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Research article

On the concentration–compactness principle for Folland–Stein spaces and for fractional horizontal Sobolev spaces†

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Abstract: In this paper we establish some variants of the celebrated concentration–compactness principle of Lions – CC principle briefly – in the classical and fractional Folland–Stein spaces. In the first part of the paper, following the main ideas of the pioneering papers of Lions, we prove the CC principle and its variant, that is the CC principle at infinity of Chabrowski, in the classical Folland–Stein space, involving the Hardy–Sobolev embedding in the Heisenberg setting. In the second part, we extend the method to the fractional Folland–Stein space. The results proved here will be exploited in a forthcoming paper to obtain existence of solutions for local and nonlocal subelliptic equations in the Heisenberg group, involving critical nonlinearities and Hardy terms. Indeed, in this type of problems a triple loss of compactness occurs and the issue of finding solutions is deeply connected to the concentration phenomena taking place when considering sequences of approximated solutions.

Keywords: Heisenberg group; concentration–compactness principles; critical exponents; Hardy terms; integro–differential operators

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Dedicated to the memory of Professor Ireneo Peral, with high feelings of admiration for his notable contributions in Mathematics and great affection

1. Introduction

In recent years, geometric analysis and partial differential equations on the Heisenberg group have attracted great attention. In this article, we investigate some concentration–compactness results related to the Hardy–Sobolev embedding on the classical and fractional Folland–Stein spaces in the Heisenberg group. Before stating the main results, let us recall some relevant contributions in the topic.

The Heisenberg group $\mathbb{H}^n$ is the Lie group which has $\mathbb{R}^{2n+1}$ as a background manifold and whose group structure is given by the non–Abelian law

$$\xi \circ \xi' = \Big(z + z', t + t' + 2\sum_{i=1}^{n}(y_i x_i' - x_i y_i')\Big)$$

for all $\xi, \xi' \in \mathbb{H}^n$, with

$$\xi = (z, t) = (x_1, \dots, x_n, y_1, \dots, y_n, t) \text{ and } \xi' = (z', t') = (x_1', \dots, x_n', y_1', \dots, y_n', t').$$

We denote by r the *Korányi norm*, defined as

$$r(\xi) = r(z, t) = (|z|^4 + t^2)^{1/4},$$

with $\xi = (z, t)$, $z = (x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, $t \in \mathbb{R}$, and $|z|$ the Euclidean norm in $\mathbb{R}^{2n}$.

A key result, whose importance is also due to its connection with the CR Yamabe problem, is the subelliptic Sobolev embedding theorem in $\mathbb{H}^n$, which is due to Folland and Stein [14]. This result is valid in the more general context of Carnot groups, but we state it in the set up of the Heisenberg group. If $1 < p < Q$, where $Q = 2n + 2$ is the homogeneous dimension of the Heisenberg group $\mathbb{H}^n$, we know by [14] that there exists a positive constant $C = C(p, Q)$ such that

$$\int_{\mathbb{H}^n} |\varphi|^{p^*} d\xi \le C \int_{\mathbb{H}^n} |D_H \varphi|_H^p d\xi \quad \text{for all } \varphi \in C_c^\infty(\mathbb{H}^n), \qquad p^* = \frac{pQ}{Q - p}, \tag{1.1}$$

and p^* is the critical exponent related to p. Moreover, the vector

$$D_H u = (X_1 u, \cdots, X_n u, Y_1 u, \cdots, Y_n u)$$

is the *horizontal gradient* of a regular function u, where $\{X_j, Y_j\}_{j=1}^n$ is the basis of horizontal left invariant vector fields on $\mathbb{H}^n$, that is

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \qquad j = 1, \dots, n. \tag{1.2}$$

Unlike the Euclidean case, cf. [34] and [2], the value of the best constant in (1.1) is unknown. In the particular case $p = 2$, the problem of the determination of the best constant in (1.1) is related to the CR Yamabe problem and it has been solved by the works of Jerison and Lee [21–24]. In the general case, existence of extremal functions of (1.1) was proved by Vassilev in [35] via the concentration–compactness method of Lions, see also [20]. This method does not allow an explicit determination of the best constant C_{p^*} of (1.1). However, we know from [35] that C_{p^*} is achieved in the *Folland–Stein space* $S^{1,p}(\mathbb{H}^n)$, which is defined, for $1 < p < Q$, as the completion of $C_c^\infty(\mathbb{H}^n)$ with respect to the norm

$$\|D_H u\|_p = \left(\int_{\mathbb{H}^n} |D_H u|_H^p d\xi\right)^{1/p}.$$

Thus, we can write the best constant C_{p^*} of the Folland–Stein inequality (1.1) as

$$C_{p^*} = \inf_{\substack{u\in S^{1,p}(\mathbb{H}^n)\\ u\neq 0}} \frac{\|D_H u\|_p^p}{\|u\|_{p^*}^p}. \tag{1.3}$$

Note that the Euler–Lagrange equation of the nonnegative extremals of (1.1) leads to the critical equation

$$-\Delta_{H,p} u = |u|^{p^*-2}u \quad \text{in } \mathbb{H}^n,$$

where the operator $\Delta_{H,p}$ is the well known p Kohn–Spencer Laplacian, which is defined as

$$\Delta_{H,p}\varphi = \mathrm{div}_H(|D_H\varphi|_H^{p-2} D_H\varphi),$$

for all $\varphi \in C^2(\mathbb{H}^n)$.

The study of critical equations is deeply connected to the concentration phenomena, which occur when considering sequences of approximated solutions. Indeed, given a weakly convergent sequence $(u_k)_k$ in $S^{1,p}(\mathbb{H}^n)$, we can infer that $(u_k)_k$ is bounded in $L^{p^*}(\mathbb{H}^n)$, but we do not have compactness properties in general. On the other hand, we know that the sequences $\mu_k = |D_H u|_H^p d\xi$ and $\nu_k = |u_k|^{p^*} d\xi$ weak* converge to some measures μ and ν in the dual space $\mathcal{M}(\mathbb{H}^n)$ of all real valued, finite, signed Radon measures on $\mathbb{H}^n$. An essential step in the concentration–compactness method is the study of the exact behavior of the limit measures in the space $\mathcal{M}(\mathbb{H}^n)$ and in the spirit of Lions. In particular, following [25, 26], Ivanov and Vassilev in [20] proved the following result.

Theorem A (Lemma 1.4.5, Ivanov and Vassilev [20])**.** *Let $(u_k)_k$ be a sequence in $S^{1,p}(\mathbb{H}^n)$ such that $u_k \rightharpoonup u$ in $S^{1,p}(\mathbb{H}^n)$ and $|u_k|^{p^*}d\xi \overset{*}{\rightharpoonup} \nu$, $|D_H u_k|_H^p d\xi \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\mathbb{H}^n)$, for some appropriate $u \in S^{1,p}(\mathbb{H}^n)$, and finite nonnegative Radon measures μ, ν on $\mathbb{H}^n$.*

Then, there exist an at most countable set J, a family of points $\{\xi_j\}_{j\in J} \subset \mathbb{H}^n$ and two families of nonnegative numbers $\{\mu_j\}_{j\in J}$ and $\{\nu_j\}_{j\in J}$ such that

$$\nu = |u|^{p^*}d\xi + \sum_{j\in J}\nu_j\delta_{\xi_j}, \quad \mu \geq |D_H u|_H^p d\xi + \sum_{j\in J}\mu_j\delta_{\xi_j} \quad \nu_j^{p/p^*} \leq \frac{\mu_j}{C_{p^*}} \quad \textit{for all } j \in J,$$

where δ_{ξ_j} are the Dirac functions at the points ξ_j of $\mathbb{H}^n$.

The aim of this paper is to extend Theorem A in two different ways. First, we want to prove a version of Theorem A suitable to deal with a combined Hardy and Sobolev embedding. Indeed, following [16], we set

$$\psi(\xi) = |D_H r(\xi)|_H = \frac{|z|}{r(\xi)} \quad \text{for } \xi = (z,t) \neq (0,0).$$

Assume from now on that $1 < p < Q$ and let $\varphi \in C_c^\infty(\mathbb{H}^n \setminus \{O\})$. Then, the Hardy inequality in the Heisenberg group states as follows

$$\int_{\mathbb{H}^n} |\varphi|^p \psi^p \frac{d\xi}{r^p} \leq \left(\frac{p}{Q-p}\right)^p \int_{\mathbb{H}^n} |D_H \varphi|_H^p d\xi. \tag{1.4}$$

Inequality (1.4) was obtained by Garofalo and Lanconelli in [16] when $p = 2$ and then extended to all $p > 1$ in [7, 29]. When $p = 2$, the optimality of the constant $(2/(Q-2))^2$ is shown in [18]. Let us also mention that a sharp inequality of type (1.4) has been derived in general Carnot–Carathéodory spaces by Danielli, Garofalo and Phuc in [8].

Obviously, inequality (1.4) remains valid in $S^{1,p}(\mathbb{H}^n)$. Moreover, inequalities (1.1) and (1.4) imply that for any $\sigma \in (-\infty, \mathcal{H}_p)$ the following best constant is well defined

$$\mathcal{I}_\sigma = \inf_{\substack{u \in S^{1,p}(\mathbb{H}^n) \\ u \neq 0}} \frac{\|D_H u\|_p^p - \sigma \|u\|_{H_p}^p}{\|u\|_{p^*}^p}, \tag{1.5}$$

where

$$\mathcal{H}_p = \inf_{\substack{u \in S^{1,p}(\mathbb{H}^n) \\ u \neq 0}} \frac{\|D_H u\|_p^p}{\|u\|_{H_p}^p}, \qquad \|u\|_{H_p}^p = \int_{\mathbb{H}^n} |u|^p \frac{\psi^p}{r^p} d\xi. \tag{1.6}$$

Note that, when $\sigma = 0$, we recover the Sobolev embedding, that is $\mathcal{I}_0 = C_{p^*}$. However, the Hardy embedding $S^{1,p}(\mathbb{H}^n) \hookrightarrow L^p(\mathbb{H}^n, \psi^p r^{-p} d\xi)$ is continuous, but not compact, even locally in any neighborhood of O, where $O = (0,0)$ denotes the origin of $\mathbb{H}^n$. A challenging problem is then to provide sufficient conditions for the existence of a nontrivial solution to critical equations with Hardy terms in the whole space $\mathbb{H}^n$, when a triple loss of compactness takes place. To overcome this difficulty, we prove in Theorem 1.1 and Theorem 1.2 some versions of the concentration–compactness principle for related to the embedding (1.5).

Theorem 1.1. *Let $\sigma \in (-\infty, \mathcal{H}_p)$ and let $(u_k)_k$ be a sequence in $S^{1,p}(\mathbb{H}^n)$ such that $u_k \rightharpoonup u$ in $S^{1,p}(\mathbb{H}^n)$, and $|u_k|^{p^*} d\xi \overset{*}{\rightharpoonup} \nu$, $|D_H u_k|_H^p d\xi \overset{*}{\rightharpoonup} \mu$, $|u_k|^p \psi^p \dfrac{d\xi}{r(\xi)^p} \overset{*}{\rightharpoonup} \omega$ in $\mathcal{M}(\mathbb{H}^n)$, for some appropriate $u \in S^{1,p}(\mathbb{H}^n)$, and finite nonnegative Radon measures μ, ν, ω on $\mathbb{H}^n$.*

Then, there exist an at most countable set J, a family of points $\{\xi_j\}_{j\in J} \subset \mathbb{H}^n$, two families of nonnegative numbers $\{\mu_j\}_{j\in J}$ and $\{\nu_j\}_{j\in J}$ and three nonnegative numbers ν_0, μ_0, ω_0, such that

$$\nu = |u|^{p^*} d\xi + \nu_0 \delta_O + \sum_{j\in J} \nu_j \delta_{\xi_j}, \tag{1.7}$$

$$\mu \geq |D_H u|_H^p d\xi + \mu_0 \delta_O + \sum_{j\in J} \mu_j \delta_{\xi_j}, \tag{1.8}$$

$$\omega = |u|^p \psi^p \frac{d\xi}{r(\xi)^p} + \omega_0 \delta_O, \tag{1.9}$$

$$\nu_j^{p/p^*} \leq \frac{\mu_j}{C_{p^*}} \quad \text{for all } j \in J, \qquad \nu_0^{p/p^*} \leq \frac{\mu_0 - \sigma\omega_0}{\mathcal{I}_\sigma}, \tag{1.10}$$

where $C_{p^*} = \mathcal{I}_0$ and $\mathcal{I}_\sigma$ are defined in (1.3) *and* (1.5)*, while δ_O, δ_{ξ_j} are the Dirac functions at the points O and ξ_j of $\mathbb{H}^n$, respectively.*

Theorem 1.1 extends Theorem A and also Theorem 1.2 of [5] to the case of unbounded domains, see also [20, 31, 32]. The strategy is the same as the one in the seminal papers of Lions [25, 26], but there are some complications due to the non Euclidean context.

The whole Heisenberg group is endowed with noncompact families of dilations and translations, which could provide a loss of compactness due to the *drifting towards infinity* of the mass, or – in other words – the concentration at infinity. In order to deal with this type of phenomena, we prove a variant of the concentration–compactness principle of Lions, that is the *concentration–compactness principle at infinity*. This variant was introduced by Bianchi, Chabrowski and Szulkin in [3, 6] and we prove an extension of their results suitable to deal with critical Hardy equations in the Heisenberg group.

Denote by $B_R(\xi)$ the *Korányi open ball* of radius R centered at ξ. For simplicity B_R is the ball of radius R centered at $\xi = O$.

Theorem 1.2. *Let $(u_k)_k$ be a sequence in $S^{1,p}(\mathbb{H}^n)$ as in Theorem* 1.1 *and define*

$$\nu_\infty = \lim_{R\to\infty} \limsup_{k\to\infty} \int_{B_R^c} |u_k|^{p^*} d\xi, \quad \mu_\infty = \lim_{R\to\infty} \limsup_{k\to\infty} \int_{B_R^c} |D_H u_k|_H^p d\xi, \tag{1.11}$$

$$\omega_\infty = \lim_{R\to\infty} \limsup_{k\to\infty} \int_{B_R^c} |u_k|^p \psi^p \frac{d\xi}{r(\xi)^p}. \tag{1.12}$$

Then,

$$\limsup_{k\to\infty} \int_{\mathbb{H}^n} |u_k|^{p^*} d\xi = \nu(\mathbb{H}^n) + \nu_\infty, \quad \limsup_{k\to\infty} \int_{\mathbb{H}^n} |D_H u_k|_H^p d\xi = \mu(\mathbb{H}^n) + \mu_\infty, \tag{1.13}$$

$$\limsup_{k\to\infty} \int_{\mathbb{H}^n} |u_k|^p \psi^p \frac{d\xi}{r(\xi)^p} = \omega(\mathbb{H}^n) + \omega_\infty, \quad \nu_\infty^{p/p^*} \le \frac{\mu_\infty - \sigma\omega_\infty}{\mathcal{I}_\sigma}, \tag{1.14}$$

where μ, ν, ω are the measures introduced in Theorem 1.1.

In the second part of the paper, we want to extend the previous results to the fractional case. Let $0 < s < 1$ and $1 < p < \infty$. We define the *fractional Sobolev space $HW^{s,p}(\mathbb{H}^n)$* as the completion of $C_c^\infty(\mathbb{H}^n)$ with respect to the norm

$$\|\cdot\|_{HW^{s,p}(\mathbb{H}^n)} = \|\cdot\|_{L^p(\mathbb{H}^n)} + [\cdot]_{H,s,p},$$

where

$$[\varphi]_{H,s,p} = \left(\iint_{\mathbb{H}^n\times\mathbb{H}^n} \frac{|\varphi(\xi) - \varphi(\eta)|^p}{r(\eta^{-1}\circ\xi)^{Q+sp}} d\xi d\eta \right)^{1/p} \quad \text{along any } \varphi \in C_c^\infty(\mathbb{H}^n). \tag{1.15}$$

The fractional Sobolev embedding in the Heinseberg group was obtained in [1] following the lines of [9] and states as follows. If $sp < Q$, then there exists a constant $C_{p_s^*}$ depending on p, Q and s such that

$$\|\varphi\|_{p_s^*}^p \le C_{p_s^*} [\varphi]_{H,s,p}^p \quad \text{for all } \varphi \in C_c^\infty(\mathbb{H}^n), \qquad p_s^* = \frac{pQ}{Q - sp}. \tag{1.16}$$

The proof of the above inequality is obtained directly, by extending the method of [9] to the Heisenberg context.

For notational simplicity, the fractional (s, p) horizontal gradient of any function $u \in HW^{s,p}(\mathbb{H}^n)$ is denoted by

$$|D_H^s u|^p(\xi) = \int_{\mathbb{H}^n} \frac{|u(\xi) - u(\eta)|^p}{r(\eta^{-1}\circ\xi)^{Q+ps}} d\eta = \int_{\mathbb{H}^n} \frac{|u(\xi\circ h) - u(\xi)|^p}{r(h)^{Q+ps}} dh. \tag{1.17}$$

Note that the (s, p) horizontal gradient of a function $u \in HW^{s,p}(\mathbb{H}^n)$ is well defined a.e. in $\mathbb{H}^n$ and $|D^s_H u|^p \in L^1(\mathbb{H}^n)$ thanks to Tonelli's theorem.

From now on we fix $0 < s < 1$, $1 < p < \infty$ with $sp < Q$. Then, the following result holds true.

Theorem 1.3. *Let $(u_k)_k$ be a sequence in $HW^{s,p}(\mathbb{H}^n)$ such that $u_k \rightharpoonup u$ in $HW^{s,p}(\mathbb{H}^n)$, and furthermore $|u_k|^{p^*_s} d\xi \overset{*}{\rightharpoonup} \nu$, $|D^s_H u_k|^p d\xi \overset{*}{\rightharpoonup} \mu$, in $\mathcal{M}(\mathbb{H}^n)$, for some appropriate $u \in HW^{1,p}(\mathbb{H}^n)$, and finite nonnegative Radon measures μ, ν on $\mathbb{H}^n$.*

Then, there exist an at most countable set J, a family of points $\{\xi_j\}_{j\in J} \subset \mathbb{H}^n$, two families of nonnegative numbers $\{\mu_j\}_{j\in J}$ and $\{\nu_j\}_{j\in J}$ such that

$$\nu = |u|^{p^*_s} d\xi + \sum_{j\in J} \nu_j \delta_{\xi_j}, \quad \mu \geq |D^s_H u|^p d\xi + \sum_{j\in J} \mu_j \delta_{\xi_j}, \tag{1.18}$$

$$\nu_j^{p/p^*_s} \leq \frac{\mu_j}{C_{p^*_s}} \quad \text{for all } j \in J, \tag{1.19}$$

where the constant $C_{p^*_s}$ is defined in (1.16).

In the Euclidean setting, the first extension of the CC method in the fractional Sobolev spaces was obtained in [30] for $p = 2$ and then in [28] for any p, with $1 < p < N/s$. We also refer to [4, 10, 12] for similar results in this context and to [33] for the vectorial fractional Sobolev spaces.

In Theorem 1.3 we extend the previous results from the Euclidean setting to the Heisenberg environment and we also widen Theorem A from the local case to the fractional setup. To the best of our knowledge Theorem 1.3 is the first extension of the method in the fractional Sobolev space in Heisenberg group.

Actually, the strategy is the same as the one in the seminal papers of Lions [25, 26], but there are several complications due to both the nonlocal and the subelliptic context. In order to overcome these difficulties, we employ the crucial Lemmas 4.4 and 4.5, proved using the key Lemma 4.2 and Corollary 4.3. To enter into details, the latter results give precise decay estimates and scaling properties for the fractional (s, p) horizontal gradients of functions of class $C^\infty_c(\mathbb{H}^n)$, with respect to the intrinsic family of dilations δ_R.

The paper is organized as follows. In Section 2, we recall some fundamental definitions and properties related to the Heisenberg group $\mathbb{H}^n$. Section 3 is devoted to the proof of Theorems 1.1 and 1.2, while the final Section 4 deals with the proof of Theorem 1.3, based on some preliminary lemmas.

2. Preliminaries

2.1. The Heisenberg group

In this section we present the basic properties of $\mathbb{H}^n$ as a Lie group. For a complete treatment, we refer to [13, 16, 17, 20, 35]. Let $\mathbb{H}^n$ be the Heisenberg group of topological dimension $2n + 1$, that is the Lie group which has $\mathbb{R}^{2n+1}$ as a background manifold and whose group structure is given by the non–Abelian law

$$\xi \circ \xi' = \left(z + z', t + t' + 2 \sum_{i=1}^{n} (y_i x'_i - x_i y'_i) \right)$$

for all $\xi, \xi' \in \mathbb{H}^n$, with

$$\xi = (z, t) = (x_1, \dots, x_n, y_1, \dots, y_n, t) \text{ and } \xi' = (z', t') = (x_1', \dots, x_n', y_1', \dots, y_n', t').$$

The inverse is given by $\xi^{-1} = -\xi$ and so $(\xi \circ \xi')^{-1} = (\xi')^{-1} \circ \xi^{-1}$.

The real Lie algebra of $\mathbb{H}^n$ is generated by the left–invariant vector fields on $\mathbb{H}^n$

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \qquad T = \frac{\partial}{\partial t},$$

for $j = 1, \dots, n$. This basis satisfies the Heisenberg canonical commutation relations

$$[X_j, Y_k] = -4\delta_{jk}T, \quad [Y_j, Y_k] = [X_j, X_k] = [Y_j, T] = [X_j, T] = 0.$$

Moreover, all the commutators of length greater than two vanish, and so $\mathbb{H}^n$ is a nilpotent graded stratified group of step two. A left invariant vector field X, which is in the span of $\{X_j, Y_j\}_{j=1}^n$, is called *horizontal*.

For each real positive number R, the dilation $\delta_R : \mathbb{H}^n \to \mathbb{H}^n$, naturally associated with the Heisenberg group structure, is defined by

$$\delta_R(\xi) = (Rz, R^2 t) \quad \text{for all } \xi = (z, t) \in \mathbb{H}^n.$$

It is easy to verify that the Jacobian determinant of the dilatation δ_R is constant and equal to R^{2n+2}, where the natural number $Q = 2n + 2$ is the *homogeneous dimension* of $\mathbb{H}^n$.

The anisotropic dilation structure on $\mathbb{H}^n$ introduces the *Korányi norm*, which is given by

$$r(\xi) = r(z, t) = (|z|^4 + t^2)^{1/4} \quad \text{for all } \xi = (z, t) \in \mathbb{H}^n.$$

Consequently, the Korányi norm is homogeneous of degree 1, with respect to the dilations δ_R, $R > 0$, that is

$$r(\delta_R(\xi)) = r(Rz, R^2 t) = (|Rz|^4 + R^4 t^2)^{1/4} = R\, r(\xi) \quad \text{for all } \xi = (z, t) \in \mathbb{H}^n.$$

Clearly, $\delta_R(\eta \circ \xi) = \delta_R(\eta) \circ \delta_R(\xi)$. The corresponding distance, the so called *Korányi distance*, is

$$d_K(\xi, \xi') = r(\xi^{-1} \circ \xi') \quad \text{for all } (\xi, \xi') \in \mathbb{H}^n \times \mathbb{H}^n.$$

Let $B_R(\xi_0) = \{\xi \in \mathbb{H}^n : d_K(\xi, \xi_0) < R\}$ be the *Korányi open ball* of radius R centered at ξ_0. For simplicity we put $B_R = B_R(O)$, where $O = (0, 0)$ is the natural origin of $\mathbb{H}^n$.

The Lebesgue measure on $\mathbb{R}^{2n+1}$ is invariant under the left translations of the Heisenberg group. Thus, since the Haar measures on Lie groups are unique up to constant multipliers, we denote by $d\xi$ the Haar measure on $\mathbb{H}^n$ that coincides with the $(2n + 1)$–Lebesgue measure and by $|U|$ the $(2n + 1)$–dimensional Lebesgue measure of any measurable set $U \subseteq \mathbb{H}^n$. Furthermore, the Haar measure on $\mathbb{H}^n$ is Q–homogeneous with respect to dilations δ_R. Consequently,

$$|\delta_R(U)| = R^Q |U|, \qquad d(\delta_R \xi) = R^Q d\xi.$$

In particular, $|B_R(\xi_0)| = |B_1| R^Q$ for all $\xi_0 \in \mathbb{H}^n$.

We define the *horizontal gradient* of a C^1 function $u : \mathbb{H}^n \to \mathbb{R}$ by

$$D_H u = \sum_{j=1}^{n} \left[(X_j u) X_j + (Y_j u) Y_j \right].$$

Clearly, $D_H u \in \operatorname{span}\{X_j, Y_j\}_{j=1}^n$. In $\operatorname{span}\{X_j, Y_j\}_{j=1}^n \simeq \mathbb{R}^{2n}$ we consider the natural inner product given by

$$(X, Y)_H = \sum_{j=1}^{n} \left(x^j y^j + \widetilde{x}^j \widetilde{y}^j \right)$$

for $X = \{x^j X_j + \widetilde{x}^j Y_j\}_{j=1}^n$ and $Y = \{y^j X_j + \widetilde{y}^j Y_j\}_{j=1}^n$. The inner product $(\cdot, \cdot)_H$ produces the Hilbertian norm

$$|X|_H = \sqrt{(X, X)_H}$$

for the horizontal vector field X.

For any horizontal vector field function $X = X(\xi)$, $X = \{x^j X_j + \widetilde{x}^j Y_j\}_{j=1}^n$, of class $C^1(\mathbb{H}^n, \mathbb{R}^{2n})$, we define the *horizontal divergence* of X by

$$\operatorname{div}_H X = \sum_{j=1}^{n} [X_j(x^j) + Y_j(\widetilde{x}^j)].$$

Similarly, if $u \in C^2(\mathbb{H}^n)$, then the *Kohn–Spencer Laplacian* in $\mathbb{H}^n$, or equivalently the *horizontal Laplacian*, or the *sub–Laplacian*, of u is

$$\Delta_H u = \sum_{j=1}^{n} (X_j^2 + Y_j^2) u = \sum_{j=1}^{n} \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + 4 y_j \frac{\partial^2}{\partial x_j \partial t} - 4 x_j \frac{\partial^2}{\partial y_j \partial t} \right) u + 4|z|^2 \frac{\partial^2 u}{\partial t^2}.$$

According to the celebrated Theorem 1.1 due to Hörmander in [19], the operator Δ_H is *hypoelliptic*. In particular, $\Delta_H u = \operatorname{div}_H D_H u$ for each $u \in C^2(\mathbb{H}^n)$. A well known generalization of the Kohn–Spencer Laplacian is the *horizontal p–Laplacian* on the Heisenberg group, $p \in (1, \infty)$, defined by

$$\Delta_{H,p} \varphi = \operatorname{div}_H (|D_H \varphi|_H^{p-2} D_H \varphi) \quad \text{for all } \varphi \in C_c^\infty(\mathbb{H}^n).$$

2.2. Classical Sobolev spaces in the Heisenberg group

Let us now review some useful facts about the classical Sobolev spaces on the Heisenberg group $\mathbb{H}^n$. We just consider the special case in which $1 \le p < Q$ and Ω is an open set in $\mathbb{H}^n$. Denote by $HW^{1,p}(\Omega)$ the *horizontal Sobolev space* consisting of the functions $u \in L^p(\Omega)$ such that $D_H u$ exists in the sense of distributions and $|D_H u|_H \in L^p(\Omega)$, endowed with the natural norm

$$\|u\|_{HW^{1,p}(\Omega)} = \left(\|u\|_{L^p(\Omega)}^p + \|D_H u\|_{L^p(\Omega)}^p \right)^{1/p}, \quad \|D_H u\|_{L^p(\Omega)} = \left(\int_\Omega |D_H u|_H^p d\xi \right)^{1/p}.$$

By [14] we know that if $1 \le p < Q$, then the embedding

$$HW^{1,p}(\Omega) \hookrightarrow L^s(\Omega) \text{ for all } s \in [p, p^*], \quad p^* = \frac{pQ}{Q - p},$$

is continuous.

Let us also briefly recall a version of the Rellich theorem in the Heisenberg group. This topic is largely treated in [13, 16, 17, 20] for vector fields satisfying the Hörmander condition. The general Hörmander vector fields have been introduced in [19] and include, as a special case, the horizontal vector fields (1.2) on the Heisenberg group. For our purposes it is sufficient to recall that for any p, with $1 \le p < Q$, and for any Korányi ball $B_R(\xi_0)$, the embedding

$$HW^{1,p}(B_R(\xi_0)) \hookrightarrow\hookrightarrow L^q(B_R(\xi_0)) \tag{2.1}$$

is compact, provided that $1 \le q < p^*$. This result holds, more in general, for bounded Poincaré–Sobolev domains Ω of $\mathbb{H}^n$ and was first established in [27], even for general Hörmander vector fields. For a complete treatment on this topic we mention, e.g., [16, 20, 25].

2.3. Fractional Sobolev spaces in the Heisenberg group

Let $s \in (0,1)$ and $1 < p < \infty$. We endow $HW^{s,p}(\mathbb{H}^n)$, defined in the Introduction, with the norm

$$\|\cdot\|_{HW^{s,p}(\mathbb{H}^n)} = \|\cdot\|_p + [\,\cdot\,]_{H,s,p}.$$

Our aim is to prove the compactness of the immersion $HW^{s,p}(\mathbb{H}^n) \hookrightarrow L^p(B_R(\xi_0))$ for all $\xi_0 \in \mathbb{H}^n$ and $R > 0$. The proof relies on a Lie group version of the celebrated Frèchet–Kolmogorov Compactness Theorem, cf. Theorem A.4.1 of [11]. First, we need the following lemma.

Lemma 2.1. *Let* $0 < s < 1$, $1 < p < \infty$. *Then, there exists a constant* $C = C(s,p,n) > 0$ *such that for any* $h \in \mathbb{H}^n$, *with* $0 < r(h) < 1/2$,

$$\|\tau_h u - u\|_p \le C r(h)^s [\,u\,]_{H,s,p} \quad \textit{for all } u \in HW^{s,p}(\mathbb{H}^n),$$

where $\tau_h u(\xi) = u(h \circ \xi)$ *for* $\xi \in \mathbb{H}^n$.

Proof. Fix $u \in HW^{s,p}(\mathbb{H}^n)$, $h \in \mathbb{H}^n$, with $0 < r(h) < 1/2$, and $\xi \in \mathbb{H}^n$. Take any $\eta \in B(\xi, r(h))$. Let us first observe that $r(\eta^{-1} \circ \xi) \le r(h)$, so that $r(\eta^{-1} \circ h \circ \xi) \le r(\eta^{-1} \circ \xi) + r(h) \le 2r(h)$ by the triangle inequality. Then,

$$|\tau_h u(\xi) - u(\xi)|^p \le 2^{p-1} \left(|u(h \circ \xi) - u(\eta)|^p + |u(\eta) - u(\xi)|^p \right).$$

Now, averaging in η over $B(\xi, r(h))$, we get

$$|\tau_h u(\xi) - u(\xi)|^p \le c \left(\frac{1}{r(h)^Q} \int_{B(\xi,r(h))} |u(h \circ \xi) - u(\eta)|^p d\eta + \frac{1}{r(h)^Q} \int_{B(\xi,r(h))} |u(\eta) - u(\xi)|^p d\eta \right),$$

with $c = c(s,p,n)$. Thus, integrating in ξ over $\mathbb{H}^n$, we obtain

$$\begin{aligned}
\|\tau_h u - u\|^p &\le c r(h)^{sp} \left(\int_{\mathbb{H}^n} \int_{B(\xi,r(h))} \frac{|u(h \circ \xi) - u(\eta)|^p}{r(h)^{Q+sp}} d\eta d\xi + \int_{\mathbb{H}^n} \int_{B(\xi,r(h))} \frac{|u(\eta) - u(\xi)|^p}{r(h)^{Q+sp}} d\eta d\xi \right) \\
&\le 2c r(h)^{sp} \left(\int_{\mathbb{H}^n} \int_{B(\xi,r(h))} \frac{|u(h \circ \xi) - u(\eta)|^p}{r(\eta^{-1} \circ h \circ \xi)^{Q+sp}} d\eta d\xi + \int_{\mathbb{H}^n} \int_{B(\xi,r(h))} \frac{|u(\eta) - u(\xi)|^p}{r(\eta^{-1} \circ \xi)^{Q+sp}} d\eta d\xi \right) \\
&\le C r(h)^{sp} [\,u\,]_{H,s,p},
\end{aligned}$$

with $C = 4c$. ◻

Theorem 2.2. *Let* $0 < s < 1$ *and* $1 < p < \infty$. *Then, for every sequence* $(u_k)_k$ *bounded in* $HW^{s,p}(\mathbb{H}^n)$ *there exists* $u \in HW^{s,p}(\mathbb{H}^n)$ *and a subsequence* $(u_{k_j})_j \subset (u_k)_k$ *such that for all* $\xi_0 \in \mathbb{H}^n$ *and* $R > 0$

$$u_{k_j} \to u \quad \text{in } L^p(B_R(\xi_0)) \quad \text{as } j \to \infty.$$

Proof. Let $M = \sup_{k\in\mathbb{N}} \|u_k\|_{HW^{s,p}(\mathbb{H}^n)}$. Clearly, if $(u_k)_k$ is bounded in $HW^{s,p}(\mathbb{H}^n)$, then is also bounded in $L^p(\mathbb{H}^n)$. Moreover, by Lemma 2.1, we know that

$$\|\tau_h u_k - u_k\|_p \le Cr(h)^s [\, u_k \,]_{H,s,p} \le CMr(h)^s.$$

Consequently,

$$\lim_{h\to O} \sup_{k\in\mathbb{N}} \|\tau_h u_k - u_k\|_p = 0.$$

Therefore, a Lie group version of the Fréchet–Kolmogorov theorem, cf. Theorem A.4.1 of [11], yields the existence of a function $u \in L^p(\mathbb{H}^n)$ and a subsequence of $(u_k)_k$, still denoted $(u_k)_k$, such that $u_k \to u$ a.e. in $\mathbb{H}^n$ and $u_k \to u$ in $L^p(B_R(\xi_0))$ for all $\xi_0 \in \mathbb{H}^n$ and $R > 0$.

It remains to prove that $u \in HW^{s,p}(\mathbb{H}^n)$. This follows straightly from an application of Fatou's Lemma. Indeed,

$$0 \le \lim_{k\to\infty} \frac{|u_k(\eta) - u_k(\xi)|^p}{r(\eta^{-1} \circ \xi)^{Q+sp}} = \frac{|u(\eta) - u(\xi)|^p}{r(\eta^{-1} \circ \xi)^{Q+sp}} \quad \text{for a.e. } (\xi, \eta) \in \mathbb{H}^n \times \mathbb{H}^n.$$

Consequently, Fatou's Lemma, together with the lower semicontinuity of $[\,\cdot\,]_{H,s,p}$, gives

$$[u]_{H,s,p} \le \lim_{k\to\infty} [u_k]_{H,s,p} \le \sup_{k\in\mathbb{N}} [u_k]_{H,s,p} < \infty.$$

This concludes the proof. □

3. Proof of Theorems 1.1 and 1.2

This section is devoted to the proof of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Let $(u_k)_k$ be a sequence in $S^{1,p}(\mathbb{H}^n)$ as in the statement of the theorem. Obviously, (1.7), (1.8) and the first part of (1.10) follow from Theorem A, see [20]. Thus, there is no reason to repeat the proof here. Let us then focus on the proof of (1.9). We proceed diving the argument into two cases.

Case 1. $u = 0$. Fix $\varphi \in C_c^\infty(\mathbb{H}^n)$. Then, since clearly $\varphi u_k \in S^{1,p}(\mathbb{H}^n)$ for all k, we get by (1.6)

$$\mathcal{H}_p \|\varphi u_k\|_{H_p}^p \le \int_{\mathbb{H}^n} |\varphi|^p |D_H u_k|_H^p d\xi + \|D_H \varphi u_k\|_p^p. \tag{3.1}$$

Now, by the subelliptic Rellich Theorem, see (2.1), we know that $u_k \to 0$ in $L^p(B_R)$ for all $R > 0$. Therefore,

$$\lim_{k\to\infty} \|D_H \varphi u_k\|_p = 0. \tag{3.2}$$

Consequently, by the weak* convergence and (3.2), letting $k \to \infty$, we obtain

$$\Big(\int_{\mathbb{H}^n} |\varphi|^p d\omega\Big)^{1/p} \le \mathcal{H}_p^{-1/p} \Big(\int_{\mathbb{H}^n} |\varphi|^p d\mu\Big)^{1/p} \qquad \text{for all } \varphi \in C_c^\infty(\mathbb{H}^n).$$

Thus, by Lemma 1.4.6 of [20], we conclude that there exist an at most countable set J, a family of points $\{\xi_j\}_{j\in J} \subset \mathbb{H}^n$ and a family of nonnegative numbers $\{\omega_j\}_{j\in J\cup\{0\}}$, such that

$$\omega = \omega_0\delta_O + \sum_{j\in J} \omega_j \delta_{\xi_j}. \tag{3.3}$$

Clearly, the set J determined in (3.3) is not necessary the same of the one obtained in the representation of ν. However, since the coefficients ν_j, μ_j, ω_j are allowed to be 0, we can replace these two sets with their union (which is still at most countable). For this reason we keep the same notation J for the index set.

In order to conclude the proof of (1.9) on Case 1, it remains to show that ω is concentrated at O, namely that $\omega_j = 0$ for any $j \in J$. But this is obvious. Indeed, fix $\varphi \in C_c^\infty(\mathbb{H}^n)$, with $O \notin \operatorname{supp}\varphi$, so that $\xi \mapsto |\varphi(\xi)|^p \psi^p r(\xi)^{-p}$ is in $L^\infty(\operatorname{supp}\varphi)$. Then, since obviously $u_k \to 0$ in $L^p(\operatorname{supp}\varphi)$, we get

$$\int_{\mathbb{H}^n} |\varphi|^p |u_k|^p \psi^p \frac{d\xi}{r(\xi)^p} = \int_{\operatorname{supp}\varphi} |\varphi|^p |u_k|^p \psi^p \frac{d\xi}{r(\xi)^p} \leq C \int_{\operatorname{supp}\varphi} |u_k|^p d\xi \to 0$$

as $k \to \infty$. This, combined with the weak* convergence, gives $\int_{\mathbb{H}^n} |\varphi|^p d\omega = 0$, that is ω is a measure concentrated in O. Hence $\omega = \omega_0\delta_O$, and so (1.9) in proved in Case 1.

Case 2. $u \neq 0$. Set $\widetilde{u}_k = u_k - u$. Clearly, $\widetilde{u}_k \rightharpoonup 0$ in $S^{1,p}(\mathbb{H}^n)$ and (3.1) still holds for $\varphi\widetilde{u}_k$ for any $\varphi \in C_c^\infty(\mathbb{H}^n)$. Moreover, thanks to Case 1, there exists a finite nonnegative Radon measure $\widetilde{\omega}$ on $\mathbb{H}^n$, such that, up to a subsequence still labelled $(\widetilde{u}_k)_k$, we have as $k \to \infty$

$$|\widetilde{u}_k|^p \psi^p \frac{d\xi}{r(\xi)^p} \overset{*}{\rightharpoonup} \widetilde{\omega} \quad \text{in } \mathcal{M}(\mathbb{H}^n), \tag{3.4}$$

where

$$\widetilde{\omega} = \omega_0 \delta_O, \tag{3.5}$$

and ω_0 is an appropriate nonnegative number as shown in Case 1. Now, by (2.1), up to a subsequence,

$$u_k \to u \quad \text{a.e. in } \mathbb{H}^n, \qquad |u_k| \leq g_R \quad \text{a.e. in } \mathbb{H}^n$$

for some $g_R \in L^p(B_R)$ and all $R > 0$. Thus, for all $\varphi \in C_c^\infty(\mathbb{H}^n)$ an application of Brézis–Lieb lemma yields

$$\lim_{k\to\infty} \left(\|\varphi u_k\|_{H_p}^p - \|\varphi\widetilde{u}_k\|_{H_p}^p\right) = \|\varphi u\|_{H_p}^p.$$

A combination of the above formulas gives for all $\varphi \in C_c^\infty(\mathbb{H}^n)$

$$|u_k|^p \psi^p \frac{d\xi}{r(\xi)^p} - |u|^p \psi^p \frac{d\xi}{r(\xi)^p} = |u_k - u|^p \psi^p \frac{d\xi}{r(\xi)^p} - o(1), \tag{3.6}$$

where $o(1) \overset{*}{\rightharpoonup} 0$ in $\mathcal{M}(\mathbb{H}^n)$. Then, computing the limit in (3.6), by the weak* convergence and (3.4), we get $\widetilde{\omega} = \omega - |u_k - u|^p \psi^p \dfrac{d\xi}{r(\xi)^p}$. Consequently, taking into account (3.5), we obtain (1.9).

It remains to prove that $\nu_0^{p/p^*} \leq (\mu_0 - \sigma\omega_0)/\mathcal{I}_\sigma$. Fix $\varphi \in C_c^\infty(\mathbb{H}^n)$ such that $0 \leq \varphi \leq 1$, $\varphi(O) = 1$ and $\operatorname{supp} \varphi = \overline{B_1}$. Take $\varepsilon > 0$ and put $\varphi_\varepsilon(\xi) = \varphi(\delta_{1/\varepsilon}(\xi))$, $\xi \in \mathbb{H}^n$. Then,

$$\begin{aligned} \mathcal{I}_\sigma \left(\int_{\mathbb{H}^n} |\varphi_\varepsilon|^{p^*} |u_k|^{p^*} d\xi \right)^{p/p^*} &\leq \int_{\mathbb{H}^n} |\varphi_\varepsilon|^p |D_H u_k|_H^p d\xi + \|D_H \varphi_\varepsilon u_k\|_p^p - \sigma \int_{\mathbb{H}^n} |\varphi_\varepsilon|^p |u_k|^p \psi^p \frac{d\xi}{r(\xi)^p} \\ &= \int_{\mathbb{H}^n} |\varphi_\varepsilon|^p |D_H u_k|_H^p d\xi + o(1) - \sigma \int_{\mathbb{H}^n} |\varphi_\varepsilon|^p |u_k|^p \psi^p \frac{d\xi}{r(\xi)^p}, \end{aligned} \tag{3.7}$$

arguing as before. Now, we know that

$$\lim_{\varepsilon \to 0^+} \lim_{k \to \infty} \int_{\mathbb{H}^n} |\varphi_\varepsilon|^{p^*} |u_k|^{p^*} d\xi = \nu_0, \tag{3.8}$$

and

$$\lim_{\varepsilon \to 0^+} \lim_{k \to \infty} \int_{\mathbb{H}^n} |\varphi_\varepsilon|^p |D_H u_k|_H^p d\xi = \mu_0. \tag{3.9}$$

Finally, from (1.9) and the fact that $\omega \geq \omega_0 \delta_O$ we get

$$\lim_{\varepsilon \to 0^+} \lim_{k \to \infty} \int_{\mathbb{H}^n} |\varphi_\varepsilon|^p |u_k|^p \psi^p \frac{d\xi}{r(\xi)^p} = \lim_{\varepsilon \to 0^+} \int_{B_\varepsilon} |\varphi_\varepsilon|^p d\omega \geq \omega_0. \tag{3.10}$$

Hence, passing to the limit as $k \to \infty$ and $\varepsilon \to 0^+$ in (3.7), by (3.8)–(3.10) we obtain that

$$\mathcal{I}_\sigma \nu_0^{p/p^*} \leq \mu_0 - \sigma\omega_0.$$

This concludes the proof. □

In Theorem 1.1 we examine the behavior of weakly convergent sequences in the Folland–Stein space in situations in which the lack of compactness occurs. However, this method does not exclude a possible loss of compactness due to the *drifting towards infinity* of the mass, or – in other words – the concentration at infinity. Let us then turn to the proof of the concentration–compactness principle at infinity, which extend the method introduced in the Euclidean setting in [3, 6].

Proof of Theorem 1.2. Fix a sequence $(u_k)_k$ in $S^{1,p}(\mathbb{H}^n)$, as in the statement of the Theorem 1.1.

Let $\Psi \in C^\infty(\mathbb{H}^n)$ be such that $0 \leq \Psi \leq 1$, $\Psi = 0$ in B_1 and $\Psi = 1$ in B_2^c. Take $R > 0$ and put $\Psi_R(\xi) = \Psi(\delta_{1/R}(\xi))$, $\xi \in \mathbb{H}^n$. Write

$$\int_{\mathbb{H}^n} |D_H u_k|_H^p d\xi = \int_{\mathbb{H}^n} |D_H u_k|_H^p |\Psi_R|^p d\xi + \int_{\mathbb{H}^n} |D_H u_k|_H^p (1 - |\Psi_R|^p) d\xi. \tag{3.11}$$

We first observe that

$$\int_{B_{2R}^c} |D_H u_k|_H^p d\xi \leq \int_{\mathbb{H}^n} |D_H u_k|_H^p |\Psi_R|^p d\xi \leq \int_{B_R^c} |D_H u_k|_H^p d\xi$$

and so by (1.11)

$$\mu_\infty = \lim_{R \to \infty} \limsup_{k \to \infty} \int_{\mathbb{H}^n} |D_H u_k|_H^p |\Psi_R|^p d\xi. \tag{3.12}$$

On the other hand, since μ is finite, $1-|\Psi_R|^p$ has compact support and $\Psi_R \to 0$ a.e. in $\mathbb{H}^n$, we have by the definition of μ and the dominated convergence theorem that

$$\lim_{R\to\infty}\limsup_{k\to\infty}\int_{\mathbb{H}^n}|D_H u_k|_H^p(1-|\Psi_R|^p)d\xi = \lim_{R\to\infty}\int_{\mathbb{H}^n}(1-|\Psi_R|^p)d\mu = \mu(\mathbb{H}^n). \tag{3.13}$$

Using (3.12) and (3.13) in (3.11) we obtain the second part of (1.13). Arguing similarly for ν and ω, we see that

$$\nu_\infty = \lim_{R\to\infty}\limsup_{k\to\infty}\int_{\mathbb{H}^n}|\Psi_R|^{p^*}|u_k|^{p^*}d\xi, \quad \omega_\infty = \lim_{R\to\infty}\limsup_{k\to\infty}\int_{\mathbb{H}^n}|\Psi_R|^p|u_k|^p\psi^p\frac{d\xi}{r(\xi)^p}, \tag{3.14}$$

and

$$\lim_{R\to\infty}\limsup_{k\to\infty}\int_{\mathbb{H}^n}(1-|\Psi_R|^{p^*})|u_k|^{p^*}d\xi = \nu(\mathbb{H}^n), \quad \lim_{R\to\infty}\limsup_{k\to\infty}\int_{\mathbb{H}^n}(1-|\Psi_R|^p)|u_k|^p\psi^p\frac{d\xi}{r(\xi)^p} = \omega(\mathbb{H}^n).$$

Thus, (1.13)–(1.14) are proved in the same way.

In order to show the last part of (1.14), let us consider again the regular function Ψ_R. Then, since $0 \le \Psi_R \le 1$, by (1.5) applied to $\Psi_R u_k \in S^{1,p}(\mathbb{H}^n)$, we get for all k

$$\mathcal{I}_\sigma\left(\int_{\mathbb{H}^n}|\Psi_R|^{p^*}|u_k|^{p^*}d\xi\right)^{p/p^*} \le \int_{\mathbb{H}^n}|\Psi_R|^p|D_H u_k|_H^p d\xi + \|D_H\Psi_R u_k\|_p^p - \sigma\int_{\mathbb{H}^n}|\Psi_R|^p|u_k|^p\psi^p\frac{d\xi}{r(\xi)^p}. \tag{3.15}$$

Finally, from the fact that $\lim_{R\to\infty}\limsup_{k\to\infty}\|D_H\Psi_R u_k\|_p^p = 0$, using (3.12) and (3.14) in (3.15) we obtain the desired conclusion. □

4. Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3. Before getting there, we need some preliminary results.

Lemma 4.1. *Let $\varphi \in C_c^\infty(\mathbb{H}^n)$, $\varepsilon > 0$ and $\xi_0 \in \mathbb{H}^n$. Define $\mathbb{H}^n \ni \xi \mapsto \varphi_\varepsilon(\xi) = \varphi(\delta_{1/\varepsilon}(\xi_0^{-1}\circ\xi))$. Then,*

$$|D_H^s\varphi_\varepsilon(\xi)|^p = \frac{1}{\varepsilon^{sp}}\left|D_H^s\varphi\left(\delta_{1/\varepsilon}(\xi_0^{-1}\circ\xi)\right)\right|^p.$$

Proof. Fix $\varphi \in C_c^\infty(\mathbb{H}^n)$, $\varepsilon > 0$ and $\xi_0 \in \mathbb{H}^n$. The proof is a simple consequence of the change of variables formula. Indeed, if we put $\eta = \delta_{1/\varepsilon}(h)$, $d\eta = \varepsilon^{-Q}dh$, then

$$\begin{aligned}|D_H^s\varphi_\varepsilon(\xi)|^p &= \int_{\mathbb{H}^n}\frac{|\varphi_\varepsilon(\xi\circ h)-\varphi_\varepsilon(\xi)|^p}{r(h)^{Q+ps}}dh\\ &= \int_{\mathbb{H}^n}\frac{\left|\varphi(\delta_{1/\varepsilon}(\xi_0^{-1}\circ\xi\circ h)) - \varphi(\delta_{1/\varepsilon}(\xi_0^{-1}\circ\xi))\right|^p}{r(h)^{Q+ps}}dh\\ &= \int_{\mathbb{H}^n}\frac{\left|\varphi(\delta_{1/\varepsilon}(\xi_0^{-1}\circ\xi)\circ\delta_{1/\varepsilon}(h)) - \varphi(\delta_{1/\varepsilon}(\xi_0^{-1}\circ\xi))\right|^p}{r(h)^{Q+ps}}dh\\ &= \frac{1}{\varepsilon^{sp}}\int_{\mathbb{H}^n}\frac{\left|\varphi(\delta_{1/\varepsilon}(\xi_0^{-1}\circ\xi\circ\eta) - \varphi(\delta_{1/\varepsilon}(\xi_0^{-1}\circ\xi))\right|^p}{r(\eta)^{Q+ps}}d\eta,\end{aligned}$$

as required thanks to (1.17). □

Note that, in general, the nonlocal (s, p) horizontal gradient of a compactly supported function does not need to have compact support. For this, we use the following lemma, which gives valuable decay estimates of the fractional (s, p) horizontal gradient of a $C_c^\infty(\mathbb{H}^n)$ function as $r(\xi) \to \infty$. The next lemma is an extension of Lemma 2.2 of [4] to the Heisenberg setting.

Lemma 4.2. *Let $\varphi \in C_c^\infty(\mathbb{H}^n)$ be such that $0 \le \varphi \le 1$, $\operatorname{supp}\varphi \subset B_R$ for some $R > 0$.Then, there exists a constant $C = C(s, p, n)$ such that for any $\xi \in \mathbb{H}^n$*

$$|D_H^s\varphi(\xi)|^p \le C \min\{1, R^Q r(\xi)^{-(Q+sp)}\}.$$

In particular, $|D_H^s\varphi|^p \in L^\infty(\mathbb{H}^n)$.

Proof. Let us first prove the global L^∞ bound. Consider for any $\xi \in \mathbb{H}^n$

$$|D^s\varphi(\xi)|^p = \int_{\mathbb{H}^n} \frac{|\varphi(\xi \circ h) - \varphi(\xi)|^p}{r(h)^{Q+ps}} dh = \left(\int_{B_1} + \int_{B_1^c}\right) \frac{|\varphi(\xi \circ h) - \varphi(\xi)|^p}{r(h)^{Q+ps}} dh$$

and compute separately the last two integrals. By the mean value theorem and [14]

$$\int_{B_1} \frac{|\varphi(\xi \circ h) - \varphi(\xi)|^p}{r(h)^{Q+ps}} dh \le C_1 \int_{B_1} \frac{1}{r(h)^{Q+sp-p}} dh \le C_2,$$

since $Q + sp - p < Q$. On the other hand,

$$\int_{B_1^c} \frac{|\varphi(\xi \circ h) - \varphi(\xi)|^p}{r(h)^{Q+ps}} dh \le 2^p \|\varphi\|_\infty^p \int_{B_1^c} \frac{1}{r(h)^{Q+sp}} dh \le C_3,$$

being obviously $Q + sp > Q$. Now, consider $\xi \in \mathbb{H}^n$ with $r(\xi) \ge 2R$. Clearly, $\varphi(\xi) = 0$ and so

$$|D^s\varphi(\xi)|^p = \int_{\mathbb{H}^n} \frac{|\varphi(\xi \circ h)|^p}{r(h)^{Q+ps}} dh = \int_{r(\xi\circ h)<R} \frac{|\varphi(\xi \circ h)|^p}{r(h)^{Q+ps}} dh.$$

Now, if $r(\xi \circ h) < R$ and $r(\xi) > 2R$, then $r(\xi) - r(h) \le r(\xi \circ h)$ so that $r(h) \ge r(\xi) - R \ge r(\xi)/2$. Therefore,

$$|D^s\varphi(\xi)|^p \le \frac{2^{Q+sp}}{r(\xi)^{Q+sp}} \|\varphi\|_\infty^p \int_{r(\xi\circ h)<R} dh \le CR^Q r(\xi)^{-(Q+sp)}.$$

This concludes the proof of the lemma. □

Combining Lemma 4.1 and Lemma 4.2, we obtain the following.

Corollary 4.3. *Let $\varphi \in C_c^\infty(\mathbb{H}^n)$ be such that $0 \le \varphi \le 1$ and $\operatorname{supp}\varphi \subset B_1$. Let $\xi_0 \in \mathbb{H}^n$ and define $\mathbb{H}^n \ni \xi \mapsto \varphi_\varepsilon(\xi) = \varphi(\delta_{1/\varepsilon}(\xi_0^{-1} \circ \xi))$. Then, there exists a constant $C = C(s, p, n)$ such that for any $\xi \in \mathbb{H}^n$*

$$|D_H^s\varphi_\varepsilon(\xi)|^p \le C \min\{\varepsilon^{-sp}, \varepsilon^Q r(\xi)^{-(Q+sp)}\}.$$

Using the previous estimates, we are able to prove the next result, which is an extension of Lemma 2.4 of [4] to the Heisenberg context.

Lemma 4.4. *Let $\varphi \in C_c^\infty(\mathbb{H}^n)$. Then, the following embedding is compact*

$$HW^{s,p}(\mathbb{H}^n) \hookrightarrow\hookrightarrow L^p(\mathbb{H}^n, |D_H^s\varphi|^p d\xi).$$

Proof. Let $(u_k)_k$ be a bounded sequence in $HW^{s,p}(\mathbb{H}^n)$, say $\sup_k \|u_k\|_{HW^{s,p}(\mathbb{H}^n)} \le M$, with $M > 0$. From the reflexivity of $HW^{s,p}(\mathbb{H}^n)$ and Theorem 2.2, there exist $u \in HW^{s,p}(\mathbb{H}^n)$ and a subsequence, still denoted by $(u_k)_k$, such that

$$u_k \rightharpoonup u \text{ in } HW^{s,p}(\mathbb{H}^n), \qquad u_k \to u \in L^p(B_R) \quad \text{for any } R > 0. \tag{4.1}$$

Fix $R > 0$ so large that $\operatorname{supp}\varphi \subset B_R$. Certainly,

$$\int_{\mathbb{H}^n} |u_k(\xi) - u(\xi)|^p |D_H^s\varphi(\xi)|^p d\xi = \left(\int_{B_{2R}} + \int_{B_{2R}^c}\right) |u_k(\xi) - u(\xi)|^p |D_H^s\varphi(\xi)|^p d\xi.$$

Now, from Lemma 4.2 and (4.1)

$$\int_{B_{2R}} |u_k(\xi) - u(\xi)|^p |D_H^s\varphi(\xi)|^p d\xi \le \|D_H^s\varphi\|_\infty^p \int_{B_{2R}} |u_k(\xi) - u(\xi)|^p d\xi = o(1) \tag{4.2}$$

as $k \to \infty$. On the other hand, using (1.16) and the Hölder inequality with $q = p_s^*/p = Q/(Q - sp)$ and $q' = Q/sp$, we get by Lemma 4.2

$$\begin{aligned}
\int_{B_{2R}^c} |u_k(\xi) - u(\xi)|^p |D_H^s\varphi(\xi)|^p d\xi &\le \|u_k - u\|_{p_s^*}^p \left(\int_{B_{2R}^c} \left|\int_{\mathbb{H}^n} \frac{|\varphi(\xi \circ h) - \varphi(\xi)|^p}{r(h)^{Q+ps}} dh\right|^{Q/s} d\xi\right)^{sp/Q} \\
&\le 2^{p^2-1}\|\varphi\|_\infty^{p^2}(\|u_k\|_{p_s^*}^p + \|u\|_{p_s^*}^p)\left(\int_{B_{2R}^c} \frac{d\xi}{r(\xi)^{Q(1+Q/sp)}}\right)^{sp/Q} \\
&\le C\left(\int_{B_{2R}^c} \frac{d\xi}{r(\xi)^{Q(1+Q/sp)}}\right)^{sp/Q}
\end{aligned} \tag{4.3}$$

where $C = 2^{p^2}\|\varphi\|_\infty^{p^2} M^p$. Now, for any $\tau > 0$ we can choose $R > 0$ ever larger, if necessary, so that

$$\left(\int_{B_{2R}^c} \frac{d\xi}{r(\xi)^{Q(1+Q/sp)}}\right)^{sp/Q} < \frac{\tau}{C}.$$

Finally, by (4.2) and (4.3), we obtain

$$\limsup_{k\to\infty} \int_{\mathbb{H}^n} |u_k(\xi) - u(\xi)|^p |D_H^s\varphi(\xi)|^p d\xi \le \tau$$

for all $\tau > 0$. Sending $\tau \to 0^+$ we get the desired conclusion. □

The proof of the next lemma is based on the precise decay rate of $|D_H^s\varphi_\varepsilon|^p$, cf. Corollary 4.3. The main difficulty here, as we already pointed out in the Introduction, is based essentially on the fact that the nonlocal (s, p) horizontal gradient $|D_H^s\varphi_\varepsilon|^p$ does not need to have compact support. The proof uses the same strategy of Lemma 4.4, which is effective thanks to the decay estimates given in Lemma 4.1 and Corollary 4.3.

Lemma 4.5. *Let $(u_k)_k$ be a bounded sequence in $HW^{s,p}(\mathbb{H}^n)$ and let $\varphi \in C_c^\infty(\mathbb{H}^n)$ be such that $0 \le \varphi \le 1$, $\varphi(O) = 1$ and supp $\varphi \subset B_1$. Take $\varepsilon > 0$, fix $\xi_0 \in \mathbb{H}^n$ and put $\mathbb{H}^n \ni \xi \mapsto \varphi_\varepsilon(\xi) = \varphi(\delta_{1/\varepsilon}(\xi_0^{-1} \circ \xi))$. Then*

$$\lim_{\varepsilon\to 0^+} \limsup_{k\to\infty} \int_{\mathbb{H}^n} |D_H^s\varphi_\varepsilon|^p |u_k|^p d\xi = 0.$$

Proof. Let $(u_k)_k$ be a bounded sequence in $HW^{s,p}(\mathbb{H}^n)$, say $\sup_k \|u_k\|_{HW^{s,p}(\mathbb{H}^n)} = M$. From the reflexivity of $HW^{s,p}(\mathbb{H}^n)$ and Theorem 2.2, there exist $u \in HW^{s,p}(\mathbb{H}^n)$ and a subsequence, still denoted by $(u_k)_k$, such that

$$u_k \rightharpoonup u \text{ in } HW^{s,p}(\mathbb{H}^n), \qquad u_k \to u \text{ in } L^p(B_R(\xi_0)), \tag{4.4}$$

for any $R > 0$ and $\xi_0 \in \mathbb{H}^n$. Clearly,

$$\int_{\mathbb{H}^n} |D^s_H \varphi_\varepsilon|^p |u_k|^p d\xi = \left(\int_{B_\varepsilon(\xi_0)} + \int_{B^c_\varepsilon(\xi_0)} \right) |D^s_H \varphi_\varepsilon|^p |u_k|^p d\xi.$$

Let us first estimate the integral over $B_\varepsilon(\xi_0)$. By Corollary 4.3 there exists $C = C(s, p, n) > 0$ such that

$$\limsup_{k\to\infty} \int_{B_\varepsilon(\xi_0)} |D^s_H \varphi_\varepsilon|^p |u_k|^p d\xi \le C \limsup_{k\to\infty} \varepsilon^{-sp} \int_{B_\varepsilon(\xi_0)} |u_k|^p = C\varepsilon^{-sp} \int_{B_\varepsilon(\xi_0)} |u|^p d\xi \underset{\varepsilon\to 0^+}{\longrightarrow} 0 \tag{4.5}$$

thanks to (4.4) and the Lebesgue theorem, being $sp < Q$.

Now we turn to the integral over $B^c_\varepsilon(\xi_0)$. Using the Hölder inequality with $q = p^*_s/p = Q/(Q - sp)$ and $q' = Q/sp$, and again Corollary 4.3, we get

$$\begin{aligned} \int_{B^c_\varepsilon(\xi_0)} |D^s_H \varphi_\varepsilon|^p |u_k|^p d\xi &\le \|u_k\|^p_{p^*_s} \left(\int_{B^c_\varepsilon(\xi_0)} |D^s_H \varphi_\varepsilon(\xi)|^{Q/s} d\xi \right)^{sp/Q} \le C\, M^p \varepsilon^Q \left(\int_{B^c_\varepsilon} \frac{d\xi}{r(\xi)^{Q(1+Q/sp)}} \right)^{sp/Q} \\ &\le C\, M^p |B_1|^{sp/Q} \varepsilon^Q \left(\varepsilon^{Q(1-Q/sp)} \right)^{sp/Q} = C\, M^p |B_1|^{sp/Q} \varepsilon^{sp}. \end{aligned}$$

Therefore, it follows that

$$\limsup_{k\to\infty} \int_{B^c_\varepsilon} |D^s_H \varphi_\varepsilon|^p |u_k|^p d\xi \le C\, M^p |B_1|^{sp/Q} \varepsilon^{sp}. \tag{4.6}$$

Finally, using (4.5) and (4.6), we conclude

$$\lim_{\varepsilon\to 0^+} \limsup_{k\to\infty} \int_{\mathbb{H}^n} |D^s_H \varphi_{\varepsilon,j}|^p |u_k|^p d\xi \le \lim_{\varepsilon\to 0^+} \left(C\varepsilon^{-sp} \int_{B_\varepsilon(\xi_0)} |u|^p d\xi + C\, M^p |B_1|^{sp/Q} \varepsilon^{sp} \right) = 0,$$

as required. □

Lemma 4.5 extends to the Heisenberg case a remark given in the proof of Theorem 1.1 of [4], stated in the Euclidean framework.

Proof of Theorem 1.3. Let $(u_k)_k$ be a sequence in $HW^{s,p}(\mathbb{H}^n)$, as in the statement of the theorem, and let us divide the proof into two cases.
Case 1. $u = 0$. Fix $\varphi \in C^\infty_c(\mathbb{H}^n)$. Then, an application of Lemma 4.4 immediately yields

$$\iint_{\mathbb{H}^n\times\mathbb{H}^n} |u_k(\xi)|^p \frac{|\varphi(\xi) - \varphi(\eta)|^p}{r(\eta^{-1} \circ \xi)^{Q+sp}} d\xi d\eta = o(1),$$

as $k \to \infty$. Consequently, since $\varphi u_k \in HW^{s,p}(\mathbb{H}^n)$ for all k, we get

$$
\begin{aligned}
C_{p_s^*}\Big(\int_{\mathbb{H}^n}|\varphi|^{p_s^*}|u_k|^{p_s^*}d\xi\Big)^{p/p_s^*} &\le [\varphi u_k]_{H,s,p}^p = \left(\iint_{\mathbb{H}^n\times\mathbb{H}^n}\frac{|(\varphi u_k)(\xi)-(\varphi u_k)(\eta)|^p}{r(\eta^{-1}\circ\xi)^{Q+sp}}d\xi d\eta\right)\\
&\le 2^{p-1}\left(\iint_{\mathbb{H}^n\times\mathbb{H}^n}|\varphi(\eta)|^p\frac{|u_k(\xi)-u_k(\eta)|^p}{r(\eta^{-1}\circ\xi)^{Q+sp}}d\xi d\eta\right.\\
&\qquad\left.+\iint_{\mathbb{H}^n\times\mathbb{H}^n}|u_k(\xi)|^p\frac{|\varphi(\xi)-\varphi(\eta)|^p}{r(\eta^{-1}\circ\xi)^{Q+sp}}d\xi d\eta\right)\\
&\le 2^{p-1}\int_{\mathbb{H}^n}|\varphi|^p|D_H^s u_k|^p d\xi + o(1)
\end{aligned}
\tag{4.7}
$$

as $k \to \infty$. Therefore, passing to the limit in (4.7), by the weak* convergence we have the following reverse Hölder inequality

$$
\Big(\int_{\mathbb{H}^n}|\varphi|^{p_s^*}d\nu\Big)^{1/p_s^*} \le C\Big(\int_{\mathbb{H}^n}|\varphi|^p d\mu\Big)^{1/q} \qquad \text{for all } \varphi \in C_c^\infty(\mathbb{H}^n).
$$

Thus, by Lemma 1.4.6 of [20], we conclude that there exist an at most countable set J, a family of points $\{\xi_j\}_{j\in J} \subset \mathbb{H}^n$ and a family of nonnegative numbers $\{\nu_j\}_{j\in J}$ such that

$$
\nu = \sum_{j\in J}\nu_j\delta_{\xi_j} \tag{4.8}
$$

Case 2. $u \neq 0$. Set $\widetilde{u}_k = u_k - u$. Clearly, $\widetilde{u}_k \rightharpoonup 0$ in $HW^{s,p}(\mathbb{H}^n)$ and (4.7) still holds for $\varphi\widetilde{u}_k$ for any $\varphi \in C_c^\infty(\mathbb{H}^n)$. Moreover, $k \mapsto |\widetilde{u}_k|^{p_s^*}d\xi$ and $k \mapsto |D_H^s\widetilde{u}_k|^p d\xi$ are still bounded sequences of measures and so by Proposition 1.202 of [15], we can conclude that there exist two bounded nonnegative Radon measure $\widetilde{\nu}$ and $\widetilde{\mu}$ on $\mathbb{H}^n$, such that, up to a subsequence, we have

$$
|D_H^s\widetilde{u}_k|^p d\xi \overset{*}{\rightharpoonup} \widetilde{\mu}, \qquad |\widetilde{u}_k|^{p_s^*}d\xi \overset{*}{\rightharpoonup} \widetilde{\nu} \quad \text{in } \mathcal{M}(\mathbb{H}^n). \tag{4.9}
$$

Thus, from Case 1 there exist an at most countable set J, a family of points $\{\xi_j\}_{j\in J} \subset \mathbb{H}^n$ and a family of nonnegative numbers $\{\nu_j\}_{j\in J}$ such that $\widetilde{\nu} = \sum_{j\in J}\nu_j\delta_{\xi_j}$. Consequently, the claimed representation (1.7) of ν follows exactly as in Theorem 1.1.

Let us now prove the first part of (1.10). Fix a test function $\varphi \in C_c^\infty(\mathbb{H}^n)$, such that $0 \le \varphi \le 1$, $\varphi(O) = 1$ and supp $\varphi \subset B_1$. Take $\varepsilon > 0$ and put $\varphi_{\varepsilon,j}(\xi) = \varphi(\delta_{1/\varepsilon}(\xi_j^{-1}\circ\xi))$, $\xi \in \mathbb{H}^n$, for any fixed $j \in J$, where $\{\xi_j\}_j$ is introduced in (1.7). Fix $j \in J$ and $\tau > 0$. Then, there exists $C_\tau > 0$ such that, by (1.16) applied to $\varphi_{\varepsilon,j}u_k$, we have

$$
\begin{aligned}
C_{p_s^*}\Big(\int_{\mathbb{H}^n}|\varphi_{\varepsilon,j}|^{p_s^*}|u_k|^{p_s^*}d\xi\Big)^{p/p_s^*} &\le \iint_{\mathbb{H}^n\times\mathbb{H}^n}\frac{\big|(\varphi_{\varepsilon,j}u_k)(\xi)-(\varphi_{\varepsilon,j}u_k)(\eta)\big|^p}{r(\eta^{-1}\circ\xi)^{Q+sp}}d\xi d\eta\\
&\le (1+\tau)\int_{\mathbb{H}^n}|\varphi_{\varepsilon,j}|^p|D_H^s u_k|^p d\xi + C_\tau\int_{\mathbb{H}^n}|D_H^s\varphi_{\varepsilon,j}|^p|u_k|^p d\xi.
\end{aligned}
\tag{4.10}
$$

We aim to pass to the limit in (4.10) as $k \to \infty$ and $\varepsilon \to 0^+$. To do this, let us observe first that from the weak* convergence and (1.7) we get

$$\begin{aligned}\lim_{\varepsilon\to0^+}\lim_{k\to\infty}\int_{\mathbb{H}^n}|\varphi_{\varepsilon,j}|^{p_s^*}|u_k|^{p_s^*}d\xi &= \lim_{\varepsilon\to0^+}\int_{B_\varepsilon(\xi_j)}|\varphi_{\varepsilon,j}|^{p_s^*}d\nu \\ &= \lim_{\varepsilon\to0^+}\left\{\int_{B_\varepsilon(\xi_j)}|\varphi_{\varepsilon,j}|^{p_s^*}|u|^{p_s^*}d\xi + \nu_j\delta_{\xi_j}(\varphi_{\varepsilon,j})\right\} \\ &= \nu_j,\end{aligned}\tag{4.11}$$

since

$$\int_{B_\varepsilon(\xi_j)}|\varphi_{\varepsilon,j}|^{p_s^*}|u|^{p_s^*}d\xi \le \int_{B_\varepsilon(\xi_j)}|u|p_s^*d\xi = o(1)$$

as $\varepsilon\to0^+$, being $0\le\varphi\le1$. On the other hand, the weak* convergence gives

$$\lim_{k\to\infty}\int_{\mathbb{H}^n}|\varphi_{\varepsilon,j}|^p|D_H^su_k|^pd\eta = \int_{\mathbb{H}^n}|\varphi_{\varepsilon,j}|^pd\mu,\tag{4.12}$$

while Lemma 4.5 yields

$$\lim_{\varepsilon\to0^+}\limsup_{k\to\infty}\int_{\mathbb{H}^n}|D_H^s\varphi_{\varepsilon,j}|^p|u_k|^pd\eta = 0.\tag{4.13}$$

Then, combining (4.11)–(4.13) and letting $\varepsilon\to0^+$ in (4.10), we find that

$$C_{p_s^*}\nu_j^{p/p_s^*}\le(1+\tau)\mu_j \quad \text{for any } j\in J,$$

where $\mu_j=\lim_{\varepsilon\to0^+}\mu(B_\varepsilon(\xi_j))$. Since $\tau>0$ is arbitrary, sending $\tau\to0^+$, we finally obtain

$$C_{p_s^*}\nu_j^{p/p_s^*}\le\mu_j, \quad j\in J.$$

Obviously,

$$\mu\ge\sum_{j\in J}\mu_j\delta_{\xi_j}.$$

Denote by $\mathcal{B}_\varepsilon(\xi_0)$ the Euclidean ball of $\mathbb{R}^{2n+1}$ of center $\xi_0\in\mathbb{H}^n$ and radius ε. By Lebesgue's differentiation theorem for measures (see for example [15]), in order to prove that $\mu\ge|D_H^su|^pd\xi$ it suffices to show that

$$\liminf_{\varepsilon\to0^+}\frac{\mu(\mathcal{B}_\varepsilon(\xi_0))}{|\mathcal{B}_\varepsilon(\xi_0)|}\ge|D_H^su|^p(\xi_0) \quad \text{for a.e. } \xi_0\in\mathbb{H}^n,\tag{4.14}$$

where $|\mathcal{B}_\varepsilon(\xi_0)|$ is the Lebesgue measure of the Euclidean ball $\mathcal{B}_\varepsilon(\xi_0)$.

Clearly, since $|D_H^su|^pd\xi\in L^1_{\mathrm{loc}}(\mathbb{H}^n)$, we know that for a.e. $\xi_0\in\mathbb{H}^n$

$$\lim_{\varepsilon\to0^+}\frac{1}{|\mathcal{B}_\varepsilon(\xi_0)|}\int_{\mathcal{B}_\varepsilon(\xi_0)}|D_H^su|^p(\xi)d\xi = |D_Hu|_H^p(\xi_0).\tag{4.15}$$

Fix $\varepsilon>0$ and $\xi_0\in\mathbb{H}^n$ such that (4.15) holds. Now, the functional $\Phi: HW^{s,p}(\mathbb{H}^n)\to\mathbb{R}$, defined as

$$\Phi u = \int_{\mathcal{B}_\varepsilon(\xi_0)}\int_{\mathbb{H}^n}\frac{|u(\xi)-u(\eta)|^p}{r(\eta^{-1}\circ\xi)^{Q+sp}}d\eta d\xi = \int_{\mathcal{B}_\varepsilon(\xi_0)}|D_H^su|^pd\xi,$$

is convex and strongly continuous on $HW^{s,p}(\mathbb{H}^n)$. Thus, since $u_k \rightharpoonup u$ in $HW^{s,p}(\mathbb{H}^n)$, we have

$$\liminf_{k\to\infty}\int_{\mathcal{B}_\varepsilon(\xi_0)}|D^s_H u_k|^p d\xi \geq \int_{\mathcal{B}_\varepsilon(\xi_0)}|D^s_H u|^p d\xi.$$

Therefore, an application of Proposition 1.203 – Part (*ii*) of [15] gives

$$\begin{aligned}\frac{\mu(\mathcal{B}_\varepsilon(\xi_0))}{|\mathcal{B}_\varepsilon(\xi_0)|} &\geq \limsup_{k\to\infty}\frac{\mu_k(\mathcal{B}_\varepsilon(\xi_0))}{|\mathcal{B}_\varepsilon(\xi_0)|} = \limsup_{k\to\infty}\frac{1}{|\mathcal{B}_\varepsilon(\xi_0)|}\int_{\mathcal{B}_\varepsilon(\xi_0)}|D^s_H u_k|^p d\xi\\ &\geq \liminf_{k\to\infty}\frac{1}{|\mathcal{B}_\varepsilon(\xi_0)|}\int_{\mathcal{B}_\varepsilon(\xi_0)}|D^s_H u_k|^p d\xi \geq \frac{1}{|\mathcal{B}_\varepsilon(\xi_0)|}\int_{\mathcal{B}_\varepsilon(\xi_0)}|D^s_H u|^p d\xi.\end{aligned}$$

Now, passing to the liminf as $\varepsilon \to 0^+$ and using (4.15), we obtain (4.14).

Finally, since $|D^s_H u|^p d\xi$ is orthogonal to $\sum_{j\in J}\mu_j\delta_{\xi_j}$, we get the desired conclusion. This concludes the proof.

□

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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