cdot\omega, |w-w_*|)\, d\omega \ \ , \ \ n = \frac{(w-w_*)}{|w-w_*|} \tag{2.5}$$

satisfies

$$\Lambda \text{ is continuous and } 0 \leq \Lambda(w, w_*) \leq c_3 \ \text{ with } c_3 > 0. \tag{2.6}$$

We note that this assumption allows to avoid the technicalities related to the singular behavior of the collision kernel in the angular variable.

Remark 2.1 (The case of simple shear)**.** *As we anticipated in Section 1, in this work, we will focus on homoenergetic flows in the case of simple shear deformations. More precisely, this corresponds to a particular choice of the operator* $\mathcal{L}(\cdot) \in C^1([0,\infty); M_{3\times 3}(\mathbb{R}))$*, namely* $\mathcal{L}(t) = L + \alpha I$ *where* $L \in M_{3\times 3}(\mathbb{R})$ *is the simple shear deformation matrix given by* (1.14) *and* $\alpha \in \mathbb{R}$*. With this choice of* $\mathcal{L}(t)$ *now* (2.1) *reads as*

$$\partial_t G - \alpha\partial_w \cdot (wG) - \partial_w \cdot (Lw\, G) = QG(w) \tag{2.7}$$

$$QG(w) = \int_{\mathbb{R}^3} dw_* \int_{S^2} d\omega B(n\cdot\omega, |w-w_*|)\left[G'G'_* - G_*G\right] \ ,$$

$$G(0,w) = G_0(w),$$

where $\partial_w \cdot (LwG) = Kw_2\partial_{w_1}G$. *We further notice that when* $\alpha = 0$ *the equation above reduces to*

$$\partial_t G - Kw_2\partial_{w_1}G = QG(w). \tag{2.8}$$

We now introduce some definitions and notation that we will use in what follows. We denote by $\mathcal{M}_+\left(\mathbb{R}^3_c\right)$ the set of nonnegative Radon measures in $\mathbb{R}^3_c$, namely the compactification of $\mathbb{R}^3$ by means of a single point ∞. This technical issue is needed in order to have convenient compactness properties for some subsets of $\mathcal{M}_+\left(\mathbb{R}^3_c\right)$. The space $C\left([0,\infty) : \mathcal{M}_+\left(\mathbb{R}^3_c\right)\right)$ is defined endowing $\mathcal{M}_+\left(\mathbb{R}^3_c\right)$ with the measure norm

$$\|\mu\|_M = \sup_{\varphi\in C\left(\mathbb{R}^3_c\right):\|\varphi\|_\infty=1} |\mu(\varphi)| = \int_{\mathbb{R}^3_c} |\mu|(dw). \tag{2.9}$$

Given $G \in C\left([0,\infty) : \mathcal{M}_+\left(\mathbb{R}^3_c\right)\right)$ we define

$$\mathbb{A}[G](t,w) = \int_{\mathbb{R}^3} dw_* \int_{S^2} d\omega B(n\cdot\omega, |w-w_*|)\, G_*(t,\cdot). \tag{2.10}$$

Given $h_0 \in \mathcal{M}_+\left(\mathbb{R}^3_c\right)$ we will denote as $S_G(t;t_0)$, $t \geq t_0 \geq 0$, the operator $S_G(t;t_0) : \mathcal{M}_+\left(\mathbb{R}^3_c\right) \to \mathcal{M}_+\left(\mathbb{R}^3_c\right)$ defined by means of

$$\partial_t h - \partial_w \cdot ([\mathcal{L}(t)w]\, h) = -\mathbb{A}[G](t,w)\, h \quad , \quad h(t_0,\cdot) = h_0 \tag{2.11}$$

$$h(t,w) = S_G(t;t_0)\, h_0. \tag{2.12}$$

The operator $S_G(t;t_0)$ is well defined, since (2.11) can be solved explicitly using the method of characteristics taking into account (2.4). The solution is given

$$S_G(t;s)\, h_0(w) = \exp\left(-\int_s^t \mathbb{A}[G](\xi, U(t,\xi)w)\, d\xi\right) \exp\left(\int_s^t \operatorname{tr}(\mathcal{L}(\xi))\, d\xi\right) h_0(U(t;s)w)$$

where:

$$\frac{\partial[U(s;t)w]}{\partial t} = -\mathcal{L}(t)U(s;t)w \quad , \quad U(s;s)w = w \in \mathbb{R}^3.$$

A relevant point is that $\mathbb{A}[G](w) \geq 0$ and as a consequence no divergences arise from large values of $|w|$. We will use the following concept of solutions of (2.1)–(2.3).

Definition 2.2. *We will say that* $G \in C\left([0,\infty] : \mathcal{M}_+\left(\mathbb{R}^3_c\right)\right)$ *is a mild solution of (2.1)–(2.3) with initial value* $G(0,\cdot) = G_0 \in \mathcal{M}_+\left(\mathbb{R}^3_c\right)$ *if* G *satisfies the following integral equation:*

$$G(t,w) = S_G(t;0)\, G_0(w) + \int_0^t S_G(t;s)\, Q^{(+)}G(s,w)\, ds, \tag{2.13}$$

where the operator $S_G(t;s)$ *is as in (2.12) and*

$$Q^{(+)}G(w) = \int_{\mathbb{R}^3} dw_* \int_{S^2} d\omega B(n\cdot\omega, |w-w_*|)\, G'G'_*, \tag{2.14}$$

$$Q^{(-)}G(w) = G\int_{\mathbb{R}^3} dw_* \int_{S^2} d\omega B(n\cdot\omega, |w-w_*|)\, G_* = \mathbb{A}[G]\, G. \tag{2.15}$$

Notice that (2.13) must be understood as an identity in the sense of measure, i.e., acting over an arbitrary test function $\varphi \in C\left(\mathbb{R}_c^3\right)$. Moreover, all the operators appearing in (2.13) are well defined for $G \in C\left([0,\infty) : \mathcal{M}_+\left(\mathbb{R}_c^3\right)\right)$ and the operator $S_G(t;s) : \mathcal{M}_+\left(\mathbb{R}_c^3\right) \to \mathcal{M}_+\left(\mathbb{R}_c^3\right)$ is bounded for $0 \le s \le t \le T < \infty$.

We further introduce the concept of weak solution of (2.1)–(2.3). The only difference between solutions in the sense of measures and the *weak solutions* defined below is that in this second case we write the collision kernel in a symmetrized form which will be convenient in forthcoming computations.

Definition 2.3. *We will say that $G \in C\left([0,\infty) : \mathcal{M}_+\left(\mathbb{R}_c^3\right)\right)$ is a weak solution of (2.1)–(2.3) with initial value $G(0,\cdot) = G_0 \in \mathcal{M}_+\left(\mathbb{R}_c^3\right)$ if for any $T \in (0,\infty)$ and any test function $\varphi \in C\left([0,T) : C^1\left(\mathbb{R}_c^3\right)\right)$ the following identity holds*

$$\begin{aligned}
&\int_{\mathbb{R}^3} \varphi(T,w)\, G(T,dw) - \int_{\mathbb{R}^3} \varphi(0,w)\, G_0(dw) \qquad (2.16)\\
&= \int_0^T dt \int_{\mathbb{R}^3} \partial_t \varphi G(t,dw) - \int_0^T dt \int_{\mathbb{R}^3} \left[\mathcal{L}(t) w \cdot \partial_w \varphi\right] G(t,dw)\\
&\quad + \frac{1}{2}\int_0^T dt \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} d\omega G(t,dw)\, G(t,dw_*)\, B(n\cdot\omega, |w-w_*|) \Big[\varphi(t,w') + \varphi(t,w'_*)\\
&\quad - \varphi(t,w) - \varphi(t,w_*)\Big].
\end{aligned}$$

We will use the following norms:

$$\|G\|_{1,s} = \int_{\mathbb{R}^3} (1+|w|^s)\, G(dw) \quad \text{for } G \in \mathcal{M}_+\left(\mathbb{R}_c^3\right), \quad s > 0. \tag{2.17}$$

We can now focus on the global well posedness for (2.1)–(2.3). We will consider the case of Maxwell molecules, namely we consider collision kernels B in (2.2) with homogeneity parameter $\gamma = 0$, i.e., $B(n\cdot\omega, |w-w_*|) = B(n\cdot\omega)$. Moreover, we will restrict to kernels satisfying the boundedness condition (2.6) since the theory is simpler.

We now state the result concerning the global existence and uniqueness of mild solutions (cf. Definition 2.2) and, as consequence, the existence of weak solutions (cf. Definition 2.4). We will not include the proofs of these results which can be found in [20], Section 4.1.

Theorem 2.4 (Existence of mild solutions)**.** *Suppose that $G_0 \in \mathcal{M}_+\left(\mathbb{R}^3\right)$ satisfies*

$$\int_{\mathbb{R}^3} G_0(dw) < \infty.$$

Then, there exists a unique mild solution $G \in C\left([0,\infty) : \mathcal{M}_+\left(\mathbb{R}_c^3\right)\right)$ in the sense of Definition 2.2 to the initial value problem (2.1)–(2.3) with $\mathcal{L}$ and B satisfying (2.4), (2.6). Moreover, the problem (2.1)–(2.3) is satisfied in the sense of measures.

We just observe that the proof of Theorem 2.4 is technical but relies on standard techniques, namely a Banach fixed point argument for the evolution operator

$$\mathcal{T}_T[G](t,w) = S_G(t;0)\, G_0(w) + \int_0^t S_G(t;s)\, Q^{(+)} G(s,w)\, ds, \quad 0 \le t \le T$$

and an extension argument.

Theorem 2.5 (Existence of weak solutions)**.** *Suppose that $G \in C\left([0,\infty) : \mathcal{M}_+\left(\mathbb{R}^3_c\right)\right)$ is a mild solution of (2.1)–(2.3) with $\mathcal{L}$ and B satisfying (2.4), (2.6) and initial value $G(0,\cdot) = G_0 \in \mathcal{M}_+\left(\mathbb{R}^3_c\right)$. Then G is also a weak solution of (2.1)–(2.3) in the sense of Definition 2.4. Suppose that in addition G_0 satisfies*

$$\|G_0\|_{1,s} < \infty \tag{2.18}$$

for some $s > 0$. Then the mild solution of (2.1)–(2.3) satisfies

$$\sup_{0\le t\le T} \|G(t,\cdot)\|_{1,s} < C(T, \|G_0\|_M)\, \|G_0\|_{1,s} < \infty \tag{2.19}$$

for any $T \in (0,\infty)$. Moreover, if $s > 2$ the identity (2.16) is satisfied for any test function $\varphi \in C\left([0,T] : C^1\left(\mathbb{R}^3_c\right)\right)$ such that $|\varphi(w)| + |w|\,|\nabla_v \varphi(w)| \le C_0\left(1 + |w|^2\right)$.

We remark that when $\mathcal{L}(t)$ has the form $\mathcal{L}(t) = L + \alpha I$ with L as in (1.14) and $\alpha = 0$ the well-posedness results presented above guarantee global existence of solutions to (2.8), i.e., homoenergetic solutions of the Boltzmann equation in the case of simple shear, for a large class of initial data. Moreover, we notice that when $\mathcal{L}(t) = L + \alpha I$ with L as in (1.14) and $\alpha \neq 0$ Theorem 2.4 will be used to prove existence of self-similar solutions of (2.7), as we will see in the next section.

3. Self-similar profiles for simple shear deformations and Maxwell molecules interactions

We consider the Boltzmann equation for homoenergetic flows in the case of simple shear

$$\partial_t g - K w_2 \partial_{w_1} g = Q[g] \tag{3.1}$$

$$Qg(w) = \int_{\mathbb{R}^3} dw_* \int_{S^2} d\omega B(n\cdot\omega, |w - w_*|)\left[g' g'_* - g_* g\right]. \tag{3.2}$$

The self-similar solutions that we will construct and study in this section are characterized by a balance between the hyperbolic (or drift) term $-L(t)\,w \cdot \partial_w g = -K w_2 \partial_{w_1} g$ and the collision term $Qg(w)$. Such a balance is only possible for specific choices of the homogeneity γ of the collision kernel B. Actually this balance occur for $\gamma = 0$, i.e., for Maxwell molecules.

We first recall that, in the case of simple shear, the mass is conserved, namely

$$\rho(t) = \rho(0) = 1 \tag{3.3}$$

where, without loss of generality, we can use the normalization $\rho(0) = 1$ rescaling the time unit. Using (1.1), the definition of $\rho(t)$ in (1.17) and (3.3) we obtain that the physical dimensions of the three terms in (3.1) are

$$\frac{[g]}{[t]},\ [g],\ [w]^\gamma [g]. \tag{3.4}$$

We now observe that if $|w|$ changes in time, we can have a balance of second and third terms in (3.4) only if $\gamma = 0$. On the other hand (3.4) indicates that we cannot obtain a balance between the first two terms of this equation with power law behaviors for $[w]$, and the only way to obtain such a balance is to assume that $[w]$ scales like an exponential of t. Hence, this argument suggests that for Maxwell molecules, i.e., when $\gamma = 0$, we consider solutions with the form

$$g(w,t) = e^{-3\beta t} G(\xi) \quad , \quad \xi = \frac{w}{e^{\beta t}}. \tag{3.5}$$

Here $\beta \in \mathbb{R}$ characterizes the behavior of the internal energy and it is an eigenvalue to be determined. The factor $e^{-3\beta t}$ in (3.5) is needed in order to have the density conservation condition (3.3). If we then plug (3.5) into (3.1) we obtain

$$-\beta\partial_{\xi}(\xi G) - K\partial_{\xi_1}(\xi_2 G) = Q[G] \tag{3.6}$$

and (3.3) implies the normalization condition

$$\int_{\mathbb{R}^3} G(\xi)\, d\xi = 1. \tag{3.7}$$

Notice that (3.6) is the stationary version of (2.8) considered in Section 2.

3.1. Existence of self-similar profiles

We now state the existence of a self-similar profile under a suitable smallness assumption on the shear parameter. We have the following result (see [20], Section 4.3).

Theorem 3.1 (Existence of a self-similar profile)**.** *Suppose that $B(n\cdot\omega, |w - w_*|) = B(n\cdot\omega)$ (i.e., $\gamma = 0$) and that $b > 0$ is defined as*

$$b = 3\pi \int_{-1}^{1} B(x)\, x^2 \left(1 - x^2\right) dx > 0. \tag{3.8}$$

There exists $k_0 > 0$ small such that for any $\zeta \geq 0$ and $K \in \mathbb{R}$ such that $\frac{K}{b} \leq k_0$ there exists $\beta \in \mathbb{R}$ and $G \in \mathcal{M}_+\left(\mathbb{R}^3_c\right)$ which solves (3.6) in the sense of measures and satisfies the normalization condition (3.7) as well as:

$$\int_{\mathbb{R}^3} w_j G(dw) = 0, \ \int_{\mathbb{R}^3} |w|^2 G(dw) = \zeta\ . \tag{3.9}$$

We notice that the assumption $\int_{\mathbb{R}^3} w_j G(dw) = 0$ is not restrictive and it can be assumed without loss of generality using a simple change of variables.

Moreover, we remark that if $\zeta = 0$ the only solution of (3.6) satisfying (3.9) is $G = \delta_{w=0}$. Thus, the main content of Theorem 3.1 is the existence of solutions of (2.7) with arbitrary values of the velocity dispersion and hence arbitrary values of the temperature.

Remark 3.2 (Physical meaning of the smallness condition)**.** *Solutions of (3.1) with the form (3.5) exist under a smallness condition assumption on the shear parameter K. More precisely, K should be sufficiently small compared with the parameter b which measures the strength of the collision term. The quantity $\frac{K}{b}$ is a nondimensional parameter with a clear physical meaning. The parameter K is, up to a multiplicative constant, the inverse of the time scale τ_{shear} in which the effect of the shear deformes a sphere into a ellipsoid for which the largest semiaxes has double length than the shortest one. On the other hand b is the inverse of the average time between collisions τ_{coll}. Then $\frac{K}{b} = \frac{\tau_{coll}}{\tau_{shear}}$ and hence the smallness condition in Theorem 3.1 means $\frac{K}{b} = \frac{\tau_{coll}}{\tau_{shear}}$ small.*

Remark 3.3 (The Thermodynamic entropy)**.** *We have already observed that, for homoenergetic solutions, the particle density $\rho = \rho(t)$ and internal energy $\varepsilon = \varepsilon(t)$ are constant in space. We now focus on another important thermodynamic quantity, namely the entropy. For homoenergetic solutions also the entropy density for particle (minus the $H-$function) is space independent and given by $\frac{s(t)}{\rho(t)} = -\frac{1}{\rho(t)} \int_{\mathbb{R}^3} g(t,w) \log(g(t,w))\, dw$. In the case of self-similar solutions* (3.5) *the entropy density for particle reads*

$$\frac{S}{\rho} = \log\left(\frac{e^{\frac{3}{2}}}{\rho}\right) - C_G \tag{3.10}$$

$$\text{where} \quad C_G = \frac{\int G(\xi) \log(G(\xi))\, d\xi}{\int G(\xi) d\xi} + \log\left[\frac{\left(\int |\xi|^2 G(\xi) d\xi\right)^{\frac{3}{2}}}{\left(\int G(\xi) d\xi\right)^{\frac{5}{2}}}\right]. \tag{3.11}$$

In these far-from-equilibrium solutions the temperature and density can be rapidly changing, and the entropy rapidly increasing. Nevertheless, the relation between entropy, density and energy expressed by (3.10) *is the same as for the equilibrium Maxwellian distribution, except for one small but interesting point: the value of the constant C_G (cf.* (3.11)*) is strictly smaller than that of the Maxwellian distribution.*

3.1.1. Proof of Theorem 3.1: strategy

In order to prove Theorem 3.1 we need to prove the existence of nontrivial steady states for the evolution equation

$$\partial_t G - \beta \partial_\xi \cdot (\xi G) - \partial_{\xi_1}(K \xi_2 G) = Q(G,G)(\xi,t) \qquad t > 0, \tag{3.12}$$

$$G(\xi,0) = G_0(\xi) \tag{3.13}$$

which is the evolution equation for time-dependent solutions $g(w,t) = e^{-3\beta t} G(\xi,t)$ with $\xi = \frac{w}{e^{\beta t}}$. As we already discussed, Eq (3.12) is a particular case of Eq (2.1) with $\mathcal{L}(t) = L + \alpha I$ and L as in (1.14), namely (2.7).

We first show that the evolution problem (3.12) has a unique mild solution $G(t,\cdot)$. This is guaranteed by the results obtained in Section 2 (cf. Theorem 2.4). We then prove that the nonlinear evolution defined by means of Theorems 2.4, 2.5 is continuous in time in the weak topology of measures. More precisely, we denote as $\mathcal{S}_\beta(t) G_0 = G(t,\cdot)$ the unique mild solution of (3.12) given by Theorem 2.4 with an initial datum $G_0 \in \mathcal{M}_+\left(\mathbb{R}^3_c\right)$ such that $\int_{\mathbb{R}^3} G_0(dw) = 1$, $\int_{\mathbb{R}^3} |w|^s G_0(dw) < \infty$ for some $s > 2$. We have that the family of operators $\mathcal{S}_\beta(t)$ define an evolution semigroup. The mapping $\mathcal{S}_\beta : [0,\infty) \times \mathcal{M}_+\left(\mathbb{R}^3_c\right) \to \mathcal{M}_+\left(\mathbb{R}^3_c\right)$ is uniformly continuous in the weak topology of $\mathcal{M}_+\left(\mathbb{R}^3_c\right)$ on any set of the form $[0,T] \times \mathcal{M}_{s,+}\left(\mathbb{R}^3_c\right)$, where $T \in (0,\infty)$ and $\mathcal{M}_{s,+}\left(\mathbb{R}^3_c\right)$ is the subset of measures $G_0 \in \mathcal{M}_+\left(\mathbb{R}^3_c\right)$ with moments of order s bounded. We do not prove here this result for the sake of shortness. We refer to [20], Lemma 4.19 for the proof.

Moreover, we prove that the evolution semigroup has an invariant set $\mathcal{U}$ which is convex, compact in the $*$- weak topology of measures. Thus, we can apply Schauder fixed point Theorem which provides existence of the desired self-similar solutions.

We start proving that the evolution semigroup $\mathcal{S}_\beta(t)$ has an invariant set.

3.1.2. Moment equations

A crucial fact that we will use is that for Maxwell molecules (i.e., collision kernels with homogeneity $\gamma = 0$) the tensor of second moments

$$M_{j,k} = \int_{\mathbb{R}^3} w_j w_k G(t, dw)$$

satisfies a linear system of equations if G is a mild solution of (2.1)–(2.3). In order to compute the evolution equations for $M_{j,k}$ we will use (2.16) with the test functions $\varphi = w_j w_k$. The resulting right-hand side can then be computed using suitable tensorial properties of the Boltzmann equation acting over quadratic functions.

We recall that we are assuming the collision kernel B in (2.1) to be such that

$$B(n \cdot \omega, |w - w_*|) = B(n \cdot \omega) \tag{3.14}$$

which corresponds to the case of Maxwell molecules.

We now compute the evolution equation for the moments $\left(M_{j,k}\right)_{j,k=1,2,3}$. In what follows we use the convention that the repeated indexes are summed.

Proposition 3.4. *Suppose that $G_0 \in \mathcal{M}_+\left(\mathbb{R}^3_c\right)$ satisfies* (2.7) *and $\int_{\mathbb{R}^3} (1 + |w|^s) G_0(dw) < \infty$ for some $s > 2$ and that Q is as in (2.2) with B given by* (3.14). *Let us assume also that*

$$\int_{\mathbb{R}^3} G_0(dw) = 1 \ \ \text{and} \ \ \int_{\mathbb{R}^3} w\, G_0(dw) = 0\,. \tag{3.15}$$

Suppose that $G \in C\left([0, \infty] : \mathcal{M}_+\left(\mathbb{R}^3_c\right)\right)$ is the unique mild solution of (2.1)–(2.3) in Theorem 2.4. Then the tensor $M = \left(M_{j,k}\right)_{j,k=1,2,3}$ defined by means of $M_{j,k} = \int_{\mathbb{R}^3} w_j w_k G(t, dw)$ is defined for $t \geq 0$ and it satisfies the linear system of ODEs with constant coefficients

$$\frac{dM_{j,k}}{dt} + \mathcal{L}_{j,\ell}(t) M_{k,\ell} + \mathcal{L}_{k,\ell}(t) M_{j,\ell} = -2b\left(M_{j,k} - m\delta_{j,k}\right), \quad j,k = 1,2,3, \quad M_{j,k} = M_{k,j} \tag{3.16}$$

where $b > 0$.

Proof. Due to Theorem 2.5 with $s > 2$ the tensor M is well defined and G is a weak solution of (2.1)–(2.3) in the sense of Definition 2.3. Moreover, Theorem 2.5 implies also that we can take as test functions in (2.16) $\varphi = 1$ and $\varphi = w_j$

$$\int_{\mathbb{R}^3} G(t, dw) = 1 \ \ \text{and} \ \ \int_{\mathbb{R}^3} w\, G(t, dw) = 0. \tag{3.17}$$

Moreover, taking in (2.16) the test functions $\varphi = W_{j,k} = w_j w_k$ we obtain

$$\frac{dM_{j,k}}{dt} + \int_{\mathbb{R}^3} \left[\mathcal{L}(t) w \cdot \partial_w \left(w_j w_k\right)\right] G(t, dw) = K_{j,k} \tag{3.18}$$

where we define $K_{j,k}$ as

$$K_{j,k} = \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} d\omega B(n \cdot \omega) \left[w'_j w'_k + w'_{j,*} w'_{k,*} - w_j w_k - w_{j,*} w_{k,*}\right] G(t, dw)\, G(t, dw_*) \tag{3.19}$$

where we have used that $B(n \cdot \omega, |w - w_*|) = B(n \cdot \omega)$ due to the fact that B is homogeneous of order zero. We now introduce

$$T_{j,k} = \frac{1}{2} \int_{S^2} d\omega B(n \cdot \omega) \left[w'_j w'_k + w'_{j,*} w'_{k,*} - w_j w_k - w_{j,*} w_{k,*} \right]. \tag{3.20}$$

Hence (3.19) becomes

$$K_{j,k} = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} T_{j,k} G(t, dw) G(t, dw_*) \tag{3.21}$$

Exploiting the tensorial properties of the Boltzmann operator acting over quadratic functions, it is possible to prove that

$$T_{j,k} = -b \left[(w - w_*)_j (w - w_*)_k - \frac{|w - w_*|^2}{3} \delta_{j,k} \right] \ , \quad j, k = 1, 2, 3. \tag{3.22}$$

with b as in (3.8), i.e.,

$$b = 3\pi \int_{-1}^{1} B(x) x^2 \left(1 - x^2\right) dx > 0 \quad (x = \cos\theta).$$

We refer to [20], Proposition 4.10, for the proof of identity (3.22).

From (3.21), using (3.22), we then obtain

$$K_{j,k} = -b \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left[(w - w_*)_j (w - w_*)_k - \frac{|w - w_*|^2}{3} \delta_{j,k} \right] G(t, dw) G(t, dw_*) .$$

Expanding the products and using (3.17) as well as the symmetry properties of the integrals, we obtain

$$K_{j,k} = -2b \left(M_{j,k} - m\delta_{j,k} \right) \ , \ m = \frac{1}{3} \operatorname{tr}(M) . \tag{3.23}$$

Notice that the trace of $K = \left(K_{j,k} \right)_{j,k=1,2,3}$ is zero, something that might be seen directly from (3.19) using that $|w'|^2 + \left|w'_*\right|^2 = |w|^2 + |w_*|^2$.

Computing the integral on the left hand side of (3.18) and using (3.23) we obtain (3.16). □

We notice that, with the choice $\mathcal{L}(t) = L + \alpha I$, (3.16) becomes a linear system of equations with constant coefficients for the second moments, namely

$$\frac{dM_{j,k}}{dt} + 2\alpha M_{j,k} + L_{j,\ell} M_{k,\ell} + L_{k,\ell} M_{j,\ell} = -2b \left(M_{j,k} - m\delta_{j,k} \right) \ , \quad j, k = 1, 2, 3, \quad M_{j,k} = M_{k,j}. \tag{3.24}$$

Moreover, using that L is as in (1.14), (3.24) becomes

$$\frac{dM_{j,k}}{dt} + 2\alpha M_{j,k} - K \left[\delta_{1,j} M_{2,k} + \delta_{1,k} M_{2,j} \right] = -2b \left(M_{j,k} - m\delta_{j,k} \right) \ , \quad j, k = 1, 2, 3. \tag{3.25}$$

Therefore, a solution to (3.25) has the form $M_{j,k} = \Gamma_{j,k} e^{2b\lambda t}$. We can formulate an equivalent problem and determine the values of α for which there is a stationary solution of (3.24) with the form $M_{j,k} = \Gamma_{j,k}$. Such values of α solve the eigenvalue problem

$$\frac{\alpha}{b} \Gamma_{j,k} + \frac{1}{2b} \left(L_{j,\ell} \Gamma_{k,\ell} + L_{k,\ell} \Gamma_{j,\ell} \right) = -\left(\Gamma_{j,k} - \Gamma\delta_{j,k} \right) \ , \quad j, \ k = 1, 2, 3, \quad \Gamma_{j,k} = \Gamma_{k,j} \tag{3.26}$$

with

$$\Gamma = \frac{1}{3}\left(\Gamma_{1,1} + \Gamma_{2,2} + \Gamma_{3,3}\right) \tag{3.27}$$

It is possible to prove that there exists $\alpha \in \mathbb{R}$ and a real symmetric, positive-definite matrix $\left(\Gamma_{j,k}\right)_{j,k=1,2,3}$ such that (3.26), (3.27) hold. Moreover, α can be chosen to be the complex number with largest real part for which (3.26), (3.27) holds for a nonzero $\left(\Gamma_{j,k}\right)_{j,k=1,2,3}$. We emphasize that when L is as in (1.14) the eigenvalue problem becomes

$$\left(\frac{\alpha}{b}+1\right)\Gamma_{1,1} + \frac{K}{b}\Gamma_{1,2} = \Gamma \ , \ \ \Gamma = \frac{1}{3}\left(\Gamma_{1,1} + \Gamma_{2,2} + \Gamma_{3,3}\right)$$
$$\left(\frac{\alpha}{b}+1\right)\Gamma_{1,2} + \frac{K}{2b}\Gamma_{2,2} = 0 \ , \ \left(\frac{\alpha}{b}+1\right)\Gamma_{1,3} + \frac{K}{2b}\Gamma_{2,3} = 0$$
$$\left(\frac{\alpha}{b}+1\right)\Gamma_{2,2} = \Gamma \ , \ \left(\frac{\alpha}{b}+1\right)\Gamma_{2,3} = 0 \ , \ \left(\frac{\alpha}{b}+1\right)\Gamma_{3,3} = \Gamma$$

with:

$$\Gamma_{j,k} = \Gamma_{k,j} \ \ , \ \ j,\ k = 1,2,3. \tag{3.28}$$

It is possible to compute explicitly the eigenvalues and to prove that the eigenvalue with largest real part is real and positive. We will denote this eigenvalue as β ($\beta \in \mathbb{R}_+$) and as $\bar{N}_{j,k}$ the corresponding eigenvector. For a detailed discussion we refer to [20], Proposition 5.3.

We also recall a standard result in Kinetic Theory (cf. [8, 31]) known as Povzner Estimates that we write here in a more convenient form (see [20]), suitable for the arguments made later.

Lemma 3.5. *Let $s > 2$. There exists a continuous function $\kappa_s : [0,1] \to \mathbb{R}$ such that $\kappa_s(y) > 0$ if $y \in [0,1)$, $\kappa_s(0) = 0$, and a constant $C_s > 0$ such that, for any $w, w_* \in \mathbb{R}^3$ the following inequality holds*

$$|w'|^s + \left|w_*'\right|^s - |w|^s - |w_*|^s \leq -\kappa_s\left(|n\cdot\omega|\right)\left(|w|^s + |w_*|^s\right) + C_s\left[|w|^{s-1}|w_*| + |w_*|^{s-1}|w|\right]. \tag{3.29}$$

where $n = \frac{(w-w_*)}{|w-w_*|}$.

3.1.3. End of the proof of Theorem 3.1

We now consider the nonlinear evolution $\mathcal{S}_\alpha(t)G_0 = G(t,\cdot)$ defined as the unique mild solution of (3.12) provided by Theorem 2.4. We notice that the evolution semigroup $\mathcal{S}_\alpha(\cdot)$ is continuous in time in the weak topology of measures (see Lemma 4.19 in [20]).

Proposition 3.6. *Let $2 < s < 3$. Suppose that $\int_{\mathbb{R}^3} G_0(dw) = 1$, $\int_{\mathbb{R}^3} |w|^s G_0(dw) < \infty$ and that the following identities hold:*

$$\int_{\mathbb{R}^3} w_j G_0(dw) = 0 \ \ , \ \ j \in \{1,2,3\} \ \ , \ \ \int_{\mathbb{R}^3} w_j w_k G_0(dw) = K\bar{N}_{j,k} \ , \ \ j,k \in \{1,2,3\} \tag{3.30}$$

where $\left(\bar{N}_{j,k}\right)$ is the eigenvector associated to the eigenvalue β with largest real part and $K \geq 0$. Then

$$\begin{gathered}\int_{\mathbb{R}^3} \mathcal{S}_\beta(t)G_0(dw) = 1 \ , \ \ \int_{\mathbb{R}^3} w_j \mathcal{S}_\beta(t)G_0(dw) = 0 \ \ , \\ \int_{\mathbb{R}^3} w_j w_k \mathcal{S}_\beta(t)(G_0)(dw) = K\bar{N}_{j,k} \ \text{ for any } t \geq 0.\end{gathered} \tag{3.31}$$

Moreover, there exists $k_0 > 0$ *sufficiently small, which depends on* B, *such that if* $K \le k_0 b$ *there exists a constant* $C_* = C_*(K) > 0$ *such that if we assume that*

$$\int_{\mathbb{R}^3} |w|^s G_0(dw) \le C_* \tag{3.32}$$

then, for any $t \ge 0$

$$\int_{\mathbb{R}^3} |w|^s \mathcal{S}(t)(G_0)(dw) \le C_* \,. \tag{3.33}$$

Proof. Due to Proposition 3.4 the moments $M_{j,k}$ satisfy (3.24). Then, choosing $\alpha = \beta$ as well as (3.26) we obtain the second group of identities in (3.31). The conservation of mass and linear momentum in (3.31) follows as in the proof of Proposition 3.4.

It only remains to prove (3.33) assuming (3.32) with C_* sufficiently large. To this end we approximate G_0 by a sequence $G_{0,m}$ which satisfies $\left\|G_{0,m}\right\|_{1,\bar{s}} < \infty$, with $\bar{s} > s$. We consider the weak formulation (2.16) with test function $\varphi(w) = |w|^s$ with $2 < s < 3$ we obtain that the function $M_s^{(m)}(t) = \int_{\mathbb{R}^3} |w|^s G_m(t, dw)$ satisfies

$$\begin{aligned}
&\partial_t M_s^{(m)}(t) = -s\beta M_s^{(m)}(t) - s\int_{\mathbb{R}^3} |w|^{s-2} w(w \cdot Lw) G_m(dw,t) \\
&+ \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} d\omega G_m(dw,t) G_m(dw_*,t) B(n\cdot\omega)\left[|w'|^s + \left|w'_*\right|^s - |w|^s - |w_*|^s\right].
\end{aligned}$$

We then estimate $\int_{\mathbb{R}^3} |w|^{s-2} w(w \cdot Lw) G_m(dw,t) = K \int_{\mathbb{R}^3} w_2 w_1 |w|^{s-2} G_m(dw,t)$ by $k_0 b M_s^{(m)}(t)$. It then follows using the fact that $|\beta| \le C k_0 b$ as well as the Povzner estimates (3.29) that

$$\begin{aligned}
\partial_t M_s^{(m)}(t) \le{} & C\, k_0 b M_s^{(m)}(t) \\
& + \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} d\omega G_m(t,dw) G_m(t,dw_*) B(n\cdot\omega) \\
& \times\left[-\kappa_s(|n\cdot\omega|)(|w|^s + |w_*|^s) + C_s\left[|w|^{s-1}|w_*| + |w_*|^{s-1}|w|\right]\right]
\end{aligned}$$

where C is just a numerical constant. The function $\kappa_s(y)$ is continuous and it vanishes only for $y = 0$. Since B is also continuous we can prove that

$$\int_{S^2} B(n\cdot\omega)\kappa_s(|n\cdot\omega|)\, d\omega \ge \mu b$$

for some $\mu > 0$ which depends only on the modulus of continuity of B. Then

$$\begin{aligned}
&\partial_t M_s^{(m)}(t) \le Ck_0 b M_s^{(m)}(t) - \frac{\mu b}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}(|w|^s + |w_*|^s) G_m(t,dw) G_m(t,dw_*) \\
&+ C_s \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left[|w|^{s-1}|w_*| + |w_*|^{s-1}|w|\right] G_m(t,dw) G_m(t,dw_*) \\
&= (Ck_0 - \mu) b M_s^{(m)}(t) + C_s\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left[|w|^{s-1}|w_*| + |w_*|^{s-1}|w|\right] G_m(t,dw) G_m(t,dw_*)\,.
\end{aligned}$$

The estimates (3.31) imply that $\int_{\mathbb{R}^3} |w|^2 G_m(t,dw) \le CK$. Then, since $s < 3$,

$$\partial_t M_s^{(m)}(t) \le (Ck_0 - \mu) b M_s^{(m)}(t) + C_s K.$$

Here C is just a numerical constant. Then, it follows that, choosing $k_0 \le \frac{\mu}{2C}$, we have $M_s^{(m)} \le C_* = 2CC_sK$. Taking the limit $m \to \infty$ we obtain $M_s^{(m)} \to M_s = \int_{\mathbb{R}^3} |w|^s G(t, dw) \le C_*$ and the result follows. □

We now have all the ingredients to prove the existence of the desired self-similar solution, as stated in the Theorem 3.1, using Schauder fixed point Theorem. A similar idea has been also used with adaptations to solve analogous problems for different kinetic equations. We mention for instance [11] (coagulation-fragmentation equations), [12,24] (Weak Turbulence equation for NLS), [17] (Boltzmann equation for granular media), [13,25–27] (Smoluchowski's coagulation equation).

Proof of Theorem 3.1. Suppose that ζ in (3.9) is strictly positive, since for $\zeta = 0$ we have $G = \delta(w)$. We define the subset $\mathcal{U}$ of $\mathcal{M}_+\left(\mathbb{R}_c^3\right)$ such that

$$\int_{\mathbb{R}^3} G(dw) = 1 \ , \quad \int_{\mathbb{R}^3} w_j G(dw) = 0, \ \int_{\mathbb{R}^3} w_j w_k G(dw) = K\bar{N}_{j,k} \tag{3.34}$$

holds, as well as the inequality $\int_{\mathbb{R}_c^3} |w|^s G(dw) \le C_* = C_*(\zeta)$. We choose K in (3.34) in order to have

$$K \sum_{j=1}^{3} \bar{N}_{j,j} = \zeta.$$

The set $\mathcal{U}$ is convex and closed in the $*-$weak topology of measures. Moreover $\mathcal{U}$ is compact in this topology. We consider the evolution semigroup $\mathcal{S}_\beta(t)$. For any $h > 0$ (arbitrarily small) we have that the operator $\mathcal{S}_\beta(h)$ transforms $\mathcal{U}$ in itself. Given that $\mathcal{S}_\beta(h)$ is compact, we can apply Schauder theorem to prove the existence of $G_*^{(h)} \in \mathcal{U}$ such that $\mathcal{S}_\beta(h) G_*^{(h)} = G_*^{(h)}$. Moreover, since $\mathcal{S}_\beta(h)$ defines a semigroup we have $\mathcal{S}_\beta(mh) G_*^{(h)} = G_*^{(h)}$ for any integer m. We then take a subsequence $\{h_k\}$ such that $h_k \to 0$ and the corresponding sequence of fixed points $\left\{G_*^{(h_k)}\right\}$. This sequence is compact in $\mathcal{U}$ and, taking a subsequence if needed (but denoted still as $\{h_k\}$), we obtain that it converges to some G_*. Given any $t > 0$ we can obtain integers m_k such that $m_k h_k \to t$. We have $\mathcal{S}_\beta(m_k h_k) G_*^{(h_k)} = G_*^{(h_k)} \to G_*$ and on the other hand

$$\mathcal{S}_\beta(m_k h_k) G_*^{(h_k)} = \left[\mathcal{S}_\beta(m_k h_k) - \mathcal{S}_\beta(t)\right] G_*^{(h_k)} + \mathcal{S}_\beta(t) G_*^{(h_k)}.$$

The last term converges to $\mathcal{S}_\beta(t) G_*$ using the weak continuity of the semigroup $\mathcal{S}_{\bar{\alpha}}(t)$. On the other hand we have that $\left[\mathcal{S}_\beta(m_k h_k) - \mathcal{S}_\beta(t)\right] G_*^{(h_k)} \to 0$ as $k \to \infty$ in the weak topology due to the uniform continuity in time. Then $\mathcal{S}_\beta(t) G_* = G_*$ for any $t > 0$. Then G_* is a stationary point for the semigroup. Notice that we can pass to the limit in (3.34). □

3.2. Stability and uniqueness of self-similar profiles

In the previous Sections we discussed the problem of existence of a self-similar solutions of (3.1) having the form $g(w, t) = e^{-3\beta t} G(we^{-\beta t})$, guaranteed by Theorem 3.1 under a smallness assumption on the shear parameter. We now want to address the following questions:

- is the self-similar profile unique?
- is this solution stable in a suitable class of general data?

Recently, in [4], it has been possible to positively answer these questions. Indeed, in addition to a different proof of existence of self-similar solutions, we obtained that these solutions are unique (in the class of probability measures $\mathcal{P}_+(\mathbb{R}^3)$) and stable. Specifically, they attract all the solutions of (3.1) having the same mass as well as the same first order moments. Actually, we proved that they are exponentially convergent in the topology of the uniform convergence for the Fourier transforms of the probability measures.

More precisely, the results obtained in [4] can be summarized in the following Theorem.

Theorem 3.7 (Stability of the self-similar profile)**.** *Let $g \in C([0,\infty);\mathcal{P}_+(\mathbb{R}^3))$ be a solution to* (3.1) *with $g(\cdot,0) = g_0(\cdot) \in \mathcal{P}_+(\mathbb{R}^3)$ s.t. $\int_{\mathbb{R}^3} dv\, g_0(v)|v|^p < \infty$ for $p > 2$. There exists an $\varepsilon_0 = \varepsilon_0(p) > 0$ such that if $|K| \le \varepsilon_0$ there exists $G \in \mathcal{P}_+(\mathbb{R}^3)$ satisfying $\displaystyle\int_{\mathbb{R}^d} dv\, G(v)|v|^p < \infty$, $2 < p \le 4$, and $\beta = \beta(L) \in \mathbb{R}$ such that*

$$e^{3\beta t}\, g\left(e^{\beta t} v + e^{-tL^T} U, t\right) \to \lambda^{-3} G(\lambda^{-1} v) \quad as \quad t \to \infty, \tag{3.35}$$

in the weak topology of $\mathcal{P}_+(\mathbb{R}^3)$. Here $\lambda = \lambda(g_0, K) > 0$ and $U = \displaystyle\int_{\mathbb{R}^3} dv\, g_0(v) v \in \mathbb{R}^3$.

The main tool that we used to prove these results is the well developed machinery available for the study of the Boltzmann equations in the case of Maxwell molecules by means of the Fourier transform method that was introduced by A. Bobylev in [1] and that we briefly summarize here. Let $g(t,\cdot) \in \mathcal{P}_+(\mathbb{R}^3)$ and set $\varphi(k,t) = \hat{g}(k,t)$ the Fourier Transform of the one particle probability density. Hence, we work in the class of characteristic functions which are the Fourier transforms of time-dependent probability measures. They are a subset Φ of the space of complex-valued continuous functions

$$\Phi = \left\{\varphi \in C(\mathbb{R}^3;\mathbb{C}) \;:\; \varphi(k) = \hat{g}(k), \quad \text{for} \quad g \in \mathcal{P}_+(\mathbb{R}^3)\right\}. \tag{3.36}$$

Thus $\varphi \in C([0,\infty);\Phi)$. It is well known that the collision operator can be simplified by the Fourier Transform ([1,2]). On the other hand the Fourier Transform of the hyperbolic term has a simpler form. From (3.1) we then obtain

$$\partial_t \varphi + (Lk)\cdot\partial_k \varphi = \int_{S^2} d\omega\, B\left(\hat{k}\cdot\omega\right)\varphi(k_+)\varphi(k_-) - \underbrace{\varphi_{|k=0}}_{=1}\varphi, \tag{3.37}$$

where $k_\pm = \frac{1}{2}(k \pm |k|\omega)$. We consider (3.37) with initial condition

$$\varphi(k,0) = \varphi_0(k) = \int_{\mathbb{R}^3} dv\, g_0(v) e^{-ik\cdot v}, \quad \varphi_0(0) = 1. \tag{3.38}$$

Note that (3.37) implies the mass conservation property that in the Fourier variables reads $\varphi(0,t) = \varphi_0(0) = 1$.

At this point, it is possible to prove existence of a self-similar profile $\Psi(k)$ with the form $\varphi(k,t) = \Psi(ke^{\beta t}), \beta \in \mathbb{R}$ and stability:

$$\lim_{t\to\infty} \varphi(e^{-\beta t}k, t) = \Psi(k) \;\text{ in }\; \|\cdot\|_\infty$$

which implies uniqueness and

$$|\varphi(k,t) - \Psi(\lambda k)| \le C e^{-\theta t}(|k|^2 + |k|^p), \quad k \in \mathbb{R}^3, \quad \lambda \ge 0,\; C > 0,\; \theta > 0. \tag{3.39}$$

Reformulating the results in terms of the original measure $g \in \mathcal{P}_{+}(\mathbb{R}^3)$ we then obtain the claim of Theorem 3.7.

Remark 3.8 (Extension of the result to the case of non-cutoff Maxwell molecules)**.** *We mention that, recently, the results (cf. Theorems 3.1 and 3.7) concerning the existence, uniqueness and stability of self-similar solutions to* (3.1) *under the assumption of sufficiently small deformations, and for pseudo-Maxwellian molecules (cf.* (2.6)*) have been extended in [23] to the case of non-cutoff Maxwell molecules removing the assumption* (2.6) *for the function* $\Lambda(w, w_*)$ *defined in (2.5).*

4. Non-self-similar behavior

We recall that, the collision operator in (1.7) is quadratic. It rescales like:

$$\rho(t)\,[w]^{\gamma}\,[g] \tag{4.1}$$

where $[w]$ is the order of magnitude of w, γ is the homogeneity of the collision kernel B (cf. (1.4)) and $[g]$ the order of magnitude of g.

The term $L(t)$ can yield different behaviors as $t \to \infty$ and here we restricted to the simple shear case (1.14). We denoted the term $L(t)\,w \cdot \partial_w g = K w_2 \partial_{w_1} g$ as hyperbolic term. It can be constant, or it can behave like a power law (increasing or decreasing). The key idea is that there are three possibilities depending on the value of the homogeneity γ and the function yielding the scaling of $[w]$. Either the hyperbolic term is larger than the collision term as $t \to \infty$, either the collision term is larger or either the hyperbolic term and the collision term have the same order of magnitude. Suppose that $L(t)$ scales like a function $\eta(t)$. The hyperbolic term scales then like $\eta(t)\,[g]$ and the collision term scales as in (4.1). Therefore, we need to compare the terms: $\eta(t)$ and $\rho(t)\,[w]^{\gamma}$.

In order to present which is the expected picture for homoenergetic solutions of the Boltzmann equation, we present here the conjectures for the cases in which the hyperbolic term and the collision term do not balance. More precisely, the cases for which the hyperbolic terms dominate, and those in which the collision term dominates.

We describe a few details on these formal results below.

4.1. Collision-dominated behavior

In this section we study the long time asymptotics of solutions of

$$\partial_t g - K w_2 \partial_{w_1} g = \int_{\mathbb{R}^3} dw_* \int_{S^2} d\omega\, B(n \cdot \omega, |w - w_*|)\,[g' g'_* - g_* g] \tag{4.2}$$

when the cross-section of the collision kernel has homogeneity larger than zero. This corresponds to taking the matrix $L(t)$ of the form (1.14) in (1.16).

More precisely, we will assume for definiteness that the kernels $B(n \cdot \omega, |v - v_*|)$ are homogeneous in $|v - v_*|$ with homogeneity $\gamma > 0$ (cf. (1.4)). A typical example of cross-section is the one of hard-sphere potentials, $B(\omega, |v - v_*|) = |\omega \cdot (v - v_*)|$, $e = \omega$.

We introduce the linearized operator

$$\mathbb{L}[H](\xi) = \int_{\mathbb{R}^3} d\xi_* \int_{S^2} d\omega B(n \cdot \omega, |\xi - \xi_*|)\, e^{-|\xi_*|^2}\,[H'_* + H' - H - H_*], \tag{4.3}$$

for any $H \in \mathcal{D}(\mathbb{L}) \subset L^2\left(\mathbb{R}^3; e^{-|\xi|^2}d\xi\right)$. We recall that the space $L^2\left(\mathbb{R}^3; e^{-|\xi|^2}d\xi\right)$ is a Hilbert space with the scalar product

$$\langle f, g\rangle_w = \int_{\mathbb{R}^3} f(\xi)g(\xi)e^{-|\xi|^2}d\xi. \tag{4.4}$$

The operator $-\mathbb{L}$ is a well studied linear operator in kinetic theory. It is a nonnegative, self-adjoint operator which has good functional analysis properties for suitable choices of the collision kernel B. Its kernel consists of collision invariants, i.e., it is the subspace spanned by the functions $\left\{1, \xi, |\xi|^2\right\}$. We define the subspace $\mathcal{W} = \left\{1, \xi, |\xi|^2\right\}^{\perp} \subset L^2\left(\mathbb{R}^3; e^{-|\xi|^2}d\xi\right)$. For further details we refer to [8].

The intuitive idea behind the asymptotic behavior computed in this case is that there is a competition between the shear term $w_2\partial_{w_1}g$ and the collision term $Qg\,(w)$ for large $|w|$. The effect of the shear term is to increase the temperature of the system. Comparing the order of magnitude of the shear term and the collision term, it turns out that for $\gamma > 0$, the collision term becomes the dominant one for large times. In such a case the asymptotics of the velocity dispersion can be computed formally by means of a suitable adaptation of the classical Hilbert expansions (see for instance [8]) around the Maxwellian equilibrium. Since the collision terms are the dominant ones, we expect that, in the long time asymptotics, the solutions should behave as a Maxwellian distribution with increasing temperature. Indeed, the effect of the shear is a small perturbation (compared to the collision term) which produces a growth of the temperature of the Maxwellian. More precisely, one can state the following Formal Theorem.

Theorem 4.1 (Formal)**.** *Suppose that the cross-section $B(\cdot,\cdot)$ satisfies condition* (1.4) *with $\gamma > 0$. Then there exists $g\,(t, w)$ a weak solution of the Boltzmann equation* (4.2) *in the sense of Definition 2.3 for which the following asymptotics as $t \to \infty$ holds:*

$$\beta(t)^{-\frac{3}{2}}g(t, \frac{\xi}{\sqrt{\beta}}) \to C_0e^{-|\xi|^2} \quad in \quad L^2\left(\mathbb{R}^3; e^{-|\xi|^2}d\xi\right),$$

with $C_0 = \pi^{-\frac{3}{2}}$. Moreover, $\beta\,(t)$ satisfies

$$\beta\,(t) = C\,t^{-\frac{2}{\gamma}}(1 + o(1)) \;\; as \;\; t \to \infty.$$

Here $C = \left(\frac{4}{3}\gamma\, b\right)^{-\frac{2}{\gamma}}$ and $b > 0$ is given by:

$$b = K^2\left\langle \xi_1\xi_2, (-\mathbb{L})^{-1}(\xi_1\xi_2)\right\rangle.$$

We notice that $\beta \to 0$ as $t \to \infty$ as expected and the temperature $T = \frac{1}{\beta}$ increases as a power law. We notice that the exponent is larger if γ approaches zero and if $\gamma = 0$ we have exponential growth.

The proof of this result relies on the fact that the asymptotics of the velocity dispersion can be otained using a suitable Hilbert expansion around the Maxwellian equilibrium. More precisely, we look for a solutions with the form

$$g(t, w) \sim C_0\,(\beta\,(t))^{\frac{3}{2}} \exp\left(-\beta\,(t)\,|w|^2\right)[1 + h_1\,(t, w) + h_2\,(t, w)\,...] \tag{4.5}$$

with

$$1 \gg |h_1| \gg |h_2| \quad \text{as} \quad t \to \infty. \tag{4.6}$$

The details of the proof can be found in [21].

4.2. Hyperbolic-dominated behavior

As mentioned before, we focus here on the case of frozen collisions, namely when the collisions term becomes so small that the effect of collisions is irrelevant as $t \to \infty$. The formal argument underlying this conjectures is based on the control of the collision rate (gain term of the Boltzmann operator) for molecular densities that satisfy the asymptotic first order hyperbolic equation $\partial_t g + \partial_w \cdot (L(t)wg) = 0$. If the resulting collision rate is decreasing in time, we refer to this case as hyperbolic-dominated behavior. More precisely, *frozen collisions* refers to the case of exponentially decreasing behavior of the collision rate as $t \to \infty$. In this case we conjecture that $g(t, w)$ converges in the sense of measures to a limit distribution that depends on the initial datum.

We consider homoenergetic flows with $L(t)$ as in (1.14) describing a simple shear deformation, namely

$$\partial_t g - K w_2 \partial_{w_1} g = \mathbb{C} g(w). \tag{4.7}$$

When the homogeneity γ of the collision kernel B is smaller than -1 it is possible to prove that there exist solutions of (4.7) for which the contribution of the collision term $\mathbb{C}g(w)$ is negligible as $t \to \infty$. For such solutions, a given particle would not collide with any other for large times, and we might expect to have w_2 approximately constant and w_1 increasing linearly in t. This suggests to look for solutions with the form

$$g(t, w) = \frac{1}{t} G(\tau, \xi) \quad , \quad \tau = \log(t), \quad \xi_1 = \frac{w_1}{t}, \quad \xi_j = w_j \ \text{ if } \ j = 2, 3. \tag{4.8}$$

Then, using the homogeneity of the kernel, we obtain that G satisfies

$$\partial_\tau G - \partial_\xi \cdot ([(\xi_1 + K\xi_2) e_1] G) = e^{(1+\gamma)\tau} \mathbb{C} G(\xi). \tag{4.9}$$

The long time asymptotics of the hyperbolic equation obtained putting the right-hand side equal to zero in (4.9) strongly depends on the regularity properties of the initial data $G_0(\xi)$. For the sake of simplicity we assume that $G_0 \in C^2$. Hence, the solution of the corresponding hyperbolic equation can be obained using the method of characteristics. We have

$$G(\tau, \xi) = e^\tau G_0(\xi_1 e^\tau + K\xi_2(e^\tau - 1), \xi_2, \xi_3),$$

and, in the original variables,

$$g(t, w) = t\, G_0(w_1 + K w_2(t - 1), w_2, w_3).$$

Suppose that G_0 is compactly supported or decreases sufficiently fast as $|w| \to \infty$. Then, integrating with respect to ξ against a smooth test function, we obtain

$$G(\tau, \xi) \rightharpoonup \left[\int_{-\infty}^{\infty} G_0(\eta, \xi_2, \xi_3)\, d\eta\right] \delta(\xi_1 + K\xi_2) \ \text{ as } \ \tau \to \infty. \tag{4.10}$$

If $\gamma < -1$ it would follow that the contribution of the collision term $e^{(1+\gamma)\tau}\mathbb{C}G(\xi)$ decreases exponentially as $\tau \to \infty$ and it yields a negligible contribution as $\tau \to \infty$. In this case the effect of the collisions becomes frozen for large times. A detailed justification for these conjectures can be found in [22]. The rigorous proof of the smallness of this term would require some careful analysis of the collision term.

We further notice that these regimes are complementary to those given by the formal Hilbert expansion and the self-similar profile, except for a gap: $-1 \leq \gamma < 0$. In this gap the collision rate is small but it still plays a significant role in the formal asymptotic behavior of the Boltzmann equation. Specifically, collisions take place with increasingly large mean free paths. The description of the long time asymptotics for the particle distribution in this case is a challenging problem. In [22] we obtained some estimates which indicate that the solutions do not behave in self-similar manner in this case. Moreover, a simplified model has been introduced in [22] which includes the effect of the shear and a simplified mechanism of collisions. The asymptotic behaviour of the solutions of this model has been studied and it yields an involved distribution of particles, which differs greatly from a self-similar distribution.

A detailed justification for these conjectures can be found in [22].

5. Conclusions

We summarize in the following table (see Table 1) the results and the conjectures presented in this paper concerning the long-time behavior for homoenergetic flows of the Boltzmann equation in the case of simple shear.

Table 1. Long time asymptotics of homoenergetic solutions in the simple shear case.

Balance case ($\gamma = 0$)	Hyperbolic-dom. case ($\gamma < 0$)	Collision-dom. case ($\gamma > 0$)
Self-similar solutions $\|w\|^2 \sim e^{bt}$, $b = b(K)$ (increasing temperature)	• $-1 < \gamma < 0$? • $\gamma < -1$ frozen collisions	Hilbert expansion $\|w\|^2 \sim t^{\frac{1}{\gamma}}$

In these notes, we provided a rigorous proof of the existence of self-similar solutions yielding a non Maxwellian distribution of velocities in the case in which the hyperbolic term and the collisions balance in the case of simple shear deformations (cf. [20]). We further discussed the global uniqueness and stability of these self-similar solutions which has been proved in [4] under suitable smallness assumptions on the deformation term. A distinctive feature of these self-similar solutions is that the corresponding particle distribution does not satisfy a detailed balance condition. In these solutions the particle velocities are given by a subtle interplay between particle collisions and shear.

In the case of collision-dominated behavior and in the case of hyperbolic-dominated behavior we proposed some conjectures for asymptotic formulas for the solutions based on formal computations. (The complete results can be found in [21, 22] respectively. In the first case we have obtained that the corresponding distribution of particle velocities for the associated homoenergetic flows can be approximated by a family of Maxwellian distributions with a changing temperature whose rate of change is obtained by means of a Hilbert expansion. It would be relevant to prove rigorously the existence of those solutions and to understand their stability properties.

In the case in which the hyperbolic terms are much larger than the collision terms the resulting solutions yield more complex behaviors than the ones that we have obtained in the previous cases. The detailed understanding of the particle distributions in this case is largely open and challenging.

Conflict of interest

The authors declare no conflict of interest.

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