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*Research article*

## The principal transmission condition

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**Abstract:** The paper treats pseudodifferential operators  $P = \text{Op}(p(\xi))$  with homogeneous complex symbol  $p(\xi)$  of order  $2a > 0$ , generalizing the fractional Laplacian  $(-\Delta)^a$  but lacking its symmetries, and taken to act on the halfspace  $\mathbb{R}_+^n$ . The operators are seen to satisfy a principal  $\mu$ -transmission condition relative to  $\mathbb{R}_+^n$ , but generally not the full  $\mu$ -transmission condition satisfied by  $(-\Delta)^a$  and related operators (with  $\mu = a$ ). However,  $P$  acts well on the so-called  $\mu$ -transmission spaces over  $\mathbb{R}_+^n$  (defined in earlier works), and when  $P$  moreover is strongly elliptic, these spaces are the solution spaces for the homogeneous Dirichlet problem for  $P$ , leading to regularity results with a factor  $x_n^\mu$  (in a limited range of Sobolev spaces). The information is then shown to be sufficient to establish an integration by parts formula over  $\mathbb{R}_+^n$  for  $P$  acting on such functions. The formulation in Sobolev spaces, and the results on strongly elliptic operators going beyond certain operators with real kernels, are new. Furthermore, large solutions with nonzero Dirichlet traces are described, and a halfways Green's formula is established, as new results for these operators. Since the principal  $\mu$ -transmission condition has weaker requirements than the full  $\mu$ -transmission condition assumed in earlier papers, new arguments were needed, relying on work of Vishik and Eskin instead of the Boutet de Monvel theory. The results cover the case of nonsymmetric operators with real kernel that were only partially treated in a preceding paper.

**Keywords:** fractional-order pseudodifferential operator;  $\alpha$ -stable Lévy process; homogeneous symbol; Dirichlet problem on the halfspace; regularity estimate; halfways Green's formula

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### 1. Introduction

Boundary value problems for fractional-order pseudodifferential operators  $P$ , in particular where  $P$  is a generalization of the fractional Laplacian  $(-\Delta)^a$  ( $0 < a < 1$ ), have currently received much interest in applications, such as in financial theory and probability (but also in mathematical physics and differential geometry), and many methods have been used, most often probabilistic or potential-

theoretic methods.

The author has studied such problems by pseudodifferential methods in [8–13], under the assumption that the operators satisfy a  $\mu$ -transmission condition at the boundary of the domain  $\Omega \subset \mathbb{R}^n$ , which allows to show regularity results for solutions of the Dirichlet problem in elliptic cases, to show integration by parts formulas, and much else.

In the present paper we consider translation-invariant pseudodifferential operators ( $\psi$ do's)  $P = \text{Op}(p(\xi))$  of order  $2a > 0$  with homogeneous symbol  $p(\xi)$ , which are only taken to satisfy the top-order equation in the  $\mu$ -transmission condition (relative to the domain  $\Omega = \mathbb{R}_+^n$ ), we call this the *principal  $\mu$ -transmission condition*. It is shown that they retain some of the features: The solution spaces for the homogeneous Dirichlet problem in the elliptic case equal the  $\mu$ -transmission spaces from [8] (in a setting of low-order Sobolev spaces), having a factor  $x_n^\mu$ . The integration by parts formula holds (even when  $P$  is not elliptic):

$$\begin{aligned} \int_{\mathbb{R}_+^n} Pu \partial_n \bar{u}' dx + \int_{\mathbb{R}_+^n} \partial_n u \overline{P^* u'} dx \\ = \Gamma(\mu + 1) \Gamma(\mu' + 1) \int_{\mathbb{R}^{n-1}} s_0 \gamma_0(u/x_n^\mu) \gamma_0(\bar{u}'/x_n^{\mu'}) dx', \end{aligned}$$

when  $u$  and  $u'$  are in  $x_n^\mu C^\infty(\overline{\mathbb{R}_+^n})$  resp.  $x_n^{\mu'} C^\infty(\overline{\mathbb{R}_+^n})$  ( $\mu' = 2a - \mu$ ) and compactly supported.

We also treat nonhomogeneous local Dirichlet problems with Dirichlet trace  $\gamma_0(u/x_n^{\mu-1})$ , and show how the above formula implies a “halfways” Green’s formula where one factor has nonzero Dirichlet trace.  $P$  can be of any positive order, and  $\mu$  can be complex.

The results apply in particular to the operator  $L = \text{Op}(\mathcal{A}(\xi) + i\mathcal{B}(\xi))$  with  $\mathcal{A}$  real, positive and even in  $\xi$ ,  $\mathcal{B}$  real and odd in  $\xi$ , which satisfies the principal  $\mu$ -transmission equation for a suitable real  $\mu$ . Hereby we can compensate for an error made in the recent publication [13] (see also [14]), where it was overlooked that  $L$  may not satisfy the full  $\mu$ -transmission condition when  $\mathcal{B} \neq 0$  (it does so for  $\mathcal{B} = 0$ ). The general  $L$  are now covered by the present work. They were treated earlier by Dipierro, Ros-Oton, Serra and Valdinoci [5] under some hypotheses on  $a$  and  $\mu$ ; they come up in applications as infinitesimal generators of  $\alpha$ -stable  $n$ -dimensional Lévy processes, see [5]. (The calculations in [13] are valid when applied to operators satisfying the full  $\mu$ -transmission condition.)

The study of  $x$ -independent  $\psi$ do's  $P$  on the half-space  $\mathbb{R}_+^n$  serves as a model case for operators on domains  $\Omega \subset \mathbb{R}^n$  with curved boundary and possible  $x$ -dependence, and can be expected to be a useful ingredient in the general treatment, as carried out for the operator  $L$  in [5].

*Plan of the paper:* In Section 2 we give an overview of the aims and results of the paper with only few technicalities. Section 3 introduces the principal transmission condition in detail for homogeneous  $\psi$ do symbols. In Section 4, the Wiener-Hopf method is applied to derive basic decomposition and factorization formulas for such symbols. This is used in Section 5 to establish mapping properties for the operators, and regularity properties for solutions of the homogeneous Dirichlet problem in strongly elliptic cases; here  $\mu$ -transmission spaces (known from [8]) defined in an  $L_2$ -framework play an important role. Section 6 gives the proof of the above-mentioned integration by parts formula on  $\mathbb{R}_+^n$ . Section 7 treats nonhomogeneous local Dirichlet conditions, and a halfways Green’s formula is established.

## 2. Presentation of the main results

The study is concerned with the so-called *model case*, where the pseudodifferential operators have  $x$ -independent symbols, hence act as simple multiplication operators in the Fourier transformed space (this frees us from using the deeper composition rules needed for  $x$ -dependent symbols), and the considered open subset  $\Omega$  of  $\mathbb{R}^n$  is simplest possible, namely  $\Omega = \mathbb{R}_+^n = \{x \in \mathbb{R}^n \mid x_n > 0\}$ . We assume  $n \geq 2$  and denote  $x = (x_1, \dots, x_n) = (x', x_n)$ ,  $x' = (x_1, \dots, x_{n-1})$ . Recall the formulas for the Fourier transform  $\mathcal{F}$  and the operator  $P = \text{Op}(p(\xi))$ :

$$\begin{aligned} \mathcal{F}u &= \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} u(x) dx, & \mathcal{F}^{-1}v &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} v(\xi) d\xi, \\ Pu &= \text{Op}(p(\xi))u = \mathcal{F}^{-1}(p(\xi)(\mathcal{F}u)(\xi)). \end{aligned} \quad (2.1)$$

We work in  $L_2(\mathbb{R}^n)$  and  $L_2(\mathbb{R}_+^n)$  and their derived  $L_2$ -Sobolev spaces (the reader is urged to consult (5.1) below for notation). On  $L_2(\mathbb{R}^n)$ , the Plancherel theorem

$$\|u\|_{L_2(\mathbb{R}^n)} = c \|\hat{u}\|_{L_2(\mathbb{R}^n)}, \quad c = (2\pi)^{-n/2}, \quad (2.2)$$

makes norm estimates of operators easy. (There is more on Fourier transforms and distribution theory e.g., in [7].) The model case serves both as a simplified special case, and as a proof ingredient for more general cases of domains with curved boundaries, and possibly  $x$ -dependent symbols.

The symbols  $p(\xi)$  we shall consider are scalar and homogeneous of degree  $m = 2a > 0$  in  $\xi$ , i.e.,  $p(t\xi) = t^m p(\xi)$  for  $t > 0$ , and are  $C^1$  for  $\xi \neq 0$ , defining operators  $P = \text{Op}(p)$ .

A typical example is the squareroot Laplacian with drift:

$$L_1 = (-\Delta)^{\frac{1}{2}} + \mathbf{b} \cdot \nabla, \quad \text{with symbol } \mathcal{L}_1(\xi) = |\xi| + i\mathbf{b} \cdot \xi, \quad (2.3)$$

where  $\mathbf{b} = (b_1, \dots, b_n)$  is a real vector. Here  $m = 1$ ,  $a = \frac{1}{2}$ . It satisfies the condition for *strong ellipticity*, which is:

$$\text{Re } p(\xi) \geq c_0 |\xi|^m \quad \text{with } c_0 > 0, \quad \text{all } \xi \in \mathbb{R}^n; \quad (2.4)$$

this is important in regularity discussions. Some results are obtained without the ellipticity hypothesis; as an example we can take the operator  $L_2$  with symbol

$$\mathcal{L}_2(\xi) = |\xi_1 + \dots + \xi_n| + i\mathbf{b} \cdot \xi, \quad (2.5)$$

whose real part is zero e.g., when  $\xi = (1, -1, 0, \dots, 0)$ .

The operators are well-defined on the Sobolev spaces over  $\mathbb{R}^n$ : When  $p$  is homogeneous of degree  $m \geq 0$ , there is an inequality

$$|p(\xi)| \leq C |\xi|^m \leq C \langle \xi \rangle^m, \quad \langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$$

(we say that  $p$  is of *order*  $m$ ); then

$$\|Pu\|_{L_2(\mathbb{R}^n)} = c \|p(\xi)\hat{u}(\xi)\|_{L_2(\mathbb{R}^n)} \leq cC \|\langle \xi \rangle^m \hat{u}\|_{L_2(\mathbb{R}^n)} = C' \|u\|_{H^m(\mathbb{R}^n)}, \quad (2.6)$$

so  $P$  maps  $H^m(\mathbb{R}^n)$  continuously into  $L_2(\mathbb{R}^n)$ . Similarly, it maps  $H^{s+m}(\mathbb{R}^n)$  continuously into  $H^s(\mathbb{R}^n)$  for all  $s \in \mathbb{R}$ .

But for these pseudodifferential operators it is not obvious how to define them relative to the subset  $\mathbb{R}_+^n$ , since they are not defined pointwise like differential operators, but by integrals (they are *nonlocal*). The convention is here to let them act on suitable linear subsets of  $L_2(\mathbb{R}_+^n)$ , where we identify  $L_2(\mathbb{R}_+^n)$  with the set of  $u \in L_2(\mathbb{R}^n)$  that are zero on  $\mathbb{R}_-^n$ , i.e., have their support  $\text{supp } u \subset \overline{\mathbb{R}_+^n}$ . (The support  $\text{supp } u$  of a function or distribution  $u$  is the complement of the largest open set where  $u = 0$ . The operator that extends functions on  $\mathbb{R}_+^n$  by zero on  $\mathbb{R}_-^n$  is denoted  $e^+$ .) Then we apply  $P$  and restrict to  $\mathbb{R}_+^n$  afterwards; this is the operator  $r^+P$ . ( $r^+$  stands for restriction from  $\mathbb{R}^n$  to  $\mathbb{R}_+^n$ .)

Aiming for the integration by parts formula mentioned in the start, we have to clarify for which functions  $u, u'$  the integrals make sense. It can be expected from earlier studies ([5, 10, 20]) that the integral will be meaningful for solutions of the so-called *homogeneous Dirichlet problem on  $\mathbb{R}_+^n$* , namely the problem

$$r^+Pu = f \text{ on } \mathbb{R}_+^n, \quad u = 0 \text{ on } \mathbb{R}_-^n \quad (2.7)$$

(where the latter condition can also be written  $\text{supp } u \subset \overline{\mathbb{R}_+^n}$ ). This raises the question of where  $r^+P$  lands; which  $f$  can be prescribed? Or, if  $f$  is given in certain space, where should  $u$  lie in order to hit the space where  $f$  lies?

Altogether, we address the following three questions on  $P$ :

- (1) Forward mapping properties. From which spaces does  $r^+P$  map into an  $H^s$ -space for  $f$ ?
- (2) Regularity properties. If  $u$  solves (2.7) with  $f$  in an  $H^s$ -space for a high  $s$ , will  $u$  then belong to a space with a similar high regularity?
- (3) Integration by parts formula for functions in spaces where  $r^+P$  is well-defined.

It turns out that the answers to all three points depend profoundly on the introduction of so-called  $\mu$ -transmission spaces. To explain their importance, we turn for a moment to the fractional Laplacian which has a well-established treatment:

For the case of  $(-\Delta)^a$ ,  $0 < a < 1$ , it was shown in [8] that the following space is relevant:

$$\mathcal{E}_a(\overline{\mathbb{R}_+^n}) = e^+ x_n^a C^\infty(\overline{\mathbb{R}_+^n}). \quad (2.8)$$

It has the property that  $(-\Delta)^a$  maps it to  $C^\infty(\overline{\mathbb{R}_+^n})$ ; more precisely,

$$r^+(-\Delta)^a \text{ maps } \mathcal{E}_a(\overline{\mathbb{R}_+^n}) \cap \mathcal{E}'(\mathbb{R}^n) \text{ into } C^\infty(\overline{\mathbb{R}_+^n}). \quad (2.9)$$

Here  $\mathcal{E}'(\mathbb{R}^n)$  is the space of distributions with compact support, so the intersection with this space means that we consider functions in  $\mathcal{E}_a$  that are zero outside a compact set.

For Sobolev spaces, it was found in [8] that the good space for  $u$  is the so-called  $a$ -transmission space  $H^{a(t)}(\overline{\mathbb{R}_+^n})$ ; here

$$r^+(-\Delta)^a \text{ maps } H^{a(t)}(\overline{\mathbb{R}_+^n}) \cap \mathcal{E}'(\mathbb{R}^n) \text{ into } \overline{H}^{t-2a}(\mathbb{R}_+^n), \quad (2.10)$$

for all  $t \geq a$  (say).  $\mathcal{E}_a(\overline{\mathbb{R}_+^n}) \cap \mathcal{E}'(\mathbb{R}^n)$  is a dense subset of  $H^{a(t)}(\overline{\mathbb{R}_+^n})$ . The definition of the space  $H^{a(t)}(\overline{\mathbb{R}_+^n})$  is recalled below in (2.15) and in more detail in Section 5.3; let us for the moment just mention that it is the sum of the space  $\dot{H}^t(\overline{\mathbb{R}_+^n})$  and a certain subspace of  $x_n^a \overline{H}^{t-a}(\mathbb{R}_+^n)$ . This also holds when  $a$  is replaced by a more general  $\mu$ .

For  $(-\Delta)^a$ , the  $a$ -transmission spaces provide the right answers to question (1), and they are likewise right for question (2) (both facts established in [8]), and there are integration by parts formulas for  $(-\Delta)^a$  applied to elements of these spaces, [10, 11].

The key to the proofs is the so-called  $a$ -transmission condition that  $(-\Delta)^a$  satisfies; it is an infinite list of equations for  $p(\xi)$  and its derivatives, linking the values on the interior normal to  $\mathbb{R}_+^n$  with the values on the exterior normal. We formulate it below with  $a$  replaced by a general  $\mu$ .

**Definition 2.1.** Let  $\mu \in \mathbb{C}$ , and let  $p(\xi)$  be homogeneous of degree  $m$ . Denote the interior resp. exterior normal to the boundary of  $\mathbb{R}_+^n$  by  $(0, \pm 1) = \{(\xi', \xi_n) \mid \xi' = 0, \xi_n = \pm 1\}$ .

1°  $p$  (and  $P = \text{Op}(p)$ ) is said to satisfy **the principal  $\mu$ -transmission condition** at  $\mathbb{R}_+^n$  if

$$p(0, -1) = e^{i\pi(m-2\mu)} p(0, 1). \quad (2.11)$$

2°  $p$  (and  $P = \text{Op}(p)$ ) is said to satisfy **the  $\mu$ -transmission condition** at  $\mathbb{R}_+^n$  if

$$\partial_\xi^\alpha p(0, -1) = e^{i\pi(m-2\mu-|\alpha|)} \partial_\xi^\alpha p(0, 1), \text{ for all } \alpha \in \mathbb{N}_0^n. \quad (2.12)$$

Note that  $\mu$  is determined from  $p$  in (2.11) up to addition of an integer, when  $p(0, 1) \neq 0$ .

The operators considered on smooth domains  $\Omega$  in [8] were assumed to satisfy (2.12) (for the top-order term  $p_0$  in the symbol) at all boundary points  $x_0 \in \partial\Omega$ , with  $(0, 1)$  replaced by the interior normal  $\nu$  at  $x_0$ , and  $(0, -1)$  replaced by  $-\nu$ . The lower-order terms  $p_j$  in the symbol, homogeneous of degree  $m - j$ , should then satisfy analogous rules with  $m - j$  instead of  $m$ .

The principal  $\mu$ -transmission condition (2.11) is of course much less demanding than the full  $\mu$ -transmission condition (2.12). What we show in the present paper is that when (2.11) holds, the  $\mu$ -transmission spaces are still relevant, and provide the appropriate answers to both questions (1) and (2), however just for  $t$  (the regularity parameter) in a limited range. This range is large enough that integration by parts formulas can be established, answering (3).

By simple geometric considerations one finds:

**Proposition 2.2.** 1° When  $p(\xi)$  is homogeneous of degree  $m$ , there is a  $\mu \in \mathbb{C}$ , uniquely determined modulo  $\mathbb{Z}$  if  $p(0, 1) \neq 0$ , such that (2.11) holds.

2° If moreover,  $p$  is strongly elliptic (2.4) and  $m = 2a > 0$ ,  $\mu$  can be chosen uniquely to satisfy  $\mu = a + \delta$  with  $|\text{Re } \delta| < \frac{1}{2}$ .

This is shown in Section 3. From here on we work under two slightly different assumptions. The symbol  $p(\xi)$  is in both cases taken homogeneous of degree  $m = 2a > 0$  and  $C^1$  for  $\xi \neq 0$ . We pose Assumption 3.1 requiring that  $p$  is strongly elliptic and  $\mu$  is chosen as in Proposition 2.2 2°. We pose Assumption 3.2 just requiring that  $\mu$  is defined according to Proposition 2.2 1°. In all cases we write  $\mu = a + \delta$ , and define  $\mu' = a - \delta = 2a - \mu$ .

**Example 2.3.** Consider  $\mathcal{L}_1 = |\xi| + i\mathbf{b} \cdot \xi$  defined in (2.3). The order is  $1 = m = 2a$ , so  $a = \frac{1}{2}$ . Here  $\mathcal{L}_1(0, 1) = 1 + ib_n$  and  $\mathcal{L}_1(0, -1) = 1 - ib_n$ . The angle  $\theta$  in  $\mathbb{C} = \mathbb{R}^2$  between the positive real axis and  $1 + ib_n$  is  $\theta = \text{Arctan } b_n$ . Set  $\delta = \theta/\pi$ , then

$$\begin{aligned} \mathcal{L}_1(0, 1) &= e^{i\pi\delta} |\mathcal{L}_1(0, 1)| = e^{i\pi\delta} (1 + |\mathbf{b}|^2)^{\frac{1}{2}}, \text{ similarly} \\ \mathcal{L}_1(0, -1) &= e^{-i\pi\delta} |\mathcal{L}_1(0, -1)| = e^{-i\pi\delta} (1 + |\mathbf{b}|^2)^{\frac{1}{2}}. \end{aligned}$$

Moreover,

$$\mathcal{L}_1(0, -1)/\mathcal{L}_1(0, 1) = e^{-2i\pi\delta} = e^{i\pi(2a-2(a+\delta))}, \text{ when } a = \frac{1}{2},$$

so (2.11) holds with  $m = 2a = 1$ ,  $\mu = \frac{1}{2} + \delta$ , where  $\delta = \frac{1}{\pi} \text{Arctan } b_n$ , and Assumption 3.1 is satisfied. Note that  $\delta \in ] -\frac{1}{2}, \frac{1}{2}[$ .

For  $\mathcal{L}_2$  in (2.5), the values at  $(0, 1)$  and  $(0, -1)$  are the same as the values for  $\mathcal{L}_1$ , so (2.11) holds with the same values, and Assumption 3.2 is satisfied. But not Assumption 3.1 since  $\mathcal{L}_2$  is not strongly elliptic.

When  $b_n \neq 0$ , hence  $\delta \neq 0$ , neither of these symbols satisfy the full  $\mu$ -transmission condition Definition 2.1 2°, since second derivatives remove the  $(i\mathbf{b} \cdot \xi)$ -term so that the resulting symbol is even (with  $\mu = a + \delta$  replaced by  $\mu = a$ ).

Our answer to (1) is now the following (achieved in Section 5.4):

**Theorem 2.4.** *Let  $P$  satisfy Assumption 3.2. For  $\text{Re } \mu - \frac{1}{2} < t < \text{Re } \mu + \frac{3}{2}$ ,  $r^+P$  defines a continuous linear mapping*

$$r^+P: H^{\mu(t)}(\overline{\mathbb{R}}_+^n) \rightarrow \overline{H}^{t-2a}(\mathbb{R}_+^n). \quad (2.13)$$

It is important to note that  $r^+P$  then also makes good sense on subsets of  $H^{\mu(t)}(\overline{\mathbb{R}}_+^n)$ . In particular, since  $\mathcal{E}_\mu(\overline{\mathbb{R}}_+^n) \cap \mathcal{E}'(\mathbb{R}^n)$  is a subset of  $H^{\mu(t)}(\overline{\mathbb{R}}_+^n)$  for all  $t$ , the operator  $r^+P$  is well-defined on  $\mathcal{E}_\mu(\overline{\mathbb{R}}_+^n) \cap \mathcal{E}'(\mathbb{R}^n)$ , mapping it into  $\bigcap_{t < \text{Re } \mu + \frac{3}{2}} \overline{H}^{t-2a}(\mathbb{R}_+^n) \subset \overline{H}^{\text{Re } \delta + \frac{3}{2} - a - \varepsilon}(\mathbb{R}_+^n)$ , any  $\varepsilon > 0$ , by (2.13). When  $\text{Re } \delta > -\frac{1}{2}$  (always true under Assumption 3.1), this is assured to be contained in  $\overline{H}^{1-a}(\mathbb{R}_+^n)$ .

Our answer to (2) is (cf. Section 5.4):

**Theorem 2.5.** *Let  $P$  satisfy Assumption 3.1. Then  $P = \widehat{P} + P'$ , where  $P'$  is of order  $2a - 1$ , and  $r^+\widehat{P}$  is a bijection from  $H^{\mu(t)}(\overline{\mathbb{R}}_+^n)$  to  $\overline{H}^{t-2a}(\mathbb{R}_+^n)$  for  $\text{Re } \mu - \frac{1}{2} < t < \text{Re } \mu + \frac{3}{2}$ . In other words, there is unique solvability of (2.7) with  $P$  replaced by  $\widehat{P}$ , in the mentioned spaces.*

*For  $r^+P$  itself, there holds the regularity property: Let  $\text{Re } \mu - \frac{1}{2} < t < \text{Re } \mu + \frac{3}{2}$ , let  $f \in \overline{H}^{t-2a}(\mathbb{R}_+^n)$ , and let  $u \in \dot{H}^\sigma(\overline{\mathbb{R}}_+^n)$  (for some  $\sigma > \text{Re } \mu - \frac{1}{2}$ ) solve the homogeneous Dirichlet problem (2.7). Then  $u \in H^{\mu(t)}(\overline{\mathbb{R}}_+^n)$ .*

The last statement shows a lifting of the regularity of  $u$  in the elliptic case, namely if it solves (2.7) lying in a low-order space  $\dot{H}^\sigma(\overline{\mathbb{R}}_+^n)$ , then it is in the best possible  $\mu$ -transmission space according to Theorem 2.4, mapping into the given range space  $\overline{H}^{t-2a}(\mathbb{R}_+^n)$ . In other words, the domain of the homogeneous Dirichlet problem with range in  $\overline{H}^{t-2a}(\mathbb{R}_+^n)$  equals  $H^{\mu(t)}(\overline{\mathbb{R}}_+^n)$ .

The strategy for both theorems is, briefly expressed, as follows: The first step is to replace  $P = \text{Op}(p(\xi))$  by  $\widehat{P} = \text{Op}(\widehat{p}(\xi))$ , where  $\widehat{p}(\xi)$  is better controlled at  $\xi' = 0$  and  $p'(\xi) = p(\xi) - \widehat{p}(\xi)$  is  $O(|\xi|^{2a-1})$  for  $|\xi| \rightarrow \infty$ . The second step is to reduce  $\widehat{P}$  to order 0 by composition with "plus/minus order-reducing operators"  $\Xi_\pm^t = \text{Op}(\langle \xi' \rangle \pm i\xi_n^t)$  ((3.11), (5.2)) geared to the value  $\mu$  (recall  $\mu' = 2a - \mu$ ):

$$\widehat{Q} = \Xi_-^{-\mu'} \widehat{P} \Xi_+^{-\mu}. \quad (2.14)$$

Then the homogeneous symbol  $q$  associated with  $\widehat{Q}$  satisfies the principal 0-transmission condition. The third step is to decompose  $\widehat{Q}$  into a sum (when Assumption 3.2 holds) or a product (when Assumption 3.1 holds) of operators whose action relative to the usual Sobolev spaces  $\dot{H}^s(\overline{\mathbb{R}}_+^n)$  and  $\overline{H}^s(\mathbb{R}_+^n)$  can be well understood, so that we can show forward mapping properties and (in the strongly elliptic case) bijectiveness properties for  $\widehat{Q}$ . The fourth step is to carry this over to forward mapping properties and (in the strongly elliptic case) bijectiveness properties for  $\widehat{P}$ . The fifth and last step is to

take  $P' = P - \widehat{P}$  back into the picture and deduce the forward mapping resp. regularity properties for the original operator  $P$ .

It is the right-hand factor  $\Xi_+^{-\mu}$  in (2.14) that is the reason why the  $\mu$ -transmission spaces, defined by

$$H^{\mu(t)}(\overline{\mathbb{R}}_+^n) = \Xi_+^{-\mu} e^+ \overline{H}^{t-\operatorname{Re}\mu}(\mathbb{R}_+^n), \quad (2.15)$$

enter. Here  $e^+ \overline{H}^{t-\operatorname{Re}\mu}(\mathbb{R}_+^n)$  has a jump at  $x_n = 0$  when  $t > \operatorname{Re}\mu + \frac{1}{2}$ , and then the coefficient  $x_n^\mu$  appears.

The analysis of  $\widehat{Q}$  is based on a Wiener-Hopf technique (cf. Section 4) explained in Eskin's book [6], instead of the involvement of the extensive Boutet de Monvel calculus used in [8].

An interesting feature of the results is that the  $\mu$ -transmission spaces have a universal role, depending only on  $\mu$  and not on the exact form of  $P$ .

Finally, we answer (3) by showing an integration by parts formula, based just on Assumption 3.2.

**Theorem 2.6.** *Let  $P$  satisfy Assumption 3.2, and assume moreover that  $\operatorname{Re}\mu > -1$ ,  $\operatorname{Re}\mu' > -1$ . For  $u \in \mathcal{E}_\mu(\overline{\mathbb{R}}_+^n) \cap \mathcal{E}'(\mathbb{R}^n)$ ,  $u' \in \mathcal{E}_{\mu'}(\overline{\mathbb{R}}_+^n) \cap \mathcal{E}'(\mathbb{R}^n)$ , there holds*

$$\begin{aligned} \int_{\mathbb{R}_+^n} Pu \partial_n \bar{u}' dx + \int_{\mathbb{R}_+^n} \partial_n u \overline{P^* u'} dx \\ = \Gamma(\mu + 1) \Gamma(\mu' + 1) \int_{\mathbb{R}^{n-1}} s_0 \gamma_0(u/x_n^\mu) \gamma_0(\bar{u}'/x_n^{\mu'}) dx', \end{aligned} \quad (2.16)$$

where  $s_0 = e^{-i\pi\delta} p(0, 1)$ . The formula extends to  $u \in H^{\mu(t)}(\overline{\mathbb{R}}_+^n)$ ,  $u' \in H^{\mu'(t')}(\overline{\mathbb{R}}_+^n)$ , for  $t > \operatorname{Re}\mu + \frac{1}{2}$ ,  $t' > \operatorname{Re}\mu' + \frac{1}{2}$ .

The integrals over  $\mathbb{R}_+^n$  in (2.16) are interpreted as dualities when needed. The basic step in the proof is the treatment of one order-reducing operator in Proposition 6.1, by an argument shown in detail in [10, Th. 3.1, Rem. 3.2], and recalled in [13, Th. 4.1].

In the proof of (2.16) in Section 6, the formula is first shown for the nicer operator  $\widehat{P}$ , and thereafter extended to  $P$ . (The formula (2.16) for  $(-\Delta)^a$  in Ros-Oton and Serra [20, Th. 1.9] should have a minus sign on the boundary contribution; this has been corrected by Ros-Oton in the survey [19, p. 350].)

The theory will be carried further, to include “large” solutions of a nonhomogeneous local Dirichlet problem, and to show regularity results and a “halfways Green's formula”, see Section 7, but we shall leave those aspects out of this preview.

The example  $L_1$  in (2.3) is a special case of the operator  $L = \operatorname{Op}(\mathcal{L}(\xi))$ , where  $\mathcal{L}(\xi) = \mathcal{A}(\xi) + i\mathcal{B}(\xi)$  with  $\mathcal{A}(\xi)$  real, *even* in  $\xi$  and positive, and  $\mathcal{B}(\xi)$  real and *odd* in  $\xi$ . There are more details below in (3.5)ff. (this stands for (3.5) and the near following text) and Examples 5.9, 6.5, 7.4.  $L$  was first studied in [5] (under certain restrictions on  $a$  and  $\mu$ ), and our results apply to it. Theorem 2.6 gives an alternative proof for the same integration by parts formula, established in [5, Prop. 1.4] by extensive real function-theoretic methods.

The result on the integral over  $\mathbb{R}_+^n$  is combined in [5] with localization techniques to get an interesting result for curved domains, and it is our hope that the present results for more general strongly elliptic operators can be used in a similar way.

### 3. The principal mu-transmission condition

#### 3.1. Analysis of homogeneous symbols

Let  $p(\xi)$  be a complex function on  $\mathbb{R}^n$  that is homogeneous of degree  $m$  in  $\xi$ , and let  $\nu \in \mathbb{R}^n$  be a unit vector. For a complex number  $\mu$ , we shall say that  $p$  satisfies *the principal  $\mu$ -transmission condition in the direction  $\nu$* , when

$$p(-\nu) = e^{i\pi(m-2\mu)} p(\nu). \quad (3.1)$$

When  $p(\nu) \neq 0$ , we can rewrite (3.1) as

$$e^{i\pi(m-2\mu)} = \frac{p(-\nu)}{p(\nu)}, \text{ i.e., } \mu = \frac{m}{2} - \frac{1}{2\pi i} \log \frac{p(-\nu)}{p(\nu)},$$

where  $\log$  is a complex logarithm. This determines the possible  $\mu$  up to addition of an integer.

The (full)  $\mu$ -transmission property defined in [8] demands much more, namely that

$$\partial_\xi^\alpha p(-\nu) = e^{i\pi(m-2\mu-|\alpha|)} \partial_\xi^\alpha p(\nu), \text{ all } \alpha \in \mathbb{N}_0^n. \quad (3.2)$$

Besides assuming infinite differentiability, this is a stronger condition than (3.1) in particular because of the requirements it puts on derivatives of  $p$  transversal to  $\nu$ .

To analyse this we observe that when a (sufficiently smooth) function  $f(t)$  on  $\mathbb{R} \setminus \{0\}$  is homogeneous of degree  $m \in \mathbb{R}$ , then it has the form, for some  $c_1, c_2 \in \mathbb{C}$ ,

$$f(t) = \begin{cases} c_1 t^m & \text{for } t > 0, \\ c_2 (-t)^m & \text{for } t < 0, \end{cases}$$

and its derivative outside  $t = 0$  is a function homogeneous of degree  $m - 1$  satisfying

$$\partial_t f(t) = \begin{cases} c_1 m t^{m-1} & \text{for } t > 0, \\ -c_2 m (-t)^{m-1} & \text{for } t < 0. \end{cases}$$

In particular, if  $c_1 \neq 0$ ,  $m \neq 0$ ,

$$f(-1)/f(1) = c_2/c_1, \quad \partial_t f(-1)/\partial_t f(1) = -c_2/c_1.$$

In the case  $m = 0$ ,  $f$  is constant for  $t > 0$  and  $t < 0$ , and the derivative is zero there.

Thus, when  $p(\xi)$  is a (sufficiently smooth) function on  $\mathbb{R}^n \setminus \{0\}$  that is homogeneous of degree  $m \neq 0$ , and we consider it on a two-sided ray  $\{t\nu \mid t \in \mathbb{R}\}$  where  $\nu$  is a unit vector and  $p(\nu) \neq 0$ , then

$$p(-\nu) = c_0 p(\nu) \implies \partial_t p(t\nu)|_{t=-1} = -c_0 \partial_t p(t\nu)|_{t=1}. \quad (3.3)$$

So for example, when  $\nu$  is the inward normal  $(0, 1) = \{(\xi', \xi_n) \mid \xi' = 0, \xi_n = 1\}$  to  $\mathbb{R}_+^n$ ,

$$p(0, -1) = c_0 p(0, 1) \implies \partial_{\xi_n} p(0, -1) = -c_0 \partial_{\xi_n} p(0, 1).$$

For  $p(\xi)$  satisfying (3.1), this means that when  $p(\nu) \neq 0$ , it will also satisfy

$$\partial_t p(t\nu)|_{t=-1} = e^{i\pi(m-2\mu-1)} \partial_t p(t\nu)|_{t=1},$$



in view of (3.3). This argument can be repeated, showing that

$$\partial_t^k p(t\nu)|_{t=-1} = e^{i\pi(m-2\mu-k)} \partial_t^k p(t\nu)|_{t=1}, \quad (3.4)$$

as long as the derivatives at  $t = 1$  do not vanish. That can happen when  $m$  is a nonnegative integer (namely from the  $(m + 1)$ 'st step on); then (3.4) is trivially satisfied. On the other hand, we cannot infer that derivatives of  $\partial_\xi^\alpha p$  for arbitrary  $\alpha$  have the property (3.2); this will be illustrated in examples below.

In general,  $\mu$  takes different values for different  $\nu$ . When  $\Omega$  is a sufficiently smooth subset of  $\mathbb{R}^n$  with interior normal  $\nu(x)$  at boundary points  $x \in \partial\Omega$ , we say that  $p$  satisfies *the principal  $\mu$ -transmission condition at  $\Omega$*  if  $\mu(x)$  is a function on  $\partial\Omega$  such that (3.1) holds with this  $\mu(x)$  at boundary points  $x \in \partial\Omega$ . For  $\Omega = \mathbb{R}_+^n$ , the normal  $\nu$  equals  $(0, 1)$  at all boundary points and  $\mu$  is a constant; this is the situation considered in the present paper.

In [13] we have studied a special class of symbols first considered by Dipierro, Ros-Oton, Serra and Valdinoci in [5]:

$$\mathcal{L}(\xi) = \mathcal{A}(\xi) + i\mathcal{B}(\xi), \quad (3.5)$$

the functions being  $C^\infty$  for  $\xi \neq 0$  and homogeneous in  $\xi$  of degree  $2a > 0$  ( $a < 1$ ), and where  $\mathcal{A}(\xi)$  is real and even in  $\xi$  (i.e.,  $\mathcal{A}(-\xi) = \mathcal{A}(\xi)$ ),  $\mathcal{B}(\xi)$  is real and odd in  $\xi$  (i.e.,  $\mathcal{B}(-\xi) = -\mathcal{B}(\xi)$ ), and  $\mathcal{L}$  is strongly elliptic (i.e.,  $\mathcal{A}(\xi) > 0$  for  $\xi \neq 0$ ). As shown in [13, Sect. 2],  $\mathcal{L}$  satisfies (3.1) on each unit vector  $\nu$ , for  $m = 2a$  and

$$\mu(\nu) = a + \delta(\nu), \quad \text{with } \delta(\nu) = \frac{1}{\pi} \text{Arctan } b, \quad b = \mathcal{B}(\nu)/\mathcal{A}(\nu); \quad (3.6)$$

this follows straightforwardly (as in Example 2.3) from the observation that  $\mathcal{L}(-\nu)/\mathcal{L}(\nu) = (1 - ib)/(1 + ib)$ ,  $b = \mathcal{B}(\nu)/\mathcal{A}(\nu)$ . It then also satisfies (3.4) with this  $\mu$ .

But the full  $\mu$ -transmission condition need not hold. For example, the symbol  $\mathcal{L}_1(\xi) = |\xi| + i\mathbf{b} \cdot \xi$  in (2.3) (with  $\mathbf{b} \in \mathbb{R}^n$ ) satisfies the principal  $\mu$ -transmission condition for  $\nu = (0, 1)$  with  $\mu = \frac{1}{2} + \delta$ ,  $\delta \neq 0$  if  $b_n \neq 0$ , whereas

$$\partial_{\xi_1}^2 \mathcal{L}_1 = (\xi_2^2 + \cdots + \xi_n^2)/|\xi|^3$$

and its derivatives satisfy the conditions in (3.2) for  $\nu = (0, 1)$  with  $\mu$  replaced by  $\frac{1}{2}$ .

The statement in [13, Th. 3.1] that solutions of the homogeneous Dirichlet problem have a structure with the factor  $x_n^\mu$ , was quoted from [8] based on the full  $\mu$ -transmission condition, and therefore applies to  $L = \text{Op}(\mathcal{L})$  when  $\mathcal{B} = 0$  (a case belonging to [8]), but not in general when  $\mathcal{B} \neq 0$ . Likewise, the integration by parts formulas for  $L$  derived in [13] using details from the Boutet de Monvel calculus are justified when  $\mathcal{B} = 0$  or when other operators  $P$  satisfying the full  $\mu$ -transmission condition are inserted, but not in general when  $\mathcal{B} \neq 0$ . Fortunately, there are cruder methods that do lead to such results, on the basis of the principal  $\mu$ -transmission condition alone, and that is what we show in this paper.

The treatment of  $\mathcal{L}$  will be incorporated in a treatment of general strongly elliptic homogeneous symbols in the following. This requires that we allow complex values of  $\mu$ .

Let  $P = \text{Op}(p(\xi))$  be defined by (2.1) from a symbol  $p(\xi)$  that is  $C^1$  for  $\xi \neq 0$ , homogeneous of order  $m = 2a > 0$ , and now also strongly elliptic (2.4). To fix the ideas, we shall consider the operator relative to the set  $\mathbb{R}_+^n$ , with interior normal  $\nu = (0, 1)$ . Denote  $p(\xi)|\xi|^{-2a} = p_1(\xi)$ ; it is homogeneous

of degree 0. Both  $p$  and  $p_1$  take values in a closed subsector of  $\{z \in \mathbb{C} \mid \operatorname{Re} \xi_n > 0\} \cup \{0\}$ . For any  $\xi' \in \mathbb{R}^{n-1}$ , one has for  $+1$  and  $-1$  respectively,

$$\lim_{\xi_n \rightarrow \pm\infty} p_1(\xi', \xi_n) = \lim_{\xi_n \rightarrow \pm\infty} p_1(\xi' / |\xi_n|, \pm 1) = p_1(0, \pm 1) = p(0, \pm 1).$$

With the logarithm  $\log z$  defined to be positive for real  $z > 1$ , with a cut along the negative real axis, denote  $\log p(0, \pm 1) = \alpha_{\pm}$ ; here  $\operatorname{Re} \alpha_{\pm} = \log |p(0, \pm 1)|$  and  $\operatorname{Im} \alpha_{\pm}$  is the argument of  $p(0, \pm 1)$ . With this notation,

$$p(0, -1)/p(0, 1) = e^{\alpha_-} / e^{\alpha_+} = e^{\alpha_- - \alpha_+},$$

so (3.1) for  $m = 2a$  holds with  $\nu = (0, 1)$  when  $\alpha_- - \alpha_+ = i\pi(2a - 2\mu)$ , i.e.,

$$\mu = a + \delta \text{ with } \delta = (\alpha_+ - \alpha_-)/2\pi i; \quad (3.7)$$

this  $\mu$  is the *factorization index*. These calculations were given in [8, Sect. 3] (with  $m = 2a$ ), and are in principle consistent with the determination of the factorization index by Eskin in [6, Ex. 6.1] (which has different plus/minus conventions because of a different definition of the Fourier transform).

Since  $p(\xi)$  takes values in  $\{\operatorname{Re} z > 0\}$  for  $\xi \neq 0$ , both  $p(0, 1)$  and  $p(0, -1)$  lie there and the difference between their arguments is less than  $\pi$ , so  $|\operatorname{Im}(\alpha_+ - \alpha_-)/2\pi| < \frac{1}{2}$ ; in other words

$$|\operatorname{Re} \delta| < \frac{1}{2}. \quad (3.8)$$

Note that  $\delta$  is real in the case (3.5).

We collect the information on  $P$  in the following description:

**Assumption 3.1.** *The operator  $P = \operatorname{Op}(p(\xi))$  is defined from a symbol  $p(\xi)$  that is  $C^1$  for  $\xi \neq 0$ , homogeneous of order  $m = 2a > 0$ , and strongly elliptic (2.4). It satisfies the principal  $\mu$ -transmission condition in the direction  $(0, 1)$ :*

$$p(0, -1) = e^{i\pi(m-2\mu)} p(0, 1),$$

with  $\mu$  equal to the factorization index  $\mu = a + \delta$  derived around (3.7), and  $|\operatorname{Re} \delta| < \frac{1}{2}$ . Denote  $\mu' = 2a - \mu = a - \delta$ .

In Eskin's book [6], the case of constant-coefficient pseudodifferential operators considered on  $\mathbb{R}_+^n$  is studied in §§4–17, and the calculations rely on the principal transmission condition up to and including §9. From §10 on, additional conditions on transversal derivatives are required (the symbol class  $D_{\alpha+i\beta}^{(0)}$  seems to correspond to our full 0-transmission condition, giving operators preserving smoothness up to the boundary). In the following, we draw on some of the points made in §§6–7 there.

For an operator  $A$  defined from a homogeneous symbol  $a(\xi)$ , the behavior at zero can be problematic to deal with. In [6, §7], there is introduced a technique that leads to a nicer operator, in the context of operators relative to  $\mathbb{R}_+^n$ : One eliminates the singularity at  $\xi' = 0$  by replacing the homogeneous symbol  $a(\xi', \xi_n)$  by

$$\widehat{a}(\xi', \xi_n) = a(\langle \xi' \rangle \xi' / |\xi'|, \xi_n), \quad (3.9)$$

the corresponding operator denoted  $\widehat{A}$ . (In comparison with [6] we have replaced the factor  $1 + |\xi'|$  used there by  $\langle \xi' \rangle = (1 + |\xi'|^2)^{\frac{1}{2}}$ .) It is shown there that when  $a(\xi)$  is homogeneous of degree  $\alpha + i\beta$ , then

$$a'(\xi) = a(\xi) - \widehat{a}(\xi) \text{ is } O(|\xi|^{\alpha-1}) \text{ for } |\xi| \geq 2, \quad (3.10)$$

hence is of lower order in a certain sense. Many results with Sobolev estimates are then shown primarily for the “hatted” version  $\widehat{A} = \text{Op}(\widehat{a})$ , and supplied afterwards with information on  $A' = \text{Op}(a')$ . Indeed, we shall see that the results we are after for our operators  $P = \text{Op}(p)$ , can be obtained in a manageable way for  $\widehat{P} = \text{Op}(\widehat{p})$ , and then extended to  $P$  by a supplementing analysis of  $P'$ . The important thing is that special properties with respect to  $\xi_n$ , such as holomorphic extendability into  $\mathbb{C}_+$  or  $\mathbb{C}_-$ , are not disturbed when  $a$  is replaced by  $\widehat{a}$ .

Some of the results that we shall show do not require ellipticity of  $P$ . We therefore introduce also a weaker assumption:

**Assumption 3.2.** *The operator  $P = \text{Op}(p(\xi))$  is defined from a symbol  $p(\xi)$  that is  $C^1$  for  $\xi \neq 0$ , homogeneous of order  $m = 2a > 0$ , and satisfies the principal  $\mu$ -transmission condition in the direction  $(0, 1)$  with  $\mu = a + \delta$  for some  $\delta \in \mathbb{C}$ . Denote  $a - \delta = \mu'$ .*

For the symbols  $p$  considered in the rest of the paper, we assume at least that Assumption 3.2 holds. As noted earlier, when  $P$  satisfies (3.1) for some  $\mu$ , it also does so with  $\mu$  replaced by  $\mu + k$ ,  $k \in \mathbb{Z}$ . The precision in Assumption 3.1, that  $\mu$  should equal the factorization index, is needed for elliptic solvability statements.

### 3.2. Reduction to symbols of order zero

Consider the symbols of “order-reducing” operators (more on them in Section 5):

$$\begin{aligned} \chi_{0,\pm}^t(\xi) &= (|\xi'| \pm i\xi_n)^t; \text{ consequently} \\ \widehat{\chi}_{0,\pm}^t(\xi) &= (|\langle \xi' \rangle \xi' / |\xi'| \pm i\xi_n)^t = (\langle \xi' \rangle \pm i\xi_n)^t = \chi_{\pm}^t(\xi); \end{aligned} \quad (3.11)$$

the last entry is the usual notation. Together with our symbol  $p(\xi)$  of order  $2a$ , we shall consider its reduction to a symbol  $q$  of order 0 defined by:

$$q(\xi) = \chi_{0,-}^{-\mu'} p(\xi) \chi_{0,+}^{-\mu}, \text{ hereby } p(\xi) = \chi_{0,-}^{\mu'} q(\xi) \chi_{0,+}^{\mu}. \quad (3.12)$$

The “hatted” version is:

$$\widehat{q}(\xi) = \chi_-^{-\mu'} \widehat{p}(\xi) \chi_+^{-\mu}, \text{ hereby } \widehat{p}(\xi) = \chi_-^{\mu'} \widehat{q}(\xi) \chi_+^{\mu}. \quad (3.13)$$

Here  $q$  is continuous and homogeneous of degree 0 for  $\xi \neq 0$ ; it is  $C^1$  in  $\xi_n$  there, and  $C^1$  in  $\xi'$  for  $\xi' \neq 0$  with bounded first derivatives on  $|\xi| = 1$ . Since  $i = e^{i\pi/2}$ ,

$$\begin{aligned} q(0, 1) &= (-i)^{\mu-2a} p(0, 1) i^{-\mu} = i^{2a-2\mu} p(0, 1) = e^{i\pi(a-\mu)} p(0, 1), \\ q(0, -1) &= (+i)^{\mu-2a} p(0, -1) (-i)^{-\mu} = i^{2\mu-2a} e^{i\pi(2a-2\mu)} p(0, 1) \\ &= e^{i\pi(a-\mu)} p(0, 1) = q(0, 1), \end{aligned}$$

so  $q$  satisfies the principal 0-transmission condition in the direction  $\nu = (0, 1)$ :

$$q(0, -1) = q(0, 1). \quad (3.14)$$

In view of (3.1)–(3.4), we have moreover when  $p(0, 1) \neq 0$  that

$$\partial_{\xi_n} q(0, -1) = -\partial_{\xi_n} q(0, 1). \quad (3.15)$$

Note that since  $\mu - a = \delta$ ,  $q(0, 1) = e^{-i\pi\delta} p(0, 1)$ . We shall denote

$$s_0 = q(0, 1) = e^{-i\pi\delta} p(0, 1). \quad (3.16)$$

In the case  $p = \mathcal{L}$  in (3.5)–(3.6),  $\mathcal{L}(0, 1) = e^{i\pi\delta} |\mathcal{L}(0, 1)|$  with  $\delta$  real, so

$$s_0 = e^{-i\pi\delta} \mathcal{L}(0, 1) = |\mathcal{L}(0, 1)| = (\mathcal{A}(0, 1)^2 + \mathcal{B}(0, 1)^2)^{\frac{1}{2}} \text{ then.} \quad (3.17)$$

## 4. The Wiener-Hopf decomposition

### 4.1. The sum decomposition

Since  $p(\xi)$  is only assumed to satisfy the principal  $\mu$ -transmission condition,  $q(\xi)$  will in general only satisfy the principal 0-transmission condition, not the full one, so the techniques of the Boutet de Monvel calculus brought forward in [8] are not available. Instead we go back to a more elementary application of the original Wiener-Hopf method [22].

When  $b(\xi_n)$  is a function on  $\mathbb{R}$ , denote

$$\begin{aligned} b_+(\xi_n + i\tau) &= \frac{i}{2\pi} \int_{\mathbb{R}} \frac{b(\eta_n)}{\eta_n - \xi_n - i\tau} d\eta_n \text{ for } \tau < 0, \\ b_-(\xi_n + i\tau) &= \frac{-i}{2\pi} \int_{\mathbb{R}} \frac{b(\eta_n)}{\eta_n - \xi_n - i\tau} d\eta_n \text{ for } \tau > 0, \end{aligned} \quad (4.1)$$

when the integrals have a sense. When  $b$  is suitably nice,  $b_+$  is holomorphic in  $\xi_n + i\tau$  for  $\tau < 0$  and extends to a continuous function on  $\overline{\mathbb{C}}_-$  (also denoted  $b_+$ ),  $b_-$  has these properties relative to  $\overline{\mathbb{C}}_+$ , and  $b(\xi_n) = b_+(\xi_n) + b_-(\xi_n)$  on  $\mathbb{R}$ . With the notation of spaces  $H$ ,  $H^\pm$  introduced by Boutet de Monvel in [4], denoted  $\mathcal{H}$ ,  $\mathcal{H}^\pm$  in our subsequent works, the decomposition holds for  $b \in \mathcal{H}$  with  $b_\pm \in \mathcal{H}^\pm$  on  $\mathbb{R}$ . Since we are presently dealing with functions with cruder properties, we shall instead apply a useful lemma shown in [6, Lemma 6.1]:

**Lemma 4.1.** *Suppose that  $b(\xi', \xi_n)$  is homogeneous of degree 0 in  $\xi$ , is  $C^1$  for  $\xi' \neq 0$ , and satisfies*

$$|b(\xi', \xi_n)| \leq C|\xi'| |\xi|^{-1}, \quad |\partial_j b(\xi', \xi_n)| \leq C|\xi|^{-1} \text{ for } j \leq n-1. \quad (4.2)$$

Then the function defined for  $\tau < 0$  by

$$b_+(\xi', \xi_n + i\tau) = \frac{i}{2\pi} \int_{\mathbb{R}} \frac{b(\xi', \eta_n)}{\eta_n - \xi_n - i\tau} d\eta_n \quad (4.3)$$

is holomorphic with respect to  $\xi_n + i\tau$  in  $\mathbb{C}_-$ , is homogeneous of degree 0, extends by continuity with respect to  $(\xi', \xi_n + i\tau) \in \overline{\mathbb{C}}_-$  for  $|\xi| + |\tau| > 0$ ,  $\tau \leq 0$ , and satisfies the estimate

$$|b_+(\xi', \xi_n + i\tau)| \leq C_\varepsilon |\xi'|^{1-\varepsilon} (|\xi| + |\tau|)^{\varepsilon-1}, \text{ any } \varepsilon > 0. \quad (4.4)$$

There is an analogous statement for  $b_-$  with  $\mathbb{C}_-$  replaced by  $\mathbb{C}_+$ .

The symbol  $q$  derived from  $p$  by (3.12) satisfies

$$q(\xi) = s_0 + f(\xi),$$

where  $f$  is likewise homogeneous of degree 0, and has  $f(0, 1) = f(0, -1) = 0$ . We make two applications of Lemma 4.1. One is, under Assumption 3.2, to apply it directly to  $f$  to get a sum decomposition  $f = f_+ + f_-$  where the terms extend holomorphically to  $\mathbb{C}_-$  resp.  $\mathbb{C}_+$  with respect to  $\xi_n$ ; this will be convenient in establishing the forward mapping properties and integration by parts formula for the present operators. The other is, under Assumption 3.1, to apply the lemma to the function  $b(\xi) = \log q(\xi)$  to get a sum decomposition of  $b$  and hence a *factorization* of  $q$ ; this is used to show that  $P$  has appropriate solvability properties (the solutions exhibiting a singularity  $x_n^\mu$  at the boundary).

We show that  $f$  has the properties required for Lemma 4.1 as follows: To see that (4.2) is verified by  $f$ , note that the second inequality follows since  $\partial_j f$  is bounded on the unit sphere  $\{|\xi| = 1\}$  and homogeneous of degree  $-1$ . For the first inequality we have, when  $\xi_n > |\xi'|$  (hence  $|\xi'/\xi_n| < 1$ ),

$$|f(\xi', \xi_n)| = \left| q\left(\frac{\xi'}{\xi_n}, 1\right) - q(0, 1) \right| \leq \sum_{j < n} \left| \frac{\xi_j}{\xi_n} \right| \sup_{|\eta'| \leq 1} |\partial_j q(\eta', 1)| \leq C \frac{|\xi'|}{|\xi_n|} \leq C' \frac{|\xi'|}{|\xi|}, \quad (4.5)$$

using the mean value theorem and the fact that  $|\xi_n| \sim |\xi|$  when  $|\xi_n| \geq |\xi'|$ . A similar estimate is found for  $\xi_n < -|\xi'|$ . For  $|\xi_n| \leq |\xi'|$ , we use that  $q$  is bounded, so that  $|q(\xi) - s_0||\xi|/|\xi'| \leq c|q(\xi) - s_0||\xi'|/|\xi'| \leq c'$ . We have obtained:

**Proposition 4.2.** *When  $p$  satisfies Assumption 3.2 and  $q$  is derived from  $p$  by (3.12), then there is a sum decomposition of  $f = q - s_0$ :*

$$q(\xi) - s_0 = f_+(\xi) + f_-(\xi),$$

where  $f_+(\xi', \xi_n)$  is holomorphic with respect to  $\xi_n + i\tau$  in  $\mathbb{C}_-$ , and continuous with respect to  $(\xi', \xi_n + i\tau) \in \overline{\mathbb{C}_-}$  for  $|\xi| + |\tau| > 0$ ,  $\tau \leq 0$ , and satisfies estimates

$$|f_+(\xi', \xi_n + i\tau)| \leq C_\varepsilon |\xi'|^{1-\varepsilon} (|\xi| + |\tau|)^{\varepsilon-1}, \quad \text{any } \varepsilon > 0, \quad (4.6)$$

and  $f_-$  has the analogous properties with  $\mathbb{C}_-$  replaced by  $\mathbb{C}_+$ .

For the corresponding hatted symbol, we then have  $\widehat{q} = s_0 + \widehat{f}_+ + \widehat{f}_-$ , with  $\widehat{f}_\pm$  defined from  $f_\pm$ . They have similar holomorphy properties, and satisfy estimates as in (4.6) with  $|\xi'|$  replaced by  $\langle \xi' \rangle$ .

#### 4.2. The product decomposition

In order to obtain a factorization for symbols satisfying Assumption 3.1, we shall study  $\log q$ . By the strong ellipticity,  $q(\xi) \neq 0$  for  $\xi \neq 0$ . Moreover,  $p(\xi)|\xi|^{-2a} = \chi_{0,-}^{-a} p(\xi) \chi_{0,+}^{-a}$  takes values in a subsector of  $\{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$  and the multiplication by  $\chi_{0,-}^{\delta}$  and  $\chi_{0,+}^{-\delta}$  gives the function  $q$  taking values in the sector  $\{z \in \mathbb{C} \mid |\arg z| \leq \pi(\frac{1}{2} + |\operatorname{Re} \delta|)\}$  disjoint from the negative real axis. So the logarithm is well-defined with inverse  $\exp$ .

Assume first that  $s_0 = 1$ ; this can simply be obtained by dividing out  $q(0, 1)$ . The function  $b(\xi) = \log q(\xi)$  is homogeneous of degree 0 and has  $b(0, 1) = b(0, -1) = 0$  and the appropriate continuity properties, and bounds on first derivatives, so the same proof as for  $f$  applies to  $b$  to give the decomposition  $b = b_+ + b_-$ . Then we define  $q^\pm = \exp(b_\pm)$ , they are homogeneous of degree 0. For example,

$$q^+ = 1 + g^+, \quad \text{where } g^+ = \sum_{k \geq 1} (b_+)^k.$$

Here  $|b_+(\xi)| \leq C_\varepsilon |\xi'|^{1-\varepsilon} |\xi|^{-1+\varepsilon}$ , and there is a constant  $C'_\varepsilon$  such that  $C_\varepsilon |\xi'|^{1-\varepsilon} |\xi|^{-1+\varepsilon} \leq \frac{1}{2}$  for  $|\xi_n| \geq C'_\varepsilon |\xi'|$ . On this set the series for  $g^+$  converges with  $|g^+| \leq |b_+|$ , hence  $g^+$  satisfies an estimate of the form (4.4) there. It likewise does so on the set  $|\xi_n| \leq C'_\varepsilon |\xi'|$  since  $|\xi'| \sim |\xi|$  there. There are similar results for  $q^- = \exp(b_-) = 1 + g^-$  with  $\mathbb{C}_-$  replaced by  $\mathbb{C}_+$ . This shows:

**Proposition 4.3.** *When  $p$  satisfies Assumption 3.1 and  $q$  is derived from  $p$  by (3.12) and satisfies  $s_0 = 1$ , then there is a factorization of  $q$ :*

$$q(\xi) = q^-(\xi)q^+(\xi),$$

where  $q^+(\xi', \xi_n)$  is holomorphic with respect to  $\xi_n + i\tau$  in  $\mathbb{C}_-$ , and continuous with respect to  $(\xi', \xi_n + i\tau) \in \overline{\mathbb{C}_-}$  for  $|\xi| + |\tau| > 0$ ,  $\tau \leq 0$ . Moreover,  $g^+ = q^+ - 1$  satisfies estimates

$$|g^+(\xi', \xi_n + i\tau)| \leq C_\varepsilon |\xi'|^{1-\varepsilon} (|\xi| + |\tau|)^{\varepsilon-1}, \text{ all } \varepsilon > 0, \quad (4.7)$$

and  $q^-$ ,  $g^- = q^- - 1$  have the analogous properties with  $\mathbb{C}_-$  replaced by  $\mathbb{C}_+$ . The symbols are homogeneous of degree 0, and  $q^+$  and  $q^-$  are elliptic.

For general  $s_0$ , we apply the factorization to  $q_0 = s_0^{-1}q$ , so that  $q_0 = q_0^- q_0^+$ ; then  $q = q^- q^+$  with  $q^- = s_0 q_0^- = s_0(1 + g^-)$  and  $q^+ = q_0^+ = 1 + g^+$ .

The ellipticity follows from the construction as  $\exp(b_\pm)$ , or one can observe that the product  $q^+ q^- = q$  is elliptic (i.e., nonzero for  $\xi \neq 0$ ).

The notation with upper index  $\pm$  is chosen here to avoid confusion with the lower  $\pm$  used later to indicate truncation,  $P_\pm = r^\pm P e^\pm$ .

Turning to the corresponding hatted symbols, we have obtained  $\widehat{q} = \widehat{q}^- \widehat{q}^+$ , with  $\widehat{q}^\pm, \widehat{g}^\pm$  defined from  $q^\pm, g^\pm$ , respectively. They have similar holomorphy properties, the  $\widehat{q}^\pm$  are elliptic, and the  $\widehat{g}^\pm$  satisfy estimates as in (4.7) with  $|\xi'|$  replaced by  $\langle \xi' \rangle$ :

$$|\widehat{g}^+(\xi', \xi_n + i\tau)| \leq C_\varepsilon \langle \xi' \rangle^{1-\varepsilon} (\langle \xi \rangle + |\tau|)^{\varepsilon-1}, \text{ all } \varepsilon > 0. \quad (4.8)$$

## 5. Mapping properties and the homogeneous Dirichlet problem

### 5.1. Some function spaces

First recall some terminology:  $\mathcal{E}'(\mathbb{R}^n)$  is the space of distributions on  $\mathbb{R}^n$  with compact support,  $\mathcal{S}(\mathbb{R}^n)$  is the Schwartz space of  $C^\infty$ -functions  $f$  on  $\mathbb{R}^n$  such that  $x^\beta D^\alpha f$  is bounded for all  $\alpha, \beta$ , and  $\mathcal{S}'(\mathbb{R}^n)$  is its dual space of temperate distributions.  $\langle \xi \rangle$  stands for  $(1 + |\xi|^2)^{\frac{1}{2}}$ . We denote by  $r^+$  the operator restricting distributions on  $\mathbb{R}^n$  to distributions on  $\mathbb{R}_+^n$ , and by  $e^+$  the operator extending functions on  $\mathbb{R}_+^n$  by zero on  $\mathbb{R}^n \setminus \mathbb{R}_+^n$ . Then  $r^+ \mathcal{S}'(\mathbb{R}^n)$  is denoted  $\mathcal{S}'(\overline{\mathbb{R}_+^n})$ . The following notation for  $L_2$ -Sobolev spaces will be used, for  $s \in \mathbb{R}$ :

$$\begin{aligned} H^s(\mathbb{R}^n) &= \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \langle \xi \rangle^s \mathcal{F}u \in L_2(\mathbb{R}^n)\}, \\ \overline{H}^s(\mathbb{R}_+^n) &= r^+ H^s(\mathbb{R}^n), \text{ the restricted space,} \\ \dot{H}^s(\overline{\mathbb{R}_+^n}) &= \{u \in H^s(\mathbb{R}^n) \mid \text{supp } u \subset \overline{\mathbb{R}_+^n}\}, \text{ the supported space,} \end{aligned} \quad (5.1)$$

as in our earlier papers on fractional-order operators. An elaborate presentation of  $L_p$ -based spaces was given in [8]. (The notation with dots and overlines stems from Hörmander [17, App. B.2] and is

practical in formulas where both types of spaces occur. There are other notations without the overline, and where the dot is replaced by a ring or twiddle.)

Here  $\overline{H}^s(\mathbb{R}_+^n)$  identifies with the dual space of  $\dot{H}^{-s}(\overline{\mathbb{R}}_+^n)$  for all  $s \in \mathbb{R}$  (the duality extending the  $L_2(\mathbb{R}_+^n)$  scalar product). When  $|s| < \frac{1}{2}$ , there is an identification of  $\dot{H}^s(\overline{\mathbb{R}}_+^n)$  with  $\overline{H}^s(\mathbb{R}_+^n)$  (more precisely with  $e^+ \overline{H}^s(\mathbb{R}_+^n)$ ). The trace operator  $\gamma_0: u \mapsto \lim_{x_n \rightarrow 0^+} u(x', x_n)$  extends to a continuous mapping  $\gamma_0: \overline{H}^s(\mathbb{R}_+^n) \rightarrow H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$  for  $s > \frac{1}{2}$ .

The order-reducing operators  $\Xi_{\pm}^t$  are defined for  $t \in \mathbb{C}$  by  $\Xi_{\pm}^t = \text{Op}(\chi_{\pm}^t)$ , where  $\chi_{\pm}^t = \langle \langle \xi' \rangle \pm i\xi_n \rangle^t$ , cf. (3.11). These operators have the homeomorphism properties:

$$\Xi_+^t: \dot{H}^s(\overline{\mathbb{R}}_+^n) \xrightarrow{\sim} \dot{H}^{s-\text{Re } t}(\overline{\mathbb{R}}_+^n), \quad r^+ \Xi_-^t e^+: \overline{H}^s(\mathbb{R}_+^n) \xrightarrow{\sim} \overline{H}^{s-\text{Re } t}(\mathbb{R}_+^n), \quad \text{all } s \in \mathbb{R}, t \in \mathbb{C}; \quad (5.2)$$

$r^+ \Xi_-^t e^+$  is often denoted  $\Xi_{-,+}^t$  for short. For each  $t \in \mathbb{C}$ , the operators  $\Xi_+^t$  and  $\Xi_{-,+}^t$  identify with each other's adjoints over  $\overline{\mathbb{R}}_+^n$  (more comments on this in [8, Rem. 1.1]). Recall also the simple composition rules (as noted e.g., in [15, Th. 1.2]):

$$\Xi_+^s \Xi_+^t = \Xi_+^{s+t}, \quad \Xi_{-,+}^s \Xi_{-,+}^t = \Xi_{-,+}^{s+t} \text{ for } s, t \in \mathbb{C}.$$

We define

$$\mathcal{E}_{\mu}(\overline{\mathbb{R}}_+^n) = e^+ x_n^{\mu} C^{\infty}(\overline{\mathbb{R}}_+^n) \text{ when } \text{Re } \mu > -1, \quad (5.3)$$

and  $\mathcal{E}_{\mu}(\overline{\mathbb{R}}_+^n)$  is defined successively as the linear hull of first-order derivatives of elements of  $\mathcal{E}_{\mu+1}(\overline{\mathbb{R}}_+^n)$  when  $\text{Re } \mu \leq -1$  (then distributions supported in the boundary can occur). The spaces were introduced in Hörmander's unpublished lecture notes [16] and are presented in [8] (and with a different notation in [17, Sect. 18.2]), and they satisfy for all  $\mu$  (cf. [8, Props. 1.7, 4.1]):

$$\mathcal{E}_{\mu}(\overline{\mathbb{R}}_+^n) \cap \mathcal{E}'(\mathbb{R}^n) \subset \Xi_+^{-\mu} e^+ \bigcap_s \overline{H}^s(\overline{\mathbb{R}}_+^n). \quad (5.4)$$

A sharper statement follows from [13, Lemma 6.1] (when  $\text{Re } \mu > -1$ ):

$$e^+ x_n^{\mu} \mathcal{S}(\overline{\mathbb{R}}_+^n) = \Xi_+^{-\mu} e^+ \mathcal{S}(\overline{\mathbb{R}}_+^n). \quad (5.5)$$

## 5.2. Mapping properties of the zero-order operator in Sobolev spaces

Let  $P$  satisfy Assumption 3.1, and consider  $\widehat{Q}^{\pm} = \text{Op}(\widehat{q}^{\pm})$ , defined from the symbols  $q^{\pm}(\xi)$  introduced in Proposition 4.3. Since  $\widehat{q}^{\pm}$  are bounded symbols with bounded inverses, and extend holomorphically in  $\xi_n$  into  $\mathbb{C}_-$  resp.  $\mathbb{C}_+$ ,

$$\widehat{Q}^+ : \dot{H}^s(\overline{\mathbb{R}}_+^n) \xrightarrow{\sim} \dot{H}^s(\overline{\mathbb{R}}_+^n) \text{ and } \widehat{Q}_-^+ = r^+ \widehat{Q}^- e^+ : \overline{H}^s(\mathbb{R}_+^n) \xrightarrow{\sim} \overline{H}^s(\mathbb{R}_+^n), \text{ for all } s \in \mathbb{R}; \quad (5.6)$$

the latter follows since  $r^+ \widehat{Q}^- e^+$  is the adjoint of  $\text{Op}(\overline{\widehat{q}^-})$  over  $\mathbb{R}_+^n$ , where  $\text{Op}(\overline{\widehat{q}^-})$  defines homeomorphisms in  $\dot{H}^s(\overline{\mathbb{R}}_+^n)$  (since  $\overline{\widehat{q}^-}$  has similar properties as  $\widehat{q}^+$ ). The inverses  $(\widehat{Q}^{\pm})^{-1} = \text{Op}((\widehat{q}^{\pm})^{-1})$  have similar homeomorphism properties. Since  $\overline{H}^s(\mathbb{R}_+^n) = \dot{H}^s(\overline{\mathbb{R}}_+^n)$  for  $|s| < \frac{1}{2}$ , it follows that we also have for  $|s| < \frac{1}{2}$ :

$$\widehat{Q}_+^+ = r^+ \widehat{Q}^+ e^+ : \overline{H}^s(\mathbb{R}_+^n) \xrightarrow{\sim} \overline{H}^s(\mathbb{R}_+^n), \quad \widehat{Q}_-^+ \widehat{Q}_+^+ : \overline{H}^s(\mathbb{R}_+^n) \xrightarrow{\sim} \overline{H}^s(\mathbb{R}_+^n).$$

If  $q$  satisfies the full 0-transmission condition, we are in the case studied in [8], and the bijectiveness in  $\overline{H}^s(\mathbb{R}_+^n)$  can be lifted to all higher  $s$  by use of elements of the Boutet de Monvel calculus, as accounted

for in the proof of [8, Th. 4.4]. The symbol  $q$  presently considered is only known to satisfy the principal 0-transmission condition (and possibly a few more identities). We shall here show that a lifting is possible in general up to  $s < \frac{3}{2}$ .

**Proposition 5.1.** *Let  $P$  satisfy Assumption 3.1, and consider  $\widehat{Q}^+ = \text{Op}(\widehat{q}^+)$  derived from it in Section 4.*

*For any  $-\frac{1}{2} < s < \frac{3}{2}$ ,  $\widehat{Q}_+^+ = r^+ \widehat{Q}^+ e^+$  is continuous*

$$r^+ \widehat{Q}^+ e^+ : \overline{H}^s(\mathbb{R}_+^n) \rightarrow \overline{H}^s(\mathbb{R}_+^n), \quad (5.7)$$

and the same holds for the operator  $((\widehat{Q}^+)^{-1})_+$  defined from its inverse  $(\widehat{Q}^+)^{-1}$ .

In fact, (5.7) is a homeomorphism, and the inverse of  $\widehat{Q}_+^+$  is  $((\widehat{Q}^+)^{-1})_+$ .

*Proof.* We already have the mapping property (5.7) for  $|s| < \frac{1}{2}$ , because  $\widehat{q}^+$  is a bounded symbol, and  $e^+ \overline{H}^s(\mathbb{R}_+^n)$  identifies with  $\overline{H}^s(\mathbb{R}_+^n)$  then. Now let  $s = \frac{3}{2} - \varepsilon$  for a small  $\varepsilon > 0$ . Here we need to show that when  $u \in \overline{H}^{\frac{3}{2}-\varepsilon}(\mathbb{R}_+^n)$ , then  $r^+ \partial_j \widehat{Q}^+ e^+ u \in \overline{H}^{\frac{1}{2}-\varepsilon}(\mathbb{R}_+^n)$  for  $j = 1, \dots, n$ . For  $j < n$ , this follows simply because  $\partial_j$  can be commuted through  $r^+$ ,  $\widehat{Q}^+$  and  $e^+$  so that we can use that  $\partial_j u \in \overline{H}^{\frac{1}{2}-\varepsilon}(\mathbb{R}_+^n)$ . For  $j = n$ , we proceed as follows:

Since  $u \in \overline{H}^{\frac{3}{2}-\varepsilon}(\mathbb{R}_+^n)$ , the extension by zero  $e^+ u$  has a jump at  $x_n = 0$ , and a rule for distributions applies:

$$\partial_n e^+ u = e^+ \partial_n u + (\gamma_0 u)(x') \otimes \delta(x_n), \quad \gamma_0 u \in H^{1-\varepsilon}(\mathbb{R}^{n-1}). \quad (5.8)$$

(The rule is obvious when  $u \in C^\infty(\overline{\mathbb{R}_+^n})$ , and extends by continuity to Sobolev spaces.) Therefore, since  $\widehat{Q}^+ = I + \widehat{G}^+$  where  $\widehat{G}^+ = \text{Op}(\widehat{g}^+(\xi))$  from Proposition 4.3,

$$\partial_n \widehat{Q}^+ e^+ u = \widehat{Q}^+ \partial_n e^+ u = \widehat{Q}^+ e^+ \partial_n u + (I + \widehat{G}^+)(\gamma_0 u \otimes \delta(x_n)).$$

In the restriction to  $\mathbb{R}_+^n$ ,  $r^+ I(\gamma_0 u \otimes \delta(x_n))$  drops out, so we are left with

$$r^+ \partial^n \widehat{Q}^+ e^+ u = r^+ \widehat{Q}^+ \partial_n e^+ u = r^+ \widehat{Q}^+ e^+ \partial_n u + K_{\widehat{g}^+} \gamma_0 u, \quad K_{\widehat{g}^+} \varphi = r^+ \widehat{G}^+(\varphi(x') \otimes \delta(x_n)).$$

Here  $K_{\widehat{g}^+}$  is a potential operator (in the terminology of Eskin [6] and Rempel-Schulze [18], generalizing the concept of Poisson operator of Boutet de Monvel [3, 4]), which acts as follows:

$$K_{\widehat{g}^+} \varphi = r^+ \mathcal{F}^{-1}[\widehat{g}^+(\xi) \widehat{\varphi}(\xi')].$$

By (4.8),

$$|\widehat{g}^+(\xi)| \leq C \langle \xi' \rangle^{1-\varepsilon/2} \langle \xi \rangle^{\varepsilon/2-1},$$

hence

$$\begin{aligned} \|K_{\widehat{g}^+} \varphi\|_{\overline{H}^{\frac{1}{2}-\varepsilon}(\mathbb{R}_+^n)}^2 &\leq \|\widehat{G}^+(\varphi \otimes \delta)\|_{\overline{H}^{\frac{1}{2}-\varepsilon}(\mathbb{R}^n)}^2 = c \int_{\mathbb{R}^n} |\widehat{g}(\xi)|^2 |\widehat{\varphi}(\xi')|^2 \langle \xi \rangle^{1-2\varepsilon} d\xi \\ &\leq C \int_{\mathbb{R}^n} |\widehat{\varphi}(\xi')|^2 \langle \xi \rangle^{1-2\varepsilon-2+\varepsilon} \langle \xi' \rangle^{2-\varepsilon} d\xi = C \int_{\mathbb{R}^n} |\widehat{\varphi}(\xi')|^2 \langle \xi \rangle^{-1-\varepsilon} \langle \xi' \rangle^{2-\varepsilon} d\xi \\ &= C' \int_{\mathbb{R}^{n-1}} |\widehat{\varphi}(\xi')|^2 \langle \xi' \rangle^{2-2\varepsilon} d\xi' = C'' \|\varphi\|_{\overline{H}^{1-\varepsilon}(\mathbb{R}^{n-1})}^2, \end{aligned}$$



since  $\int_{\mathbb{R}} \langle \xi \rangle^{-1-\varepsilon} d\xi_n = \langle \xi' \rangle^{-\varepsilon} \int_{\mathbb{R}} \langle \eta_n \rangle^{-1-\varepsilon} d\eta_n$ . Inserting  $\varphi = \gamma_0 u$ , we thus have

$$\|K_{\widehat{g}^+} \gamma_0 u\|_{\overline{H}^{\frac{1}{2}-\varepsilon}(\mathbb{R}_+^n)} \leq C_1 \|\gamma_0 u\|_{H^{1-\varepsilon}(\mathbb{R}^{n-1})} \leq C_2 \|u\|_{\overline{H}^{\frac{3}{2}-\varepsilon}(\mathbb{R}_+^n)}.$$

Thus

$$\|r^+ \partial^n \widehat{Q}^+ e^+ u\|_{\overline{H}^{\frac{1}{2}-\varepsilon}} \leq \|r^+ \widehat{Q}^+ e^+ \partial_n u\|_{\overline{H}^{\frac{1}{2}-\varepsilon}} + \|K_{\widehat{g}^+} \gamma_0 u\|_{\overline{H}^{\frac{1}{2}-\varepsilon}} \leq C_3 \|u\|_{\overline{H}^{\frac{3}{2}-\varepsilon}}.$$

Altogether, this shows the desired mapping property for  $s = \frac{3}{2} - \varepsilon$ , and the property for general  $\frac{1}{2} \leq s < \frac{3}{2}$  follows by interpolation with the case  $s = 0$ .

The mapping property (5.7) holds for the inverse  $(\widehat{Q}^+)^{-1}$ , since its symbol  $(q^+)^{-1}$  equals  $1 + \sum_{k \geq 1} (-b_+)^k$  with essentially the same structure.

The identity  $((\widehat{Q}^+)^{-1})_+ \widehat{Q}_+^+ = I = \widehat{Q}_+^+ ((\widehat{Q}^+)^{-1})_+$  valid on  $L_2(\mathbb{R}_+^n)$ , holds a fortiori on  $\overline{H}^s(\mathbb{R}_+^n)$  for  $0 < s < \frac{3}{2}$ , and extends by continuity to  $\overline{H}^s(\mathbb{R}_+^n)$  for  $-\frac{1}{2} < s < 0$ .  $\square$

When  $P$  merely satisfies Assumption 3.2, we can still show a useful forward mapping property of  $\widehat{Q}$ , based on the decomposition in Proposition 4.2.

**Proposition 5.2.** *Let  $P$  satisfy Assumption 3.2, and consider  $\widehat{Q}$  and  $\widehat{F}_{\pm} = \text{Op}(\widehat{f}_{\pm})$  derived from it in Section 4.*

*The operator  $\widehat{F}_{+,+} = r^+ \widehat{F}_+ e^+$  is continuous*

$$r^+ \widehat{F}_+ e^+ : \overline{H}^s(\mathbb{R}_+^n) \rightarrow \overline{H}^s(\mathbb{R}_+^n) \text{ for any } -\frac{1}{2} < s < \frac{3}{2}. \quad (5.9)$$

*The operator  $\widehat{F}_{-,+} = r^+ \widehat{F}_- e^+$  is continuous from  $\overline{H}^s(\mathbb{R}_+^n)$  to  $\overline{H}^s(\mathbb{R}_+^n)$  for any  $s \in \mathbb{R}$ .*

*The operator  $\widehat{Q}_+ = r^+ \widehat{Q} e^+$  is continuous*

$$r^+ \widehat{Q} e^+ : \overline{H}^s(\mathbb{R}_+^n) \rightarrow \overline{H}^s(\mathbb{R}_+^n) \text{ for any } -\frac{1}{2} < s < \frac{3}{2}. \quad (5.10)$$

*Proof.* Since  $\widehat{F}_+$  has bounded symbol, it maps  $\dot{H}^s(\overline{\mathbb{R}_+^n})$  into  $H^s(\mathbb{R}^n)$  for all  $s$ , so for  $|s| < \frac{1}{2}$ , (5.9) follows since  $\dot{H}^s(\overline{\mathbb{R}_+^n}) = e^+ \overline{H}^s(\mathbb{R}_+^n)$  then. For  $\frac{1}{2} < s < \frac{3}{2}$ , we proceed as in the proof of Proposition 5.1, using that

$$r^+ \partial^n \widehat{F}_+ e^+ u = r^+ \widehat{F}_+ \partial_n e^+ u = r^+ \widehat{F}_+ e^+ \partial_n u + K_{\widehat{f}_+} \gamma_0 u, \quad K_{\widehat{f}_+} \varphi = r^+ \widehat{F}_+ (\varphi(x') \otimes \delta(x_n)),$$

where  $K_{\widehat{f}_+}$  satisfies similar estimates as  $K_{\widehat{g}^+}$  by Proposition 4.2.

For  $r^+ \widehat{F}_- e^+$ , the statement follows since it is on  $\mathbb{R}_+^n$  the adjoint of  $\text{Op}(\widehat{f}_-)$ , which preserves support in  $\overline{\mathbb{R}_+^n}$  and therefore maps  $\dot{H}^s(\overline{\mathbb{R}_+^n})$  into itself for all  $s \in \mathbb{R}$ . For  $\widehat{Q}$ , the statement now follows since it equals  $s_0 + \widehat{F}_- + \widehat{F}_+$ .  $\square$

This is as far as we get by applying Lemma 4.1 to  $f$ . To obtain the mapping property for higher  $s$  would require a control over the potential operators

$$\varphi \mapsto r^+ \text{Op}(\xi_n^j \widehat{f}_+(\xi))(\varphi(x') \otimes \delta(x_n))$$

for  $j \geq 1$  as well. At any rate, the property shown in Proposition 5.2 will be sufficient for the integration by parts formulas we are aiming for.

In the elliptic case, we conclude from Proposition 5.1 for the operator  $\widehat{Q}$ :

**Corollary 5.3.** *Let  $P$  satisfy Assumption 3.1, and consider the operators  $\widehat{Q}$ ,  $\widehat{Q}^+$ ,  $\widehat{Q}^-$  with symbols  $\widehat{q}$ ,  $\widehat{q}^+$ ,  $\widehat{q}^-$  derived from it in Section 4. The operator  $\widehat{Q}_+ \equiv r^+ \widehat{Q} e^+$  acts like  $r^+ \widehat{Q}^- e^+ r^+ \widehat{Q}^+ e^+ = \widehat{Q}_+ \widehat{Q}_+$ , mapping continuously and bijectively*

$$\widehat{Q}_+ = r^+ \widehat{Q} e^+ : \overline{H}^s(\mathbb{R}_+^n) \xrightarrow{\sim} \overline{H}^s(\mathbb{R}_+^n) \text{ for } -\frac{1}{2} < s < \frac{3}{2}, \quad (5.11)$$

and the inverse (continuous in the opposite direction) equals

$$(r^+ \widehat{Q} e^+)^{-1} = r^+ (\widehat{Q}^+)^{-1} e^+ r^+ (\widehat{Q}^-)^{-1} e^+. \quad (5.12)$$

*Proof.* We have for  $u \in \dot{H}^s(\overline{\mathbb{R}_+^n}) \simeq \overline{H}^s(\mathbb{R}_+^n)$ ,  $|s| < \frac{1}{2}$ , that

$$r^+ \widehat{Q} e^+ u = r^+ \widehat{Q}^- \widehat{Q}^+ e^+ u = r^+ \widehat{Q}^- (e^+ r^+ + e^- r^-) \widehat{Q}^+ e^+ u = r^+ \widehat{Q}^- e^+ r^+ \widehat{Q}^+ e^+ u,$$

since  $r^- \widehat{Q}^+ e^+ u = 0$ ; this identity is also valid on the subspaces  $\overline{H}^s(\mathbb{R}_+^n)$  with  $s \geq \frac{1}{2}$ . Combining the homeomorphism property of  $r^+ \widehat{Q}^+ e^+$  shown in Proposition 5.1 with the known homeomorphism property of  $r^+ \widehat{Q}^- e^+$  on  $\overline{H}^s(\mathbb{R}_+^n)$ -spaces (cf. (5.6)), we get (5.11). The inverse is pinned down by using that  $r^+ \widehat{Q}^- e^+$  has inverse  $r^+ (\widehat{Q}^-)^{-1} e^+$  on  $\overline{H}^s(\mathbb{R}_+^n)$  for all  $s$ , and  $r^+ \widehat{Q}^+ e^+$  has inverse  $r^+ (\widehat{Q}^+)^{-1} e^+$  on  $\overline{H}^s(\mathbb{R}_+^n)$  for  $-\frac{1}{2} < s < \frac{3}{2}$  in view of Proposition 5.1.  $\square$

### 5.3. Mapping properties of the modified $P$ , transmission spaces

Now turn the attention to  $\widehat{P}$ , which is related to  $\widehat{Q}$  by

$$\widehat{P} = \Xi_-^{\mu'} \widehat{Q} \Xi_+^{\mu'}, \quad \widehat{Q} = \Xi_-^{-\mu'} \widehat{P} \Xi_+^{-\mu'}, \quad (5.13)$$

cf. (3.12)–(3.13).

We shall describe the solutions of the homogeneous Dirichlet problem (in the strongly elliptic case)

$$r^+ \widehat{P} u = f, \quad \text{supp } u \subset \overline{\mathbb{R}_+^n}, \quad (5.14)$$

with  $f$  given in a space  $\overline{H}^s(\mathbb{R}_+^n)$ , and  $u$  assumed a priori to lie in a space  $\dot{H}^\sigma(\overline{\Omega})$  for low  $\sigma$ , e.g., with  $\sigma = a$ .

First we observe for  $\Xi_-^{-\mu'} = r^+ \Xi_-^{-\mu'} e^+$  that

$$\Xi_-^{-\mu'} r^+ \widehat{P} = r^+ \Xi_-^{-\mu'} \widehat{P}, \quad (5.15)$$

since, as accounted for in [8, Rem. 1.1, (1.13)], the action of  $r^+ \Xi_-^{-\mu'}$  is independent of how  $r^+ \widehat{P}$  is extended into  $\overline{\mathbb{R}_+^n}$ . Thus, in view of the mapping properties (5.2) of  $\Xi_-^{-\mu'}$ ,

$$\|r^+ \widehat{P} u\|_{\overline{H}^s(\mathbb{R}_+^n)} \simeq \|\Xi_-^{-\mu'} r^+ \widehat{P} u\|_{\overline{H}^{s+\text{Re } \mu'}(\mathbb{R}_+^n)} = \|r^+ \Xi_-^{-\mu'} \widehat{P} u\|_{\overline{H}^{s+\text{Re } \mu'}(\mathbb{R}_+^n)}. \quad (5.16)$$

Composing the equation in (5.14) with  $\Xi_-^{-\mu'}$  to the left, we can therefore write it as

$$r^+ \Xi_-^{-\mu'} P u = g, \quad \text{where } g = \Xi_-^{-\mu'} f \in \overline{H}^{s+\text{Re } \mu'}(\mathbb{R}_+^n). \quad (5.17)$$

Next, we shall also replace  $u$ . Because of the right-hand factor  $\Xi_+^{-\mu'}$  in the expression for  $\widehat{Q}$  in (5.13), we need to introduce the  $\mu$ -transmission spaces

$$H^{\mu(t)}(\overline{\mathbb{R}_+^n}) \equiv \Xi_+^{-\mu} e^+ \overline{H}^{t-\text{Re } \mu}(\mathbb{R}_+^n) \text{ for } t > \text{Re } \mu - \frac{1}{2}, \quad (5.18)$$

defined in [8]; they are Hilbert spaces. (For  $t \leq \text{Re } \mu - \frac{1}{2}$ , the convention is to take  $H^{\mu(t)}(\overline{\mathbb{R}_+^n}) = \dot{H}^t(\overline{\mathbb{R}_+^n})$ , but this is rarely used.) The following properties were shown in [8]:

**Theorem 5.4.** [8] Let  $t > \operatorname{Re} \mu - \frac{1}{2}$ .

1° The mapping  $r^+ \Xi_+^\mu$  is a homeomorphism of  $H^{\mu(t)}(\overline{\mathbb{R}}_+^n)$  onto  $\overline{H}^{t-\operatorname{Re} \mu}(\mathbb{R}_+^n)$  with inverse  $\Xi_+^{-\mu} e^+$ .

2° For  $|t - \operatorname{Re} \mu| < \frac{1}{2}$ ,  $H^{\mu(t)}(\overline{\mathbb{R}}_+^n) = \dot{H}^t(\overline{\mathbb{R}}_+^n)$ .

3° Assume  $\operatorname{Re} \mu > -1$  and  $t > \operatorname{Re} \mu + \frac{1}{2}$ . Then

$$H^{\mu(t)}(\overline{\mathbb{R}}_+^n) \subset \dot{H}^t(\overline{\mathbb{R}}_+^n) + x_n^\mu e^+ \overline{H}^{t-\operatorname{Re} \mu}(\mathbb{R}_+^n), \quad (5.19)$$

where  $\dot{H}^t(\overline{\mathbb{R}}_+^n)$  is replaced by  $\dot{H}^{t-\varepsilon}(\overline{\mathbb{R}}_+^n)$  if  $t - \operatorname{Re} \mu - \frac{1}{2} \in \mathbb{N}$ . Moreover, the trace of  $u/x_n^\mu$  is well-defined on  $H^{\mu(t)}(\overline{\mathbb{R}}_+^n)$  and satisfies

$$\Gamma(1 + \mu) \gamma_0(u/x_n^\mu) = \gamma_0 \Xi_+^\mu u \in H^{t-\operatorname{Re} \mu - \frac{1}{2}}(\mathbb{R}^{n-1}). \quad (5.20)$$

Rule 1° is shown in [8, Prop. 1.7]. Rule 2°, shown in [8, (1.26)], holds because of the mapping property (5.2) for  $\Xi_+^{-\mu}$  and the identification of  $e^+ \overline{H}^{t-\operatorname{Re} \mu}(\mathbb{R}_+^n)$  with  $\dot{H}^{t-\operatorname{Re} \mu}(\overline{\mathbb{R}}_+^n)$  when  $t - \operatorname{Re} \mu \in ] -\frac{1}{2}, \frac{1}{2}[$ .

Rule 3° is shown in [8, Th. 5.1, Cor. 5.3, Th. 5.4]; it deals with a higher  $t$ , where  $e^+ \overline{H}^{t-\operatorname{Re} \mu}(\mathbb{R}_+^n)$  has a jump at  $x_n = 0$ , and the coefficient  $x_n^\mu$  appears. Let us just mention the key formula

$$\mathcal{F}_{\xi_n \rightarrow x_n}^{-1} [(\langle \xi' \rangle + i\xi_n)^{-\mu} (\langle \xi' \rangle + i\xi_n)^{-1}] = \frac{1}{\Gamma(\mu+1)} e^+ r^+ x_n^\mu e^{-\langle \xi' \rangle x_n},$$

which indicates how  $\Xi_+^{-\mu} = \operatorname{Op}(\langle \xi' \rangle + i\xi_n)^{-\mu}$  is connected with the factor  $x_n^\mu$ . Besides in [8, Sect. 5], explicit calculations are carried out e.g., in [12, Lemma 3.3] (and [9, Appendix]).

We note in passing that in the definition (5.18), one can equivalently replace the order-reducing operator family  $\Xi_+^t = \operatorname{Op}(\langle \xi' \rangle + i\xi_n^t)$  by  $\operatorname{Op}([\xi'] + i\xi_n^t)$ , or by  $\Lambda_+^t$ , as defined in [8].

Now continue the discussion of (5.17): In view of Theorem 5.4 1°, we can set  $v = r^+ \Xi_+^\mu u$ , where  $u = \Xi_+^{-\mu} e^+ v$ , and hereby

$$r^+ \Xi^{-\mu'} \widehat{P} u = r^+ \Xi^{-\mu'} \widehat{P} \Xi_+^{-\mu} e^+ v = r^+ \widehat{Q} e^+ v = \widehat{Q}_+ v.$$

Then the Eq (5.17) reduces to an equivalent equation

$$\widehat{Q}_+ v = g,$$

with  $g$  given in  $\overline{H}^{s+\operatorname{Re} \mu'}(\mathbb{R}_+^n)$  and  $v$  a priori taken in  $\dot{H}^{\sigma-\operatorname{Re} \mu}(\overline{\mathbb{R}}_+^n)$ . We shall denote  $s = t - 2a$ , so  $s + \operatorname{Re} \mu' = t - 2a + 2a - \operatorname{Re} \mu = t - \operatorname{Re} \mu$ . The equation was solved in Corollary 5.3 and we find for  $r^+ \widehat{P}$ :

**Theorem 5.5.** Let  $P$  satisfy Assumption 3.1. For  $\operatorname{Re} \mu - \frac{1}{2} < t < \operatorname{Re} \mu + \frac{3}{2}$ ,  $r^+ \widehat{P}$  defines a homeomorphism (continuous bijective operator with continuous inverse)

$$r^+ \widehat{P}: H^{\mu(t)}(\overline{\mathbb{R}}_+^n) \xrightarrow{\sim} \overline{H}^{t-2a}(\mathbb{R}_+^n). \quad (5.21)$$

Furthermore, if  $u$  is in  $\dot{H}^\sigma(\overline{\mathbb{R}}_+^n)$  for some  $\sigma > \operatorname{Re} \mu - \frac{1}{2}$  (this includes the value  $\sigma = a$ ) and solves (5.14) with  $f \in \overline{H}^{t-2a}(\mathbb{R}_+^n)$ , then  $u \in H^{\mu(t)}(\overline{\mathbb{R}}_+^n)$ .

Here  $\operatorname{Re} \mu > -\frac{1}{2}$  since  $a > 0$  and  $|\operatorname{Re} \delta| < \frac{1}{2}$ , so the rules in Theorem 5.4 3° apply.

*Proof.* In view of (5.15) and (5.16), and the mapping property of  $\widehat{Q}$  established in Corollary 5.3,  $r^+\widehat{P}$  has the forward mapping property in (5.21).

To solve (5.14), let  $\sigma = \operatorname{Re} \mu - \frac{1}{2} + \varepsilon$  for a small  $\varepsilon$ , set  $g = \Xi_{-,+}^{-\mu'} f \in \overline{H}^{t-2a+\operatorname{Re} \mu'}(\mathbb{R}_+^n) = \overline{H}^{t-\operatorname{Re} \mu}(\mathbb{R}_+^n)$  and  $v = r^+\Xi_+^\mu u \in \dot{H}^{-\frac{1}{2}+\varepsilon}(\overline{\mathbb{R}}_+^n)$ . Then (5.14) reduces to solving

$$\widehat{Q}_+ v = g, \quad (5.22)$$

with  $g$  given in  $\overline{H}^{t-\operatorname{Re} \mu}(\mathbb{R}_+^n)$  and  $v$  a priori lying in  $\dot{H}^{-\frac{1}{2}+\varepsilon}(\overline{\mathbb{R}}_+^n)$ . By Corollary 5.3, (5.22) has a unique solution  $v \in \overline{H}^{t-\operatorname{Re} \mu}(\mathbb{R}_+^n)$ , so  $u$  must lie in  $H^{\mu(t)}(\overline{\mathbb{R}}_+^n)$ , and the mapping (5.21) is bijective.  $\square$

**Remark 5.6.** This theorem differs from the strategy pursued in [6], and gives a new insight. The technique in [6, Th. 7.3] for showing solvability in a higher-order Sobolev space, say with  $\frac{1}{2} < t - \operatorname{Re} \mu < \frac{3}{2}$ ,  $f$  given in  $\overline{H}^{t-2a}(\mathbb{R}_+^n)$ , is to supplement  $\widehat{P}$  with a potential operator  $K_{\widehat{P}}$  constructed from  $\widehat{P}$  such that the solutions are of the form  $u = u_+ + K_{\widehat{P}}\varphi$  with  $u_+ \in \dot{H}^t(\overline{\mathbb{R}}_+^n)$ ,  $\varphi$  a generalized trace derived from  $f$ . Our aim is to show that there is a *universal description* of the space of solutions  $u$  of (5.14) with right-hand side in  $\overline{H}^{t-2a}(\mathbb{R}_+^n)$ , that depends only on  $\mu$ , and applies to any  $P$  of the given type. The  $\mu$ -transmission spaces (5.18) serve this purpose. In [8], they are shown to have this role for arbitrarily high  $t$  when the full  $\mu$ -transmission condition holds.

One more important property of  $\mu$ -transmission spaces is that the spaces with  $C^\infty$ -ingredients  $\mathcal{E}_\mu(\overline{\mathbb{R}}_+^n) \cap \mathcal{E}'(\mathbb{R}^n)$  and  $e^+ x_n^\mu \mathcal{S}(\overline{\mathbb{R}}_+^n)$  are dense subsets of  $H^{\mu(t)}(\overline{\mathbb{R}}_+^n)$  for all  $t > \operatorname{Re} \mu - \frac{1}{2}$ ,  $\operatorname{Re} \mu > -1$  (cf. [8, Prop. 4.1] and [13, Lemma 7.1]). Recall also (5.5), which makes the statement for  $e^+ x_n^\mu \mathcal{S}(\overline{\mathbb{R}}_+^n)$  rather evident, since  $\mathcal{S}(\overline{\mathbb{R}}_+^n)$  is dense in  $\overline{H}^s(\mathbb{R}_+^n)$  for all  $s \in \mathbb{R}$ . Hence  $r^+\widehat{P}$  applies nicely to these spaces.

When  $P$  merely satisfies Assumption 3.2, we have at least the forward mapping part of (5.21):

**Theorem 5.7.** *Let  $P$  satisfy Assumption 3.2. For  $\operatorname{Re} \mu - \frac{1}{2} < t < \operatorname{Re} \mu + \frac{3}{2}$ ,  $r^+\widehat{P}$  maps continuously*

$$r^+\widehat{P}: H^{\mu(t)}(\overline{\mathbb{R}}_+^n) \rightarrow \overline{H}^{t-2a}(\mathbb{R}_+^n). \quad (5.23)$$

*Proof.* This follows as in the preceding proof, now using the mapping property of  $r^+\widehat{Q}e^+$  established in Proposition 5.2.  $\square$

#### 5.4. Consequences for the given operator $P$

The following consequences can be drawn for the original operator  $P$ :

**Theorem 5.8.** *1° Let  $P$  satisfy Assumption 3.2. Then  $P = \widehat{P} + P'$ , where  $\widehat{P}$  is defined by (3.9) and  $P'$  is of order  $2a - 1$ . For  $\operatorname{Re} \mu - \frac{1}{2} < t < \operatorname{Re} \mu + \frac{3}{2}$ ,  $r^+P$  maps continuously*

$$r^+P: H^{\mu(t)}(\overline{\mathbb{R}}_+^n) \rightarrow \overline{H}^{t-2a}(\mathbb{R}_+^n). \quad (5.24)$$

*2° Let  $P$  satisfy Assumption 3.1. Then in the decomposition  $P = \widehat{P} + P'$ ,  $r^+\widehat{P}$  is invertible, as described in Theorem 5.5.*

*Let  $\operatorname{Re} \mu - \frac{1}{2} < t < \operatorname{Re} \mu + \frac{3}{2}$ , let  $f \in \overline{H}^{t-2a}(\mathbb{R}_+^n)$ , and let  $u \in \dot{H}^\sigma(\overline{\mathbb{R}}_+^n)$  (for some  $\sigma > \operatorname{Re} \mu - \frac{1}{2}$ ) solve the homogeneous Dirichlet problem*

$$r^+Pu = f \text{ on } \mathbb{R}_+^n, \quad \operatorname{supp} u \subset \overline{\mathbb{R}}_+^n. \quad (5.25)$$

*Then  $u \in H^{\mu(t)}(\overline{\mathbb{R}}_+^n)$ .*

*Proof.* The original operator  $P$  equals  $\text{Op}(p(\xi))$  with  $p(\xi)$  homogeneous on  $\mathbb{R}^n$  of degree  $2a > 0$ ; in particular it is continuous at 0. It is decomposed into

$$p(\xi) = \widehat{p}(\xi) + p'(\xi). \quad (5.26)$$

where  $p'(\xi)$  is  $O(\langle \xi \rangle^{2a-1})$  for  $|\xi| \geq 2$  by (3.10) and continuous, hence

$$|p'(\xi)| \leq C' \langle \xi \rangle^{2a-1} \text{ for } \xi \in \mathbb{R}^n.$$

This implies that  $P' = \text{Op}(p')$  maps  $H^s(\mathbb{R}^n)$  continuously into  $H^{s-2a+1}(\mathbb{R}^n)$  for all  $s \in \mathbb{R}$ , and hence

$$r^+ P' : \dot{H}^s(\overline{\mathbb{R}}_+^n) \rightarrow \overline{H}^{s-2a+1}(\overline{\mathbb{R}}_+^n) \text{ for all } s \in \mathbb{R}. \quad (5.27)$$

1°. The forward mapping property (5.23) holds for  $r^+ \widehat{P}$  by Theorem 5.7. To show that it holds for  $r^+ P'$ , let  $\text{Re } \mu - \frac{1}{2} < t < \text{Re } \mu + \frac{3}{2}$ .

If  $t - \text{Re } \mu < \frac{1}{2}$ , then  $H^{\mu(t)}(\overline{\mathbb{R}}_+^n) = \dot{H}^t(\overline{\mathbb{R}}_+^n)$ , and  $r^+ P' \dot{H}^t(\overline{\mathbb{R}}_+^n) \subset \overline{H}^{t-2a+1}(\overline{\mathbb{R}}_+^n) \subset \overline{H}^{t-2a}(\overline{\mathbb{R}}_+^n)$  by (5.27), matching the mapping property of  $\widehat{P}$ .

If  $\frac{1}{2} \leq t - \text{Re } \mu < \frac{3}{2}$ , we use the definition of  $H^{\mu(t)}(\overline{\mathbb{R}}_+^n)$  to see that for small  $\varepsilon > 0$ ,

$$\begin{aligned} r^+ P' H^{\mu(t)}(\overline{\mathbb{R}}_+^n) &= r^+ P' \Xi_+^{-\mu} e^+ \overline{H}^{t-\text{Re } \mu}(\overline{\mathbb{R}}_+^n) \subset r^+ P' \Xi_+^{-\mu} \dot{H}^{\frac{1}{2}-\varepsilon}(\overline{\mathbb{R}}_+^n) \\ &= r^+ P' \dot{H}^{\frac{1}{2}-\varepsilon+\text{Re } \mu}(\overline{\mathbb{R}}_+^n) \subset \overline{H}^{\frac{3}{2}-\varepsilon+\text{Re } \mu-2a}(\overline{\mathbb{R}}_+^n) \subset \overline{H}^{t-2a}(\overline{\mathbb{R}}_+^n), \end{aligned}$$

also matching the mapping property of  $\widehat{P}$ .

Now (5.24) follows by adding the statements for  $P'$  and  $\widehat{P}$ . This shows 1°.

2°. The first statement registers what we already know about  $r^+ \widehat{P}$ . Proof of the regularity statement: With  $u$  and  $f$  as defined there, denote  $\sigma = \text{Re } \mu - \frac{1}{2} + \varepsilon$ ; here  $\varepsilon > 0$ . Then

$$r^+ \widehat{P}u = r^+ Pu - r^+ P'u \in \overline{H}^{t-2a}(\overline{\mathbb{R}}_+^n) + \overline{H}^{\text{Re } \mu + \frac{1}{2} + \varepsilon - 2a}(\overline{\mathbb{R}}_+^n).$$

If  $t \leq \text{Re } \mu + \frac{1}{2} + \varepsilon$ ,  $r^+ \widehat{P}u \in \overline{H}^{t-2a}(\overline{\mathbb{R}}_+^n)$ , and we conclude from Theorem 5.5 that  $u \in H^{\mu(t)}(\overline{\mathbb{R}}_+^n)$ .

If  $t > \text{Re } \mu + \frac{1}{2} + \varepsilon$ ,  $r^+ \widehat{P}u \in \overline{H}^{\text{Re } \mu + \frac{1}{2} + \varepsilon - 2a}(\overline{\mathbb{R}}_+^n)$ ; here Theorem 5.5 applies to give the intermediate information that  $u \in H^{\mu(\text{Re } \mu + \frac{1}{2} + \varepsilon)}(\overline{\mathbb{R}}_+^n)$ . From this follows that

$$u \in \Xi_+^{-\mu} e^+ \overline{H}^{\frac{1}{2} + \varepsilon}(\overline{\mathbb{R}}_+^n) \subset \Xi_+^{-\mu} \dot{H}^{\frac{1}{2} - \varepsilon'}(\overline{\mathbb{R}}_+^n) = \dot{H}^{\text{Re } \mu + \frac{1}{2} - \varepsilon'}(\overline{\mathbb{R}}_+^n),$$

for any  $\varepsilon' > 0$ . Then  $r^+ P'u \in \overline{H}^{\text{Re } \mu + \frac{3}{2} - \varepsilon' - 2a}(\overline{\mathbb{R}}_+^n)$ . Choosing  $\varepsilon'$  so small that  $\text{Re } \mu + \frac{3}{2} - \varepsilon' \geq t$ , we have that  $r^+ P'u \in \overline{H}^{t-2a}(\overline{\mathbb{R}}_+^n)$ ; hence  $r^+ \widehat{P}u \in \overline{H}^{t-2a}(\overline{\mathbb{R}}_+^n)$ , so it follows from Theorem 5.5 that  $u \in H^{\mu(t)}(\overline{\mathbb{R}}_+^n)$ . This ends the proof of 2°.  $\square$

**Example 5.9.** Theorem 5.8 applies to the operator  $L = \text{Op}(\mathcal{L}(\xi))$  described in (3.5)ff., showing that it maps  $H^{\mu(t)}(\overline{\mathbb{R}}_+^n)$  to  $\overline{H}^{t-2a}(\overline{\mathbb{R}}_+^n)$  for  $-\frac{1}{2} < t - \mu < \frac{3}{2}$ , and that solutions of the homogeneous Dirichlet problem with  $f \in \overline{H}^{t-2a}(\overline{\mathbb{R}}_+^n)$  are in  $H^{\mu(t)}(\overline{\mathbb{R}}_+^n)$  for these  $t$ . The appearance of the factor  $x_n^\mu$  (cf. (5.19)) is consistent with the regularity shown in terms of Hölder spaces in [5].

In particular, the result provides a valid basis for applying  $r^+ L$  to  $\mathcal{E}_\mu(\overline{\mathbb{R}}_+^n) \cap \mathcal{E}(\mathbb{R}^n)$  or  $e^+ x_n^\mu \mathcal{S}(\overline{\mathbb{R}}_+^n)$ , mapping these spaces into  $\bigcap_{\varepsilon > 0} \overline{H}^{\frac{3}{2} - a + \delta - \varepsilon}(\overline{\mathbb{R}}_+^n)$ .

**Remark 5.10.** The domain spaces  $H^{\mu(t)}(\overline{\mathbb{R}_+^n})$  entering in Theorem 5.8 can be precisely described: For  $|t - \operatorname{Re} \mu| < \frac{1}{2}$ , we already know from Theorem 5.4 2° that  $H^{\mu(t)}(\overline{\mathbb{R}_+^n}) = \dot{H}^t(\overline{\mathbb{R}_+^n})$ . For  $\frac{1}{2} < t - \operatorname{Re} \mu < \frac{3}{2}$ , we have by [12, Lemma 3.3] that  $u \in H^{\mu(t)}(\overline{\mathbb{R}_+^n})$  if and only if

$$u = v + w, \text{ where } w \in \dot{H}^t(\overline{\mathbb{R}_+^n}) \text{ and } v = e^+ x_n^\mu K_0 \gamma_0(u/x_n^\mu);$$

here  $K_0$  is the Poisson operator  $K_0: \varphi \mapsto z$  solving the Dirichlet problem for  $1 - \Delta$ ,

$$(1 - \Delta)z = 0 \text{ on } \mathbb{R}_+^n, \quad \gamma_0 z = \varphi \text{ at } x_n = 0,$$

with  $\varphi \in H^{t - \operatorname{Re} \mu - \frac{1}{2}}(\mathbb{R}^{n-1})$ . For  $t - \operatorname{Re} \mu = \frac{1}{2}$ , we have the information  $u \in \bigcap_{\varepsilon > 0} \dot{H}^{t-\varepsilon}(\overline{\mathbb{R}_+^n})$ . As a concrete example, the elements  $u$  of  $H^{\frac{1}{2}(\frac{3}{2})}(\overline{\mathbb{R}_+^n})$  are the functions  $u = v + w$ , where  $w \in \dot{H}^{\frac{3}{2}}(\overline{\mathbb{R}_+^n})$  and  $v = x_n^{\frac{1}{2}} K_0 \varphi$  for some  $\varphi \in H^{\frac{1}{2}}(\mathbb{R}^{n-1})$ ; this  $\varphi$  equals  $\gamma_0(u/x_n^{\frac{1}{2}})$ .

## 6. The integration by parts formula

### 6.1. An integration by parts formula for the modified $P$

It will now be shown that the operators  $P$  satisfying merely the principal  $\mu$ -transmission condition (Assumption 3.2) have an integration by parts formula over  $\mathbb{R}_+^n$ , involving traces  $\gamma_0(u/x_n^\mu)$ . The study will cover the special operator  $L$  in Example 5.9 (regardless of whether a full  $\mu$ -transmission condition might hold, as assumed in [13]). It also covers more general strongly elliptic operators, and it covers operators that are not necessarily elliptic.

The basic observation is:

**Proposition 6.1.** *Let  $\mu \in \mathbb{C}$ . Let  $w \in \bigcap_s \overline{H}^s(\mathbb{R}_+^n)$ , and let  $u' \in \mathcal{E}_{\bar{\mu}}(\overline{\mathbb{R}_+^n}) \cap \mathcal{E}'(\mathbb{R}^n)$ . Denote  $w' = r^+ \Xi_+^{\bar{\mu}} u' \in \bigcap_s \overline{H}^s(\mathbb{R}_+^n)$ ; correspondingly  $u' = \Xi_+^{-\bar{\mu}} e^+ w'$  in view of Theorem 5.4 1°. Then*

$$(I \equiv) \int_{\mathbb{R}_+^n} \Xi_-^\mu e^+ w \partial_n \bar{u}' dx = (\gamma_0 w, \gamma_0 w')_{L_2(\mathbb{R}^{n-1})} + (w, \partial_n w')_{L_2(\mathbb{R}_+^n)}. \tag{6.1}$$

The left-hand side is interpreted as in (6.2) below when  $\operatorname{Re} \mu \leq 0$ .

The formula extends to  $w \in \overline{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+^n)$  and  $u' \in H^{\bar{\mu}(t)}(\overline{\mathbb{R}_+^n})$  with  $t \geq \operatorname{Re} \mu + \frac{1}{2} - \varepsilon$  (for small  $\varepsilon > 0$ ), using the representation (6.2).

*Proof.* This was proved in [10, Th. 3.1] for  $\mu = a > 0$  (see also Remark 3.2 there with the elementary case  $a = 1$ ), and in [13, Th. 4.1] for real  $\mu > -\frac{1}{2}$ , so the main task is to check that the larger range of complex  $\mu$  is allowed. We write  $\bar{u}'$  as  $\bar{u}'$  for short.

Note that when  $\operatorname{Im} \mu \neq 0$ ,  $\mathcal{E}_\mu(\overline{\mathbb{R}_+^n})$  is different from  $\mathcal{E}_{\operatorname{Re} \mu}(\overline{\mathbb{R}_+^n})$ , e.g., since  $x_n^\mu/x_n^{\operatorname{Re} \mu} = x_n^{i \operatorname{Im} \mu} = e^{i \operatorname{Im} \mu \log x_n}$  has absolute value 1 and is  $C^\infty$  for  $x_n > 0$ , but oscillates when  $x_n \rightarrow 0$ .

By the mapping properties of  $\Xi_{\pm}^\mu$  (cf. (5.2)),  $r^+ \Xi_-^\mu e^+ w \in \bigcap_s \overline{H}^s(\mathbb{R}_+^n)$ , hence is integrable. When  $\operatorname{Re} \mu > 0$ , the function  $\partial_n u'$  is  $O(x_n^{\operatorname{Re} \mu - 1})$  and compactly supported, so the left-hand side of (6.1) makes sense as an integral of an  $L_1$ -function. When  $\mu$  is general, we observe that for any small  $\varepsilon > 0$ ,

$$\partial_n u' \in \mathcal{E}_{\bar{\mu}-1}(\overline{\mathbb{R}_+^n}) \cap \mathcal{E}'(\mathbb{R}^n) \subset \Xi_+^{1-\bar{\mu}} e^+ \bigcap_s \overline{H}^s(\mathbb{R}_+^n) \subset \Xi_+^{1-\bar{\mu}} \dot{H}^{\frac{1}{2}-\varepsilon}(\overline{\mathbb{R}_+^n}) = \dot{H}^{-\frac{1}{2}+\operatorname{Re} \mu - \varepsilon}(\overline{\mathbb{R}_+^n}),$$

so the integral  $I$  makes sense as the duality

$$I = \langle r^+ \Xi_-^\mu e^+ w, \partial_n u' \rangle_{\overline{H}^{\frac{1}{2}-\operatorname{Re}\mu+\varepsilon}(\mathbb{R}_+^n), \dot{H}^{-\frac{1}{2}+\operatorname{Re}\mu-\varepsilon}(\overline{\mathbb{R}}_+^n)}. \quad (6.2)$$

Since the adjoint of  $r^+ \Xi_-^\mu e^+$  equals  $\Xi_+^{\bar{\mu}}$ ,  $I$  is by transposition turned into

$$I = \langle w, \Xi_+^{\bar{\mu}} \partial_n u' \rangle_{\overline{H}^{\frac{1}{2}+\varepsilon}, \dot{H}^{-\frac{1}{2}-\varepsilon}} = \langle w, \partial_n \Xi_+^{\bar{\mu}} \Xi_+^{-\bar{\mu}} e^+ w' \rangle_{\overline{H}^{\frac{1}{2}+\varepsilon}, \dot{H}^{-\frac{1}{2}-\varepsilon}} = \langle w, \partial_n e^+ w' \rangle_{\overline{H}^{\frac{1}{2}+\varepsilon}, \dot{H}^{-\frac{1}{2}-\varepsilon}}.$$

Note that  $\partial_n e^+ w$  satisfies an equation like (5.8), which fits in here since the space  $\dot{H}^{-\frac{1}{2}-\varepsilon}(\overline{\mathbb{R}}_+^n)$  contains distributions of the form  $\varphi(x') \otimes \delta(x_n)$ . The expression is analysed as in [10, Th. 3.1] (and [13, Th. 4.1]), leading to

$$I = (\gamma_0 w, \gamma_0 w')_{L_2(\mathbb{R}^{n-1})} + (w, e^+ \partial_n w')_{L_2(\mathbb{R}_+^n)}, \quad (6.3)$$

which shows (6.1).

For the whole analysis, it suffices that  $w \in \overline{H}^s(\mathbb{R}_+^n)$  with  $s = \frac{1}{2} + \varepsilon$ , since  $\Xi_{-,+}^\mu w \in \overline{H}^{\frac{1}{2}-\operatorname{Re}\mu+\varepsilon}(\mathbb{R}_+^n)$  then. For  $u'$ , it then suffices that  $u' \in H^{\bar{\mu}(t)}(\overline{\mathbb{R}}_+^n) = \Xi_+^{-\bar{\mu}} e^+ \overline{H}^{t-\operatorname{Re}\mu}(\mathbb{R}_+^n)$  with  $t \geq \operatorname{Re}\mu + \frac{1}{2} - \varepsilon$  (assuming  $0 < \varepsilon < 1$ ), since

$$\partial_n u' \in \Xi_+^{1-\bar{\mu}} e^+ \overline{H}^{t-\operatorname{Re}\mu}(\mathbb{R}_+^n) \subset \Xi_+^{1-\bar{\mu}} \dot{H}^{\frac{1}{2}-\varepsilon}(\overline{\mathbb{R}}_+^n) = \dot{H}^{-\frac{1}{2}+\operatorname{Re}\mu-\varepsilon}(\overline{\mathbb{R}}_+^n)$$

then, so that the duality in (6.2) is well-defined.  $\square$

We shall now show:

**Theorem 6.2.** *Let  $P$  satisfy Assumption 3.2; it is of order  $2a$  and satisfies the principal  $\mu$ -transmission condition in the direction  $(0, 1)$  for some  $\mu = a + \delta \in \mathbb{C}$ , and we denote  $a - \delta = \mu'$ . Assume moreover that  $\operatorname{Re}\mu > -1$ ,  $\operatorname{Re}\mu' > -1$ . Consider  $\widehat{P} = \operatorname{Op}(\widehat{p}(\xi))$ , as defined by (3.9). For  $u \in \mathcal{E}_\mu(\overline{\mathbb{R}}_+^n) \cap \mathcal{E}'(\mathbb{R}^n)$ ,  $u' \in \mathcal{E}_{\bar{\mu}'}(\overline{\mathbb{R}}_+^n) \cap \mathcal{E}'(\mathbb{R}^n)$ , there holds*

$$\begin{aligned} \int_{\mathbb{R}_+^n} \widehat{P} u \partial_n \bar{u}' dx + \int_{\mathbb{R}_+^n} \partial_n u \overline{\widehat{P}^* u'} dx \\ = \Gamma(\mu + 1) \Gamma(\mu' + 1) \int_{\mathbb{R}^{n-1}} s_0 \gamma_0(u/x_n^\mu) \gamma_0(\bar{u}'/x_n^{\mu'}) dx', \end{aligned} \quad (6.4)$$

where  $s_0 = e^{-i\pi\delta} p(0, 1)$ . The formula extends to  $u \in H^{\bar{\mu}(t)}(\overline{\mathbb{R}}_+^n)$ ,  $u' \in H^{\bar{\mu}'(t')}(\overline{\mathbb{R}}_+^n)$ , for  $t > \operatorname{Re}\mu + \frac{1}{2}$ ,  $t' > \operatorname{Re}\mu' + \frac{1}{2}$ .

The integrals over  $\mathbb{R}_+^n$  are interpreted as dualities (as in Proposition 6.1) when  $\operatorname{Re}\mu$  or  $\operatorname{Re}\mu' \leq 0$ , and when extended to general  $u, u'$ .

*Proof.* Since integration over  $\mathbb{R}_+^n$  in itself indicates that the functions behind the integration sign are restricted to  $\mathbb{R}_+^n$ , we can leave out the explicit mention of  $r^+$ . Recall that

$$\widehat{p} = \chi_-^{\mu'} \widehat{q} \chi_+^\mu, \quad \widehat{P} = \Xi_-^{\mu'} \widehat{Q} \Xi_+^\mu,$$

cf. (3.13). The adjoint is  $\widehat{P}^* = \Xi_-^{\bar{\mu}} \widehat{Q} \Xi_+^{\bar{\mu}'}$ . Recall from Proposition 4.2 that

$$q(\xi) = s_0 + f_+(\xi) + f_-(\xi), \quad \text{hence } \widehat{Q} = s_0 + \widehat{F}_+ + \widehat{F}_-,$$

where  $\widehat{f}_\pm(\xi)$  extend holomorphically in  $\xi_n + i\tau$  into  $\mathbb{C}_-$  resp.  $\mathbb{C}_+$ , estimated as in (4.8).

Accordingly,  $\widehat{P}$  splits up in three terms

$$\widehat{P} = \widehat{P}_1 + \widehat{P}_2 + \widehat{P}_3, \text{ where } \widehat{P}_1 = s_0 \Xi_-^{\mu'} \Xi_+^{\mu}, \quad \widehat{P}_2 = \Xi_-^{\mu'} \widehat{F}_+ \Xi_+^{\mu}, \quad \widehat{P}_3 = \Xi_-^{\mu'} \widehat{F}_- \Xi_+^{\mu}. \quad (6.5)$$

Consider the contribution from  $\widehat{P}_1$ :

$$\int_{\mathbb{R}_+^n} \widehat{P}_1 u \partial_n \bar{u}' dx + \int_{\mathbb{R}_+^n} \partial_n u \overline{\widehat{P}_1^* u'} dx = s_0 \int_{\mathbb{R}_+^n} \Xi_-^{\mu'} \Xi_+^{\mu} u \partial_n \bar{u}' dx + s_0 \int_{\mathbb{R}_+^n} \partial_n u \overline{\Xi_-^{\mu'} \Xi_+^{\mu} u'} dx.$$

Recall that  $s_0 = q(0, 1) = e^{-i\pi\delta} p(0, 1)$  by (3.16); this constant is left out of the next calculations.

When  $u \in \mathcal{E}_\mu(\mathbb{R}_+^n) \cap \mathcal{E}'(\mathbb{R}^n)$ , then  $w = r^+ \Xi_+^{\mu} u \in \cap_s \overline{H^s}(\mathbb{R}_+^n)$ . Similarly as in (5.15),  $r^+ \Xi_-^{\mu'} \Xi_+^{\mu} u = r^+ \Xi_-^{\mu'} e^+ r^+ \Xi_+^{\mu} u$ , which equals  $r^+ \Xi_-^{\mu'} e^+ w$ , hence lies in  $\cap_s \overline{H^s}(\mathbb{R}_+^n)$  by (5.2). An application of Proposition 6.1 with  $\mu$  replaced by  $\mu'$  gives:

$$\int_{\mathbb{R}_+^n} \Xi_-^{\mu'} \Xi_+^{\mu} u \partial_n \bar{u}' dx = \int_{\mathbb{R}_+^n} r^+ \Xi_-^{\mu'} e^+ w \partial_n \bar{u}' dx = (\gamma_0 w, \gamma_0 w')_{L_2(\mathbb{R}^{n-1})} + (w, \partial_n w')_{L_2(\mathbb{R}_+^n)},$$

where  $w' = r^+ \Xi_+^{\mu'} u'$ .

We can apply the analogous argument to show that the conjugate of  $\int_{\mathbb{R}_+^n} \partial_n u \overline{\Xi_-^{\mu'} \Xi_+^{\mu} u'} dx$  satisfies

$$\int_{\mathbb{R}_+^n} \Xi_-^{\mu'} \Xi_+^{\mu} u' \partial_n \bar{u} dx = (\gamma_0 w', \gamma_0 w)_{L_2(\mathbb{R}^{n-1})} + (w', \partial_n w)_{L_2(\mathbb{R}_+^n)};$$

here  $w' = r^+ \Xi_+^{\mu'} u'$  and  $w = r^+ \Xi_+^{\mu} u$  are the same as the functions defined in the treatment of the first integral.

It follows by addition that

$$\begin{aligned} & \int_{\mathbb{R}_+^n} \Xi_-^{\mu'} \Xi_+^{\mu} u \partial_n \bar{u}' dx + \int_{\mathbb{R}_+^n} \partial_n u \overline{\Xi_-^{\mu'} \Xi_+^{\mu} u'} dx \\ &= 2(\gamma_0 w, \gamma_0 w')_{L_2(\mathbb{R}^{n-1})} + (w, \partial_n w')_{L_2(\mathbb{R}_+^n)} + (\partial_n w, w')_{L_2(\mathbb{R}_+^n)} = (\gamma_0 w, \gamma_0 w')_{L_2(\mathbb{R}^{n-1})}; \end{aligned}$$

in the last step we used that  $\int_{\mathbb{R}_+^n} (w \partial_n \bar{w}' + \partial_n w \bar{w}') dx = - \int_{\mathbb{R}^{n-1}} \gamma_0 w \gamma_0 \bar{w}' dx'$ . Insertion of  $\gamma_0 w = \gamma_0 \Xi_+^{\mu} u = \Gamma(1 + \mu) \gamma_0(u/x_n^\mu)$  (cf. (5.20)), and similarly  $\gamma_0 w' = \gamma_0 \Xi_+^{\mu'} u' = \Gamma(1 + \mu') \gamma_0(u'/x_n^{\mu'})$ , gives (6.4) with  $\widehat{P}$  replaced by  $\widehat{P}_1$  (using also that  $\overline{\Gamma(1 + \mu')} = \Gamma(1 + \mu)$ ).

As for extension of the formula to larger spaces, we note that by Proposition 6.1, the calculations for the first integral allow  $w \in \overline{H}^{\frac{1}{2} + \varepsilon}(\mathbb{R}_+^n)$ , corresponding to  $u \in H^{\mu(t)}(\overline{\mathbb{R}_+^n})$  with  $t = \text{Re } \mu + \frac{1}{2} + \varepsilon$ , and  $u' \in H^{\mu'(t')}(\overline{\mathbb{R}_+^n})$  with  $t' \geq \text{Re } \mu' + \frac{1}{2} - \varepsilon$ . With the analogous conditions for the calculations of the second integral, we find altogether that  $t > \text{Re } \mu + \frac{1}{2}$ ,  $t' > \text{Re } \mu' + \frac{1}{2}$ , is allowed.

The contributions from  $\widehat{P}_2$  and  $\widehat{P}_3$  will be treated by variants of this proof, where we show that their boundary integrals give zero.

Consider  $\widehat{P}_2$ . As in (5.15), we have:

$$\begin{aligned} r^+ \widehat{P}_2 u &= r^+ \Xi_-^{\mu'} \widehat{F}_+ \Xi_+^{\mu} u = r^+ \Xi_-^{\mu'} e^+ r^+ (\widehat{F}_+ \Xi_+^{\mu} u), \\ r^+ \widehat{P}_2^* u' &= r^+ \Xi_-^{\mu'} \widehat{F}_+^* \Xi_+^{\mu'} u' = r^+ (\Xi_-^{\mu'} \widehat{F}_+^*) e^+ r^+ \Xi_+^{\mu'} u', \end{aligned}$$



where  $\widehat{F}_+^* = \text{Op}(\widehat{f}_+)$ . Set

$$w = r^+ \Xi_+^\mu u, \quad w_1 = r^+ \widehat{F}_+ \Xi_+^\mu u, \quad w' = r^+ \Xi_+^{\mu'} u'. \quad (6.6)$$

Here when  $u \in H^{\mu(t)}(\overline{\mathbb{R}}_+^n)$ ,  $w \in \overline{H}^{t-\text{Re}\mu}(\mathbb{R}_+^n)$ , and when  $u' \in H^{\mu'(t')}(\overline{\mathbb{R}}_+^n)$ ,  $w' \in \overline{H}^{t'-\text{Re}\mu'}(\mathbb{R}_+^n)$ . For  $w_1$  we have since  $u = \Xi_+^{-\mu} e^+ w$  (by Theorem 5.4 1°), that

$$w_1 = r^+ \widehat{F}_+ \Xi_+^\mu u = r^+ \widehat{F}_+ \Xi_+^\mu \Xi_+^{-\mu} e^+ w = r^+ \widehat{F}_+ e^+ w \in \overline{H}^{t-\text{Re}\mu}(\mathbb{R}_+^n),$$

when  $-\frac{1}{2} < t - \text{Re}\mu < \frac{3}{2}$ , by the mapping property for  $\widehat{F}_+$  established in Proposition 5.2.

We can then apply Proposition 6.1 to the first integral for  $\widehat{P}_2$ , with  $\mu$  replaced by  $\mu'$ , giving when  $t - \text{Re}\mu > \frac{1}{2}$ :

$$\begin{aligned} \int_{\mathbb{R}_+^n} \widehat{P}_2 u \partial_n \bar{u}' dx &= \int_{\mathbb{R}_+^n} \Xi_-^{\mu'} \widehat{F}_+ \Xi_+^\mu u \partial_n \bar{u}' dx = \int_{\mathbb{R}_+^n} \Xi_-^{\mu'} e^+ w_1 \partial_n \bar{u}' dx \\ &= (\gamma_0 w_1, \gamma_0 w')_{L_2(\mathbb{R}^{n-1})} + (w_1, \partial_n w')_{L_2(\mathbb{R}_+^n)}. \end{aligned} \quad (6.7)$$

There is a general formula for the trace, entering in Vishik and Eskin's calculus as well as that of Boutet de Monvel,

$$\gamma_0 v = (2\pi)^{-n} \int_{\mathbb{R}^{n-1}} e^{ix' \cdot \xi'} \int_{\mathbb{R}} \mathcal{F}(e^+ v) d\xi_n d\xi',$$

where the integral over  $\mathbb{R}$  is read either as an ordinary integral or, if necessary, as the integral  $\int^+$  defined e.g., in [7, (10.85)] (also recalled in [11, (A.1), (A.15)]). Applying this to  $w_1$ , we find:

$$\gamma_0 w_1 = \gamma_0(\widehat{F}_{+,+} w) = (2\pi)^{-n} \int_{\mathbb{R}^{n-1}} e^{ix' \cdot \xi'} \int_{\mathbb{R}} \widehat{f}_+(\xi', \xi_n) \mathcal{F}(e^+ w) d\xi_n d\xi'. \quad (6.8)$$

This integral gives 0 for the following reason: It suffices to take  $w$  in the dense subspace of  $\overline{H}^{t-\text{Re}\mu}(\mathbb{R}_+^n)$  of compactly supported functions in  $C^\infty(\overline{\mathbb{R}}_+^n)$ . Both  $\widehat{f}_+$  and  $\mathcal{F}(e^+ w)$  are holomorphic in  $\mathbb{C}_-$  as functions of  $\xi_n$ ,  $\widehat{f}_+$  being  $O(\langle \xi_n \rangle^{-1+\varepsilon})$  and  $\mathcal{F}(e^+ w)$  being  $O(\langle \xi_n \rangle^{-1})$  on  $\mathbb{C}_-$ , whereby the integrand is  $O(\langle \xi_n \rangle^{-2+\varepsilon})$  there (and is in  $L_1$  on  $\mathbb{R}$ ); then the integral over  $\mathbb{R}$  can be transformed to a closed contour in  $\mathbb{C}_-$  and gives 0.

We can then conclude:

$$\int_{\mathbb{R}_+^n} \widehat{P}_2 u \partial_n \bar{u}' dx = \int_{\mathbb{R}_+^n} \Xi_-^{\mu'} \widehat{F}_+ \Xi_+^\mu u \partial_n \bar{u}' dx = (w_1, \partial_n w')_{L_2(\mathbb{R}_+^n)}. \quad (6.9)$$

The other contribution from  $\widehat{P}_2$  is, in conjugated form,

$$\begin{aligned} \int_{\mathbb{R}_+^n} \widehat{P}_2^* u' \partial_n \bar{u} dx &= \int_{\mathbb{R}_+^n} \Xi_-^{\mu'} \widehat{F}_+^* e^+ r^+ \Xi_+^{\mu'} u' \partial_n \bar{u} dx \\ &= \langle r^+ \Xi_-^{\mu'} \widehat{F}_+^* e^+ r^+ \Xi_+^{\mu'} u', \partial_n \bar{u} \rangle_{\overline{H}^{\frac{1}{2}-\text{Re}\mu+\varepsilon}, \dot{H}^{-\frac{1}{2}+\text{Re}\mu-\varepsilon}} \\ &= \langle r^+ \Xi_+^{\mu'} u', \widehat{F}_+ \Xi_+^\mu \partial_n \bar{u} \rangle_{\overline{H}^{\frac{1}{2}+\varepsilon}, \dot{H}^{-\frac{1}{2}-\varepsilon}} = \langle r^+ \Xi_+^{\mu'} u', \partial_n \widehat{F}_+ \Xi_+^\mu \bar{u} \rangle_{\overline{H}^{\frac{1}{2}+\varepsilon}, \dot{H}^{-\frac{1}{2}-\varepsilon}} \\ &= \langle r^+ \Xi_+^{\mu'} u', \partial_n \widehat{F}_+ e^+ w \rangle_{\overline{H}^{\frac{1}{2}+\varepsilon}, \dot{H}^{-\frac{1}{2}-\varepsilon}} = \langle w', \partial_n e^+ w_1 \rangle_{\overline{H}^{\frac{1}{2}+\varepsilon}, \dot{H}^{-\frac{1}{2}-\varepsilon}} \\ &= (\gamma_0 w', \gamma_0 w_1)_{L_2(\mathbb{R}^{n-1})} + (w', \partial_n w_1)_{L_2(\mathbb{R}_+^n)} = (w', \partial_n w_1)_{L_2(\mathbb{R}_+^n)}, \end{aligned}$$

where we used Proposition 6.1 in a similar way, and at the end used that  $\gamma_0 w_1 = 0$ , cf. (6.8)ff. Finally, taking the contributions from  $\widehat{P}_2$  together, we get

$$\begin{aligned} \int_{\mathbb{R}_+^n} \widehat{P}_2 u \partial_n \bar{u}' \, dx + \int_{\mathbb{R}_+^n} \partial_n u \overline{\widehat{P}_2^* u'} \, dx &= (w_1, \partial_n w')_{L_2(\mathbb{R}_+^n)} + (\partial_n w_1, w')_{L_2(\mathbb{R}_+^n)} \\ &= -(\gamma_0 w_1, \gamma_0 w')_{L_2(\mathbb{R}^{n-1})} = 0, \end{aligned}$$

using again that  $\gamma_0 w_1 = 0$ .

It is found in a similar way, using that  $\widehat{F}_-^*$  is of plus-type, that  $\widehat{P}_3$  contributes with zero. □

### 6.2. An integration by parts formula for the original $P$

To extend the formula to the original operator  $P$ , we shall show that  $P' = P - \widehat{P}$  (cf. Theorem 5.8 1°) gives a zero boundary contribution.

**Lemma 6.3.** *Let  $a > 0$  and let  $S = \text{Op}(s(\xi))$ , where  $s(\xi)$  is  $O(\langle \xi \rangle^{2a-1})$ . Then*

$$\int_{\mathbb{R}_+^n} S u \partial_n \bar{u}' \, dx + \int_{\mathbb{R}_+^n} \partial_n u \overline{S^* u'} \, dx = 0, \tag{6.10}$$

for any  $u, u' \in \dot{H}^a(\overline{\mathbb{R}_+^n})$ .

*Proof.* Since  $u \in \dot{H}^a(\overline{\mathbb{R}_+^n})$ ,  $r^+ S u \in \overline{H}^{1-a}(\mathbb{R}_+^n)$ ; moreover  $\partial_n u' \in \dot{H}^{a-1}(\overline{\mathbb{R}_+^n})$ , so we can write the first integral as

$$\langle r^+ S u, \partial_n u' \rangle_{\overline{H}^{1-a}(\mathbb{R}_+^n), \dot{H}^{a-1}(\overline{\mathbb{R}_+^n})}.$$

Approximate  $u'$  in  $\dot{H}^a(\overline{\mathbb{R}_+^n})$  by a sequence of functions  $\varphi_k \in C_0^\infty(\mathbb{R}_+^n)$ ,  $k \in \mathbb{N}$ ; then

$$\langle r^+ S u, \partial_n \varphi_k \rangle_{\overline{H}^{1-a}, \dot{H}^{a-1}} = -\langle r^+ \partial_n S u, \varphi_k \rangle_{\overline{H}^{-a}, \dot{H}^a} \rightarrow -\langle r^+ \partial_n S u, u' \rangle_{\overline{H}^{-a}, \dot{H}^a}.$$

With a similar argument for the second integral, we have

$$\begin{aligned} \langle r^+ S u, \partial_n u \rangle_{\overline{H}^{1-a}, \dot{H}^{a-1}} + \langle \partial_n u, r^+ S^* u' \rangle_{\dot{H}^{a-1}, \overline{H}^{1-a}} \\ &= -\langle r^+ \partial_n S u, u' \rangle_{\overline{H}^{-a}, \dot{H}^a} - \langle u, r^+ \partial_n S^* u' \rangle_{\dot{H}^a, \overline{H}^{-a}} \\ &= -\langle r^+ \partial_n S u, u' \rangle_{\overline{H}^{-a}, \dot{H}^a} + \langle u, r^+ (\partial_n S)^* u' \rangle_{\dot{H}^a, \overline{H}^{-a}} = 0, \end{aligned}$$

since  $\partial_n S^* = S^* \partial_n = -(\partial_n S)^*$ , and it is well-known that the operator  $S_1 = \partial_n S$  of order  $2a$  satisfies  $\langle r^+ S_1 u, u' \rangle = \langle u, r^+ S_1^* u' \rangle$  for  $u, u' \in \dot{H}^a(\overline{\mathbb{R}_+^n})$ . □

We can then conclude:

**Theorem 6.4.** *Let  $P, \mu, \mu'$  be as in Theorem 6.2. For  $u \in \mathcal{E}_\mu(\overline{\mathbb{R}_+^n}) \cap \mathcal{E}'(\mathbb{R}^n)$ ,  $u' \in \mathcal{E}_{\mu'}(\overline{\mathbb{R}_+^n}) \cap \mathcal{E}'(\mathbb{R}^n)$ , there holds*

$$\begin{aligned} \int_{\mathbb{R}_+^n} P u \partial_n \bar{u}' \, dx + \int_{\mathbb{R}_+^n} \partial_n u \overline{P^* u'} \, dx \\ &= \Gamma(\mu + 1) \Gamma(\mu' + 1) \int_{\mathbb{R}^{n-1}} s_0 \gamma_0(u/x_n^\mu) \gamma_0(\bar{u}'/x_n^{\mu'}) \, dx', \end{aligned} \tag{6.11}$$

where  $s_0 = e^{-i\pi\delta} p(0, 1)$ . The formula extends to  $u \in H^{\mu(t)}(\overline{\mathbb{R}_+^n})$ ,  $u' \in H^{\mu'(t')}(\overline{\mathbb{R}_+^n})$ , for  $t > \operatorname{Re} \mu + \frac{1}{2}$ ,  $t' > \operatorname{Re} \mu' + \frac{1}{2}$ , with  $t, t' \geq a$ .

The integrals over  $\mathbb{R}_+^n$  are interpreted as dualities (as in Proposition 6.1 and Lemma 6.3) when  $\operatorname{Re} \mu$  or  $\operatorname{Re} \mu' \leq 0$ , and when extended to general  $u, u'$ .

*Proof.* Recall that  $P = \widehat{P} + P'$ , where  $P' = \operatorname{Op}(p')$ ,  $|p'(\xi)| \leq C\langle \xi \rangle^{2a-1}$  (cf. Theorem 5.8 1°). We have the identities (6.10) with  $S = P'$  and (6.4) for  $u \in H^{\mu(t)}(\overline{\mathbb{R}_+^n})$ ,  $u' \in H^{\mu'(t')}(\overline{\mathbb{R}_+^n})$  with  $t, t' \geq a$ ,  $t > \operatorname{Re} \mu + \frac{1}{2}$ ,  $t' > \operatorname{Re} \mu' + \frac{1}{2}$ . Adding the identities for  $\widehat{P}$  and  $P'$  we obtain (6.11). It holds a fortiori for  $u \in \mathcal{E}_\mu(\overline{\mathbb{R}_+^n}) \cap \mathcal{E}'(\mathbb{R}^n)$ ,  $u' \in \mathcal{E}_{\mu'}(\overline{\mathbb{R}_+^n}) \cap \mathcal{E}'(\mathbb{R}^n)$ .  $\square$

**Example 6.5.** The theorem applies in particular to  $L = \operatorname{Op}(\mathcal{L}(\xi))$  studied in (3.5)–(3.6) and Example 5.9, showing that

$$\begin{aligned} \int_{\mathbb{R}_+^n} Lu \partial_n \bar{u}' dx + \int_{\mathbb{R}_+^n} \partial_n u \overline{L^* u'} dx \\ = \Gamma(\mu + 1)\Gamma(\mu' + 1) \int_{\mathbb{R}^{n-1}} |\mathcal{L}(0, 1)| \gamma_0(u/x_n^\mu) \gamma_0(\bar{u}'/x_n^{\mu'}) dx', \end{aligned} \tag{6.12}$$

The value  $s_0 = |\mathcal{L}(0, 1)| = (\mathcal{A}(0, 1)^2 + \mathcal{B}(0, 1)^2)^{\frac{1}{2}}$  is found in (3.17).

This result was proved in [5, Prop. 1.4] by completely different, real methods, for  $\mu \in ]0, 2a[ \cap ]2a - 1, 1[$ .

The result is one of the key ingredients in the proof of integration by parts formulas for operators  $L$  on bounded domains  $\Omega \subset \mathbb{R}^n$  in [5], where  $\mu(\nu)$  varies as the normal  $\nu$  varies along the boundary. It would be interesting to extend this knowledge to general strongly elliptic operators  $P$  on bounded domains by similar applications of Theorem 6.4.

**Example 6.6.** Here is an example of an application to a nonelliptic operator satisfying Assumption 3.2. Let

$$P = |\partial_1 + \partial_2|^m + |\partial_3|^{m-1} \partial_3, \text{ with symbol } p(\xi) = |\xi_1 + \xi_2|^m + i \operatorname{sign} \xi_3 |\xi_3|^m$$

on  $\mathbb{R}^3$ , for some  $1 < m < 2$ . For  $\mathbb{R}_+^3 = \{x_3 > 0\}$  we have the normal  $\nu = (0, 0, 1)$ , where

$$p(0, 0, 1) = i, \quad p(0, 0, -1) = -i, \text{ so (3.1) holds with } m - 2\mu = 1,$$

i.e.,  $\mu = (m - 1)/2$ ,  $\mu' = (m + 1)/2$ . Then by Theorem 6.4,

$$\int_{x_3 > 0} (Pu \bar{v} - u \overline{P^* v}) dx = \Gamma(\frac{m}{2} + \frac{1}{2})\Gamma(\frac{m}{2} + \frac{3}{2}) \int_{x_3=0} \gamma_0(\frac{u}{x_3^{(m-1)/2}}) \gamma_0(\frac{\bar{v}}{x_3^{(m+1)/2}}) dx',$$

for functions  $u \in x_3^{(m-1)/2} \mathcal{S}(\overline{\mathbb{R}_+^3})$ ,  $v \in x_3^{(m+1)/2} \mathcal{S}(\overline{\mathbb{R}_+^3})$ .

The halfspace  $\{x_2 > 0\}$  has the normal  $\nu' = (0, 1, 0)$  and

$$p(\nu') = 1, \quad p(-\nu') = 1, \text{ so (3.1) holds with } m - 2\mu = 0,$$

i.e.,  $\mu = m/2$ ,  $\mu' = m/2$ . Here by Theorem 6.4,

$$\int_{x_2 > 0} (Pu \bar{v} - u \overline{P^* v}) dx = \Gamma(\frac{m}{2} + 1)^2 \int_{x_2=0} \gamma_0(\frac{u}{x_2^{m/2}}) \gamma_0(\frac{\bar{v}}{x_2^{m/2}}) dx_1 dx_3,$$

for functions with a factor  $x_2^{m/2}$ .

## 7. Large solutions and a halfways Green's formula

### 7.1. Large solutions, a nonhomogeneous Dirichlet problem

Let  $P$  satisfy Assumption 3.1, and assume  $\operatorname{Re} \mu > 0$ . Along with the homogeneous Dirichlet problem (5.25), one can consider a nonhomogeneous local Dirichlet problem if the scope is expanded to allow so-called “large solutions”, behaving like  $x_n^{\mu-1}$  near the boundary of  $\mathbb{R}_+^n$ ; such solutions blow up at the boundary when  $\operatorname{Re} \mu < 1$ . Namely, one can pose the nonhomogeneous Dirichlet problem

$$r^+ P u = f \text{ on } \mathbb{R}_+^n, \quad \gamma_0(u/x_n^{\mu-1}) = \varphi \text{ on } \mathbb{R}^{n-1}, \quad \operatorname{supp} u \subset \overline{\mathbb{R}_+^n}. \quad (7.1)$$

Problem (7.1) was studied earlier for operators satisfying the  $a$ -transmission property in [8, 9] (including the fractional Laplacian  $(-\Delta)^a$ ), and a halfways Green's formula was shown in [11, Cor. 4.5]. The problem (7.1) for the fractional Laplacian, and the halfways Green's formula — with applications to solution formulas — were also studied in Abatangelo [1] (independently of [8]); the boundary condition there is given in a less explicit way except when  $\Omega$  is a ball. There have been further studies of such problems, see e.g., Abatangelo, Gomez-Castro and Vazquez [2] and its references.

Note that the boundary condition in (7.1) is *local*. There is a different problem which is also regarded as a nonhomogeneous Dirichlet problem, namely to prescribe nonzero values of  $u$  in the exterior of  $\Omega$ ; it has somewhat different solution spaces (a link between this and the homogeneous Dirichlet problem is described in [9]).

For the general operators  $P$  considered here, we shall now show that problem (7.1) has a good sense for  $u \in H^{(\mu-1)(t)}(\overline{\mathbb{R}_+^n})$  with suitable  $t$ .

More precisely, since  $P$  also satisfies the principal  $(\mu-1)$ -transmission condition (as remarked after Definition 2.1), Theorem 5.8 1° can be applied with  $\mu$  replaced by  $\mu-1$ , implying that  $r^+ P$  maps

$$r^+ P: H^{(\mu-1)(t)}(\overline{\mathbb{R}_+^n}) \rightarrow \overline{H}^{t-2a}(\mathbb{R}_+^n) \text{ for } \operatorname{Re} \mu - \frac{3}{2} < t < \operatorname{Re} \mu + \frac{1}{2}. \quad (7.2)$$

This is also valid in the case where  $P$  is only assumed to satisfy Assumption 3.2.

From Theorem 5.4 we have (note that  $\operatorname{Re} \mu - 1 > -1$ )

$$H^{(\mu-1)(t)}(\overline{\mathbb{R}_+^n}) \begin{cases} = \dot{H}^t(\overline{\mathbb{R}_+^n}) & \text{when } -\frac{3}{2} < t - \operatorname{Re} \mu < -\frac{1}{2}, \\ \subset \dot{H}^t(\overline{\mathbb{R}_+^n}) + x_n^{\mu-1} e^+ \overline{H}^{t-\operatorname{Re} \mu+1}(\mathbb{R}_+^n) & \text{when } -\frac{1}{2} < t - \operatorname{Re} \mu < \frac{1}{2}. \end{cases} \quad (7.3)$$

When  $t - \operatorname{Re} \mu > -\frac{1}{2}$ , the weighted boundary value is well-defined, cf. (5.20):

$$\gamma_0^{\mu-1} u \equiv \Gamma(\mu) \gamma_0(u/x_n^{\mu-1}) = \gamma_0(\Xi_+^{\mu-1} u) \in H^{t-\operatorname{Re} \mu+\frac{1}{2}}(\mathbb{R}^{n-1}). \quad (7.4)$$

The following regularity result holds for the nonhomogeneous Dirichlet problem:

**Theorem 7.1.** *Let  $P$  satisfy Assumption 3.1 with  $\operatorname{Re} \mu > 0$ , and let  $-\frac{1}{2} < t - \operatorname{Re} \mu < \frac{1}{2}$ . When  $f \in \overline{H}^{t-2a}(\mathbb{R}_+^n)$  and  $\varphi \in H^{t-\operatorname{Re} \mu+\frac{1}{2}}(\mathbb{R}^{n-1})$  are given, and  $u$  solves the nonhomogeneous Dirichlet problem (7.1) with  $u \in H^{(\mu-1)(\sigma)}(\overline{\mathbb{R}_+^n})$  for some  $-\frac{1}{2} < \sigma - \operatorname{Re} \mu < \frac{1}{2}$ , then in fact  $u \in H^{(\mu-1)(t)}(\overline{\mathbb{R}_+^n})$ .*

*Proof.* It is known from [8, Th. 6.1] that  $H^{\mu(\sigma)}(\overline{\mathbb{R}}_+^n)$  is a closed subspace of  $H^{(\mu-1)(\sigma)}(\overline{\mathbb{R}}_+^n)$ , equal to the set of  $v \in H^{(\mu-1)(\sigma)}(\overline{\mathbb{R}}_+^n)$  for which  $\gamma_0(v/x_n^{\mu-1}) = 0$ . From the given  $\varphi$  we define

$$w = \Gamma(\mu)\Xi_+^{-\mu+1}e^+K_0\varphi \in H^{(\mu-1)(t)}(\overline{\mathbb{R}}_+^n),$$

where  $K_0$  is the standard Poisson operator  $\varphi \mapsto K_0\varphi = \mathcal{F}_{\xi' \rightarrow x'}^{-1}[\hat{\varphi}(\xi')e^{-\langle \xi' \rangle x_n}]$ ,  $x_n > 0$ . Then in view of (7.4),

$$\gamma_0(w/x_n^{\mu-1}) = \Gamma(\mu)^{-1}\gamma_0(\Xi_+^{\mu-1}w) = \gamma_0(\Xi_+^{\mu-1}\Xi_+^{1-\mu}e^+K_0\varphi) = \gamma_0K_0\varphi = \varphi,$$

so that  $v = u - w$  solves (7.1) with  $f$  replaced by  $f - r^+Pw \in \overline{H}^{t-2a}(\mathbb{R}_+^n)$ ,  $\varphi$  replaced by 0. This is a homogeneous Dirichlet problem as in (5.25). Since  $v \in H^{(\mu-1)(\sigma)}(\overline{\mathbb{R}}_+^n)$  with  $\gamma_0(v/x_n^{\mu-1}) = 0$ , it is in  $H^{\mu(\sigma)}(\overline{\mathbb{R}}_+^n)$ . It then follows from Theorem 5.8 that  $v \in H^{\mu(t)}(\overline{\mathbb{R}}_+^n)$ , and hence  $u = v + w \in H^{(\mu-1)(t)}(\overline{\mathbb{R}}_+^n)$ .  $\square$

For the hatted version  $\widehat{P}$  there is even an existence and uniqueness result in these spaces.

**Theorem 7.2.** *Let  $P$  satisfy Assumption 3.1 with  $\operatorname{Re} \mu > 0$ , and let  $-\frac{1}{2} < t - \operatorname{Re} \mu < \frac{1}{2}$ . Then  $r^+\widehat{P}$  together with  $\gamma_0^{\mu-1}$  defines a homeomorphism:*

$$\{r^+\widehat{P}, \gamma_0^{\mu-1}\}: H^{(\mu-1)(t)}(\overline{\mathbb{R}}_+^n) \xrightarrow{\sim} \overline{H}^{t-2a}(\mathbb{R}_+^n) \times H^{t-\operatorname{Re} \mu + \frac{1}{2}}(\mathbb{R}^{n-1}). \quad (7.5)$$

*Proof.* The forward mapping properties are accounted for above. The existence of a unique solution  $u \in H^{(\mu-1)(t)}(\overline{\mathbb{R}}_+^n)$  of

$$r^+\widehat{P}u = f \text{ on } \mathbb{R}_+^n, \quad \gamma_0^{\mu-1}u = \varphi \text{ on } \mathbb{R}^{n-1}, \quad \operatorname{supp} u \subset \overline{\mathbb{R}}_+^n, \quad (7.6)$$

for given  $f \in \overline{H}^{t-2a}(\mathbb{R}_+^n)$ ,  $\varphi \in H^{t-\operatorname{Re} \mu + \frac{1}{2}}(\mathbb{R}^{n-1})$ , is shown as in Theorem 7.1, now referring to Theorem 5.5 instead of Theorem 5.8.  $\square$

These theorems show that  $H^{(\mu-1)(t)}(\overline{\mathbb{R}}_+^n)$  is the correct domain space for the nonhomogeneous Dirichlet problem, at least in the small range  $-\frac{1}{2} < t - \operatorname{Re} \mu < \frac{1}{2}$ . Recall that  $\mathcal{E}_{\mu-1}(\overline{\mathbb{R}}_+^n) \cap \mathcal{E}'(\mathbb{R}^n)$  and  $e^+x_n^{\mu-1}\mathcal{S}(\overline{\mathbb{R}}_+^n)$  are dense subsets of  $H^{(\mu-1)(t)}(\overline{\mathbb{R}}_+^n)$  for all  $t > \operatorname{Re} \mu - \frac{3}{2}$ .

### 7.2. An integration by parts formula involving the nontrivial Dirichlet trace

We now show a ‘‘halfways Green’s formula’’, where one factor  $u$  is in the domain of the nonhomogeneous Dirichlet problem for  $P$  and the other factor  $v$  is in the domain of the homogeneous Dirichlet problem for  $P^*$ :

**Theorem 7.3.** *Let  $P$  satisfy Assumption 3.2, and assume moreover that  $0 < \operatorname{Re} \mu < a + \frac{1}{2}$ ,  $\operatorname{Re} \mu' > 0$ .*

*For  $u \in \mathcal{E}_{\mu-1}(\overline{\mathbb{R}}_+^n) \cap \mathcal{E}'(\mathbb{R}^n)$  and  $v \in \mathcal{E}_{\mu'}(\overline{\mathbb{R}}_+^n) \cap \mathcal{E}'(\mathbb{R}^n)$ , there holds*

$$\int_{\mathbb{R}_+^n} Pu \bar{v} dx - \int_{\mathbb{R}_+^n} u \overline{P^*v} dx = -\Gamma(\mu)\Gamma(\mu' + 1) \int_{\mathbb{R}^{n-1}} s_0 \gamma_0(u/x_n^{\mu-1}) \gamma_0(\bar{v}/x_n^{\mu'}) dx', \quad (7.7)$$

where  $s_0 = e^{-in\delta}p(0, 1)$ . The formula extends to  $u \in H^{(\mu-1)(t)}(\overline{\mathbb{R}}_+^n)$  with  $t > \operatorname{Re} \mu - \frac{1}{2}$ ,  $v \in H^{\mu'(t')}\overline{(\mathbb{R}}_+^n)$  with  $t' > \operatorname{Re} \mu' + \frac{1}{2}$ .

The left-hand side is interpreted as follows, for small  $\varepsilon > 0$ :

$$\langle r^+Pu, v \rangle_{\overline{H}^{-\operatorname{Re} \mu' - \frac{1}{2} + \varepsilon}(\mathbb{R}_+^n), \dot{H}^{\operatorname{Re} \mu' + \frac{1}{2} - \varepsilon}(\overline{\mathbb{R}}_+^n)} - \langle u, P^*v \rangle_{\dot{H}^{\operatorname{Re} \mu - \frac{1}{2} - \varepsilon}(\overline{\mathbb{R}}_+^n), \overline{H}^{-\operatorname{Re} \mu + \frac{1}{2} + \varepsilon}(\mathbb{R}_+^n)}. \quad (7.8)$$

*Proof.* We shall show how the result can be derived from Theorem 6.4. Let  $u \in \mathcal{E}_{\mu-1}(\overline{\mathbb{R}}_+^n) \cap \mathcal{E}'(\mathbb{R}^n)$  and  $v \in \mathcal{E}_{\mu'}(\overline{\mathbb{R}}_+^n) \cap \mathcal{E}'(\mathbb{R}^n)$ . As shown in [8, p. 494], there exist functions  $U$  and  $u_1$  in  $\mathcal{E}_{\mu}(\overline{\mathbb{R}}_+^n) \cap \mathcal{E}'(\mathbb{R}^n)$  such that  $u = \partial_n U + u_1$ .

In terms of the Hilbert spaces: When  $u \in H^{(\mu-1)(t)}(\overline{\mathbb{R}}_+^n)$  with  $|t - \operatorname{Re} \mu| < \frac{1}{2}$ , let  $z = r^+ \Xi_+^{\mu-1} u \in \overline{H}^{t-\operatorname{Re} \mu+1}(\mathbb{R}_+^n)$ , then (denoting  $\operatorname{Op}(\langle \xi' \rangle) = \langle D' \rangle$ )

$$\begin{aligned} u &= \Xi_+^{-\mu+1} e^+ z = (\langle D' \rangle + \partial_n) \Xi_+^{-\mu} e^+ z = u_1 + \partial_n U, \text{ with} \\ u_1 &= \langle D' \rangle \Xi_+^{-\mu} e^+ z \in \langle D' \rangle H^{\mu(t+1)}(\overline{\mathbb{R}}_+^n) \subset H^{\mu(t)}(\overline{\mathbb{R}}_+^n), \\ U &= \Xi_+^{-\mu} e^+ z \in H^{\mu(t+1)}(\overline{\mathbb{R}}_+^n), \quad \partial_n U \in H^{\mu(t)}(\overline{\mathbb{R}}_+^n). \end{aligned} \quad (7.9)$$

Here  $H^{\mu(t)}(\overline{\mathbb{R}}_+^n) = \dot{H}^t(\overline{\mathbb{R}}_+^n)$  since  $|t - \operatorname{Re} \mu| < \frac{1}{2}$ . Moreover, when  $t = \operatorname{Re} \mu - \frac{1}{2} + \varepsilon$  for a small  $\varepsilon > 0$ , then

$$r^+ P u = r^+ P u_1 + r^+ P \partial_n U = r^+ P u_1 + \partial_n r^+ P U \quad (7.10)$$

where both terms are in  $\overline{H}^{t-2a}(\mathbb{R}_+^n) = \overline{H}^{\operatorname{Re} \mu - \frac{1}{2} + \varepsilon - 2a}(\mathbb{R}_+^n) = \overline{H}^{-\operatorname{Re} \mu' - \frac{1}{2} + \varepsilon}(\mathbb{R}_+^n)$ ; we here use Theorem 5.8 1°.

For  $v$ , we note that when  $v \in H^{\mu'(t')}(\overline{\mathbb{R}}_+^n)$  with  $t' = \operatorname{Re} \mu' + \frac{1}{2} + \varepsilon$ , then

$$\begin{aligned} v &\in H^{\mu'(t')}(\overline{\mathbb{R}}_+^n) = \Xi_+^{-\mu'} e^+ \overline{H}^{t'-\operatorname{Re} \mu'}(\mathbb{R}_+^n) \subset \Xi_+^{-\mu'} \dot{H}^{\frac{1}{2}-\varepsilon}(\overline{\mathbb{R}}_+^n) = \dot{H}^{\operatorname{Re} \mu' + \frac{1}{2}-\varepsilon}(\overline{\mathbb{R}}_+^n), \\ r^+ P^* v &\in \overline{H}^{t'-2a}(\mathbb{R}_+^n) = \overline{H}^{\operatorname{Re} \mu' + \frac{1}{2} + \varepsilon - 2a}(\mathbb{R}_+^n) = \overline{H}^{-\operatorname{Re} \mu + \frac{1}{2} + \varepsilon}(\mathbb{R}_+^n). \end{aligned}$$

Then the dualities in (7.8) are well-defined and serve as an interpretation of the left-hand side in (7.7).

The formula (7.7) will first be proved for  $u \in \mathcal{E}_{\mu-1}(\overline{\mathbb{R}}_+^n) \cap \mathcal{E}'(\mathbb{R}^n)$  and  $v \in \mathcal{E}_{\mu'}(\overline{\mathbb{R}}_+^n) \cap \mathcal{E}'(\mathbb{R}^n)$ , and afterwards be extended by continuity to general  $u, v$ . We use the decomposition (7.9), that leads to elements of  $\mathcal{E}_{\mu}(\overline{\mathbb{R}}_+^n)$  for  $t \rightarrow \infty$ . When  $u$  is supported in a ball  $\{|\xi| \leq R\}$ , we can cut  $u_1$  and  $U$  down to have support in  $\{|\xi| \leq 2R\}$ .

Consider the contribution from  $u_1$ . Here there holds

$$\langle r^+ P u_1, v \rangle_{\overline{H}^{-a}, \dot{H}^a} - \langle u_1, r^+ P^* v \rangle_{\dot{H}^a, \overline{H}^{-a}} = 0, \quad (7.11)$$

when  $u_1$  and  $v$  are in  $\dot{H}^a(\overline{\mathbb{R}}_+^n)$ , since  $P$  is of order  $2a$ . This gives the contribution 0 to (7.7) since  $t = a$  is allowed in the definition of  $u_1$  (recall that  $a > \operatorname{Re} \mu - \frac{1}{2}$  by hypothesis), and  $t' \geq a$  holds for the values of  $t'$  allowed in the definition of  $v$  (where  $t' > \operatorname{Re} \mu' + \frac{1}{2} = 2a - \operatorname{Re} \mu + \frac{1}{2} > 2a - a = a$ ). Thus  $u_1$  contributes to the boundary integral with 0.

For the contribution from  $\partial_n U$ , we note that, writing  $U = x_n^\mu w$  for  $x_n > 0$ ,  $w \in C^\infty(\overline{\mathbb{R}}_+^n)$ ,

$$\partial_n U = \partial_n(x_n^\mu w) = \mu x_n^{\mu-1} w + x_n^\mu \partial_n w \text{ for } x_n > 0,$$

so the weighted boundary value for  $x_n \rightarrow 0+$  satisfies (since  $x_n^\mu \partial_n w / x_n^{\mu-1} = x_n \partial_n w \rightarrow 0$ )

$$\gamma_0(\partial_n U / x_n^{\mu-1}) = \mu \gamma_0 w = \mu \gamma_0(U / x_n^\mu). \quad (7.12)$$

Moreover, by a simple integration by parts,

$$\langle r^+ P \partial_n U, v \rangle = \langle r^+ \partial_n P U, v \rangle = -\langle r^+ P U, \partial_n v \rangle,$$

since  $\gamma_0 v = 0$  because of  $\operatorname{Re} \mu' > 0$ . Thus, by use of Theorem 6.4 and (7.12),

$$\begin{aligned} \langle r^+ P \partial_n U, v \rangle - \langle \partial_n U, r^+ P^* v \rangle &= -\langle r^+ P U, \partial_n v \rangle - \langle \partial_n U, r^+ P^* v \rangle \\ &= -\Gamma(\mu + 1)\Gamma(\mu' + 1)s_0 \int_{\mathbb{R}^{n-1}} \gamma_0(U/x_n^\mu) \gamma_0(\bar{v}/x_n^{\mu'}) dx' \\ &= -\Gamma(\mu)\Gamma(\mu' + 1)s_0 \int_{\mathbb{R}^{n-1}} \gamma_0(\partial_n U/x_n^{\mu-1}) \gamma_0(\bar{v}/x_n^{\mu'}) dx'. \end{aligned}$$

Since  $u_1 \in \mathcal{E}_\mu(\overline{\mathbb{R}_+^n})$ ,  $\gamma_0(u_1/x_n^{\mu-1}) = 0$ , so  $u_1$  can be added to  $\partial_n U$  in the last integral. Adding also (7.11) to the left-hand side, we find (7.7).

Since the expressions depend continuously on  $u, v$  in the presented norms, the formula extends to the indicated spaces.  $\square$

**Example 7.4.** Theorems 7.1 and 7.2 apply in particular to the operator  $L$  considered in (3.5)–(3.6) and Examples 5.9 and 6.5, when  $\mu > 0$  (this holds automatically if  $a \geq \frac{1}{2}$ , since  $|\delta| < \frac{1}{2}$ ). Theorem 7.3 applies to  $L$  when  $\mu$  and  $\mu' > 0$  (again automatically satisfied when  $a \geq \frac{1}{2}$ ).

**Remark 7.5.** The transmission spaces can also be defined in terms of other scales of function spaces. The case of Bessel-potential spaces  $H_p^s$ ,  $1 < p < \infty$ , is a main subject in our preceding papers. There is also the Hölder-Zygmund scale  $C_*^s(\mathbb{R}^n)$ , coinciding with the Hölder scale  $C^s(\mathbb{R}^n)$  when  $s \in \mathbb{R}_+ \setminus \mathbb{N}$ , with spaces over  $\mathbb{R}_+^n$  defined as in (5.1). Here since  $C_*^{s+\varepsilon}(\mathbb{R}^n) \subset H^s(\mathbb{R}^n)$ , also  $C_*^{\mu(s+\varepsilon)}(\overline{\mathbb{R}_+^n}) \subset H^{\mu(s)}(\overline{\mathbb{R}_+^n})$  for  $\varepsilon > 0$ . (More details on such spaces in our earlier papers, e.g., in [12].) So the results dealing with forward mapping properties of  $r^+ P$  have useful consequences involving these spaces as well. Namely, Theorem 5.8 1° implies that  $r^+ P$  maps

$$r^+ P: C_*^{\mu(t+\varepsilon)}(\overline{\mathbb{R}_+^n}) \rightarrow \overline{H}^{t-2a}(\mathbb{R}_+^n) \text{ for } \operatorname{Re} \mu - \frac{1}{2} < t < \operatorname{Re} \mu + \frac{3}{2},$$

and the integration by parts formulas in Sections 6 and 7 hold for functions in  $C_*^{\mu(t)}$ -type spaces, for the same  $t$ .

In the opposite direction, an inclusion of an  $H^s$ -space in a Hölder spaces loses  $n/2$  in the regularity parameter, hence does not give very good results. For better regularity results, it would be interesting to extend the above theory to  $H_p^s$ -spaces with general  $1 < p < \infty$ , possibly under further hypotheses; this remains to be done. More smoothness than  $C^1$  is needed for a symbol  $q(\xi)$  to be a Fourier multiplier in  $L_p$  (some well-known conditions are recalled in [15, Sect. 1.3]). There is an extension of Vishik and Eskin's work to  $L_p$ -based spaces by Shargorodsky [21], which should be useful. It is there pointed out that [6, Lemma 17.1] shows how smoothness properties carry over to the factors in the Wiener-Hopf factorization.

## Conflict of interest

The author declares no conflict of interest.

## References

1. N. Abatangelo, Large  $s$ -harmonic functions and boundary blow-up solutions for the fractional Laplacian, *Discrete Contin. Dyn. Syst.*, **35** (2015), 5555–5607.

2. N. Abatangelo, D. Gómez-Castro, J. L. Vázquez, Singular boundary behaviour and large solutions for fractional elliptic equations, *arXiv:1910.00366*.
3. L. Boutet de Monvel, Comportement d'un opérateur pseudo-différentiel sur une variété à bord, I–II, *J. Anal. Math.*, **17** (1966), 241–304.
4. L. Boutet de Monvel, Boundary problems for pseudo-differential operators, *Acta Math.*, **126** (1971), 11–51.
5. S. Dipierro, X. Ros-Oton, J. Serra, E. Valdinoci, Non-symmetric stable operators, regularity theory and integration by parts, *arXiv:2012.04833*.
6. G. Eskin, *Boundary value problems for elliptic pseudodifferential equations*, Providence, R.I.: Amer. Math. Soc., 1981.
7. G. Grubb, *Distributions and operators*, New York: Springer, 2009.
8. G. Grubb, Fractional Laplacians on domains, a development of Hörmander's theory of  $\mu$ -transmission pseudodifferential operators, *Adv. Math.*, **268** (2015), 478–528.
9. G. Grubb, Local and nonlocal boundary conditions for  $\mu$ -transmission and fractional elliptic pseudodifferential operators, *Anal. PDE*, **7** (2014), 1649–1682.
10. G. Grubb, Integration by parts and Pohozaev identities for space-dependent fractional-order operators, *J. Differ. Equations*, **261** (2016), 1835–1879.
11. G. Grubb, Green's formula and a Dirichlet-to-Neumann operator for fractional-order pseudodifferential operators, *Commun. Part. Diff. Eq.*, **43** (2018), 750–789.
12. G. Grubb, Limited regularity of solutions to fractional heat and Schrödinger equations, *Discrete Contin. Dyn. Syst.*, **39** (2019), 3609–3634.
13. G. Grubb, Integration by parts for nonsymmetric fractional-order operators on a halfspace, *J. Math. Anal. Appl.*, **499** (2021), 125012.
14. G. Grubb, Corrigendum to: "Integration by parts for nonsymmetric fractional-order operators on a halfspace" [*J. Math. Anal. Appl.* 499 (2021) 125012], *J. Math. Anal. Appl.*, **505** (2022), 125462.
15. G. Grubb, N. J. Kokholm, A global calculus of parameter-dependent pseudodifferential boundary problems in  $L_p$  Sobolev spaces, *Acta Math.*, **171** (1993), 165–229.
16. L. Hörmander, *Seminar notes on pseudo-differential operators and boundary problems*, Lectures at IAS Princeton 1965-66. Available from: <https://lup.lub.lu.se/search/>.
17. L. Hörmander, *The analysis of linear partial differential operators, III*, Berlin: Springer Verlag, 1985.
18. S. Rempel, B. W. Schulze, *Index theory of elliptic boundary problems*, Berlin: Akademie Verlag, 1982.
19. X. Ros-Oton, Boundary regularity, Pohozaev identities and nonexistence results, In: *Recent developments in nonlocal theory*, Berlin, Boston: De Gruyter, 2018, 335–358.
20. X. Ros-Oton, J. Serra, The Pohozaev identity for the fractional Laplacian, *Arch. Rational Mech. Anal.*, **213** (2014), 587–628.
21. E. Shargorodsky, An  $L_p$ -analogue of the Vishik-Eskin theory, *Memoirs on Differential Equations and Mathematical Physics*, **2** (1994), 41–146.



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22. N. Wiener, E. Hopf, Über eine Klasse singulärer Integralgleichungen, *Sitzungsber. Preuss. Acad. Wiss.* **31** (1931), 696–706.



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