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Research article

Poincaré inequalities and Neumann problems for the variable exponent setting

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Abstract: In an earlier paper, Cruz-Uribe, Rodney and Rosta proved an equivalence between weighted Poincaré inequalities and the existence of weak solutions to a family of Neumann problems related to a degenerate p-Laplacian. Here we prove a similar equivalence between Poincaré inequalities in variable exponent spaces and solutions to a degenerate $p(\cdot)$-Laplacian, a non-linear elliptic equation with nonstandard growth conditions.

Keywords: degenerate Sobolev spaces; p-Laplacian; Poincaré inequalities; variable exponent; nonstandard growth conditions

1. Introduction

Poincaré inequalities play a central role in the study of regularity for elliptic equations. For specific degenerate elliptic equations, an important problem is to show the existence of such an inequality; however, an extensive theory has been developed by assuming their existence. See, for example, [17, 18]. In [5], the first and third authors, along with E. Rosta, gave a characterization of the existence of a weighted Poincaré inequality, adapted to the solution space of degenerate elliptic equations, in terms of the existence and regularity of a weak solution to a Neumann problem for a degenerate p-Laplacian equation.

The goal of the present paper is to extend this result to the setting of variable exponent spaces. Here, the relevant equations are degenerate $p(\cdot)$-Laplacians. The basic operator is the $p(\cdot)$-Laplacian: given an exponent function $p(\cdot)$ (see Section 2 below), let

$$\Delta_{p(\cdot)}u = -\operatorname{div}(|\nabla u|^{p(\cdot)-2}\nabla u).$$

This operator arises in the calculus of variations as an example of nonstandard growth conditions, and has been extensively studied by a number of authors: see [7,9,14,16] and the extensive references they contain. We are interested in a degenerate version of this operator,

$$Lu = -\operatorname{div}(|\sqrt{Q}\nabla u|^{p(\cdot)-2}Q\nabla u),$$

where Q is a $n \times n$, positive semi-definite, self-adjoint, measurable matrix function. These operators have also been studied, though nowhere nearly as extensively: see, for instance, [10–12]. This paper is part of an ongoing project to develop a general regularity theory for these operators.

In order to state our main result, we first give some definitions and notation that will be used throughout our work. Let $\Omega \subset \mathbb{R}^n$ be a fixed domain (open and connected), and let E be a bounded subdomain with $\overline{E} \subseteq \Omega$. Given an exponent function $p(\cdot)$, we let $L^{p(\cdot)}(E)$ denote the associated variable Lebesgue space; for a precise definition, see Section 2 below.

Let $\mathcal{S}_n$ denote the collection of all positive semi-definite, $n\times n$ self-adjoint matrices. Let $Q : \Omega \to \mathcal{S}_n$ be a measurable, matrix-valued function whose entries are Lebesgue measurable. We define

$$\gamma(x) = |Q(x)|_{\mathrm{op}} = \sup_{|\xi|=1} |Q(x)\xi|$$

to be the pointwise operator norm of $Q(x)$; this function will play an important role in our results. We will generally assume that $\gamma^{1/2}$ lies in the variable Lebesgue space $L^{p(\cdot)}(E)$. More generally, let v be a weight on Ω: i.e., a non-negative function in $L^1_{\mathrm{loc}}(\Omega)$. Given a function f on E, we define the weighted average of f on E by

$$f_{E,v} = \frac{1}{v(E)} \int_E f(x)v(x)dx = \fint_E f dv.$$

If $v = 1$ we write simply f_E. Again, we will generally assume that $v \in L^{p(\cdot)}(E)$.

Remark 1.1. *In this paper we do not assume any connection between weight v and the matrix Q. However, in many situations it is common to assume that v is the largest eigenvalue of Q: that is, $v = |Q|_{\mathrm{op}}$. See, for instance, [3, 4].*

The next two definitions are central to our main result.

Definition 1.2. *Given $p(\cdot) \in \mathcal{P}(E)$, a weight v and a measurable, matrix-valued function Q, suppose v, $\gamma^{1/2} \in L^{p(\cdot)}(E)$. Then the pair (v, Q) is said to have the Poincaré property of order $p(\cdot)$ on E if there is a positive constant $C_0 = C_0(E, p(\cdot))$ such that for all $f \in C^1(\overline{E})$,*

$$\|f - f_{E,v}\|_{L^{p(\cdot)}(v;E)} \leq C_0 \|\nabla f\|_{\mathcal{L}^{p(\cdot)}_Q(E)}. \tag{1.1}$$

Remark 1.3. *The assumption that v, $\gamma^{1/2} \in L^{p(\cdot)}(E)$ ensures that both sides of inequality* (1.1) *are finite.*

Definition 1.4. *Given $p(\cdot) \in \mathcal{P}(E)$, a weight v and a measurable matrix-valued function Q, suppose v, $\gamma^{1/2} \in L^{p(\cdot)}(E)$. Then the pair (v, Q) is said to have the $p(\cdot)$-Neumann property on E if the following hold:*

1). *Given any $f \in L^{p(\cdot)}(v;E)$, there exists a weak solution $(u,\mathbf{g})_f \in \tilde{H}^{1,p(\cdot)}_Q(v;E)$ to the degenerate Neumann problem*

$$\begin{cases} \operatorname{div}\left(\left|\sqrt{Q(x)}\nabla u(x)\right|^{p(x)-2} Q(x)\nabla u(x)\right) & = |f(x)|^{p(x)-2} f(x) v(x)^{p(x)} \text{ in } E \\ \mathbf{n}^T \cdot Q(x)\nabla u(x) & = 0 \text{ on } \partial E, \end{cases} \tag{1.2}$$

where $\mathbf{n}$ is the outward unit normal vector of ∂E.

2). *Any weak solution $(u,\mathbf{g})_f \in \tilde{H}^{1,p(\cdot)}_Q(v;E)$ of (1.2) is regular: that is, there is a positive constant $C_1 = C_1(p(\cdot), v, E)$ such that*

$$\|u\|_{L^{p(\cdot)}(v;E)} \leq C_1 \|f\|_{L^{p(\cdot)}(v;E)}^{\frac{r_*-1}{p_*-1}}, \tag{1.3}$$

where p_* and r_* are defined by

$$p_* = \begin{cases} p_+ & \text{if } \|\mathbf{g}\|_{\mathcal{L}^{p(\cdot)}_Q(E)} < 1 \\ p_- & \text{if } \|\mathbf{g}\|_{\mathcal{L}^{p(\cdot)}_Q(E)} \geq 1 \end{cases} \quad \text{and} \quad r_* = \begin{cases} p_+ & \text{if } \|f\|_{L^{p(\cdot)}(v;E)} \geq 1 \\ p_- & \text{if } \|f\|_{L^{p(\cdot)}(v;E)} < 1 \end{cases}. \tag{1.4}$$

Remark 1.5. *The degenerate, variable exponent Sobolev space, $\tilde{H}^{1,p(\cdot)}_Q(v;E)$, will be defined in Section 2. Here we note that the definition will require the assumption that v, $\gamma^{1/2} \in L^{p(\cdot)}(E)$.*

Remark 1.6. *The vector function $\mathbf{g}$ should be thought of as a weak gradient of f; we avoid the notation ∇f since in the degenerate setting it is often not a weak derivative in the classical sense. See the discussion after Definition 2.15.*

Remark 1.7. *While the PDE in (1.2) is stated in terms of a classical Neumann problem, we make no assumptions about the regularity of the boundary ∂E in our definition of a weak solution. In the constant exponent case, as noted in [5, Remark 2.10], our definition of weak solution is equivalent to this classical formulation if we assume sufficient regularity.*

Our main result shows that these two properties are equivalent under certain minimal assumptions on the exponent function $p(\cdot)$, the weight v, and the operator norm of the matrix function Q.

Theorem 1.8. *Let $p(\cdot) \in \mathcal{P}(E)$ with $1 < p_- \leq p_+ < \infty$. Suppose v is a weight in Ω and Q is a measurable matrix function with v, $\gamma^{1/2} \in L^{p(\cdot)}(E)$. Then the pair (v, Q) has the Poincaré property of order $p(\cdot)$ on E if and only if (v, Q) has the $p(\cdot)$-Neumann property on E.*

This result is a generalization of the main result in [5]; when $p(\cdot)$ is constant Theorem 1.8 is equivalent to it. However, this is not immediately clear. In the constant exponent case, the exponent on the right-hand side of the regularity estimate corresponding to (1.3) is 1. This is because in the constant exponent case the PDE is homogeneous and we can normalize the equation, but this is no longer possible in the variable exponent case. But, in the constant exponent case, we have that $\frac{r_*-1}{p_*-1} = 1$.

More significantly, there is also a difference in the definition of weighted spaces and the formulation of the Poincaré inequality. Denote the weight that appears in [5] by w; there we assumed that $w \in L^1(E)$ and defined a function f to be in $L^p(w)$ if

$$\int_E |f|^p w \, dx < \infty,$$

However, in the present case, if $p(\cdot) = p$ is a constant exponent, then we have that $f \in L^p(v; E)$ if

$$\int_E |fv|^p\,dx = \int_E |f| v^p\,dx < \infty.$$

Therefore, to pass between our current setting and that in [5], we need to define w by $v = w^{1/p}$.

This leads to a substantial difference in the statement of the Poincaré inequality. In [5] the left-hand side of the Poincaré inequality is (assuming $v = w^{1/p}$)

$$\left(\int_E |f(x) - f_{E,w}|^p w\,dx\right)^{1/p} = \|f - f_{E,w}\|_{L^p(v;E)};$$

on the other hand, in Definition 1.2 the left-hand side is

$$\left(\int_E |f(x) - f_{E,v}|^p w\,dx\right)^{1/p} = \|f - f_{E,v}\|_{L^p(v;E)}.$$

These would appear to be different conditions, but, in fact, these two versions of the Poincaré inequality are equivalent. Moreover, we have that if we use the more standard, unweighted average f_E in the Poincaré inequality, then this implies Definition 1.2. The converse, however, requires a additional assumption on v. Versions of the following result are part of the folklore of PDEs; we first encountered it as a passing remark in [8]. For completeness, we give a proof in the appendix.

Proposition 1.9. *Given $1 < p < \infty$ and a bounded set E, suppose $v \in L^p(E)$ and set $w = v^p$. Then,*

$$\|f - f_{E,v}\|_{L^p(v;E)} \approx \|f - f_{E,w}\|_{L^p(v;E)},$$

where the implicit constants depend on E, p and v. Moreover, we also have that

$$\|f - f_{E,v}\|_{L^p(v;E)} \lesssim \|f - f_E\|_{L^p(v;E)}.$$

Finally, if we assume that $v^{-1} \in L^{p'}(E)$, then

$$\|f - f_E\|_{L^p(v;E)} \lesssim \|f - f_{E,v}\|_{L^p(v;E)}.$$

Remark 1.10. *The hypothesis that $v^{-1} \in L^{p'}(E)$ is satisfied, for instance, if we assume that v^p is in the Muckenhoupt class A_p.*

The remainder of this paper is organized as follows. In Section 2 we first state the basic definitions and properties of exponent functions and variable Lebesgue spaces needed for our results. We then define matrix weighted variable exponent spaces, and use these to define the degenerate Sobolev spaces where our solutions live. An important technical step is proving that these spaces have the requisite properties. We then give the precise definition of weak solutions used in Definition 1.4. In Sections 3 and 4 we prove Theorem 1.8, each section dedicated to one implication. The proof is similar in outline to the proof in [5], but differs significantly in detail as we address the problems that arise from working in variable exponent spaces. Finally, in Appendix A we prove Proposition 1.9.

2. Preliminaries

We begin this section by reviewing the basic definitions, notation, and properties of exponent functions and variable Lebesgue spaces. For complete information, we refer the interested reader to [2].

Definition 2.1. *An exponent function is a Lebesgue measurable function* $p(\cdot) : E \to [1, \infty]$. *Denote the collection of all exponent functions on E by* $\mathcal{P}(E)$. *Define the set* $E_\infty = \{x \in E : p(x) = \infty\}$ *and let*

$$p_-(E) = p_- = \operatorname*{ess\,inf}_{x\in E} p(x), \text{ and } p_+(E) = p_+ = \operatorname*{ess\,sup}_{x\in E} p(x).$$

Definition 2.2. *Given* $p(\cdot) \in \mathcal{P}(E)$ *and a Lebesgue measurable function* f, *define the modular functional (or simply the modular) associated with* $p(\cdot)$ *by*

$$\rho_{p(\cdot),E}(f) = \int_{E\setminus E_\infty} |f(x)|^{p(x)} dx + \|f\|_{L^\infty(E_\infty)}.$$

If f *is unbounded on* E_∞ *or* $f(\cdot)^{p(\cdot)} \notin L^1(E\setminus E_\infty)$ *then we define* $\rho_{p(\cdot),E}(f) = +\infty$. *When* $|E_\infty| = 0$ *we let* $\|f\|_{L^\infty(E_\infty)} = 0$; *when* $|E\setminus E_\infty| = 0$, *then* $\rho_{p(\cdot),E}(f) = \|f\|_{L^\infty(E_\infty)}$. *In situations where there is no ambiguity we will simply write* $\rho_{p(\cdot)}(f)$ *or* $\rho(f)$.

Definition 2.3. *Let* $p(\cdot) \in \mathcal{P}(E)$ *and let* v *be a weight on* E.

1). We define the variable Lebesgue space $L^{p(\cdot)}(E)$ *to be the collection of all Lebesgue measurable functions* $f : E \to \mathbb{R}$ *satisfying*

$$\|f\|_{L^{p(\cdot)}(E)} = \inf\left\{\mu > 0 : \rho\left(\frac{f}{\mu}\right) \le 1\right\} < \infty.$$

2). We define the weighted variable Lebesgue space $L^{p(\cdot)}(v; E)$ *to be the collection of all Lebesgue measurable functions satisfying*

$$\|f\|_{L^{p(\cdot)}(v;E)} = \|fv\|_{L^{p(\cdot)}(E)} < \infty.$$

Theorem 2.4. *[2] Let* $p(\cdot) \in \mathcal{P}(E)$. *Then* $L^{p(\cdot)}(E)$ *is a Banach space. The space* $L^{p(\cdot)}(E)$ *is separable if and only if* $p_+ < \infty$, *and* $L^{p(\cdot)}(E)$ *is reflexive if and only if* $1 < p_- \le p_+ < \infty$.

The previous theorem can be extended to weighted variable Lebesgue spaces. This will be useful when proving facts variable exponent spaces of vector-valued functions. The following result was proved in [6]. The setting there is slightly different as they considered the spaces $L^{p(\cdot)}(\mu)$ where μ is a measure. However, if we let $d\mu = v^{p(\cdot)}dx$, then their results immediately transfer into our setting, since with our hypothesis μ is a σ-finite measure when $p_+ < \infty$ (which is needed to prove separability).

Theorem 2.5. *Let* $p(\cdot) \in \mathcal{P}(E)$ *and suppose* $v \in L^{p(\cdot)}(E)$. *Then:*

1). $L^{p(\cdot)}(v; E)$ *is a Banach space.*
2). $L^{p(\cdot)}(v; E)$ *is separable if* $p_+ < \infty$.
3). $L^{p(\cdot)}(v; E)$ *is reflexive if* $1 < p_- \le p_+ < \infty$.

A useful result about variable Lebesgue spaces is the extension of Hölder's inequality to the variable exponent norm.

Theorem 2.6. *[2, Theorem 2.26], Hölder's inequality Given $p(\cdot), p'(\cdot) \in \mathcal{P}(E)$ with $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ for a.e. $x \in E$, if $f \in L^{p(\cdot)}(E)$ and $g \in L^{p'(\cdot)}(E)$, then $fg \in L^1(E)$ and*

$$\int_E |f(x)g(x)|dx \leq K_{p(\cdot)} \|f\|_{L^{p(\cdot)}(E)} \|g\|_{L^{p'(\cdot)}(E)},$$

where $K_{p(\cdot)} \leq 4$ is a constant depending only on $p(\cdot)$.

The next two results are technical lemmas that we will need in the proof of our main result.

Proposition 2.7. *[2, Corollary 2.23] Given $p(\cdot) \in \mathcal{P}(E)$, suppose $|E_\infty| = 0$. If $\|f\|_{p(\cdot)} \geq 1$, then*

$$\|f\|_{p(\cdot)}^{p_-} \leq \rho(f) \leq \|f\|_{p(\cdot)}^{p_+}.$$

If $0 \leq \|f\|_{p(\cdot)} < 1$, then

$$\|f\|_{p(\cdot)}^{p_+} \leq \rho(f) \leq \|f\|_{p(\cdot)}^{p_-}.$$

Proposition 2.8. *[2, Proposition 2.21] Given $p(\cdot) \in \mathcal{P}(E)$, for all nontrivial $f \in L^{p(\cdot)}(E)$, $\rho(f/\|f\|_{p(\cdot)}) = 1$ if and only if $p_+(E/E_\infty) < \infty$.*

The next result generalizes the trivial identity $\|f^{p-1}\|_{p'} = \|f\|_p^{p-1}$ to the setting of variable Lebesgue spaces.

Theorem 2.9. *Let $E \subseteq \mathbb{R}^n$ and $p(\cdot) \in \mathcal{P}(E)$ with $1 < p_- \leq p_+ < \infty$, and f be measurable on E. Then, $\||f|^{p(\cdot)-1}\|_{L^{p'(\cdot)}(E)}$ is finite if and only if $\|f\|_{L^{p(\cdot)}(E)}$ is finite. In particular,*

$$\|f\|_{L^{p(\cdot)}(E)}^{l_*-1} \leq \||f|^{p(\cdot)-1}\|_{L^{p'(\cdot)}(E)} \leq \|f\|_{L^{p(\cdot)}(E)}^{b_*-1} \tag{2.1}$$

where l_* and $b*$ are given by

$$l_* = \begin{cases} p_+ & \text{if } \|f\|_{L^{p(\cdot)}(E)} < 1 \\ p_- & \text{if } \|f\|_{L^{p(\cdot)}(E)} \geq 1 \end{cases} \qquad b_* = \begin{cases} p_- & \text{if } \|f\|_{L^{p(\cdot)}(E)} < 1 \\ p_+ & \text{if } \|f\|_{L^{p(\cdot)}(E)} \geq 1. \end{cases}.$$

Proof. Let $\mu_{p'} = \||f|^{p(\cdot)-1}\|_{L^{p'(\cdot)}(E)}$ and assume $\mu_{p'} < \infty$. Then $\mu_{p'}^{1/(l_*-1)} \geq \mu_{p'}^{1/(p(x)-1)}$ for almost every x, and so

$$\int_E \left(\frac{|f(x)|}{\mu_{p'}^{1/(l_*-1)}} \right)^{p(x)} dx \leq \int_E \left(\frac{|f(x)|}{\mu_{p'}^{1/(p(x)-1)}} \right)^{p(x)} dx \leq \int_E \left(\frac{|f(x)|^{p(x)-1}}{\mu_{p'}} \right)^{p'(x)} dx = \rho_{p'(\cdot),E} \left(\frac{|f|^{p(\cdot)-1}}{\mu_{p'}} \right).$$

Since $p_- > 1$, we have $\operatorname{ess\,sup} p'(x) < \infty$. Thus, by Proposition 2.8, the modular above equals 1. Hence, by definition of the $L^{p(\cdot)}(E)$ norm, $\|f\|_{L^{p(\cdot)}(E)} \leq \mu_{p'}^{1/(l_*-1)}$, or equivalently,

$$\|f\|_{L^{p(\cdot)}(E)}^{l_*-1} \leq \||f|^{p(\cdot)-1}\|_{L^{p'(\cdot)}(E)} < \infty.$$

Now let $\mu_p = \|f\|_{L^{p(\cdot)}(E)}$, and assume $\mu_p < \infty$. Then the proof is essentially the same: $\mu_p^{b_*-1} \geq \mu_p^{1/(p'(x)-1)}$ a.e., and so

$$\int_E \left(\frac{|f(x)|^{p(x)-1}}{\mu_p^{b_*-1}}\right)^{p'(x)} dx \leq \int_E \left(\frac{|f(x)|^{p(x)-1}}{\mu_p^{1/(p'(x)-1)}}\right)^{p'(x)} dx = \int_E \left(\frac{|f(x)|}{\mu_p}\right)^{p(x)} dx = \rho_{p(\cdot),E}\left(\frac{f}{\mu}\right).$$

Since $p_+ < \infty$, by Proposition 2.8 the above modular equals 1. Hence,

$$\||f|^{p(\cdot)-1}\|_{L^{p'(\cdot)}(E)} \leq \mu_p^{b_*-1} = \|f\|_{L^{p(\cdot)}(E)}^{b_*-1} < \infty.$$

□

Remark 2.10. *The definitions of the exponents l_* and b_* clearly depend on the given function. It will be clear from context what function these exponents are dependent on, so we will not express this explicitly in our proofs.*

We now define the matrix-weighted, vector-valued Lebesgue space $\mathcal{L}_Q^{p(\cdot)}(E)$.

Definition 2.11. *Given a measurable matrix function $Q : E \to \mathcal{S}_n$ and $p(\cdot) \in \mathcal{P}(E)$, define the matrix-weighted variable Lebesgue space $\mathcal{L}_Q^{p(\cdot)}(E)$ to be the collection of all measurable, vector-valued functions $\mathbf{f} : E \to \mathbb{R}^n$ satisfying*

$$\|\mathbf{f}\|_{\mathcal{L}_Q^{p(\cdot)}(E)} = \||\sqrt{Q}\mathbf{f}|\|_{L^{p(\cdot)}(E)} < \infty.$$

To construct the Q-weighted Sobolev spaces, and to prove existence results for the PDE (1.2), we show that $\mathcal{L}_Q^{p(\cdot)}(E)$ is a separable, reflexive Banach space.

Theorem 2.12. *Let $Q : E \to \mathcal{S}_n$ be a positive semi-definite, self-adjoint, measurable, matrix-valued function on E such that $\gamma^{1/2} \in L^{p(\cdot)}(E)$. Then $\mathcal{L}_Q^{p(\cdot)}(E)$ is a Banach space. Moreover, it is separable if $p_+ < \infty$ and and reflexive if $1 < p_- \leq p_+ < \infty$.*

The proof of Theorem 2.12 requires some basic facts from linear algebra, as well as some results about matrix functions. If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $1 \leq r \leq \infty$, we recall the ℓ^r norms on $\mathbb{R}^n$:

$$|x|_r = \left(\sum_{j=1}^n |x_j|^r\right)^{1/r} \quad \text{and } |x|_\infty = \sup_{1\leq j\leq n} |x_j|.$$

When $r = 2$, $|x|_r$ is the Euclidean norm and we denote it by $|\cdot|_2 = |\cdot|$. Recall that in finite dimensions, all norms are equivalent. In particular, we have that for all $x \in \mathbb{R}^n$,

$$|x|_2 \leq |x|_1 \leq \sqrt{n}|x|_2, \quad |x|_\infty \leq |x|_2 \leq \sqrt{n}|x|_\infty, \quad |x|_\infty \leq |x|_1 \leq n|x|_\infty. \tag{2.2}$$

We say that an $n \times n$ matrix function $Q(\cdot)$ is positive semi-definite on E if for every nonzero $\xi \in \mathbb{R}^n$, $\xi^T Q(x)\xi \geq 0$ for almost every $x \in \Omega$. We say Q is self-adjoint if $q_{ij} = q_{ji}$ for $1 \leq i, j \leq n$. Recall that every finite, self-adjoint matrix is diagonalizable; for matrix functions this can be done measurably.

Lemma 2.13. *[15, Lemma 2.3.5] Let Q be a finite, self-adjoint matrix whose entries are Lebesgue measurable functions on some domain E. Then for every $x \in E$, $Q(x)$ is diagonalizable, i.e., there exists a matrix U whose entries are Lebesgue measurable functions on E such that $U^T QU$ is a diagonal matrix and $U(x)$ is orthogonal for every $x \in E$.*

Equivalently, there is a measurable diagonal matrix function $D(x)$ (whose entries are the non-negative eigenvalues of $Q(x)$) and an orthogonal matrix function $U(x)$ such that for almost every $x \in E$

$$Q(x) = U^T(x)D(x)U(x)$$

In particular, given such a matrix Q, we define its square root by

$$\sqrt{Q(x)} = U^T(x)\sqrt{D(x)}U(x),$$

where $\sqrt{D(x)}$ takes the square root of each entry of $D(x)$ along the diagonal.

Remark 2.14. *As mentioned in [18, Remark 5] and as a consequence of the proof of Lemma 2.13, the eigenvalues $\{\lambda_j(x)\}_{j=1}^n$ and eigenvectors $\{\mathbf{v}_j(x)\}_{j=1}^n$ associated to a self-adjoint, positive semi-definite measurable matrix function, $Q : E \to \mathcal{S}_n$ are also measuarable functions on E.*

Proof of Theorem 2.12. Since $Q(x)$ self-adjoint, by Lemma 2.13, $Q(x)$ is diagonalizable. By Remark 2.14, let $\lambda_1(x), \ldots, \lambda_n(x)$ be the measurable eigenvalues of $Q(x)$ and let $\mathbf{v}_1(x), \ldots, \mathbf{v}_n(x)$ be measurable eigenvectors with $|\mathbf{v}_j(x)| = 1$ for almost every $x \in E$, $1 \leq j \leq n$. Hence, $\{\mathbf{v}_j(x)\}_{j=1}^n$ forms a basis for $\mathbb{R}^n$ for almost every $x \in E$. Fix $\mathbf{f} \in \mathcal{L}_Q^{p(\cdot)}(E)$; then we can write $\mathbf{f}$ as

$$\mathbf{f}(x) = \sum_{j=1}^n \tilde{f}_j(x)\mathbf{v}_j(x), \tag{2.3}$$

where $\tilde{f}_j = \mathbf{f}^T\mathbf{v}_j$ is the jth component of $\mathbf{f}$ with respect to the basis $\{\mathbf{v}_j\}_{j=1}^n$. Completeness, separability and reflexivity are a consequence of the following equivalence of norms: for all $\mathbf{f} \in \mathcal{L}_Q^{p(\cdot)}(E)$,

$$\frac{1}{n}\sum_{j=1}^n \|\tilde{f}_j\|_{L^{p(\cdot)}(\lambda_j^{1/2};E)} \leq \|\mathbf{f}\|_{\mathcal{L}_Q^{p(\cdot)}(E)} \leq \sum_{j=1}^n \|\tilde{f}_j\|_{L^{p(\cdot)}(\lambda_j^{1/2};E)}. \tag{2.4}$$

Suppose for the moment that (2.4) holds. To show that $\mathcal{L}_Q^{p(\cdot)}(E)$ is complete, let $\{\mathbf{f}_k\}_{k=1}^\infty$ be a Cauchy sequence in $\mathcal{L}_Q^{p(\cdot)}(E)$. Let $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that for every $l, m > N$, $\|\mathbf{f}_l - \mathbf{f}_m\|_{\mathcal{L}_Q^{p(\cdot)}(E)} < \epsilon/n$. Inequality (2.4) then shows for $l, m > N$,

$$\sum_{j=1}^n \|(\mathbf{f}_l - \mathbf{f}_m)^T\mathbf{v}_j\|_{L^{p(\cdot)}(\lambda_j^{1/2};E)} \leq n\|\mathbf{f}_l - \mathbf{f}_m\|_{\mathcal{L}_Q^{p(\cdot)}(E)} < \epsilon.$$

Thus, for $1 \leq j \leq n$, $\{\mathbf{f}_k^T\mathbf{v}_j\}_{k=1}^\infty$ is Cauchy in $L^{p(\cdot)}(\lambda_j^{1/2};E)$. By Theorem 2.5, $L^{p(\cdot)}(\lambda_j^{1/2};E)$ is complete. Thus, there exists $\tilde{g}_j \in L^{p(\cdot)}(\lambda_j^{1/2};E)$ such that, as $k \to \infty$,

$$\|\mathbf{f}_k^T\mathbf{v}_j - \tilde{g}_j\|_{L^{p(\cdot)}(\lambda^{1/2};E)} \to 0. \tag{2.5}$$

Define $\mathbf{g} : E \to \mathbb{R}^n$ by setting $\mathbf{g}(x) = \sum_{j=1}^n \tilde{g}_j(x)\mathbf{v}_j(x)$. Since $\tilde{g}_j \in L^{p(\cdot)}(\lambda_j^{1/2};E)$, $1 \leq j \leq n$, by (2.4), $\mathbf{g} \in \mathcal{L}_Q^{p(\cdot)}(E)$. Furthermore, we have that

$$\|\mathbf{f}_k - \mathbf{g}\|_{\mathcal{L}_Q^{p(\cdot)}(E)} \leq \sum_{j=1}^n \|\mathbf{f}_k^T\mathbf{v}_j - \tilde{g}_j\|_{L^{p(\cdot)}(\lambda_j^{1/2};E)}.$$

If we combine this with (2.5), we get that $\mathbf{f}_k \to \mathbf{g}$ in $\mathcal{L}_Q^{p(\cdot)}(E)$. Therefore, $\mathcal{L}_Q^{p(\cdot)}(E)$ is complete.

Similarly, (2.4) implies $\mathcal{L}_Q^{p(\cdot)}(E)$ is separable when $p_+ < \infty$. Fix $\epsilon > 0$. Since $\lambda_j^{1/2} \leq \gamma^{1/2} \in L^{p(\cdot)}(E)$, again by Theorem 2.5, $L^{p(\cdot)}(\lambda_j^{1/2}; E)$ is separable, and so for each j there is a countable, dense subset $D_j \subseteq L^{p(\cdot)}(\lambda_j^{1/2}; E)$. Thus, for each $j = 1, \ldots, n$, there exists $d_j \in D_j$ such that

$$\|\tilde{f}_j - d_j\|_{L^{p(\cdot)}(\lambda_j^{1/2};E)} < \frac{\epsilon}{n}.$$

Define $\mathbf{d} \in D_1 \times \cdots \times D_n$ by setting $\mathbf{d} = \sum_{j=1}^n d_j \mathbf{v}_j$. Then by (2.4),

$$\|\mathbf{f} - \mathbf{d}\|_{\mathcal{L}_Q^{p(\cdot)}(E)} \leq \sum_{j=1}^n \|\tilde{f}_j - d_j\|_{L^{p(\cdot)}(\lambda_j^{1/2};E)} < \epsilon.$$

Thus, $D_1 \times \cdots \times D_n$ is a countable dense subset of $\mathcal{L}_Q^{p(\cdot)}(E)$, and so $\mathcal{L}_Q^{p(\cdot)}(E)$ is separable.

Finally, (2.4) implies $\mathcal{L}_Q^{p(\cdot)}(E)$ is reflexive when $1 < p_- \leq p_+ < \infty$. Equation (2.3) induces the map

$$T : \mathcal{L}_Q^{p(\cdot)}(E) \to \prod_{j=1}^n L^{p(\cdot)}(\lambda_j^{1/2}; E),$$

defined by $T(\mathbf{f}) = (\tilde{f}_1, \ldots, \tilde{f}_j)$. Clearly, T is linear. T is also bijective because of the norm equivalence (2.4).

Finally, T is continuous: by the norm equivalence (2.4) we have that

$$\|T(\mathbf{f}_k) - T(\mathbf{f})\|_{\prod_j L^{p(\cdot)}(\lambda_j^{1/2};E)} = \sum_{j=1}^n \|\tilde{f}_{jk} - \tilde{f}_j\|_{L^{p(\cdot)}(\lambda_j^{1/2});E} \leq n\|\mathbf{f}_k - \mathbf{f}\|_{\mathcal{L}_Q^{p(\cdot)}(E)}.$$

In the same way we have that T^{-1} is continuous since

$$\|\mathbf{f}_k - \mathbf{f}\|_{\mathcal{L}_Q^{p(\cdot)}(E)} \leq \sum_{j=1}^n \|\tilde{f}_{jk} - \tilde{f}_j\|_{L^{p(\cdot)}(\lambda_j^{1/2};E)}.$$

Therefore, $\mathcal{L}_Q^{p(\cdot)}(E)$ is isomorphic to the product space $\prod_{j=1}^n L^{p(\cdot)}(\lambda_j^{1/2}; E)$. Finite products of reflexive spaces are reflexive; hence, $\mathcal{L}_Q^{p(\cdot)}(E)$ is a reflexive Banach space.

To complete the proof we need to prove inequality (2.4). Since $|\sqrt{Q(x)}\xi|^2 = \xi^T Q(x)\xi$ for any $\xi \in \mathbb{R}^n$ and almost every $x \in E$, we have that

$$|\sqrt{Q(x)}\mathbf{f}(x)|^2 = \sum_{j=1}^n |\tilde{f}_j(x)\sqrt{Q(x)}\mathbf{v}_j(x)|^2 = \sum_{j=1}^n |\tilde{f}_j(x)|^2 \mathbf{v}_j^T(x) Q(x) \mathbf{v}_j(x) = \sum_{j=1}^n |\tilde{f}_j(x)|^2 \lambda_j(x).$$

Hence,

$$|\sqrt{Q(x)}\mathbf{f}(x)| = \left(\sum_{j=1}^n |\tilde{f}_j(x)|^2 \lambda_j(x) \right)^{1/2}$$

almost everywhere in E.

Inequality (2.4) is now straightforward to prove. Define $\tilde{\mathbf{F}} : E \to \mathbb{R}^n$ by

$$\tilde{\mathbf{F}}(x) = (|\tilde{f}_1(x)|\lambda_1^{1/2}(x), \dots, |\tilde{f}_n(x)|\lambda_n^{1/2}(x)).$$

By (2.3) we have that

$$\|\mathbf{f}\|_{\mathcal{L}_Q^{p(\cdot)}(E)} = \| |\sqrt{Q(x)}\mathbf{f}(x)| \|_{L^{p(\cdot)}(E)} = \||\tilde{\mathbf{F}}(x)|\|_{L^{p(\cdot)}(E)}.$$

By (2.2) and the triangle inequality,

$$\||\tilde{\mathbf{F}}(x)|\|_{L^{p(\cdot)}(E)} \le \||\tilde{\mathbf{F}}(x)|_1\|_{L^{p(\cdot)}(E)} \le \sum_{j=1}^{n} \||\tilde{f}_j\lambda_j^{1/2}|\|_{L^{p(\cdot)}(E)} = \sum_{j=1}^{n} \|\tilde{f}_j\|_{L^{p(\cdot)}(\lambda_j^{1/2};E)}.$$

To show the reverse inequality, we again use (2.2) and the definition of $|\cdot|_\infty$ to get

$$\||\tilde{\mathbf{F}}(x)|\|_{L^{p(\cdot)}(E)} \ge \frac{1}{n}\sum_{j=1}^{n} \||\tilde{\mathbf{F}}(x)|_\infty\|_{L^{p(\cdot)}(E)} \ge \frac{1}{n}\sum_{j=1}^{\infty} \||\tilde{f}_j(x)\lambda_j^{1/2}(x)|\|_{L^{p(\cdot)}(E)} = \frac{1}{n}\sum_{j=1}^{n} \|\tilde{f}_j\|_{L^{p(\cdot)}(\lambda_j^{1/2};E)}.$$

This completes the proof of (2.4). □

We now use these variable exponent spaces to define the degenerate Sobolev spaces where solutions in Definition 1.4 will live. Initially, we will give them as collections of equivalence classes of Cauchy sequences of $C^1(\overline{E})$ functions.

Definition 2.15. *Given $p(\cdot) \in \mathcal{P}(E)$, a weight v, and a matrix function Q, suppose v, $\gamma^{1/2} \in L^{p(\cdot)}(E)$. Define the Sobolev space $H_Q^{1,p(\cdot)}(v; E)$ to be the abstract completion of $C^1(\overline{E})$ with respect to the norm*

$$\|f\|_{H_Q^{1,p(\cdot)}(v;E)} = \|f\|_{L^{p(\cdot)}(v;E)} + \|\nabla f\|_{\mathcal{L}_Q^{p(\cdot)}(E)}. \tag{2.6}$$

Remark 2.16. *With our hypotheses on v and γ this definition makes sense, since they guarantee that for any $f \in C^1(\overline{E})$ the right-hand side of* (2.6) *is finite.*

While this space is defined abstractly, we can give a concrete representation of each equivalence class in it. Since we assume v, $\gamma^{1/2} \in L^{p(\cdot)}(E)$, by Theorems 2.4 and 2.12, the spaces $L^{p(\cdot)}(v; E)$ and $\mathcal{L}_Q^{p(\cdot)}(E)$ are complete. Therefore, if $\{u_n\}_n$ is a sequence of $C^1(\overline{E})$ functions that is Cauchy with respect to the norm in (2.6), we have that this sequence is Cauchy in $L^{p(\cdot)}(v; E)$ and $\mathcal{L}_Q^{p(\cdot)}(E)$ and so converges to a unique pair of functions $(u, \mathbf{g}) \in L^{p(\cdot)}(v; E) \times \mathcal{L}_Q^{p(\cdot)}(E)$. We stress that while the function $\mathbf{g}$ plays the role of ∇u, it cannot in general be identified with a weak derivative of u in the classical sense, even in the constant exponent case. For additional details, see [5].

Theorem 2.17. *Let $p(\cdot) \in \mathcal{P}(E)$ and suppose v, $\gamma^{1/2} \in L^{p(\cdot)}(E)$. Then $H_Q^{1,p(\cdot)}(v; E)$ is a Banach space. If $p_+ < \infty$, then it is separable, and if $1 < p_- \le p_+ < \infty$, it is reflexive.*

Proof. Recall that a closed subspace of a separable, reflexive Banach space is also a separable, reflexive Banach space. Hence, by Theorems 2.4 and 2.12, it suffices to show that $H_Q^{1,p(\cdot)}(v; E)$ is isometrically isomorphic to a closed subspace of $L^{p(\cdot)}(v; E) \times \mathcal{L}_Q^{p(\cdot)}(E)$. Given a sequence $\{u_n\}_n$ of $C^1(\overline{E})$ functions that is Cauchy with respect to (2.6), denote its associated equivalence class in $H^{1,p(\cdot)}_Q(E)$ by $[\{u_n\}_n]$. Then we have that

$$\|[\{u_n\}_n]\|_{H^{1,p(\cdot)}_Q(E)} = \lim_{n\to\infty}(\|u_n\|_{L^{p(\cdot)}(v;E)} + \|\nabla u_n\|_{\mathcal{L}^{p(\cdot)}_Q(E)}) = \|u\|_{L^{p(\cdot)}(v;E)} + \|\mathbf{g}\|_{\mathcal{L}^{p(\cdot)}_Q(E)},$$

where the pair $(u, \mathbf{g}) \in L^{p(\cdot)}(v; E) \times \mathcal{L}^{p(\cdot)}_Q(E)$ is the unique limit described above. (Note that this limit does not depend on the representative chosen from the equivalence class.)

The existence of this pair lets us define a natural map

$$I : H^{1,p(\cdot)}_Q(v; E) \to L^{p(\cdot)}(v; E) \times \mathcal{L}^{p(\cdot)}_Q(E)$$

by $I([\{u_n\}_n]) = (u, \mathbf{g})$. Clearly, I is linear and an isometry by construction. Finally, if $(u, \mathbf{g})$ is a limit point of the image, then by a diagonalization argument we can construct a sequence $\{u_n\}_n$ in $C^1(\overline{E})$ that converges to it in the product norm. But then the sequence is Cauchy in $H^{1,p(\cdot)}_Q(v; E)$ norm, and so $(u, \mathbf{g})$ is contained in the image of $H^{1,p(\cdot)}_Q(v; E)$. Thus, $H^{1,p(\cdot)}_Q(v; E)$ is isometrically isomorphic to a closed subspace of $L^{p(\cdot)}(v; E) \times \mathcal{L}^{p(\cdot)}_Q(E)$ and our proof is complete. □

It is well known that when considering Neumann boundary value problems, any solution is unique only up to addition of constants. In other words if u were a solution of the Neumann problem (1.2), then we should have that $u + c$ is also a solution for any constant c. Therefore, in defining weak solutions we will restrict our attention to the "mean-zero" subspace of $H^{1,p(\cdot)}_Q(v; E)$.

Definition 2.18. *Given the space $H^{1,p(\cdot)}_Q(v; E)$ of Definition 2.15, we define*

$$\tilde{H}^{1,p(\cdot)}_Q(v; E) = \left\{(u, \mathbf{g}) \in H^{1,p(\cdot)}_Q(v; E) : \int_E u(x)v(x)dx = 0\right\}$$

For our analysis we will need to prove that $\tilde{H}^{1,p(\cdot)}_Q(v; E)$ inherits the properties of its parent space from Theorem 2.17.

Theorem 2.19. *Given $p(\cdot) \in \mathcal{P}(E)$, suppose $v, \gamma^{1/2} \in L^{p(\cdot)}(E)$. Then $\tilde{H}^{1,p(\cdot)}_Q(v; E)$ is a Banach space. Furthermore, $\tilde{H}^{1,p(\cdot)}_Q(v; E)$ is separable if $p_+ < \infty$, and is reflexive if $1 < p_- \le p_+ < \infty$.*

Proof. To show that $\tilde{H}^{1,p(\cdot)}_Q(v; E)$ is a Banach space, it will suffice to show that $\tilde{H}^{1,p(\cdot)}_Q(v; E)$ is a closed subspace of the Banach space $H^{1,p(\cdot)}_Q(v; E)$. Let $\{(u_j, \mathbf{g}_j)\}_{j=1}^\infty$ be a Cauchy sequence in $\tilde{H}^{1,p(\cdot)}_Q(v; E)$. Since $H^{1,p(\cdot)}_Q(v; E)$ is complete, there is an element $(u, \mathbf{g}) \in H^{1,p(\cdot)}_Q(v; E)$ such that $u_j \to u$ in $L^{p(\cdot)}(v; E)$ and $\mathbf{g}_j \to \mathbf{g}$ in $\mathcal{L}^{p(\cdot)}_Q(E)$. Since $u_j \in \tilde{H}^{1,p(\cdot)}_Q(v; E)$ for each j, we have that $\int_E u_j(x)v(x)dx = 0$. Thus, by Hölder's inequality (Theorem 2.6) we get

$$\begin{aligned}\left|\int_E u(x)v(x)dx\right| &= \left|\int_E (u(x) - u_j(x))v(x)dx\right| \\ &\le \int_E |u(x) - u_j(x)|v(x)dx \le K_{p(\cdot)}\|(u - u_j)v\|_{L^{p(\cdot)}(E)}\|1\|_{L^{p'(\cdot)}(E)}.\end{aligned}$$

Since E is bounded, $\|1\|_{L^{p'(\cdot)}(E)} < \infty$. This follows at once from [2, Corollary 2.48]. Since $u_j \to u$ in $L^{p(\cdot)}(v; E)$, it follows that the right-hand side converges to 0. Hence,

$$\int_E u(x)v(x)dx = 0$$

and so $(u, \mathbf{g}) \in \tilde{H}^{1,p(\cdot)}_Q(v; E)$. Thus, $\tilde{H}^{1,p(\cdot)}_Q(v; E)$ is a closed subspace of $H^{1,p(\cdot)}_Q(v; E)$.

If $p_+ < \infty$, then $H^{1,p(\cdot)}_Q(v; E)$ is separable, and so every closed subspace, in particular $\tilde{H}^{1,p(\cdot)}_Q(v; E)$, is also separable. Finally, if $1 < p_- \le p_+ < \infty$, then $H^{1,p(\cdot)}_Q(v; E)$ is reflexive, and since every closed subspace of a reflexive Banach space is reflexive, $\tilde{H}^{1,p(\cdot)}_Q(v; E)$ is as well. □

As part of the proof of Theorem 1.8, we will need to apply the Poincaré inequality to any element of $\tilde{H}^{1,p(\cdot)}_Q(v; E)$ and not just to $C^1(\overline{E})$ functions. To prove we can do this, we need the following lemma.

Lemma 2.20. *Given $p(\cdot) \in \mathcal{P}(E)$ with $p_+ < \infty$, suppose $v, \gamma^{1/2} \in L^{p(\cdot)}(E)$. Then the set $C^1(\overline{E}) \cap \tilde{H}^{1,p(\cdot)}_Q(v; E)$ is dense in $\tilde{H}^{1,p(\cdot)}_Q(v; E)$.*

Proof. Fix $(u, \mathbf{g}) \in \tilde{H}^{1,p(\cdot)}_Q(v; E)$. Since $C^1(\overline{E})$ is dense in $\tilde{H}^{1,p(\cdot)}_Q(v; E) \subseteq H^{1,p(\cdot)}_Q(v; E)$, there exists a sequence of functions $u_k \in C^1(\overline{E})$ such that $(u_k, \nabla u_k) \to (u, \mathbf{g})$ in norm. Let $y_k = u_k - (u_k)_{E,v} \in C^1(\overline{E}) \cap \tilde{H}^{1,p(\cdot)}_Q(v; E)$; then $\nabla y_k = \nabla u_k$, and so to prove $(y_k, \nabla y_k) \to (u, \mathbf{g})$ it will suffice to show $u_k - y_k = (u_k)_{E,v} \to 0$ in $L^{p(\cdot)}(v; E)$. Since $(u, \mathbf{g}) \in \tilde{H}^{1,p(\cdot)}_Q(v; E)$, we have $u_{E,v} = 0$, and so by Hölder's inequality (Theorem 2.6)

$$(u_k)_{E,v} = \frac{1}{v(E)} \int_E (u_k - u)v\, dx \le K_{p(\cdot)} \|u_k - u\|_{L^{p(\cdot)}(v;E)} \|1\|_{L^{p'(\cdot)}(E)}.$$

Since E is bounded, $\|1\|_{L^{p'(\cdot)}(v;E)} < \infty$ as in the previous proof. Thus $(u_k)_{E,v} \to 0$. Consequently,

$$\|(u_k)_{E,v}\|_{L^{p(\cdot)}(v;E)} = |(u_k)_{E,v}| \|1\|_{L^{p(\cdot)}(v;E)}$$

converges to zero since $1 \in L^{p(\cdot)}(v; E)$. □

Theorem 2.21. *If Definition 1.2 holds, then the Poincaré inequality*

$$\|u\|_{L^{p(\cdot)}(v;E)} \le C_0 \|\mathbf{g}\|_{\mathcal{L}^{p(\cdot)}_Q(E)}$$

holds for every pair $(u, \mathbf{g}) \in \tilde{H}^{1,p(\cdot)}_Q(v; E)$.

Proof. By Lemma 2.20, for every $(u, \mathbf{g}) \in \tilde{H}^{1,p(\cdot)}_Q(v; E)$, there exists a sequence of functions $\{u_k\}_{k=1}^\infty \subseteq C^1(\overline{E}) \cap \tilde{H}^{1,p(\cdot)}_Q(v; E)$ such that $\|u_k\|_{L^{p(\cdot)}(v;E)} \to \|u\|_{L^{p(\cdot)}(v;E)}$ and $\|\nabla u_k\|_{\mathcal{L}^{p(\cdot)}_Q(v;E)} \to \|\mathbf{g}\|_{\mathcal{L}^{p(\cdot)}_Q(v;E)}$ as $k \to \infty$. Since $u_k \in \tilde{H}^{1,p(\cdot)}_Q(v; E)$, $(u_k)_{E,v} = 0$ for $k \in \mathbb{N}$. Hence, by Definition 1.2,

$$\|u\|_{L^{p(\cdot)}(v;E)} = \lim_{k\to\infty} \|u_k\|_{L^{p(\cdot)}(v;E)} = \lim_{k\to\infty} \|u_k - (u_k)_{E,v}\|_{L^{p(\cdot)}(v;E)} \le C_0 \lim_{k\to\infty} \|\nabla u_k\|_{\mathcal{L}^{p(\cdot)}_Q(v;E)} = C_0 \|\mathbf{g}\|_{\mathcal{L}^{p(\cdot)}_Q(v;E)}.$$

□

Finally, we define a weak solution to the degenerate $p(\cdot)$-Laplacian from Definition 1.4.

Definition 2.22. *Let $E \subseteq \mathbb{R}^n$ be a bounded open set, $p(\cdot) \in \mathcal{P}(E)$, and $v, \gamma^{1/2} \in L^{p(\cdot)}(E)$. Given $f \in L^{p(\cdot)}(v; E)$, the pair $(u, \mathbf{g})_f \in \tilde{H}^{1,p(\cdot)}_Q(v; E)$ is a weak solution to the Neumann problem* (1.2) *if for all test functions $\varphi \in C^1(\overline{E}) \cap \tilde{H}^{1,p(\cdot)}_Q(v; E)$,*

$$\int_E |\sqrt{Q(x)}\mathbf{g}(x)|^{p(x)-2} (\nabla\varphi(x))^T Q(x)\mathbf{g}(x) dx = -\int_E |f(x)|^{p(x)-2} f(x)\varphi(x)(v(x))^{p(x)} dx.$$

3. $p(\cdot)$-Neumann implies $p(\cdot)$-Poincaré

In this section we will give the first half of the proof of Theorem 1.8. Fix $p(\cdot) \in \mathcal{P}(E)$, $1 < p_- \leq p_+ < \infty$, let v be a weight in Ω with $v \in L^{p(\cdot)}(E)$, and Q a measurable matrix function with $\gamma^{1/2} \in L^{p(\cdot)}(E)$. Assume that the Definition 1.4 holds. We will show that the Poincaré inequality in Definition 1.2 holds.

We begin by showing that the regularity condition (1.3) in Definition 1.4 actually implies a stronger condition.

Lemma 3.1. *Let $p(\cdot)$, v, Q be as defined above. Then there exists a constant $C = C(p(\cdot), v, E)$ such that for any $f \in L^{p(\cdot)}(v; E)$ and any corresponding weak solution $(u, \mathbf{g})_f \in \tilde{H}^{1,p(\cdot)}_Q(v; E)$ of* (1.2)*,*

$$\|\mathbf{g}\|^{p_*-1}_{\mathcal{L}^{p(\cdot)}_Q(E)} \leq C\|f\|^{r_*-1}_{L^{p(\cdot)}(v;E)},$$

where p_* and r_* are defined by (1.4).

Proof. Let $f \in L^{p(\cdot)}(v; E)$ and $(u, \mathbf{g})_f$ be a weak solution of (1.2) with data f. By Proposition 2.7, Hölder's inequality, the regularity estimate (1.3), and Theorem 2.9, and using the weak solution $(u, \mathbf{g})_f$ itself as a test function in the definition of weak solution, we have that

$$\begin{aligned}
\|\mathbf{g}\|^{p_*}_{\mathcal{L}^{p(\cdot)}_Q(E)} &\leq \int_E |\sqrt{Q(x)}\mathbf{g}(x)|^{p(x)}dx \\
&= \int_E |\sqrt{Q(x)}\mathbf{g}(x)|^{p(x)-2}\mathbf{g}(x)^T Q(x)\mathbf{g}(x)dx \\
&= -\int_E |f(x)|^{p(x)-2} f(x)u(x)v(x)^{p(x)}dx \\
&\leq \int_E |f(x)|^{p(x)-1} v(x)^{p(x)-1}|u(x)|v(x)dx \\
&\leq K_{p(\cdot)}\|(fv)^{p(\cdot)-1}\|_{L^{p'(\cdot)}(E)}\|uv\|_{L^{p(\cdot)}(E)} \\
&\leq K_{p(\cdot)}\|f\|^{r_*-1}_{L^{p(\cdot)}(v;E)}\|u\|_{L^{p(\cdot)}(v;E)}. \\
&\leq K_{p(\cdot)}C_1\|f\|^{r_*-1}_{L^{p(\cdot)}(v;E)}\|f\|^{\frac{r_*-1}{p_*-1}}_{L^{p(\cdot)}(v;E)}.
\end{aligned}$$

Note that in the second to last inequality, we used that fact that in this case the exponent b_* in Theorem 2.9 equals r_*. Therefore, if we raise both sides to the power of $(p_* - 1)/p_*$, we get

$$\|\mathbf{g}\|^{p_*-1}_{\mathcal{L}^{p(\cdot)}_Q(E)} \leq C\|f\|^{r_*-1}_{L^{p(\cdot)}(v;E)},$$

where $C = C(p(\cdot), v, E)$. □

To prove that the Poincaré inequality holds, fix $f \in C^1(\overline{E})$. We will first consider the special case where $f_{E,v} = 0$ and $\|f\|_{L^{p(\cdot)}(v;E)} = 1$. Then by Proposition 2.8, the definition of weak solution with f as our test function, Hölder's inequality, and Theorem 2.9,

$$\|f\|_{L^{p(\cdot)}(v;E)} = \int_E |f(x)v(x)|^{p(x)}dx$$

$$
\begin{aligned}
&= \int_E |f(x)|^{p(x)-2} f(x) f(x) v(x)^{p(x)} dx \\
&\le \int_E |\sqrt{Q(x)}\mathbf{g}(x)|^{p(x)-2} |\nabla f(x)^T Q(x)\mathbf{g}(x)| dx \\
&\le \int_E |\sqrt{Q(x)}\mathbf{g}(x)|^{p(x)-1} |\sqrt{Q(x)}\nabla f(x)| dx \\
&\le K_{p(\cdot)} \| |\sqrt{Q}\mathbf{g}|^{p(\cdot)-1} \|_{p'(\cdot)} \|\nabla f\|_{\mathcal{L}_Q^{p(\cdot)}(E)} \\
&\le K_{p(\cdot)} \|\mathbf{g}\|^{b_*-1}_{\mathcal{L}_Q^{p(\cdot)}(E)} \|\nabla f\|_{\mathcal{L}_Q^{p(\cdot)}(E)}.
\end{aligned}
$$

By Lemma 3.1 and our assumption that $\|f\|_{L^{p(\cdot)}(v;E)} = 1$, we find

$$
\|f\|_{L^{p(\cdot)}(v;E)} \le C \|\nabla f\|_{\mathcal{L}_Q^{p(\cdot)}(E)},
$$

where $C = C(p(\cdot), v, E)$. This is what we wanted to prove.

To prove the general case, let $f_0 = f - f_{v,E}$, and $f_1 = f_0 / \|f_0\|_{L^{p(\cdot)}(v;E)}$. Then f_1 has zero mean and $\|f_1\|_{L^{p(\cdot)}(v;E)} = 1$, so by the previous case f_1 satisfies the Poincaré inequality. But by the homogeneity of this inequality, and since $\|f_0\|_{L^{p(\cdot)}(v;E)} \nabla f_1 = \nabla f_0 = \nabla f$, we have that f satisfies the Poincaré inequality as well. This completes the proof.

4. $p(\cdot)$-Poincaré implies $p(\cdot)$-Neumann

In this section we will give the second half of the proof of Theorem 1.8. Fix $p(\cdot) \in \mathcal{P}(E)$, $1 < p_- \le p_+ < \infty$, let v be a weight in Ω with $v \in L^{p(\cdot)}(E)$, and Q a measurable matrix function with $\gamma^{1/2} \in L^{p(\cdot)}(E)$. Assume that the Poincaré inequality in Definition 1.2 holds. We will show that Definition 1.4 holds by showing that a weak solution to (1.2) exists and that the regularity estimate (1.3) is satisfied.

To show the existence of a weak solution to the Neumann problem (1.2), we will apply Minty's theorem [19]. To state it, we introduce some notation. Given a reflexive Banach space $\mathcal{B}$, denote its dual space by $\mathcal{B}^*$. Given a functional $\alpha \in \mathcal{B}^*$, write its value at $\varphi \in \mathcal{B}$ as $\alpha(\varphi) = \langle \alpha, \varphi \rangle$. Thus, if $\beta : \mathcal{B} \to \mathcal{B}^*$ and $u \in \mathcal{B}$, then we have $\beta(u) \in \mathcal{B}^*$ and so its value at φ is denoted by $\beta(u)(\varphi) = \langle \beta(u), \varphi \rangle$.

Theorem 4.1. *(Minty's Theorem, [19]) Let $\mathcal{B}$ be a reflexive, separable Banach space and fix $\Gamma \in \mathcal{B}^*$. Suppose that $\mathcal{T} : \mathcal{B} \to \mathcal{B}^*$ is a bounded operator that is:*

1). Monotone: $\langle \mathcal{T}(u) - \mathcal{T}(\varphi), u - \varphi \rangle \ge 0$ for all $u, \varphi \in \mathcal{B}$;
2). Hemicontinuous: for $z \in \mathbb{R}$, the mapping $z \to \langle \mathcal{T}(u + z\varphi), \varphi \rangle$ is continuous for all $u, \varphi \in \mathcal{B}$;
3). Almost Coercive: there exists a constant $\lambda > 0$ so that $\langle \mathcal{T}(u), u \rangle > \langle \Gamma, u \rangle$ for any $u \in \mathcal{B}$ satisfying $\|u\|_{\mathcal{B}} > \lambda$.

Then the set of $u \in \mathcal{B}$ such that $\mathcal{T}(u) = \Gamma$ is non-empty.

To apply Minty's theorem to prove the existence of a weak solution, let $\mathcal{B} = \tilde{H}_Q^{1,p(\cdot)}(v; E)$. Note that with our hypotheses, by Theorem 2.19, $\tilde{H}_Q^{1,p(\cdot)}(v; E)$ is a reflexive, separable Banach space. We now define the operators Γ and $\mathcal{T}$ using the right and left-hand sides of the equation in Definition 2.22.

Definition 4.2. *Given* $f \in L^{p(\cdot)}(v;E)$*, define* $\Gamma = \Gamma_f : \tilde{H}^{1,p(\cdot)}_Q(v;E) \to \mathbb{R}$ *by setting*

$$\langle \Gamma, \mathbf{w} \rangle = -\int_E |f(x)|^{p(x)-2} f(x) w(x)(v(x))^{p(x)} dx$$

for any $\mathbf{w} = (w, \mathbf{h}) \in \tilde{H}^{1,p(\cdot)}_Q(v;E)$.

Remark 4.3. Γ_f *clearly depends on* $f \in L^{p(\cdot)}(v;E)$*. But for ease of notation we will simply write* Γ *where* f *is understood in context.*

Definition 4.4. *Define* $\mathcal{T} : \tilde{H}^{1,p(\cdot)}_Q(v;E) \to \left(\tilde{H}^{1,p(\cdot)}_Q(v;E)\right)^*$ *by setting*

$$\langle \mathcal{T}(\mathbf{u}), \mathbf{w} \rangle = \int_E \left| \sqrt{Q(x)} \mathbf{g}(x) \right|^{p(x)-2} \mathbf{h}^T(x) Q(x) \mathbf{g}(x) dx$$

for $\mathbf{u} = (u, \mathbf{g}),\ \mathbf{w} = (w, \mathbf{h}) \in \tilde{H}^{1,p(\cdot)}_Q(v;E)$,

Clearly, $\mathbf{u} = (u, \mathbf{g})$ is a weak solution of (1.2) if and only if $\langle \mathcal{T}(\mathbf{u}), \mathbf{w} \rangle = \langle \Gamma, \mathbf{w} \rangle$ for all $\mathbf{w} \in \tilde{H}^{1,p(\cdot)}_Q(v;E)$. Therefore, we will have shown that a weak solution exists if we can show that the operators Γ_f and $\mathcal{T}$ satisfy the hypotheses of Minty's Theorem.

Lemma 4.5. *Given* $f \in L^{p(\cdot)}(v;E)$*,* $\Gamma = \Gamma_f$ *is a bounded, linear functional on* $\tilde{H}^{1,p(\cdot)}_Q(v;E)$.

Proof. We first show that Γ is linear. Let $\mathbf{u} = (u, \mathbf{g}),\ \mathbf{w} = (w, \mathbf{h})$ be in $\tilde{H}^{1,p(\cdot)}_Q(v;E)$. Then for all $\alpha, \beta \in \mathbb{R}$,

$$\langle \Gamma, \alpha\mathbf{u} + \beta\mathbf{w} \rangle = -\int_E |f(x)|^{p(x)-2} f(x)(\alpha u(x) + \beta w(x))(v(x))^{p(x)} dx = \alpha\langle \Gamma, \mathbf{u} \rangle + \beta\langle \Gamma, \mathbf{w} \rangle.$$

To show that Γ is bounded, it will suffice to show that there exists a constant $C = C(f, v, p(\cdot))$ such that

$$|\langle \Gamma, \mathbf{w} \rangle| \le C \|wv\|_{L^{p(\cdot)}(E)}, \tag{4.1}$$

since $\|wv\|_{L^{p(\cdot)}(E)} = \|w\|_{L^{p(\cdot)}(v;E)} \le \|\mathbf{w}\|_{\tilde{H}^{1,p(\cdot)}_Q(v;E)}$. By Hölder's inequality,

$$\begin{aligned} |\langle \Gamma, \mathbf{w} \rangle| &= \left| \int_E |f(x)|^{p(x)-2} f(x) w(x) v(x)^{p(x)}\, dx \right| \\ &\le \int_E |f(x)|^{p(x)-1} v(x)^{p(x)-1} w(x) v(x)\, dx \le K_{p(\cdot)} \| |fv|^{p(\cdot)-1} \|_{L^{p'(\cdot)}(E)} \|wv\|_{L^{p(\cdot)}(E)}. \end{aligned}$$

Since $f \in L^{p(\cdot)}(v;E)$, by Theorem 2.9 we have that

$$\| |fv|^{p(\cdot)-1} \|_{L^{p'(\cdot)}(E)} \le \|f\|^{b_*-1}_{L^{p(\cdot)}(v;E)} < \infty.$$

Therefore, if we let $C = K_{p(\cdot)} \| |fv|^{p(\cdot)-1} \|_{L^{p'(\cdot)}(E)}$, we have that (4.1) holds. □

We now prove that $\mathcal{T}$ is bounded, monotone, hemicontinuous.

Lemma 4.6. $\mathcal{T}$ *is a bounded operator.*

Proof. We will prove that $\mathcal{T}$ is bounded by showing the operator norm of $\mathcal{T}$ is uniformly bounded. The norm of $\mathcal{T} : \tilde{H}_Q^{1,p(\cdot)}(v;E) \to \left(\tilde{H}_Q^{1,p(\cdot)}(v;E)\right)^*$ is given by

$$\|\mathcal{T}\|_{\mathrm{op}} = \sup\{|\mathcal{T}(\mathbf{u})|_{op} : \|\mathbf{u}\|_{H_Q^{1,p(\cdot)}(v;E)} = 1\},$$

where $|\mathcal{T}(\mathbf{u})|_{op} = \sup\{|\langle\mathcal{T}(\mathbf{u}),\mathbf{w}\rangle| : \|\mathbf{w}\|_{H_Q^{1,p(\cdot)}(v;E)} = 1\}$. Thus, it will suffice to show that there exists a constant C such that for all $\mathbf{u},\mathbf{w} \in \tilde{H}_Q^{1,p(\cdot)}(v;E)$ with $\|\mathbf{u}\|_{H_Q^{1,p(\cdot)}(v;E)} = \|\mathbf{w}\|_{H_Q^{1,p(\cdot)}(v;E)} = 1$, $|\langle\mathcal{T}(\mathbf{u}),\mathbf{w}\rangle| \le C$. By Theorem 2.9, for any $f \in L^{p(\cdot)}(v;E)$, $\||f|^{p(\cdot)-1}\|_{L^{p'(\cdot)}(E)} \le \|f\|_{L^{p(\cdot)}(E)}^{p_*-1}$. Therefore, by Hölder's inequality we have that

$$\begin{aligned}
|\langle\mathcal{T}(\mathbf{u}),\mathbf{w}\rangle| &= \left|\int_E |\sqrt{Q(x)}\mathbf{g}(x)|^{p(x)-2}(\mathbf{h}(x))^T Q(x)\mathbf{g}(x)dx\right| \\
&\le \int_E |\sqrt{Q(x)}\mathbf{g}(x)|^{p(x)-1}|\sqrt{Q(x)}\mathbf{h}(x)|dx \\
&\le K_{p(\cdot)}\||\sqrt{Q}\mathbf{g}|^{p(\cdot)-1}\|_{L^{p'(\cdot)}(E)}\||\sqrt{Q}\mathbf{h}\|_{L^{p(\cdot)}(E)} \\
&\le K_{p(\cdot)}\||\sqrt{Q}\mathbf{g}|\|_{L^{p(\cdot)}(E)}^{b_*-1}\||\sqrt{Q}\mathbf{h}|\|_{L^{p(\cdot)}(E)} \\
&= K_{p(\cdot)}\|\mathbf{g}\|_{\mathcal{L}_Q^{p(\cdot)}(E)}^{b_*-1}\|\mathbf{h}\|_{\mathcal{L}_Q^{p(\cdot)}(E)} \\
&\le K_{p(\cdot)}.
\end{aligned}$$

Thus, $\mathcal{T}$ is bounded. □

Lemma 4.7. *$\mathcal{T}$ is Monotone.*

Proof. Let $\mathbf{u} = (u,\mathbf{g})$ and $\mathbf{w} = (w,\mathbf{h})$ be in $\tilde{H}_Q^{1,p(\cdot)}(v;E)$. Then

$$\begin{aligned}
&\langle\mathcal{T}(\mathbf{u}) - \mathcal{T}(\mathbf{w}),\mathbf{u}-\mathbf{w}\rangle \\
&\quad= \langle\mathcal{T}(\mathbf{u}),\mathbf{u}-\mathbf{w}\rangle - \langle\mathcal{T}(\mathbf{w}),\mathbf{u}-\mathbf{w}\rangle \\
&\quad= \int_E |\sqrt{Q}\mathbf{g}|^{p(\cdot)-2}(\mathbf{g}-\mathbf{h})^T Q\mathbf{g} - |\sqrt{Q}\mathbf{h}|^{p(\cdot)-2}(\mathbf{g}-\mathbf{h})^T Q\mathbf{h}\,dx \\
&\quad= \int_E (\sqrt{Q}(\mathbf{g}-\mathbf{h}))^T(|\sqrt{Q}\mathbf{g}|^{p(\cdot)-2}\sqrt{Q}\mathbf{g}) - (\sqrt{Q}(\mathbf{g}-\mathbf{h}))^T(|\sqrt{Q}\mathbf{h}|^{p(\cdot)-2}\sqrt{Q}\mathbf{h})\,dx \\
&\quad= \int_E (\sqrt{Q}(\mathbf{g}-\mathbf{h}))^T\left[|\sqrt{Q}\mathbf{g}|^{p(\cdot)-2}\sqrt{Q}\mathbf{g} - |\sqrt{Q}\mathbf{h}|^{p(\cdot)-2}\sqrt{Q}\mathbf{h}\right]dx \\
&\quad= \int_E \langle|\sqrt{Q}\mathbf{g}|^{p(\cdot)-2}\sqrt{Q}\mathbf{g} - |\sqrt{Q}\mathbf{h}|^{p(\cdot)-2}\sqrt{Q}\mathbf{h},\ \sqrt{Q}\mathbf{g} - \sqrt{Q}\mathbf{h}\rangle_{\mathbb{R}^n}\,dx,
\end{aligned}$$

where $\langle\cdot,\cdot\rangle_{\mathbb{R}^n}$ denotes the inner product on $\mathbb{R}^n$. For each $x \in E$, the integrand is of the form

$$\langle|s|^{p-2}s - |r|^{p-2}r, s-r\rangle_{\mathbb{R}^n},$$

where $s,r \in \mathbb{R}^n$ and $p > 1$. But as noted in [13, p. 74] (see also [1, Section 4]), this expression is nonnegative. Thus, $\mathcal{T}$ is monotone. □

Lemma 4.8. *$\mathcal{T}$ is Hemicontinuous.*

Proof. Let $z,\, y \in \mathbb{R}$ and let $\mathbf{u} = (u, \mathbf{g})$, $\mathbf{w} = (w, \mathbf{h})$ be in $\tilde{H}^{1,p(\cdot)}_Q(v; E)$. Define $\psi = \mathbf{g} + z\mathbf{h}$ and $\gamma = \mathbf{g} + y\mathbf{h}$. Then

$$\begin{aligned}
&\langle \mathcal{T}(\mathbf{u} + z\mathbf{w}) - \mathcal{T}(\mathbf{u}u + y\mathbf{w}), \mathbf{w}\rangle \\
&\qquad = \int_E |\sqrt{Q}\psi|^{p(\cdot)-2}\mathbf{h}^T Q\psi - |\sqrt{Q}\gamma|^{p(\cdot)-2}\mathbf{h}^T Q\gamma \, dx \\
&\qquad = \int_E (\sqrt{Q}\mathbf{h})^T \left[|\sqrt{Q}\psi|^{p(\cdot)-2} \sqrt{Q}\psi - |\sqrt{Q}\gamma|^{p(\cdot)-2} \sqrt{Q}\gamma \right] dx \\
&\qquad = \int_E (\sqrt{Q}\mathbf{h})^T \left[|\mathbf{r}|^{p(\cdot)-2}\mathbf{r} - |\mathbf{s}|^{p(\cdot)-2}\mathbf{s} \right] dx
\end{aligned} \tag{4.2}$$

where $\mathbf{r} = \sqrt{Q}\psi$ and $\mathbf{s} = \sqrt{Q}\gamma$. Define $E^+ = \{x \in E : p(x) > 2\}$ and $E^- = \{x \in E : p(x) \leq 2\}$. We will show that the integral (4.2) tends to 0 as $z \to y$ by estimating it over E^+ and E^- separately.

Observe that our choice of $\mathbf{r}, \mathbf{s}$ gives

$$\mathbf{r} - \mathbf{s} = \sqrt{Q}(\psi - \gamma) = \sqrt{Q}(z\mathbf{h} - y\mathbf{h}) = (z - y)\sqrt{Q}\mathbf{h}. \tag{4.3}$$

Hence,

$$\|\,|\mathbf{r} - \mathbf{s}|\,\|_{L^{p(\cdot)}(E)} = |z - y|\|\,|\sqrt{Q}\mathbf{h}|\,\|_{L^{p(\cdot)}(E)} \leq |z - y|\|\mathbf{w}\|_{H^{1,p(\cdot)}_Q(v;E)} \tag{4.4}$$

Furthermore, by an inequality from [13, pp. 43, 73] (see also [1, Section 4]), we have that for $\mathbf{r}, \mathbf{s} \in \mathbb{R}^n$ and $p > 2$,

$$\left| |\mathbf{r}|^{p-2}\mathbf{r} - |\mathbf{s}|^{p-2}\mathbf{s} \right| \leq (p - 1)|\mathbf{r} - \mathbf{s}|(|\mathbf{s}|^{p-2} + |\mathbf{r}|^{p-2}).$$

If we combine this inequality with (4.3) and apply them to (4.2) over E^+, we get that

$$\begin{aligned}
&\left| \int_{E^+} (\sqrt{Q}\mathbf{h})^T \left[|\mathbf{r}|^{p(x)-2}\mathbf{r} - |\mathbf{s}|^{p(x)-2}\mathbf{s} \right] dx \right| \\
&\qquad \leq \int_{E^+} |\sqrt{Q}\mathbf{h}||p(x) - 1||\mathbf{r} - \mathbf{s}|(|\mathbf{s}|^{p(x)-2} + |\mathbf{r}|^{p(x)-2})\, dx \\
&\qquad \leq |z - y||p_+ - 1| \int_{E^+} |\sqrt{Q}\mathbf{h}|^2 \left| |\mathbf{s}|^{p(x)-2} + |\mathbf{r}|^{p(x)-2} \right| dx.
\end{aligned} \tag{4.5}$$

Since $p(x) > 2$ on E^+, by Hölder's inequality, Theorem 2.6, with exponents $\frac{p(\cdot)}{2}, \frac{p(\cdot)}{p(\cdot)-2}$ we have

$$\int_{E^+} |\sqrt{Q}\mathbf{h}|^2 \left| |\mathbf{s}|^{p(\cdot)-2} + |\mathbf{r}|^{p(\cdot)-2} \right| dx \leq K_{p(\cdot)/2} \|\,|\sqrt{Q}\mathbf{h}|^2\|_{L^{p(\cdot)/2}(E^+)} \left\| |\mathbf{s}|^{p(\cdot)-2} + |\mathbf{r}|^{p(\cdot)-2} \right\|_{L^{\frac{p(\cdot)}{p(\cdot)-2}}(E^+)}.$$

By Proposition 2.7, since $\|\,|\sqrt{Q}\mathbf{h}|\,\|_{L^{p(\cdot)}(E)}$ is finite, we have that

$$\int_{E^+} \left| |\sqrt{Q}\mathbf{h}|^2 \right|^{p(x)/2} dx = \int_{E^+} |\sqrt{Q}\mathbf{h}|^{p(x)} dx < \infty,$$

and so by the same result we have that $\|\,|\sqrt{Q}\mathbf{h}|^2\|_{p(\cdot)/2} < \infty$.

By the triangle inequality,

$$\|\,|\mathbf{s}|^{p(\cdot)-2} + |\mathbf{r}|^{p(\cdot)-2}\|_{L^{\frac{p(\cdot)}{p(\cdot)-2}}(E^+)} \leq \|\,|\mathbf{s}|^{p(\cdot)-2}\|_{L^{\frac{p(\cdot)}{p(\cdot)-2}}(E^+)} + \|\,|\mathbf{r}|^{p(\cdot)-2}\|_{L^{\frac{p(\cdot)}{p(\cdot)-2}}(E^+)}.$$

Observe that

$$|\mathbf{s}|^{(p(\cdot)-2)\cdot p(\cdot)/(p(\cdot)-2)} = |\mathbf{s}|^{p(\cdot)} = |\sqrt{Q}\gamma|^{p(\cdot)} = |\sqrt{Q}(\mathbf{g}+y\mathbf{h})|^{p(\cdot)}.$$

Since $\mathbf{g},\mathbf{h} \in \mathcal{L}_Q^{p(\cdot)}(E)$, $\||\sqrt{Q}(\mathbf{g}+y\mathbf{h})|\|_{L^{p(\cdot)}(E^+)} < \infty$. But then, again by Proposition 2.7,

$$\int_{E^+} \left| |\sqrt{Q}(\mathbf{g}+y\mathbf{h})|^{p(x)-2} \right|^{p(x)/(p(x)-2)} dx = \int_{E^+} |\sqrt{Q}(\mathbf{g}+y\mathbf{h})|^{p(x)} dx < \infty,$$

and so $\||\mathbf{s}|^{p(\cdot)-2}\|_{L^{\frac{p(\cdot)}{p(\cdot)-2}}(E^+)} < \infty$. Similarly, $\||\mathbf{r}|^{p(\cdot)-2}\|_{L^{\frac{p(\cdot)}{p(\cdot)-2}}(E^+)}$ is finite. Therefore,

$$\int_{E^+} |\sqrt{Q}\mathbf{h}|^2 \left| |\mathbf{s}|^{p(x)-2} + |\mathbf{r}|^{p(x)-2} \right| dx < \infty,$$

and so (4.5) converges to 0 as $z \to y$.

Now consider the domain E^-. By [13, p. 43] (see also [1, Section 4]) we have that for $\mathbf{r}, \mathbf{s} \in \mathbb{R}^n$ and $1 < p \leq 2$,

$$||\mathbf{s}|^{p-2}\mathbf{s} - |\mathbf{r}|^{p-2}\mathbf{r}| \leq C(p)|\mathbf{s}-\mathbf{r}|^{p-1}.$$

The constant $C(p)$ varies continuously in p; since for $x \in E^-$, $1 < p_- \leq p(x) \leq 2$, we must have that

$$C = \sup_{x \in E^-} C(p(x)) < \infty.$$

If we apply this estimate, Hölder's inequality, Theorem 2.9, and (4.4) to (4.2), we get

$$\begin{aligned}
&\left| \int_{E^-} (\sqrt{Q}\mathbf{h})^T \left[|\mathbf{r}|^{p(x)-2}\mathbf{r} - |\mathbf{s}|^{p(x)-2}\mathbf{s} \right] dx \right| \\
&\qquad \leq \int_{E^-} |\sqrt{Q}\mathbf{h}| \left| |\mathbf{r}|^{p(x)-2}\mathbf{r} - |\mathbf{s}|^{p(x)-2}\mathbf{s} \right| dx \\
&\qquad \leq C \int_{E^-} |\sqrt{Q}\mathbf{h}||\mathbf{s}-\mathbf{r}|^{p(x)-1} dx \\
&\qquad \leq CK_{p(\cdot)}\|\mathbf{h}\|_{\mathcal{L}_Q^{p(\cdot)}(E^-)} \left\| |\mathbf{s}-\mathbf{r}|^{p(\cdot)-1} \right\|_{L^{p'(\cdot)}(E)} \\
&\qquad \leq CK_{p(\cdot)}\|\mathbf{h}\|_{\mathcal{L}_Q^{p(\cdot)}(E^-)} \||\mathbf{s}-\mathbf{r}|\|_{L^{p(\cdot)}(E)}^{b_*-1} \\
&\qquad \leq CK_{p(\cdot)}\|\mathbf{h}\|_{\mathcal{L}_Q^{p(\cdot)}(E^-)} (|z-y|\|\mathbf{w}\|_{H_Q^{1,p(\cdot)}(v;E)})^{b_*-1}.
\end{aligned}$$

Thus, the integral (4.2) converges to 0 on E^- as $z \to y$, and so $\mathcal{T}$ is hemicontinuous.

□

Lemma 4.9. *$\mathcal{T}$ is almost coercive.*

Remark 4.10. *The proof of Lemma 4.9 is the only part of the proof that requires the Poincaré inequality* (1.1).

Proof. Fix $f \in L^{p(\cdot)}(v;E)$ and let $\Gamma = \Gamma_f$. We need to find $\lambda > 0$ sufficiently large that for any $\mathbf{u} = (u, \mathbf{g}) \in \tilde{H}_Q^{1,p(\cdot)}(v;E)$ such that $\|\mathbf{u}\|_{H_Q^{1,p(\cdot)}(v;E)} > \lambda$, $\langle \mathcal{T}(\mathbf{u}), \mathbf{u} \rangle > \langle \Gamma, \mathbf{u} \rangle$. Suppose first that $\lambda > 1 + C_0$ where C_0 is as in Poincaré inequality (1.1). By Lemma 2.20, and since u has mean zero, we have that

$$\lambda < \|\mathbf{u}\|_{H_Q^{1,p(\cdot)}(v;E)} = \|u\|_{L^{p(\cdot)}(v;E)} + \|\mathbf{g}\|_{\mathcal{L}_Q^{p(\cdot)}(E)} \leq (C_0+1)\|\mathbf{g}\|_{\mathcal{L}_Q^{p(\cdot)}(E)}.$$

Hence, $\|\mathbf{g}\|_{\mathcal{L}_Q^{p(\cdot)}(E)} > 1$.

By Proposition 2.7 and the Poincaré inequality (1.1) we have that

$$\langle \mathcal{T}(\mathbf{u}), \mathbf{u} \rangle = \int_E |\sqrt{Q}\mathbf{g}|^{p(x)-2}\mathbf{g}^T Q\mathbf{g}dx = \int_E |\sqrt{Q}\mathbf{g}|^{p(x)}dx \geq \|\mathbf{g}\|^{p_-}_{\mathcal{L}_Q^{p(\cdot)}(E)} \geq \frac{1}{C_0^{p_-}}\|u\|^{p_-}_{L^{p(\cdot)}(v;E)}.$$

Consequently,

$$(C_0^{p_-} + 1)\langle \mathcal{T}(\mathbf{u}), \mathbf{u} \rangle \geq \|\mathbf{g}\|^{p_-}_{\mathcal{L}_Q^{p(\cdot)}(E)} + \|u\|^{p_-}_{L^{p(\cdot)}(v;E)} \geq 2^{1-p_-}\|\mathbf{u}\|^{p_-}_{H_Q^{1,p(\cdot)}(v;E)}. \tag{4.6}$$

By Hölder's inequality,

$$\begin{aligned}|\langle \Gamma, \mathbf{u} \rangle| &= \left| \int_E |f(x)|^{p(x)-2} f(x)u(x)v(x)^{p(x)}dx \right| \leq \int_E |f(x)|^{p(x)-1}v(x)^{p(x)-1}|u(x)|v(x)dx \\ &\leq K_{p(\cdot)}\|(fv)^{p(\cdot)-1}\|_{L^{p'(\cdot)}(E)}\|u\|_{L^{p(\cdot)}(v;E)} \leq K_{p(\cdot)}\|(fv)^{p(\cdot)-1}\|_{L^{p'(\cdot)}(E)}\|\mathbf{u}\|_{H_Q^{1,p(\cdot)}(v;E)}.\end{aligned}$$

Since $f \in L^{p(\cdot)}(v;E)$, $\|fv\|_{L^{p(\cdot)}(E)} < \infty$, and so by Theorem 2.9, $\||fv|^{p(\cdot)-1}\|_{L^{p'(\cdot)}(E)} < \infty$. Let $C(f) = K_{p(\cdot)}\|(fv)^{p(\cdot)-1}\|_{L^{p'(\cdot)}(E)}$; then we have

$$C(f)\|\mathbf{u}\|_{H_Q^{1,p(\cdot)}(v;E)} = C(f)\|\mathbf{u}\|^{1-p_-}_{H_Q^{1,p(\cdot)}(v;E)}\|\mathbf{u}\|^{p_-}_{H_Q^{1,p(\cdot)}(v;E)} \leq C(f)\|\mathbf{u}\|^{1-p_-}_{H_Q^{1,p(\cdot)}(v;E)}\frac{C_0^{p_-}+1}{2^{1-p_-}}\langle \mathcal{T}(\mathbf{u}), \mathbf{u} \rangle.$$

Let $C = C(f)\frac{C_0^{p_-}+1}{2^{1-p_-}}$; then

$$|\langle \Gamma, \mathbf{u} \rangle| \leq C\|\mathbf{u}\|^{1-p_-}_{H_Q^{1,p(\cdot)}(v;E)}\langle \mathcal{T}(\mathbf{u}), \mathbf{u} \rangle.$$

Therefore, if we further assume that $\lambda > C^{1/(p_- -1)}$, then

$$\|\mathbf{u}\|_{H_Q^{1,p(\cdot)}(v;E)} > \lambda > C^{1/(p_- -1)}.$$

This in turn implies $C\|\mathbf{u}\|^{1-p_-}_{H_Q^{1,p(\cdot)}(v;E)} < 1$. Thus, we have that

$$|\langle \Gamma, \mathbf{u} \rangle| < \langle \mathcal{T}(\mathbf{u}), \mathbf{u} \rangle$$

and our proof is complete. □

We have now shown that the hypotheses of Minty's theorem are satisfied, and so a weak solution exists. To complete the proof, we need to prove that the regularity estimate (1.3) holds. This is established in the next lemma.

Lemma 4.11. *There is a positive constant $C = C(p(\cdot), E)$ such that for any $f \in L^{p(\cdot)}(v;E)$ and any corresponding weak solution $(u, \mathbf{g})_f \in \tilde{H}_Q^{1,p(\cdot)}(v;E)$,*

$$\|u\|_{L^{p(\cdot)}(v;E)} \leq C\|f\|^{\frac{r_*-1}{p_*-1}}_{L^{p(\cdot)}(v;E)},$$

where p_* and r_* are defined by (1.4).

Proof. By Definition 2.22, Proposition 2.7, Theorem 2.9 and the Poincaré inequality (1.1), we have that

$$\begin{aligned}
\|\mathbf{g}\|^{p_*}_{\mathcal{L}^{p(\cdot)}_Q(E)} &\leq \int_E |\sqrt{Q}\mathbf{g}|^{p(x)-2}\mathbf{g}^T Q\mathbf{g}dx \\
&= -\int_E |f|^{p(x)-2} f u v^{p(x)} dx \\
&\leq \int_E |f|^{p(x)-1} v^{p(x)-1}|u| v dx \\
&\leq K_{p(\cdot)} \| |fv|^{p(\cdot)-1}\|_{L^{p'(\cdot)}(E)} \|uv\|_{L^{p(\cdot)}(E)} \\
&\leq K_{p(\cdot)} C_0 \| |fv|^{p(\cdot)-1}\|_{L^{p'(\cdot)}(E)} \|\mathbf{g}\|_{\mathcal{L}^{p(\cdot)}_Q(E)} \\
&\leq K_{p(\cdot)} C_0 \|fv\|^{r_*-1}_{L^{p(\cdot)}(E)} \|\mathbf{g}\|_{\mathcal{L}^{p(\cdot)}_Q(E)} \\
&= K_{p(\cdot)} C_0 \|f\|^{r_*-1}_{L^{p(\cdot)}(v;E)} \|\mathbf{g}\|_{\mathcal{L}^{p(\cdot)}_Q(E)}.
\end{aligned}$$

If we combine this inequality with the Poincaré inequality, we get

$$\|u\|_{L^{p(\cdot)}(v;E)} \leq C_0\|\mathbf{g}\|_{\mathcal{L}^{p(\cdot)}_Q(E)} \leq C_0(K_{p(\cdot)}C_0)^{1/(p_*-1)}\|f\|^{(r_*-1)/(p_*-1)}_{L^{p(\cdot)}(v;E)} \leq C_0(K_{p(\cdot)}C_0)^{1/(p_--1)}\|f\|^{(r_*-1)/(p_*-1)}_{L^{p(\cdot)}(v;E)},$$

which is the desired inequality. □

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Conflict of interest

The authors declare no conflict of interest.

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A. Estimates for the weighted Poincaré inequality

In this section we prove Proposition 1.9. As we noted above, versions of this result appear to be known, but we have not found the proof in the literature. Recall that E is bounded, $v \in L^p(E)$, and $w = v^p$, so $w \in L^1(E)$. Fix $f \in C^1(\overline{E})$; then

$$\left|f_{E,w} - f_{E,v}\right| = \left|\frac{1}{v(E)} \int_E (f_{E,w} - f) w^{1/p}\, dx\right| \leq K_1 \left(\int_E |f - f_{E,w}|^p w\, dx\right)^{1/p} = K_1 \|f - f_{E,w}\|_{L^p(v;E)},$$

where $K_1 = \frac{|E|^{1/p'}}{v(E)}$. But then we have that

$$\|f - f_{E,v}\|_{L^p(v;E)} \leq \|f - f_{E,w}\|_{L^p(v;E)} + \|f_{E,w} - f_{E,v}\|_{L^p(v;E)} \leq (1 + K_1 w(E)^{1/p}) \|f - f_{E,w}\|_{L^p(v;E)}.$$

Conversely, if we switch the roles of v and w in the first calculation above, we get that

$$\left|f_{E,w} - f_{E,v}\right| = \frac{1}{w(E)} \int_E (f_{E,v} - f) v v^{p-1}\, dx \le K_2 \|f - f_{E,v}\|_{L^p(v;E)},$$

where $K_2 = w(E)^{-1/p}$. Then we can argue as we did before to get

$$\|f - f_{E,w}\|_{L^p(v;E)} \le (1 + K_2 w(E)^{1/p}) \|f - f_{E,v}\|_{L^p(v;E)} = 2\|f - f_{E,v}\|_{L^p(v;E)}.$$

Similarly, if we take $w = 1$ in the first argument, we get that

$$\|f - f_{E,v}\|_{L^p(v;E)} \le (1 + K_3)\|f - f_E\|_{L^p(v;E)},$$

where $K_3 = \frac{|E|}{v(E)}$. On the other hand, to prove the converse inequality, we have

$$|f_{E,v} - f_E| = \left|\frac{1}{|E|} \int_E (f - f_{E,v}) v v^{-1}\, dx\right| \le K_4 \|f - f_{E,v}\|_{L^p(v;E)},$$

where

$$K_4 = \frac{1}{|E|} \left(\int_E v^{-p'}\, dx \right)^{1/p'} < \infty$$

by our assumption that $v^{-1} \in L^{p'}(E)$. The argument then continues as before.